Integral mixed Cayley graphs over abelian groups

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Abstract

A mixed graph is said to be *integral* if all the eigenvalues of its Hermitian adjacency matrix are integer. Let Γ be an abelian group. The *mixed Cayley graph* $Cay(\Gamma, S)$ is a mixed graph on the vertex set Γ and edge set $\{(a, b) : b - a \in S\}$, where $0 \not\in S$. We characterize integral mixed Cayley graph $Cay(\Gamma, S)$ over an abelian group Γ in terms of its connection set S.

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1 Introduction

We only consider graphs without loops and multi-edges. A *graph* G is denoted by $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively. Here $E(G) \subset V(G) \times V(G) \setminus \{(u, u)|u \in V(G)\}$ such that $(u, v) \in E(G)$ if and only if $(v, u) \in E(G)$. A graph G is said to be *oriented* if $(u, v) \in E(G)$ implies that $(v, u) \not\in E(G)$. A graph G is said to be *mixed* if $(u, v) \in E(G)$ does not always imply that $(v, u) \in E(G)$, see [15] for details. In a mixed graph G, we call an edge with end vertices u and v to be *undirected* (resp. *directed*) if both (u, v) and (v, u) belong to $E(G)$ (resp. only one of (u, v) and (v, u) belongs to $E(G)$). An undirected edge (u, v) is denoted by $u \leftrightarrow v$, and a directed edge (u, v) is denoted by $u \rightarrow v$. A mixed graph can have both directed and undirected edges. Note that, if all edges of a mixed graph G are directed (resp. undirected) then G is an oriented graph (resp. a simple graph). For a mixed graph G, the underlying graph G_U of G is the simple undirected graph in which all edges of G are considered undirected. By the terms of order, size, number of components, degree of a vertex, distance between two vertices etc., we mean that they are the same as in their underlying graphs.

The *Hermitian adjacency matrix* of a mixed graph G is denoted by $H(G) = (h_{uv})_{n\times n}$, where h_{uv} is given by

$$h_{uv} = \begin{cases} 1 & \text{if } (u,v) \in E \text{ and } (v,u) \in E, \\ i & \text{if } (u,v) \in E \text{ and } (v,u) \notin E, \\ -i & \text{if } (u,v) \notin E \text{ and } (v,u) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Here $i = \sqrt{-1}$ is the imaginary number unit. This matrix was introduced by Liu and Li [15] in the study of hermitian energies of mixed graphs, and also independently by Guo and Mohar [9]. Hermitian adjacency matrix of a mixed graph incorporates both adjacency matrix of simple graph and skew adjacency matrix of an oriented graph. The Hermitian spectrum of G, denoted by $Sp_H(G)$, is the multi set of the eigenvalues of $H(G)$. It is easy to see that $H(G)$ is a Hermitian matrix and so $Sp_H(G) \subseteq \mathbb{R}$.

A mixed graph is said to be *integral* if all the eigenvalues of its Hermitian adjacency matrix are integers. Integral graphs were first defined by Harary and Schwenk in 1974 [10] and proposed a classification of integral graphs. See [5] for a survey on integral graphs.

Let Γ be a group, $S \subseteq \Gamma$ and S does not contain the identity element of Γ. The set S is said to be *symmetric* (resp. *skew-symmetric*) if S is closed under inverse (resp. $a^{-1} \notin S$ for all $a \in S$). Define $\overline{S} = \{u \in S : u^{-1} \notin S\}$. Clearly $S \setminus \overline{S}$ is symmetric and $\overline{S}$ is skew-symmetric. The *mixed Cayley graph* $G = Cay(\Gamma, S)$ is a mixed graph, where $V(G) = \Gamma$ and $E(G) = \{(a,b) : a, b \in \Gamma, a^{-1}b \in S\}$. Since we have not assumed that S is symmetric, so a mixed Cayley graph can have directed edges. If S is symmetric, then G is a (simple) *Cayley graph*. If S is skew-symmetric then G is an *oriented Cayley graph*.

