Thin distance-regular graphs with classical parameters $(D, q, q, \frac{q^t-1}{q-1} - 1)$ with $t > D$ are the Grassmann graphs

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Abstract

In a survey paper by Van Dam, Koolen and Tanaka (2016), it was asked to classify the thin Q-polynomial distance-regular graphs. In this paper, we show that a thin distance-regular graph with the same intersection numbers as a Grassmann graph $J_q(n, D)$ ($n \geqslant 2D$) is the Grassmann graph if D is large enough.

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1 Introduction

A finite connected graph Γ with vertex set $V(\Gamma)$ and path-length distance function ∂ is called *distance-regular* if, for any vertices $x, y \in V(\Gamma)$ and any non-negative integers i, j, the number p^h_{ij} of vertices at distance i from x and distance j from y depends only on i, j and $h := \partial(x, y)$, and does not depend on the particular choice of x and y. The numbers p^h_{ij} are called the *intersection numbers* of Γ.

A distance-regular graph Γ of diameter D ($D := \max\{\partial(x, y) \mid x, y \in V(\Gamma)\}$) is said to have *classical parameters* (D, b, α, β) if its intersection numbers can be expressed in terms of these four classical parameters (see Subsection 2.4).

Let $\mathbb{F}_q$ be the finite field with q elements and V be the vector space of dimension $n \geqslant 2$ over $\mathbb{F}_q$. For an integer D, $1 \leqslant D \leqslant n-1$, let $\mathcal{G}_D$ denote the set of all D-dimensional subspaces of V. The *Grassmann graph* $J_q(n, D)$ has $\mathcal{G}_D$ as the vertex set with two vertices being adjacent if and only if they intersect in a subspace of dimension $D-1$. Note that the graphs $J_q(n, D)$ and $J_q(n, n-D)$ are isomorphic (an isomorphism defined by mapping each subspace to its orthogonal complement). Without loss of generality, we further assume that $n \geqslant 2D$. The Grassmann graph $J_q(n, D)$ is a distance-regular graph with classical parameters $(D, q, q, \frac{q^{n-D+1}-1}{q-1} - 1)$.

The main result of this paper is as follows. We define *μ-graph-regular* in Subsection 2.1 and 1-*thin*, *thin* in Subsection 2.3. For a natural number $q \geqslant 2$, define the function $\chi(q)$ by:

$$\chi(q) = \begin{cases} 13 & \text{if } q = 2, \\ 10 & \text{if } q = 3, \\ 9 & \text{if } q = 4, \\ 8 & \text{if } q \in \{5, 6, 7\}, \\ 7 & \text{if } q \geqslant 8. \end{cases} \tag{1}$$

Theorem 1. *Let Γ be a 1-thin distance-regular graph with classical parameters $(D, q, q, \frac{q^t-1}{q-1} - 1)$ with $q \geqslant 2$, $t > D$ integers. Assume further that Γ is μ-graph-regular (with parameter ℓ). If $D \geqslant \chi(q)$, then Γ is the Grassmann graph $J_q(D + t - 1, D)$.*

As a thin distance-regular graph with classical parameters (D, b, α, β) and $D \geqslant 5$ is μ-graph-regular (see Lemma 8), we obtain the following corollary immediately.

Corollary 2 ([16, Lecture 40])**.** *Let Γ be a thin distance-regular graph with classical parameters $(D, q, q, \frac{q^t-1}{q-1} - 1)$ with $q \geqslant 2$, $t > D$ integers. If $D \geqslant \chi(q)$, then Γ is the Grassmann graph $J_q(D + t - 1, D)$.*

Remark 3.

(i) The twisted Grassmann graph $\tilde{J}_q(2D+1, D)$, see [6], have the same intersection numbers as the Grassmann graph $J_q(2D + 1, D)$. The Terwilliger algebra $\mathcal{T}(x)$ of the twisted Grassmann graph $\tilde{J}_q(2D + 1, D)$ depends on the base vertex x. For certain base vertices x, $\mathcal{T}(x)$ is thin and for other base vertices x, $\mathcal{T}(x)$ is not even 1-thin.

(ii) In the survey paper by Van Dam, Koolen and Tanaka [7, Problem 59], it was asked to classify the thin Q-polynomial distance-regular graphs. This paper shows that the Grassmann graphs with large diameter are characterized by their intersection numbers as thin distance-regular graphs. Note that in Koolen, Lee and Tan [13], they discussed a slightly more restricted problem.

Metsch [14], Gavrilyuk and Koolen [8] showed that the Grassmann graph $J_q(n, D)$ is uniquely determined by its intersection numbers in many cases. To state the results, we need to define the function $\xi(q)$ as follows. For a natural number $q \geqslant 2$, the function $\xi(q)$ is defined by:

$$\xi(q) = \begin{cases} 9 & \text{if } q = 2, \\ 8 & \text{if } q = 3, \\ 7 & \text{if } q \in \{4, 5, 6\}, \\ 6 & \text{if } q \geqslant 7. \end{cases}$$

Theorem 4. *Let Γ be a distance-regular graph with classical parameters $(D, q, q, \frac{q^t-1}{q-1} - 1)$ with $q \geqslant 2$, $t > D$ integers.*

(1) (Metsch [14]) *If $t \geqslant \max\{D + 3, D + 7 - q\}$ and $D \geqslant 3$, then Γ is the Grassmann graph $J_q(D + t - 1, D)$.*

(2) (Gavrilyuk and Koolen [8]) *If $t = D + 1$ and $D \geqslant \xi(q)$, then Γ is the Grassmann graph $J_q(2D, D)$.*

Therefore, in view of Theorem 4, in order to show Theorem 1, it suffices to show the following result.

Theorem 5. *Let Γ be a 1-thin distance-regular graph with classical parameters $(D, q, q, \frac{q^{D+e+1}-1}{q-1} - 1)$, where $q \geqslant 2$ and $e \in \{1, 2, 3\}$ are integers. Assume further that Γ is μ-graph-regular (with parameter ℓ). If $D \geqslant \chi(q)$, then Γ is the Grassmann graph $J_q(2D + e, D)$.*

This paper is organized as follows. In Section 2, we give the definitions and preliminaries. In Section 3, we give some spectral characterizations of the s-clique extension of the $(t_1 \times t_2)$-grid. We will use those results later in the paper to show the main result. In Section 4, we prepare for the proof of Theorem 5. Of particular interest is a sufficient condition for a distance-regular graph to contain a Delsarte clique. In Section 5, we give a proof of Theorem 5.

2 Definitions and preliminaries

The main purpose of this section is to recall some basic terminologies and notation from algebraic graph theory and algebraic combinatorics. For more comprehensive background on distance-regular graphs and the Terwilliger algebra, we refer the reader to [2], [7] and [17].

2.1 Graphs and their eigenvalues

All graphs considered in this paper are finite, undirected and simple. Let Γ be a graph with the vertex set $V(\Gamma)$. For two distinct vertices x and y, we write $x \sim y$ if they are adjacent to each other. Assume that Γ is connected. The *distance* $\partial(x, y)$ between two vertices $x, y \in V(\Gamma)$ is the length of a shortest path between x and y of Γ. By *diameter* of Γ, denoted by $D := D(\Gamma)$, we mean the maximum distance between any two vertices of Γ. For each vertex x of Γ, let $\Gamma_i(x)$ be the set of vertices of Γ at distance i from x for $0 \leqslant i \leqslant D$. For the sake of simplicity, we denote $\Gamma_1(x)$ by $\Gamma(x)$. The subgraph induced on $\Gamma(x)$ is called the *local graph* of Γ at x, denoted by $\Delta(x)$, and the number $|\Gamma(x)|$ is called the *valency* of x in Γ. In particular, Γ is *regular with valency* k (or k*-regular*) if $k = |\Gamma(x)|$ holds for all $x \in V(\Gamma)$.

A k-regular graph Γ with v vertices is called *edge-regular* with parameters (v, k, a) if any two adjacent vertices have exactly a common neighbors, and is called *co-edge-regular* with parameters (v, k, c), if any two nonadjacent vertices have precisely c common neighbors.

For two vertices x and y of a graph Γ with $\partial(x, y) = 2$, the subgraph induced on $\Gamma(x) \cap \Gamma(y)$ is called the $\mu(x, y)$*-graph* of Γ. If it does not depend on the choice of x and y, then we call it the μ*-graph*. If each $\mu(x, y)$-graph is a regular graph with valency ℓ, then we say that Γ is μ*-graph-regular* (with parameter ℓ).

Lemma 6. *Let Γ be a graph that is edge-regular with parameters (v, k, a) and μ-graph-regular with parameter ℓ. Then, for any vertex x of Γ, the local graph $\Delta(x)$ of Γ at x is co-edge-regular with parameters (k, a, ℓ).*

Proof. Fix a vertex x of Γ and let y, z be distinct non-adjacent vertices of the local graph $\Delta(x)$. Now by the definition of μ-graph-regularity we have $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)| = \ell$, see Figure 1. This means that in $\Delta(x)$ the vertices y and z have exactly ℓ common neighbors. □

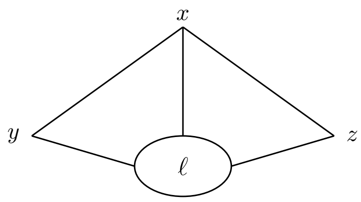

Figure 1: $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)| = \ell$

By the *eigenvalues* of a graph Γ, we mean the eigenvalues of its adjacency matrix $A := A(\Gamma)$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ be the distinct eigenvalues of A and m_i be the multiplicity of θ_i $(0 \leqslant i \leqslant D)$. Then the set $\{[\theta_0]^{m_0}, [\theta_1]^{m_1}, \ldots, [\theta_D]^{m_D}\}$ is called the *spectrum* of Γ. Remark that two graphs are called *cospectral*, if they have the same spectrum. For an eigenvalue θ of Γ, if its eigenspace contains a vector orthogonal to the all-ones vector, then we say that θ is *non-principal.* If Γ is a k-regular graph, then all its eigenvalues are non-principal unless the graph is connected and then the only principal eigenvalue is k.

We next recall the so-called *interlacing* in the following lemma.

Lemma 7 ([4, Section 2.5]). *Let N be a real symmetric $n \times n$ matrix with eigenvalues $\theta_1 \geqslant \ldots \geqslant \theta_n$. For some $m < n$, let R be a real $n \times m$ matrix with orthogonal columns i.e. $R^\top R = I$,and set $M = R^\top NR$ with eigenvalues $\eta_1 \geqslant \ldots \geqslant \eta_m$. Then the eigenvalues of M interlace those of N, that is, $\theta_i \geqslant \eta_i \geqslant \theta_{n-m+i}$, $i = 1, \ldots, m$.*

The *complement* $\overline{\Gamma}$ of a graph Γ is the graph with the same vertex set as Γ, where two distinct vertices are adjacent whenever they are nonadjacent in Γ. So, if Γ has the adjacency matrix A, then the adjacency matrix of $\overline{\Gamma}$ is $\overline{A} = J - I - A$, where J is the all-ones matrix and I is the identity matrix. If Γ is a k-regular graph with v vertices and eigenvalues $\theta_0 = k \geqslant \theta_1 \geqslant \ldots \geqslant \theta_v$, then the eigenvalues of the complement $\overline{\Gamma}$ are $v - k - 1, -1 - \theta_v, \ldots, -1 - \theta_1$.

A graph is called *clique* (or *complete*) if any two of its vertices are adjacent, and is called *coclique* (or *empty*) if any two of its vertices are nonadjacent.

Let Γ be a k-regular graph with v vertices and smallest eigenvalue $\theta_{\min}$. Then the order of a coclique C of Γ is bounded by

$$|V(C)| \leqslant \frac{v}{1 - \frac{k}{\theta_{\min}}}. \tag{2}$$

Moreover, equality implies that every vertex outside C is adjacent to exactly $-\theta_{\min}$ vertices of C (cf. [2, Proposition 1.3.2 and Proposition 3.7.2]). We call the bound (2) the *Hoffman bound*. Note that a coclique of Γ is a clique of $\overline{\Gamma}$, the complement of Γ, so this bound holds for a clique in the complement of Γ.

2.2 Distance-regular graphs and the Bose-Mesner algebra

A connected graph Γ of diameter D is said to be *distance-regular* if and only if, for all integers h, i, j with $0 \leqslant h, i, j \leqslant D$ and all vertices $x, y \in V(\Gamma)$ with $\partial(x, y) = h$, the number

$$p^h_{ij} := |\{z \in V(\Gamma) \mid \partial(x, z) = i,\ \partial(y, z) = j\}| = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent on the choice of x and y. The constants p^h_{ij} are called the *intersection numbers* of Γ. We abbreviate $c_i = p^i_{1i-1}$ $(1 \leqslant i \leqslant D)$, $a_i = p^i_{1i}$ $(0 \leqslant i \leqslant D)$, and $b_i = p^i_{1i+1}$ $(0 \leqslant i \leqslant D-1)$. Observe that Γ is regular with valency $k = b_0$, and $c_i + a_i + b_i = k$ for $0 \leqslant i \leqslant D$, where we define $c_0 = b_D = 0$. The array $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ is called the *intersection array* of the distance-regular graph Γ.

Let Γ be a distance-regular graph of diameter D. For each integer i with $0 \leqslant i \leqslant D$, define the *ith distance matrix* A_i of Γ whose rows and columns are indexed by the vertices of Γ, by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i, \end{cases} \quad (x, y \in V(\Gamma)).$$

Then $A := A_1$ is the adjacency matrix of Γ. Observe that $A_0 = I$; $A_i^\top = A_i$ $(0 \leqslant i \leqslant D)$; $\sum_{i=0}^{D} A_i = J$, the all-ones matrix; and

$$A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \quad (0 \leqslant i, j \leqslant D).$$

By these facts, we find that $\{A_0, A_1, \ldots, A_D\}$ is a basis for a commutative subalgebra $\mathcal{M}$ of the matrix algebra over $\mathbb{R}$. We call $\mathcal{M}$ the *Bose-Mesner algebra* of Γ. It is known that A generates $\mathcal{M}$. Since the algebra $\mathcal{M}$ is semi-simple and commutative, $\mathcal{M}$ also has a basis of pairwise orthogonal idempotents $E_0,\ E_1, \ldots,\ E_D$ (the so-called *primitive idempotents* of $\mathcal{M}$) satisfying

$$E_0 = \frac{1}{|V(\Gamma)|} J, \quad \sum_{i=0}^{D} E_i = I, \quad E_i^\top = E_i, \quad E_i E_j = \delta_{ij} E_i \quad (0 \leqslant i, j \leqslant D),$$

where δ_{ij} is the Kronecker delta. Since $\mathcal{M}$ has two bases $\{A_i\}_{i=0}^D$ and $\{E_i\}_{i=0}^D$, there are real scalars $\{\theta_j\}_{j=0}^D$ such that

$$A = \sum_{j=0}^{D} \theta_j E_j.$$

Observe that θ_j, $0 \leqslant j \leqslant D$ are exactly the distinct eigenvalues of A (of Γ), since $AE_j = E_j A = \theta_j E_j$.

At the end of this subsection, we recall the Delsarte bound in distance-regular graphs (cf. [2, Proposition 4.4.6]). Let Γ be a distance-regular graph of diameter $D \geqslant 2$ with distinct eigenvalues $\theta_0 = k > \theta_1 > \cdots > \theta_D$. A clique C of Γ contains at most $1 - \frac{k}{\theta_D}$ vertices, i.e.

$$|V(C)| \leqslant 1 - \frac{k}{\theta_D}. \tag{3}$$

If a clique C has order that meets the bound with equality, we call it a *Delsarte clique* of Γ, and the bound is called the *Delsarte bound*.