In 1982, Bridge and Mena [6] introduced a characterization of integral Cayley graphs over abelian groups. Later on, the exact characterization was rediscovered by Wasin So [17] for cyclic groups in 2005. In 2009, Abdollahi and Vatandoost [1] proved that there are exactly seven connected cubic integral Cayley graphs. In the same year, Klotz and Sander [13] proved that if the Cayley graph $Cay(\Gamma, S)$ over abelian group Γ is integral, then S belongs to the Boolean algebra $\mathbb{B}(\Gamma)$ generated by the subgroups of Γ, and its converse proved by Alperin and Peterson [3]. In 2013, DeVos et al. [8] gave a sufficient condition for the integrality of Cayley multigraphs and proved the necessary part for abelian groups, which in turn, is an alternative, character-theoretic proof of the characterization of Bridges and Mena [6]. In 2014, Cheng et al. [14] proved that normal Cayley graphs (its generating set S is closed under conjugation) of symmetric groups are integral. Alperin [2] gave a characterization of integral Cayley graphs over finite groups. In 2017, Lu et al. [16] gave necessary and sufficient conditions for the integrality of Cayley graphs over dihedral groups D_n. In particular, they completely determined all integral Cayley graphs of the dihedral group D_p for a prime p. In 2019, Cheng et al. [7] obtained several simple sufficient conditions for the integrality of Cayley graphs over the dicyclic group $T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$. In particular, they also completely determined all integral Cayley graphs over the dicyclic group T_{4p} for a prime p. In [12], the authors have characterized integral mixed circulant graphs in terms of their connection set. In this paper, we give a characterization of integral mixed Cayley graphs over abelian groups in terms if its connection set. In what follows, Γ is always taken to be a finite abelian group.

This paper is organized as follows. In second section, we express the eigenvalues of a mixed Cayley graph as a sum of eigenvalues of a simple Cayley graph and an oriented Cayley graph. In third section, we obtain a sufficient condition on the connection set S for integrality of the mixed Cayley graph $Cay(\Gamma, S)$ over an abelian group Γ. In fourth section, we prove the necessity of the sufficient condition obtained in Section 3.

2 Mixed Cayley graph and group characters

A *representation* of a finite group Γ is a homomorphism $\rho : \Gamma \to GL(V)$, where $GL(V)$ is the group of automorphisms of a finite dimensional vector space V over the complex field $\mathbb{C}$. The dimension of V is called the *degree* of ρ. Two representations ρ_1 and ρ_2 of Γ on V_1 and V_2, respectively, are *equivalent* if there is an isomorphism $T : V_1 \to V_2$ such that $T\rho_1(g) = \rho_2(g)T$ for all $g \in \Gamma$.

Let $\rho : \Gamma \to GL(V)$ be a representation. The *character* $\chi_\rho : \Gamma \to \mathbb{C}$ of ρ is defined by setting $\chi_\rho(g) = Tr(\rho(g))$ for $g \in \Gamma$, where $Tr(\rho(g))$ is the trace of the representation matrix of $\rho(g)$. By degree of χ_ρ we mean the degree of ρ which is simply $\chi_\rho(1)$. If W is a $\rho(g)$-invariant subspace of V for each $g \in \Gamma$, then we say W a $\rho(\Gamma)$-invariant subspace of V. If the only $\rho(\Gamma)$-invariant subspaces of V are $\{0\}$ and V, we say ρ an *irreducible representation* of Γ, and the corresponding character χ_ρ an *irreducible character* of Γ.

For a group Γ, we denote by $\mathrm{IRR}(\Gamma)$ and $\mathrm{Irr}(\Gamma)$ the complete set of non-equivalent irreducible representations of Γ and the complete set of non-equivalent irreducible characters of Γ, respectively.

Let Γ be a finite abelian group under addition with n elements, and S be a subset of Γ with $0 \notin S$, where 0 is the additive identity of Γ. Then Γ is isomorphic to the direct product of cyclic groups of prime power order, *i.e.*

$$\Gamma \cong \mathbb{Z}_{n_1} \otimes \cdots \otimes \mathbb{Z}_{n_k},$$

where $n = n_1 \cdots n_k$, and n_j is a power of a prime number for each $j = 1, \ldots, k$. We consider an abelian group Γ as $\mathbb{Z}_{n_1} \otimes \cdots \otimes \mathbb{Z}_{n_k}$ of order $n = n_1 \cdots n_k$. The *exponent* of Γ, denoted by $exp(\Gamma)$, is defined to be the least common multiple of $n_1, n_2, \ldots, n_k$. We consider the elements $x \in \Gamma$ as elements of the cartesian product $\mathbb{Z}_{n_1} \otimes \cdots \otimes \mathbb{Z}_{n_k}$, *i.e.*

$$x = (x_1, x_2, \ldots, x_k), \text{ where } x_j \in \mathbb{Z}_{n_j} \text{ for all } 1 \leqslant j \leqslant k.$$

Addition in Γ is done coordinate-wise modulo n_j. For a positive integer k and $a \in \Gamma$, we denote by ka or a^k the k-fold sum of a to itself, $(-k)a = k(-a)$, $0a = 0$, and inverse of a by $-a$.