2.3 Q-polynomial distance-regular graphs and the Terwilliger algebra

Let Γ be a distance-regular graph of diameter D with the Bose-Mesner algebra $\mathcal{M}$. Let $\circ$ denote the entrywise (or Hadamard or Schur) matrix multiplication. Since $A_i \circ A_j = \delta_{ij} A_i$ $(0 \leqslant i, j \leqslant D)$, the Bose-Mesner algebra $\mathcal{M}$ is closed under $\circ$. As $\{E_i\}_{i=0}^D$ is a basis for $\mathcal{M}$, there are real scalars q_{ij}^h such that

$$E_i \circ E_j = \frac{1}{|V(\Gamma)|} \sum_{h=0}^{D} q_{ij}^h E_h, \quad (0 \leqslant i, j \leqslant D).$$

In fact, the scalars q_{ij}^h are nonnegative, which are called the *Krein parameters* of Γ (cf.[2, p.49]). We say Γ is *Q-polynomial* (with respect to the ordering $E_0, E_1, \ldots, E_D$ or equivalently with respect to the ordering $\theta_0, \theta_1, \ldots, \theta_D$) if for all integers $0 \leqslant h, i, j \leqslant D$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two.

Assume now that Γ is a Q-polynomial distance-regular graph of diameter D. Fix a (base) vertex $x \in V(\Gamma)$, and for each i $(0 \leqslant i \leqslant D)$, define $E_i^* := E_i^*(x)$ to be the diagonal matrix whose rows and columns are indexed by $V(\Gamma)$, by

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i, \end{cases} \quad (y \in V(\Gamma)).$$

Observe that $\sum_{i=0}^{D} E_i^* = I$ and $E_i^* E_j^* = \delta_{ij} E_i^*$ for $0 \leqslant i, j \leqslant D$. Hence $E_0^*, E_1^*, \ldots, E_D^*$ is a basis for a commutative subalgebra $\mathcal{M}^* := \mathcal{M}^*(x)$ of the matrix algebra over $\mathbb{R}$, which is called the *dual Bose-Mesner algebra* with respect to the (base) vertex x of Γ. The matrix algebra generated by the Bose-Mesner algebra $\mathcal{M}$ and the dual Bose-Mesner algebra $\mathcal{M}^*$ is called the *Terwilliger (or subconstituent) algebra with respect to* x, denoted by $\mathcal{T} := \mathcal{T}(x)$. Note that the Terwilliger algebra $\mathcal{T}$ depends on the choice of base vertex x and it is semi-simple.

Let $\mathbf{V} = \mathbb{R}^{V(\Gamma)}$ denote the vector space over $\mathbb{R}$ of columns whose coordinates are indexed by $V(\Gamma)$, and endowed with the inner product $\langle , \rangle$, where $\langle u, v \rangle = u^\top v$ for all $u, v \in \mathbf{V}$. A *$\mathcal{T}$-module* W is a subspace of $\mathbf{V}$ such that $Tw \in W$ for any $T \in \mathcal{T}$ and $w \in W$. A $\mathcal{T}$-module W is called *irreducible* if it is non-zero, and contains no $\mathcal{T}$-submodule besides $0, W$. Since $\mathcal{T}$ is semi-simple, each $\mathcal{T}$-module is an orthogonal direct sum of irreducible $\mathcal{T}$-modules, and $\mathbf{V}$ decomposes into an orthogonal direct sum of irreducible $\mathcal{T}$-modules (cf. [17]).

Let W be an irreducible $\mathcal{T}$-module. With respect to the ordering $E_0^*, E_1^*, \ldots, E_D^*$ (corresponding to the ordering $A_0, A_1, \ldots, A_D$), we define the *endpoint* of W by $\min\{i \mid E_i^* W \neq 0\}$, and the *diameter* of W by $|\{i \mid E_i^* W \neq 0\}| - 1$. An irreducible $\mathcal{T}$-module W is said to be *thin* if $\dim E_i^* W \leqslant 1$ for all i $(0 \leqslant i \leqslant D)$. Note that there is a unique irreducible $\mathcal{T}$-module of endpoint 0, called the *principal $\mathcal{T}$-module*, which is thin and has basis $\{E_i^* \mathbf{1} \mid 0 \leqslant i \leqslant D\}$, where $\mathbf{1}$ is the all-ones vector. The graph Γ is called *i-thin* if, for any vertex x of Γ, each irreducible $\mathcal{T}(x)$-module of endpoint at most i is thin. The graph Γ is called *thin* if it is i-thin for all i $(0 \leqslant i \leqslant D)$ (cf. [7, Section 4.3]).

We now recall some facts about irreducible $\mathcal{T}$-modules of endpoint 1, see [16]. Keeping in mind the notation from the above, let us denote $\tilde{A} := E_1^* A E_1^*$. For notational convenience, we also set $\tilde{A}^0 = E_1^*$ and $\tilde{J} := E_1^* J E_1^*$. With an appropriate ordering of the vertices of Γ, one can see that

$$\tilde{A} = \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix},$$

where the principal submatrix N is, in fact, the adjacency matrix of $\Delta(x)$, the local graph at x of Γ.

Let U_1^* be the subspace of $E_1^*\mathbf{V}$, which is orthogonal to the all-ones vector $\mathbf{1}$. Let W be an irreducible $\mathcal{T}$-module of endpoint 1. Then E_1^*W is a one-dimensional subspace of U_1^*. (Note that E_1^*W always has dimension 1 even if W is not thin (see [9, Theorem 4.5])). In particular, any non-zero vector $w \in E_1^*W$ is an eigenvector of $\tilde{A}$, and $W = \mathcal{T}w$. Conversely, for an eigenvector w of $\tilde{A}$ with $E_1^*w \neq 0$, the subspace $W = \mathcal{T}w$ is an irreducible $\mathcal{T}$-module of endpoint 1. Let $a_0(W)$ denote the corresponding eigenvalue of $\tilde{A}$. Note that $a_0(W)$ is a non-principal eigenvalue of the local graph $\Delta(x)$ at x of Γ.

The following essential lemma says that a thin Q-polynomial distance-regular graph of diameter $D \geqslant 5$ is μ-graph-regular, and hence each local graph is co-edge-regular, according to Lemma 6.

Lemma 8 ([16, Lecture 40])**.** *Let Γ be a thin Q-polynomial distance-regular graph of diameter $D \geqslant 5$. Then Γ is μ-graph-regular.*

Note that Corollary 2 follows immediately from Theorem 1 by the above lemma.

2.4 Distance-regular graphs with classical parameters

Recall that the *q-ary Gaussian binomial coefficient* is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-m+1} - 1)}{(q^m - 1)(q^{m-1} - 1) \cdots (q - 1)}. \tag{4}$$

We say that a distance-regular graph Γ of diameter D has *classical parameters* (D, b, α, β) if the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b (1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b), \tag{5}$$

$$b_i = (\begin{bmatrix} D \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b)(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b), \tag{6}$$

where $\begin{bmatrix} j \\ 1 \end{bmatrix}_b = 1 + b + b^2 + \cdots b^{j-1}$ for $j \geqslant 1$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}_b = 0$. We notice that $b \neq 0, -1$ by the following result.

Lemma 9 ([2, Proposition 6.2.1])**.** *Let Γ be a distance-regular graph with classical parameters (D, b, α, β) and the diameter $D \geqslant 3$. Then b is an integer such that $b \neq 0, -1$.*

We remark that, if Γ is a distance-regular graph with classical parameters (D, b, α, β), then Γ is Q-polynomial, see [2, Corollary 8.4.2]. We next recall some facts about the local graphs of a distance-regular graph with classical parameters.

Proposition 10 ([8, Theorem 3.3])**.** *Let Γ be a distance-regular graph with classical parameters (D, b, α, β), diameter $D \geqslant 3$ and $b \neq 1$. For $2 \leqslant i \leqslant D - 1$, let $T_i(\zeta)$ be a polynomial of degree 4 defined by*

$$T_i(\zeta) = -(b^i - 1)(b^{i-1} - 1)(\zeta - \beta + \alpha + 1)(\zeta + 1)(\zeta + b + 1)(\zeta - \alpha b\frac{b^{D-1} - 1}{b - 1} + 1).$$

Then, for each vertex x of Γ and a non-principal eigenvalue η of its local graph $\Delta(x)$, $T_i(\eta) \geqslant 0$ holds.