Lemma 1. *[18] Let* $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ *be a cyclic group of order* n. *Then* $IRR(\mathbb{Z}_n) = \{\phi_k : 0 \leqslant k \leqslant n-1\}$, *where* $\phi_k(j) = \omega_n^{jk}$ *for all* $0 \leqslant j, k \leqslant n-1$, *and* $\omega_n = \exp(\frac{2\pi i}{n})$.

Lemma 2. *[18] Let Γ_1 and Γ_2 be two abelian groups of order m, n, respectively. Let $IRR(\Gamma_1) = \{\phi_1, \dots, \phi_m\}$, and $IRR(\Gamma_2) = \{\rho_1, \dots, \rho_n\}$. Then*

$$IRR(\Gamma_1 \times \Gamma_2) = \{\psi_{kl} : 1 \leqslant k \leqslant m, 1 \leqslant l \leqslant n\},$$

where $\psi_{kl} : \Gamma_1 \times \Gamma_2 \to \mathbb{C}^*$ and $\psi_{kl}(g_1, g_2) = \phi_k(g_1)\rho_l(g_2)$ for all $g_1 \in \Gamma_1, g_2 \in \Gamma_2$.

Consider $\Gamma = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$. By Lemma 1 and Lemma 2, $\text{IRR}(\Gamma) = \{\psi_\alpha : \alpha \in \Gamma\}$, where

$$\psi_\alpha(x) = \prod_{j=1}^{k} \omega_{n_j}^{\alpha_j x_j} \text{ for all } \alpha = (\alpha_1, \dots, \alpha_k), x = (x_1, \dots, x_k) \in \Gamma, \tag{1}$$

and $\omega_{n_j} = \exp(\frac{2\pi i}{n_j})$. Since Γ is an abelian group, every irreducible representation of Γ is 1-dimensional and thus it can be identified with its characters. Hence $\text{IRR}(\Gamma) = \text{Irr}(\Gamma)$. For $x \in \Gamma$, let $\text{ord}(x)$ denote the order of x. The following lemma can be easily proved.

Lemma 3. *Let Γ be an abelian group of order n, and $Irr(\Gamma) = \{\psi_\alpha : \alpha \in \Gamma\}$ be the set of all n characters of Γ. Then the following statements are true.*

(i) $\psi_\alpha(x) = \psi_x(\alpha)$ *for all* $x, \alpha \in \Gamma$.

(ii) $(\psi_\alpha(x))^{ord(x)} = (\psi_\alpha(x))^{ord(\alpha)} = 1$ *for all* $x, \alpha \in \Gamma$.

(iii) $\psi_\alpha(x)^l = 1$ *for all* $x, \alpha \in \Gamma$*, where* $l = exp(\Gamma)$.

Let $f : \Gamma \to \mathbb{C}$ be a function. The *Cayley color digraph* of Γ with *color function* f, denoted by $Cay(\Gamma, f)$, is defined to be the directed graph with vertex set Γ and arc set $\{(x, y) : x, y \in \Gamma\}$ such that each arc (x, y) is colored by $f(x^{-1}y)$. The *adjacency matrix* of $Cay(\Gamma, f)$ is defined to be the matrix whose rows and columns are indexed by the elements of Γ, and the (x, y)-entry is equal to $f(x^{-1}y)$. The eigenvalues of $Cay(\Gamma, f)$ are simply the eigenvalues of its adjacency matrix.

Theorem 4. *[4] Let Γ be a finite abelian group. Then the spectrum of the Cayley color digraph $Cay(\Gamma, f)$ is $\{\gamma_\alpha : \alpha \in \Gamma\}$, where*

$$\gamma_\alpha = \sum_{y \in \Gamma} f(y)\psi_\alpha(y) \quad \text{for all } \alpha \in \Gamma.$$