Note that $T_i(\zeta)$ is independent of i up to a scalar multiple ($2 \leqslant i \leqslant D-1$) and is called the *Terwilliger polynomial* of Γ. Actually Proposition 10 was first shown by Terwilliger in his "Lecture note on Terwilliger algebra" (edited by Suzuki) [16]. The explicit formula of the Terwilliger polynomial was given in [9]. Also note that, for any $x \in V(\Gamma)$, $T_i(\eta) = 0$ if and only if $W := \mathcal{T}(x)w$ is a thin irreducible $\mathcal{T}(x)$-module of endpoint 1, where w is an eigenvector of $\tilde{A} = E_1^*(x)AE_1^*(x)$ with eigenvalue $\eta = a_0(W)$.

The following lemma shows that all possible non-principal eigenvalues of any local graph of Γ are the roots of a Terwilliger polynomial of Γ, and it will play a key role in this paper.

Lemma 11 ([9]). *Let Γ be a 1-thin distance-regular graph with classical parameters (D, b, α, β), diameter $D \geqslant 3$ and $b \neq 1$. Then the possible non-principal eigenvalues of any local graph of Γ are*

$$\beta - \alpha - 1, \quad -1, \quad -b-1, \; \alpha b \frac{b^{D-1}-1}{b-1} - 1. \tag{7}$$

Note that these possible non-principal eigenvalues of the local graph $\Delta(x)$ at x of Γ corresponding to thin irreducible $\mathcal{T}(x)$-modules of endpoint 1 are the roots of Terwilliger polynomial $T_i(\zeta)$ for all i ($2 \leqslant i \leqslant D-1$).

At the end of this subsection, we mention some facts about the classical parameters for a Grassmann graph. By [2, Table 6.1, Theorem 9.3.3] or [8, Result 2.5], we have the following lemma.

Lemma 12 ([8, Result 2.5]). *A Grassmann graph $J_q(n, D)$, $n \geqslant 2D$, has classical parameters*

$$(D, b, \alpha, \beta) = (D, q, q, \begin{bmatrix} n-D+1 \\ 1 \end{bmatrix}_q - 1). \tag{8}$$

A distance-regular graph with these classical parameters has intersection array given by

$$b_j = q^{2j+1} \begin{bmatrix} n-D-j \\ 1 \end{bmatrix}_q \begin{bmatrix} D-j \\ 1 \end{bmatrix}_q, \quad 0 \leqslant j \leqslant D-1,$$
$$c_j = \begin{bmatrix} j \\ 1 \end{bmatrix}_q^2, \quad 1 \leqslant j \leqslant D,$$

and its eigenvalues and their respective multiplicities are given by

$$\theta_j = q^{j+1} \begin{bmatrix} n-D-j \\ 1 \end{bmatrix}_q \begin{bmatrix} D-j \\ 1 \end{bmatrix}_q - \begin{bmatrix} j \\ 1 \end{bmatrix}_q, \quad 0 \leqslant j \leqslant D,$$
$$m_j = \begin{bmatrix} n \\ j \end{bmatrix}_q - \begin{bmatrix} n \\ j-1 \end{bmatrix}_q, \quad 0 \leqslant j \leqslant D.$$

2.5 Partial linear spaces

Recall that a *partial linear space* is an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ and $\mathcal{L}$ are sets (whose elements are called *points* and *lines*, respectively) and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ is the *incidence relation* such that every line is incident with at least two points and there exists at most one line through any two distinct points. The *point graph* of the incidence structure $(\mathcal{P},\mathcal{L},\mathcal{I})$ is a graph defined with $\mathcal{P}$ as its vertex set, with two points being adjacent, if they are collinear.

The following lemma is from Ray-Chaudhuri and Sprague [15], and is also given in Theorem 9.3.9 of [2]. It is an important ingredient for our proof of Theorem 5.

Lemma 13 ([2, Theorem 9.3.9])**.** *Let $(\mathcal{P},\mathcal{L},\in)$ be a partial linear space such that for an integer $q \geqslant 2$:*

(1) *each line has at least q^2+q+1 points;*

(2) *each point is on more than $q+1$ lines;*

(3) *if $P \in \mathcal{P}$, $l \in \mathcal{L}$ and l is incident with P, then there are exactly $q+1$ lines on P meeting l;*

(4) *if the points P and P' have distance 2 in the point graph Γ, then there are precisely $q+1$ lines l on P such that they are incident with P';*

(5) *the point graph Γ of $(\mathcal{P},\mathcal{L},\in)$ is connected.*

Then q is a prime power, and $(\mathcal{P},\mathcal{L},\in)$ is isomorphic to $\left(\begin{bmatrix} V \\ D \end{bmatrix}_q, \begin{bmatrix} V \\ D+1 \end{bmatrix}_q, \subseteq\right)$ for some vector space V of dimension n over $\mathbb{F}_q$, where n and D are integers and satisfy $3 \leqslant D \leqslant \frac{n}{2}$. In particular, Γ is the Grassmann graph $J_q(n,D)$.

2.6 Walk-regular graphs

A graph G is called *walk-regular*, if, for all integers $r \geqslant 0$, the number of closed walks of length r (or closed r-walks) from a given vertex x is independent of the choice of x. Since this number equals $(A^r)_{xx}$, it is the same as saying that A^r has a constant diagonal for all $r \geqslant 0$, where A is the adjacency matrix of G. It is clear that A^r has a constant diagonal for $r = 0,1$, and A^2 has a constant diagonal if and only if G is regular.

We now introduce the Hoffman polynomial that is useful to prove Lemma 15.

Lemma 14 ([1, Corollary 3.3])**.** *Let G be a regular connected graph with v vertices and distinct eigenvalues $\eta_0 = a > \eta_1 > \cdots > \eta_d$. Then if the polynomial $q(\eta) = \prod_{i=1}^{d}(\eta-\eta_i)$, we have*

$$q(A) = \frac{q(a)}{v}J, \tag{9}$$

where A is the adjacency matrix of G and J is the all-ones matrix. The equality (9) is the so-called Hoffman polynomial *of G.*

Lemma 15. *Let G be a connected co-edge-regular graph with parameters (v,a,μ). If G has at most five distinct eigenvalues, then G is walk-regular.*

Proof. By the Hoffman polynomial (9), one can check that a connected regular graph with at most four distinct eigenvalues is walk-regular (also shown in [5]). Therefore, in order to show this lemma, it suffices to show that G is walk-regular if G is connected and co-edge-regular and has exactly five distinct eigenvalues.

Let A be the adjacency matrix of G. Since G is connected and co-edge-regular with parameters (v, a, μ), we obtain that, for any vertex $x \in V(G)$,

$$(A^2)_{xx} = a, \tag{10}$$
$$(A^3)_{xx} = a(a-1) - (v-a-1)\mu. \tag{11}$$

This implies that A^2 and A^3 have constant diagonals. Suppose G has exactly five distinct eigenvalues $\eta_0 = a > \eta_1 > \cdots > \eta_4$. Then A satisfies the Hoffman polynomial (9):

$$A^4 - (\sum_{i=1}^{4} \eta_i)A^3 + (\sum_{1\leqslant i<j\leqslant 4} \eta_i\eta_j)A^2 + (\sum_{1\leqslant i<j<k\leqslant 4} \eta_i\eta_j\eta_k)A + \eta_1\eta_2\eta_3\eta_4 I = \frac{\prod_{i=1}^{4}(a-\eta_i)}{v}J. \tag{12}$$

where I is the identity matrix and J is the all-ones matrix. It means that A^4 has a constant diagonal, and thus so does A^r, $r = 5, 6, \cdots$, which implies that G is walk-regular. □

We finally mention the following result that will be applied to the proof of Proposition 20. Let G be a graph with spectrum $\{[\eta_0]^{m_0}, [\eta_1]^{m_1}, \ldots, [\eta_d]^{m_d}\}$. Then

$$\mathrm{Tr}(A^r) = \sum_{i=0}^{d} m_i\eta_i^r = \text{the number of closed } r\text{-walks in } G, \tag{13}$$

where A is the adjacency matrix of G and $\mathrm{Tr}(A^r)$ is the trace of matrix A^r (i.e. the sum of the diagonal entries of A^r), see [1, Lemma 2.5].

3 Spectral characterizations of the s-clique extension of the $(t_1 \times t_2)$-grid

In this section, we give a spectral characterization of the s-clique extension of the $(t_1 \times t_2)$-grid. We will use these results in Section 4.

3.1 Clique extensions of the $(t_1 \times t_2)$-grid graphs

The *Kronecker product* $M_1 \otimes M_2$ of two matrices M_1 and M_2 is obtained by replacing the (i, j)-entry of M_1 by $(M_1)_{ij}M_2$ for all i and j. Note that, if τ and η are eigenvalues of M_1 and M_2 respectively, then $\tau\eta$ is an eigenvalue of $M_1 \otimes M_2$ (cf. [11, Section 9.7]).

Given graphs G and H with vertex sets X and Y, respectively, their *Cartesian product* $G \square H$ is the graph with the vertex set $X \times Y$, where $(x, y) \sim (x', y')$ when either $x = x'$ and $y \sim y'$ or $x \sim x'$ and $y = y'$. For the adjacency matrix we have $A(G \square H) =$

$A(G)\otimes I_{|Y|}+I_{|X|}\otimes A(H)$, where $I_{|X|}$ (resp. $I_{|Y|}$) is the identity matrix of order $|X|$ (resp. $|Y|$), see [4, Section 1.4.6].

A t-*clique* is a clique with t vertices and is denoted by K_t, where t is a positive integer. For positive integers t_1, t_2, the $(t_1\times t_2)$-*grid* is the Cartesian product $K_{t_1}\square K_{t_2}$ of K_{t_1} and K_{t_2}. The spectrum of the $(t_1\times t_2)$-grid is $\{[t_1+t_2-2]^1, [t_1-2]^{t_2-1}, [t_2-2]^{t_1-1}, [-2]^{(t_1-1)(t_2-1)}\}$ (cf. [12]).

For a positive integer s, the s-*clique extension* of a graph G is the graph $\widetilde{G}$ obtained from G by replacing each vertex $x\in V(G)$ by a clique $\widetilde{X}$ with s vertices, such that $\widetilde{x}\sim\widetilde{y}$ (for $\widetilde{x}\in\widetilde{X}$, $\widetilde{y}\in\widetilde{Y}$) in $\widetilde{G}$ if and only if $x\sim y$ in G. If $\widetilde{G}$ is the s-clique extension of G, then $\widetilde{G}$ has adjacency matrix $(A+I_{|V(G)|})\otimes J_s - I_{s\times|V(G)|}$, where A is the adjacency matrix of G and J_s is the all-ones matrix of order s and $I_{|V(G)|}$ is the identity matrix of order $|V(G)|$. In particular, if G has spectrum

$$\{[\eta_0]^{m_0}, [\eta_1]^{m_1}, \ldots, [\eta_d]^{m_d}\}, \tag{14}$$

then it follows that the spectrum of $\widetilde{G}$ (cf. [12]) is

$$\{[s(\eta_0+1)-1]^{m_0}, [s(\eta_1+1)-1]^{m_1}, \ldots, [s(\eta_d+1)-1]^{m_d}, [-1]^{(s-1)(m_0+m_1+\cdots+m_d)}\}. \tag{15}$$

For the rest of this subsection, we assume that G is a graph that is cospectral with the s-clique extension of the $(t_1\times t_2)$-grid, where $s\geqslant 2,\ t_1>t_2>1$ are integers. By (14) and (15), the graph G has spectrum

$$\{[s(t_1+t_2-1)-1]^1, [s(t_1-1)-1]^{t_2-1}, [s(t_2-1)-1]^{t_1-1}, [-1]^{(s-1)t_1t_2}, [-s-1]^{(t_1-1)(t_2-1)}\}. \tag{16}$$

By using (12), we obtain

$$\begin{aligned} &A^4+(4-s(t_1+t_2-3))A^3+(s^2(3-2(t_1+t_2)+t_1t_2)+3s(3-t_1-t_2)+6)A^2\\ &\quad+(-s^3(t_1-1)(t_2-1)+2s^2(2(t_1+t_2)-t_1t_2-3)+3s(t_1+t_2-3)-4)A\\ &\quad+((s+1)(s(t_2-1)-1)(s(t_1-1)-1))I\\ &=s^3(t_1+t_2)(t_1+t_2-1)J. \end{aligned} \tag{17}$$

We will use it in the next subsection. The following lemma shows an upper bound on the order of a clique of G.

Lemma 16. *Let G be a graph that is cospectral with the s-clique extension of the $(t_1\times t_2)$-grid, where $s\geqslant 2,\ t_1>t_2>1$ are integers. Then, for any clique C of G, we have $|V(C)|\leqslant st_1$. Moreover, if the equality holds, then every vertex outside C has exactly s neighbors in C.*

Proof. We will take advantage of the Hoffman bound (see (2)) to prove this lemma, so we consider a coclique C in the complement graph $\overline{G}$ of G. (Note that the graph induced on $V(C)$ in $\overline{G}$ is a clique of G). It is known that a graph cospectral with a connected k-regular graph is connected and k-regular. As the s-clique extension of the $(t_1\times t_2)$-grid is regular with valency $s(t_1+t_2-1)-1$, so does G. In terms of the properties of complement, one easily verify that $\overline{G}$ is regular with valency $s(t_1-1)(t_2-1)$ and with smallest eigenvalue $-s(t_1-1)$. Then, by the Hoffman bound, we infer that

$$|V(C)| \leqslant \frac{st_1t_2}{1+\frac{s(t_1-1)(t_2-1)}{s(t_1-1)}} = st_1.$$

Moreover, the equality implies that every vertex $x \notin V(C)$ is nonadjacent to s vertices of C in $\overline{G}$, which also means that x is adjacent to exactly s vertices of C in G. □

3.2 Grand cliques in G

As in the above subsection, we assume that G is a graph that is cospectral with the s-clique extension of the $(t_1 \times t_2)$-grid, where $s \geqslant 2, t_1 > t_2 > 1$ are integers. For this subsection, we further assume that G is co-edge-regular with parameters $(st_1t_2, s(t_1+t_2-1)-1, \mu)$. In Lemma 15, it was shown that G is walk-regular. This implies that $(A^r)_{xx}$ is constant for any $r \geqslant 0$ and $x \in V(G)$, where A is the adjacency matrix of G. By (10), (11) and (17), we have

$$\begin{aligned}(A^4)_{xx} =& (-s^2(t_1-1)(t_2-1)(t_1+t_2-3)+4s(t_1-1)(t_2-1))\mu \\ &+ s^3(t_1(t_1+t_2-1)^2+(t_1+t_2)(t_2-1)^2+t_2-1) \\ &- 4s^2(t_1+t_2-1)^2+6s(t_1+t_2-1)-3. \end{aligned} \tag{18}$$

In addition, the parameter μ only depends on the spectrum of G. As G is cospectral with the s-clique extension of the $(t_1 \times t_2)$-grid, we find $\mu = 2s$.

Let $G(\infty)$ be the local graph of G at vertex ∞. Assume that the vertices of $G(\infty)$ have valencies $d_1, \ldots, d_a$, where $a = s(t_1+t_2-1)-1$. Then, as G is co-edge-regular and walk-regular, we obtain that the number of walks of length two inside $G(\infty)$ is the same as the number of walks of length two in the local graph of the s-clique extension of the $(t_1 \times t_2)$-grid. Using (18), the sum of valencies and the sum of square of valencies of vertices in $G(\infty)$ are given by the following equations, where ε is the number of edges inside $G(\infty)$.

$$2\varepsilon = \sum_{i=1}^{a} d_i = s^2((t_1-1)^2+(t_2-1)^2)+(2s^2-3s)(t_1+t_2-2)+s^2-3s+2. \tag{19}$$

$$\begin{aligned}\sum_{i=1}^{a}(d_i)^2 =& s^3((t_1+t_2-1)^2+t_1^2(t_1-1)+t_2^2(t_2-1)) - s^2((t_1+t_2-1)^2 \\ &+ 4(t_1^2+t_2^2-1)) + 8s(t_1+t_2-1) - 4 \\ =& (s-1)(s(t_1+t_2-1)-2)^2 + s(t_1-1)(st_1-2)^2 + s(t_2-1)(st_2-2)^2. \end{aligned} \tag{20}$$

It is straightforward to show the following lemma that will be used later.