Lemma 5. *[4] Let Γ be an abelian group. Then the spectrum of the mixed Cayley graph $Cay(\Gamma, S)$ is $\{\gamma_\alpha : \alpha \in \Gamma\}$, where $\gamma_\alpha = \lambda_\alpha + \mu_\alpha$ and*

$$\lambda_\alpha = \sum_{s \in S \setminus \overline{S}} \psi_\alpha(s), \;\; \mu_\alpha = i \sum_{s \in \overline{S}} \Big(\psi_\alpha(s) - \psi_\alpha(-s)\Big), \text{ for all } \alpha \in \Gamma.$$

Proof. Define $f_S : \Gamma \to \{0, 1, i, -i\}$ such that

$$f_S(s) = \begin{cases} 1 & \text{if } s \in S \setminus \overline{S}, \\ i & \text{if } s \in \overline{S}, \\ -i & \text{if } s \in \overline{S}^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix of Cayley color digraph $Cay(\Gamma, f_S)$ agrees with the Hermitian adjacency matrix of mixed Cayley graph $Cay(\Gamma, S)$. Thus the result follows from Theorem 4. □

Next two corollaries are special cases of Lemma 5.

Corollary 6. *[13] Let Γ be an abelian group. Then the spectrum of the Cayley graph $Cay(\Gamma, S)$ is $\{\lambda_\alpha : \alpha \in \Gamma\}$, where $\lambda_\alpha = \lambda_{-\alpha}$ and*

$$\lambda_\alpha = \sum_{s \in S} \psi_\alpha(s) \text{ for all } \alpha \in \Gamma.$$

Proof. Note that $\overline{S} = \varnothing$, and so $s \in S$ if and only if $-s \in S$. Using Lemma 5, we have

$$\lambda_\alpha = \sum_{s \in S} \psi_\alpha(s) = \sum_{s \in S} \psi_{-\alpha}(-s) = \sum_{s \in S} \psi_{-\alpha}(s) = \lambda_{-\alpha}.$$ □

Corollary 7. *Let Γ be an abelian group. Then the spectrum of the oriented Cayley graph $Cay(\Gamma, S)$ is $\{\mu_\alpha : \alpha \in \Gamma\}$, where $\mu_\alpha = -\mu_{-\alpha}$ and*

$$\mu_\alpha = i \sum_{s \in S} \Big(\psi_\alpha(s) - \psi_\alpha(-s) \Big) \text{ for all } \alpha \in \Gamma.$$

Proof. Note that $S \setminus \overline{S} = \varnothing$. Using Lemma 5, we have

$$\mu_\alpha = i \sum_{s \in S} \Big(\psi_\alpha(s) - \psi_\alpha(-s) \Big) = i \sum_{s \in S} \Big(\psi_{-\alpha}(-s) - \psi_{-\alpha}(s) \Big) = -\mu_{-\alpha}.$$ □

Theorem 8. *Let Γ be an abelian group. The mixed Cayley graph $Cay(\Gamma, S)$ is integral if and only if both Cayley graph $Cay(\Gamma, S \setminus \overline{S})$ and oriented Cayley graph $Cay(\Gamma, \overline{S})$ are integral.*

Proof. Assume that the mixed Cayley graph $Cay(\Gamma, S)$ is integral. Let γ_α be an eigenvalue of mixed Cayley graph $Cay(\Gamma, S)$. By Lemma 5, Corollary 6 and Corollary 7, we have $\gamma_\alpha = \lambda_\alpha + \mu_\alpha$ and $\gamma_{-\alpha} = \lambda_\alpha - \mu_\alpha$ for all $\alpha \in \Gamma$, where λ_α is an eigenvalue of the Cayley graph $Cay(\Gamma, S \setminus \overline{S})$ and μ_α is an eigenvalue of the oriented Cayley graph $Cay(\Gamma, \overline{S})$. Thus $\lambda_\alpha = \frac{\gamma_\alpha + \gamma_{-\alpha}}{2} \in \mathbb{Q}$ and $\mu_\alpha = \frac{\gamma_\alpha - \gamma_{-\alpha}}{2} \in \mathbb{Q}$. As λ_α and μ_α are rational algebraic integers, so $\lambda_\alpha, \mu_\alpha \in \mathbb{Q}$ implies that λ_α and μ_α are integers. Thus the Cayley graph $Cay(\Gamma, S \setminus \overline{S})$ and the oriented Cayley graph $Cay(\Gamma, \overline{S})$ are integral.