Lemma 17. *Let u, v, p and q be some integers satisfying $0 \leqslant q \leqslant p \leqslant u$ and $v \leqslant u$. If $p+q = u+v$, then $p^2+q^2 \leqslant u^2+v^2$ and equality holds if and only if $p = u$ and $q = v$.*

We call a maximal clique in G a *grand clique*, if it contains at least $\frac{19}{36}a$ vertices. The following proposition shows the existence of grand cliques in G.

Proposition 18. *Let G be a co-edge-regular graph with parameters (v, a, μ), that is cospectral with the s-clique extension of the $(t_1 \times t_2)$-grid, where $s \geqslant 2, t_1 > 2t_2$ are integers. If $t_2 > \frac{36(s+1)^6}{s^2}$, then each vertex of G lies on a unique grand clique.*

Proof. Note that G is regular with valency $a = s(t_1+t_2-1)-1$, and co-edge-regular with parameter $\mu = 2s$. Let ∞ be a vertex of G and $G(\infty)$ be the local graph of G at ∞. As before, let ε be the number of edges inside $G(\infty)$. We first obtain a lower bound on ε. By (19), we obtain

$$2\varepsilon \geqslant s^2(t_1-1)^2 + s^2(t_2-1)^2 \tag{21}$$

as $s \geqslant 2$. Now we derive an upper bound on ε. Let B be a coclique with maximum order in $G(\infty)$ and with vertex set $\{x_1, x_2, \ldots, x_p\}$. By interlacing (see Lemma 7), we have $p \leqslant (s+1)^2$, since the smallest eigenvalue of a complete bipartite graph with parts of size 1 and p is $-\sqrt{p}$ and the smallest eigenvalue of G is $-s-1$. We define

$$\mathcal{R} := \{y \sim \infty \mid y \text{ has at least two neighbors in } B\}.$$

Let r be the cardinality of $\mathcal{R}$. Then

$$r \leqslant (2s-1)\binom{p}{2} \leqslant sp(p-1) \leqslant s^2(s+1)^2(s+2) \leqslant (s+1)^5, \tag{22}$$

as $\mu = 2s$. We also define the sets U_i such that

$$U_i = \{y \sim \infty \mid y \text{ has only } x_i \text{ as its neighbor in } B\} \cup \{x_i\},$$

where $u_i := |U_i|$. Note that $\mathcal{U} := \bigcup_i U_i$, then $\mathcal{U} \cup \mathcal{R} = V(G(\infty))$. Note further that any two vertices of U_i are adjacent, because B is maximum. Thus, for every i, the graph induced on U_i is a clique of $G(\infty)$.

Now, inside $\mathcal{U}$, we have at most $\frac{1}{2}(\sum_i u_i(u_i-1) + (p-1)(2s-1)\sum_i u_i)$ edges, inside $\mathcal{R}$, we have at most $\frac{1}{2}r(r-1)$ edges and between $\mathcal{U}$ and $\mathcal{R}$, we have at most $r(a-r)$ edges. Then, we obtain

$$2\varepsilon \leqslant \sum_i u_i(u_i-1) + r(r-1) + 2r(a-r) + (p-1)(2s-1)\sum_i u_i.$$

Without loss of generality, we may assume that $u_1 \geqslant u_2 \geqslant \ldots \geqslant u_p \geqslant 1$. Now suppose that $u_1 \leqslant \frac{19}{36}a$. Then we obtain

$$2\varepsilon \leqslant (u_1-1)\sum_i u_i + 2r(a-1) + (p-1)(2s-1)\sum_i u_i$$

$$
\begin{aligned}
&\leqslant (\frac{19}{36}a-1)a+2(s+1)^5a+(s^2+2s)(2s-1)a \quad (\text{as } \sum_i u_i = a-r \leqslant a)\\
&\leqslant \frac{19}{36}a^2+3(s+1)^5a\\
&< \frac{19}{36}s^2(t_1+t_2)^2+3s(s+1)^5(t_1+t_2) \quad (\text{as } a=s(t_1+t_2-1)-1<s(t_1+t_2)) \quad (23)
\end{aligned}
$$

Combining (21), (23), we have

$$s^2(t_1-1)^2+s^2(t_2-1)^2 \leqslant\ 2\varepsilon\ <\ \frac{19}{36}s^2(t_1+t_2)^2+3s(s+1)^5(t_1+t_2),$$

which implies that

$$\frac{19}{36}s^2(t_1^2+t_2^2)+\frac{19}{18}s^2t_1t_2+3s(s+1)^5(t_1+t_2)\ >\ s^2(t_1^2+t_2^2)-2s^2(t_1+t_2-1).$$

Then, by simplifying we see

$$38s^2t_1t_2+108(s+1)^6(t_1+t_2)\ > 17s^2(t_1^2+t_2^2) > 17\cdot\frac{5}{2}s^2t_1t_2,$$

because of $(t_1-t_2)^2-\frac{1}{2}t_1t_2 > \frac{1}{4}t_1^2-\frac{1}{2}t_1t_2 > 0$, as $t_1 > 2t_2$. Also, we obtain

$$\frac{24(s+1)^6}{s^2} > \frac{t_1t_2}{t_1+t_2} > \frac{2}{3}t_2 \quad (\text{as } t_1 > 2t_2),$$

and then $t_2 < \frac{36(s+1)^6}{s^2}$, a contradiction. This implies that there exists a grand clique containing ∞. If a vertex in G lies on more than one grand clique, then the intersection of their two grand cliques is at least $2\cdot\frac{19}{36}a-a-1=\frac{1}{18}a-1>2s$, since $t_1>2t_2$ and $t_2>\frac{36(s+1)^6}{s^2}$. However, their intersection is at most $2s$, a contradiction. This shows that every vertex of G lies on a unique grand clique. □

3.3 A spectral characterization of G

In this subsection, we give the following spectral characterization of the s-clique extension of the $(t_1\times t_2)$-grid.

Theorem 19. *Let G be a co-edge-regular graph with parameters (v,a,μ), that is cospectral with the s-clique extension of the $(t_1\times t_2)$-grid, where $s\geqslant 2, t_1>2t_2>2$ are integers. If G has a clique of order st_1, then G is the s-clique extension of the $(t_1\times t_2)$-grid.*

Proof. Let C denote a clique of order st_1 in G. From Lemma 16, we know that every vertex in $V(G)-V(C)$ has exactly s neighbors in $V(C)$. Fix a vertex $\infty\in V(C)$, now we consider the local graph $G(\infty)$ of G at ∞. Let D_∞ be the set of the neighbors of ∞ not in $V(C)$. It is easy to obtain $|D_\infty|=s(t_2-1)$, since G is regular with valency

$a = s(t_1 + t_2 - 1) - 1$. For any $x \in V(G(\infty))$, let d_x denote the valency of x in $G(\infty)$. Choosing $x \in D_\infty$, we see that $d_x \leqslant s(t_2 - 1) - 1 + s - 1 = st_2 - 2$. Hence,

$$\sum_{x \in D_\infty} d_x \leqslant s(t_2 - 1)(st_2 - 2), \tag{24}$$

with equality, if and only if, the graph induced on D_∞ is a clique of order $s(t_2 - 1)$. In addition, we know that

$$\sum_{x \in V(C) - \{\infty\}} d_x = (st_1 - 1)(st_1 - 2) + s(t_2 - 1)(s - 1).$$

By Equation (19), we deduce that

$$\sum_{x \in D_\infty} d_x = \sum_{i=1}^{a} d_i - \sum_{x \in V(C) - \{\infty\}} d_x = s(t_2 - 1)(st_2 - 2).$$

It follows that the equality in (24) holds, which implies that the graph induced on D_∞ is a clique of order $s(t_2 - 1)$. Because of

$$\sum_{x \in D_\infty} d_x^2 = s(t_2 - 1)(st_2 - 2)^2,$$

and, by (20), we have

$$\sum_{x \in V(C) - \{\infty\}} d_x^2 = \sum_{i=1}^{a} d_i^2 - \sum_{x \in D_\infty} d_x^2 = (s - 1)(s(t_1 + t_2 - 1) - 2)^2 + s(t_1 - 1)(st_1 - 2)^2.$$

Then

$$\sum_{x \in V(C) - \{\infty\}} (d_x - (st_1 - 2))^2 = (s - 1)s^2(t_2 - 1)^2.$$

It follows that there are s vertices (including ∞) of C adjacent to all of vertices of D_∞, and we infer that these s vertices together with the vertices of D_∞ induce a clique with vertex set C_∞. Note that $|C_\infty| = st_2$.

Next, we fix any $x \in D_\infty$ and let E_x be the set of the neighbors of x not in C_∞. It is also easy to obtain $|E_x| = s(t_1 - 1)$ because x has valency a. For any y, a neighbor of x, we let d_y denote the valency of y in $G(x)$. Fix $y \in E_x$, as G is co-edge-regular with parameter $\mu = 2s$ and y has s neighbors in C, by Lemma 16, we deduce that y has s neighbors in C_∞ as $y \nsim \infty$. This implies that $d_y \leqslant s(t_1 - 1) - 1 + s - 1 = st_1 - 2$. Hence,

$$\sum_{y \in E_x} d_y \leqslant s(t_1 - 1)(st_1 - 2), \tag{25}$$

with equality, if and only if, the graph induced on E_x is a clique of order $s(t_1-1)$. Similar to the proof for D_∞, we check that the equality in (25) holds. This implies the graph induced on E_x is a clique of order $s(t_1-1)$. Note that

$$\sum_{y\in C_\infty-\{x\}}(d_y-(st_2-2))^2=(s-1)s^2(t_1-1)^2.$$

It follows that there are s vertices (including x) in C_∞ adjacent to all of vertices of E_x. We infer that these s vertices together with the vertices of E_x induce a clique of order st_1.

Similarly, for any $x\in V(C)$, we have C_x, and for any $y\in D_x$, we have E_y. It is easy to check G is the s-clique extension of the $(t_1\times t_2)$-grid. □

4 Distance-regular graphs with classical parameters

In this section, we assume that Γ is a distance-regular graph with classical parameters $(D,b,\alpha,\beta)=(D,q,q,\left[{D+e+1 \atop 1}\right]_q-1)$, where $q\geqslant 2$ and $e\in\{1,2,3\}$ are integers.

Proposition 20. *Let Γ be a 1-thin distance-regular graph with classical parameters*

$$(D,b,\alpha,\beta)=(D,q,q,\left[{D+e+1 \atop 1}\right]_q-1),$$

where $q\geqslant 2$ and $e\in\{1,2,3\}$ are integers and $D\geqslant 5$. Assume further that Γ is μ-graph-regular with parameter ℓ. Let $G:=\Delta(x)$ be the local graph of Γ at a vertex $x\in V(\Gamma)$. Then

(i) *$\ell=2q$,*

(ii) *G is cospectral with the q-clique extension of the $(t_1\times t_2)$-grid, where*

$$t_1=\left[{D+e \atop 1}\right]_q,\ t_2=\left[{D \atop 1}\right]_q.$$

Proof. For integers $q\geqslant 2$ and $e\in\{1,2,3\}$, let $t_1=\left[{D+e \atop 1}\right]_q$, $t_2=\left[{D \atop 1}\right]_q$. As Γ is μ-graph-regular with parameter ℓ, we see that G is co-edge-regular with parameters (k,a_1,ℓ), where k is the valency of Γ. By Formulas (5) and (6), one can calculate that $k=b_0=qt_1t_2$ and $a_1=b_0-b_1-c_1=q(t_1+t_2-1)-1$. Let

$$\begin{aligned}
\eta_0&=a_1=q(t_1+t_2-1)-1,\\
\eta_1&=\beta-\alpha-1=q(t_1-1)-1,\\
\eta_2&=\alpha b\frac{b^{D-1}-1}{b-1}-1=q(t_2-1)-1,\\
\eta_3&=-1,\\
\eta_4&=-q-1.
\end{aligned}$$

Then, by Lemma 11, any eigenvalue of G is in $\{\eta_0, \eta_1, \ldots, \eta_4\}$, where the multiplicity of η_0 is equal to 1. For $1 \leqslant i \leqslant 4$, let m_i be the multiplicity of eigenvalue η_i of G, where $m_i = 0$ if η_i is not an eigenvalue of G. Let α denote the average valency of vertices in the local graph of G at a fixed vertex. One can check $\alpha = a_1 - 1 - \frac{\ell(qt_1t_2 - a_1 - 1)}{a_1}$, since G is co-edge-regular with parameter ℓ. Then, by Formula (13), we obtain

$$\begin{cases} m_1 + m_2 + m_3 + m_4 = qt_1t_2 - 1 \\ m_1\eta_1 + m_2\eta_2 + m_3\eta_3 + m_4\eta_4 = -a_1 \\ m_1\eta_1^2 + m_2\eta_2^2 + m_3\eta_3^2 + m_4\eta_4^2 = a_1(qt_1t_2 - a_1) \\ m_1\eta_1^3 + m_2\eta_2^3 + m_3\eta_3^3 + m_4\eta_4^3 = qt_1t_2(a_1^2 - a_1 - \ell(qt_1t_2 - a_1 - 1)) - a_1^3. \end{cases} \tag{26}$$

Let f_i denote the multiplicity of η_i $(0 \leqslant i \leqslant 4)$ in the q-clique extension of the $(t_1 \times t_2)$-grid. Note that $f_0 = 1$. By (16) and Formula (13) similarly, we gain that

$$\begin{cases} f_1 + f_2 + f_3 + f_4 = qt_1t_2 - 1 \\ f_1\eta_1 + f_2\eta_2 + f_3\eta_3 + f_4\eta_4 = -a_1 \\ f_1\eta_1^2 + f_2\eta_2^2 + f_3\eta_3^2 + f_4\eta_4^2 = a_1(qt_1t_2 - a_1) \\ f_1\eta_1^3 + f_2\eta_2^3 + f_3\eta_3^3 + f_4\eta_4^3 = qt_1t_2(a_1^2 - a_1 - 2q(qt_1t_2 - a_1 - 1)) - a_1^3. \end{cases} \tag{27}$$

Now, we set $m'_i = m_i - f_i$ for $1 \leqslant i \leqslant 4$, and compare the systems of linear equations (26) and (27) to obtain

$$\begin{cases} m'_1 + m'_2 + m'_3 + m'_4 = 0 \\ m'_1\eta_1 + m'_2\eta_2 + m'_3\eta_3 + m'_4\eta_4 = 0 \\ m'_1\eta_1^2 + m'_2\eta_2^2 + m'_3\eta_3^2 + m'_4\eta_4^2 = 0 \\ m'_1\eta_1^3 + m'_2\eta_2^3 + m'_3\eta_3^3 + m'_4\eta_4^3 = qt_1t_2(qt_1t_2 - a_1 - 1)(2q - \ell). \end{cases} \tag{28}$$

It is easy to see the coefficient determinant, denoted by $\det M$, of the system of linear equations (28) is a Vandermonde determinant, i.e.,

$$\det M = (\eta_4 - \eta_3)(\eta_4 - \eta_2)(\eta_4 - \eta_1)(\eta_3 - \eta_2)(\eta_3 - \eta_1)(\eta_2 - \eta_1).$$

Let $\det M_i$ denote a determinant by replacing the i-column of $\det M$ by the vector $(0, 0, 0, qt_1t_2(qt_1t_2 - a_1 - 1)(2q - \ell))^\top$. Hence, we obtain

$$\det M_1 = -qt_1t_2(qt_1t_2 - a_1 - 1)(2q - \ell)(\eta_4 - \eta_3)(\eta_4 - \eta_2)(\eta_3 - \eta_2).$$

Thus, by Cramer's Rule,

$$\begin{aligned} m'_1 &= \frac{\det M_1}{\det M} \\ &= \frac{-qt_1t_2(qt_1t_2 - a_1 - 1)(2q - \ell)}{(\eta_4 - \eta_1)(\eta_3 - \eta_1)(\eta_2 - \eta_1)} \end{aligned}$$

$$= \frac{-q^2 t_1 t_2 (t_1 - 1)(t_2 - 1)(2q - \ell)}{-q^3 t_1 (t_1 - 1)(t_1 - t_2)}$$
$$= \frac{t_2 (t_2 - 1)(2q - \ell)}{q(t_1 - t_2)}$$
$$= \frac{(\frac{q^D - 1}{q - 1})(q^{D-1} - 1)(2q - \ell)}{q^D (q^e - 1)}.$$

As $m'_1 \in \mathbb{Z}_{\geqslant 0}$, we obtain q^D divides $2q - \ell$. Since $\ell \leqslant c_2 - 1 = (q+1)^2 - 1 \leqslant q^3$ (as $q \geqslant 2$) and $D \geqslant 5$, we obtain that $\ell = 2q$, and $m'_i = 0$ for $1 \leqslant i \leqslant 4$, so $m_i = f_i$ for all i. □

Now we give a sufficient condition for a distance-regular graph to have a Delsarte clique.

Lemma 21. *Let Γ be a distance-regular graph with smallest eigenvalue $\theta_{\min} = -m \in \mathbb{Z}$. If Γ is the point graph of a partial linear space $(\mathcal{P}, \mathcal{L}, \in)$ such that $|\mathcal{P}| > |\mathcal{L}|$. Then there exists a Delsarte clique of Γ.*

Proof. Let N be the point-line incidence matrix. For $x \in V(\Gamma) = \mathcal{P}$, define τ_x as the number of lines through the point x. So $NN^\top = A + T$, where $^\top$ denotes the transpose and T is a diagonal matrix such that $T_{xx} = \tau_x$ for all $x \in V(\Gamma)$. Assume that $\tau_x \geqslant m + 1$ for all $x \in V(\Gamma)$. Then

$$A + T - (A + (m+1)I)$$

is positive semidefinite. As $|\mathcal{P}| > |\mathcal{L}|$, the matrix $NN^\top = A + T$ has an eigenvalue 0 and this implies that $\theta_{\min} \leqslant -m - 1$, a contradiction. It follows that there exists a vertex x such that $\tau_x \leqslant m$. Hence, by the Delsarte bound, we require

$$k = |\Gamma(x)| \leqslant \frac{k}{m} \cdot m \leqslant k, \tag{29}$$

which shows that each line through x is a Delsarte clique. □

Remark 22. The twisted Grassmann graphs $\tilde{J}_q(2D+1, D)$ that were discovered by Van Dam and Koolen [6] are the point graph of a partial linear space $(\mathcal{P}, \mathcal{L}, \in)$ with $|\mathcal{P}| = |\mathcal{L}|$. None of the lines in a Delsarte clique, although $\tilde{J}_q(2D+1, D)$ contains Delsarte cliques.

5 Proof of Theorem 5

In this section, we completes the proof of Theorem 5.

Proof of Theorem 5. We assume that Γ is a 1-thin distance-regular graph with classical parameters $(D, b, \alpha, \beta) = (D, q, q, \begin{bmatrix} D+e+1 \\ 1 \end{bmatrix}_q - 1)$ for some integers $q \geqslant 2$, $e \in \{1, 2, 3\}$ and $D \geqslant \chi(q)$, as defined in (1). Assume further that Γ is μ-graph-regular with parameter ℓ, and thus any local graph of Γ is co-edge-regular with parameters (k, a_1, ℓ), where

$k = b_0 = q\begin{bmatrix} D+e \\ 1 \end{bmatrix}_q \begin{bmatrix} D \\ 1 \end{bmatrix}_q$, $a_1 = b_0 - b_1 - c_1 = q(\begin{bmatrix} D+e \\ 1 \end{bmatrix}_q + \begin{bmatrix} D \\ 1 \end{bmatrix}_q - 1)$. Set $t_1 = \begin{bmatrix} D+e \\ 1 \end{bmatrix}_q$, $t_2 = \begin{bmatrix} D \\ 1 \end{bmatrix}_q$. Then, from Proposition 20, any local graph of Γ is cospectral with the q-clique extension of the $(t_1 \times t_2)$-grid.

We define a *line* of Γ as a maximal clique that contains at least $\frac{19}{36}a_1 + 1$ vertices. Let $\mathcal{L}$ be the set consisting of all lines in Γ. As $D \geqslant \chi(q)$ and, by Proposition 18, we obtain that for any two adjacent vertices $x, y \in V(\Gamma)$, there exists a unique line $l \in \mathcal{L}$ such that $x, y \in l$. One can see that $(V(\Gamma), \mathcal{L}, \in)$ is a partial linear space such that Γ is its point graph. Moreover, the inequality $|V(\Gamma)| > |\mathcal{L}|$ holds, see the following claim.

Claim 23. $|V(\Gamma)| > |\mathcal{L}|$.

Proof. Let τ_x denote the number of lines through the point x and σ_l denote the number of points on the line l. We have that $\sigma_l \geqslant \frac{19}{36}a_1 + 1$ for any line l by the definition of a line of Γ, and $\tau_x \leqslant \frac{k}{\frac{19}{36}a_1}$ for any vertex x of Γ by the Delsarte bound. We can show that $\frac{k}{\frac{19}{36}a_1} < \frac{19}{36}a_1 + 1$ holds, since

$$\begin{aligned}
\frac{19}{36}a_1(\frac{19}{36}a_1 + 1) &= \frac{19}{36}(q(t_1 + t_2 - 1) - 1)(\frac{19}{36}(q(t_1 + t_2 - 1) - 1) + 1) \\
&> \frac{19}{36}(q(t_1 + t_2 - 1) - 1)(\frac{19}{36}(q(t_1 + t_2 - 1)) \\
&= \frac{19^2}{36^2}q^2(t_1 + t_2 - 1)^2 - \frac{19}{36}q(t_1 + t_2 - 1) \\
&> \frac{19^2}{36^2}q^2(t_1^2 + t_2^2 + t_1(t_2 - 1)) - \frac{19}{36}q(t_1 + (t_2 - 1)) \\
&> \frac{19^2}{36^2}q^2(t_1^2 + t_2^2) \quad (\text{as } q \geqslant 2 \text{ and } t_1 > 2t_2 > 6) \\
&> \frac{19^2}{36^2} \cdot 2q \cdot \frac{5}{2}t_1 t_2 > q t_1 t_2 = k.
\end{aligned}$$

Hence, we find that $\tau_x < \sigma_l$ holds for any vertex x and line l in Γ. Because

$$|\{(x, l) \mid x \in V(\Gamma), l \in \mathcal{L}, x \in l\}| = \sum_{x \in V(\Gamma)} \tau_x = \sum_{l \in \mathcal{L}} \sigma_l, \tag{30}$$

we acquire that $|V(\Gamma)| > |\mathcal{L}|$. □

Now, by Lemma 21, we obtain that there exits a Delsarte clique in Γ, say C, which is a clique containing $q\begin{bmatrix} D+e \\ 1 \end{bmatrix}_q + 1 = qt_1 + 1$ vertices. Let x be a vertex of C. Then, from Theorem 19, $\Delta(x)$ is the q-clique extension of the $(t_1 \times t_2)$-grid. Therefore, for any neighbor y of x, $\Delta(y)$ is again the q-clique extension of the $(t_1 \times t_2)$-grid. As Γ is connected, it follows that for any vertex x of Γ, the local graph at x is the q-clique extension of the $(t_1 \times t_2)$-grid. This implies that Γ is the point graph of the partial linear space $(V(\Gamma), \mathcal{L}, \in)$, where $\mathcal{L}$ is the set of Delsarte cliques of Γ. As every edge lies in a unique Delsarte clique and any vertex outside a Delsarte clique C has either $q+1$ or none neighbors in C, it follows by Lemma 13 that Γ is the Grassmann graph $J_q(2D + e, D)$.

This completes the proof of Theorem 5. □

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