Conversely, assume that both Cayley graph $Cay(\Gamma, S \setminus \overline{S})$ and oriented Cayley graph $Cay(\Gamma, \overline{S})$ are integral. Then Lemma 5 implies that $Cay(\Gamma, S)$ is integral. □

Let $n \geqslant 2$ be a fixed integer. Define $G_n(d) = \{k : 1 \leqslant k \leqslant n-1, \gcd(k,n) = d\}$. It is clear that $G_n(d) = dG_{\frac{n}{d}}(1)$.

Alperin and Peterson [3] considered a Boolean algebra generated by a class of subgroups of a group in order to determine the integrality of Cayley graphs over abelian groups. Suppose Γ is a finite group, and $\mathcal{F}_\Gamma$ is the family of all subgroups of Γ. The Boolean algebra $\mathbb{B}(\Gamma)$ generated by $\mathcal{F}_\Gamma$ is the set whose elements are obtained by arbitrary finite intersections, unions, and complements of the elements in the family $\mathcal{F}_\Gamma$. The minimal non-empty elements of this algebra are called *atoms*. Thus each element of $\mathbb{B}(\Gamma)$ is the union of some atoms. Consider the equivalence relation $\sim$ on Γ such that $x \sim y$ if and only if $y = x^k$ for some $k \in G_m(1)$, where $m = \text{ord}(x)$.

Lemma 9. *[3] The equivalence classes of $\sim$ are the atoms of $\mathbb{B}(\Gamma)$.*

For $x \in \Gamma$, let $[x]$ denote the equivalence class of x with respect to the relation $\sim$. Also, let $\langle x \rangle$ denote the cyclic group generated by x.

Lemma 10. *[3] The atoms of the Boolean algebra $\mathbb{B}(\Gamma)$ are the sets $[x] = \{y : \langle y \rangle = \langle x \rangle\}$.*

By Lemma 10, each element of $\mathbb{B}(\Gamma)$ is a union of some sets of the form $[x]$. Thus, for all $S \in \mathbb{B}(\Gamma)$, we have $S = [x_1] \cup \cdots \cup [x_k]$ for some $x_1, \ldots, x_k \in \Gamma$.

The next result provides a complete characterization of integral Cayley graphs over an abelian group Γ in terms of the atoms of $\mathbb{B}(\Gamma)$.

Theorem 11. *([3], [6]) Let Γ be an abelian group. The Cayley graph $Cay(\Gamma, S)$ is integral if and only if $S \in \mathbb{B}(\Gamma)$.*

3 A sufficient condition for integrality of mixed Cayley graphs over abelian groups

Unless otherwise stated, we consider Γ to be an abelian group of order n. Due to Theorem 8, to find characterization of the integral mixed Cayley graph $Cay(\Gamma, S)$, it is enough to find characterization of the integral Cayley graph $Cay(\Gamma, S \setminus \overline{S})$ and the integral oriented Cayley graph $Cay(\Gamma, \overline{S})$. The integral Cayley graph $Cay(\Gamma, S \setminus \overline{S})$ is characterized by Theorem 11. So our attempt is to characterize the integral oriented Cayley graph $Cay(\Gamma, \overline{S})$.

Define $\Gamma(4)$ to be the set of all $x \in \Gamma$ which satisfies $\text{ord}(x) \equiv 0 \pmod 4$. It is clear that $exp(\Gamma) \equiv 0 \pmod 4$ if and only if $\Gamma(4) \neq \varnothing$. For all $x \in \Gamma(4)$ and $r \in \{0, 1, 2, 3\}$, define

$$M_r(x) := \{x^k : 1 \leqslant k \leqslant \text{ord}(x), k \equiv r \pmod 4\}.$$

For all $a \in \Gamma$ and $S \subseteq \Gamma$, define $a + S := \{a + s : s \in S\}$ and $-S := \{-s : s \in S\}$. Note that $-s$ denotes the inverse of s, that is $-s = s^{m-1}$, where $m = \text{ord}(s)$.

Lemma 12. *Let Γ be an abelian group and $x \in \Gamma(4)$. Then the following statements are true.*

(i) $\bigcup_{r=0}^{3} M_r(x) = \langle x \rangle$.

(ii) Both $M_1(x)$ *and* $M_3(x)$ *are skew-symmetric subsets of* Γ.

(iii) $-M_1(x) = M_3(x)$ *and* $-M_3(x) = M_1(x)$.

(iv) $a + M_1(x) = M_3(x)$ *and* $a + M_3(x) = M_1(x)$ *for all* $a \in M_2(x)$.

(v) $a + M_1(x) = M_1(x)$ *and* $a + M_3(x) = M_3(x)$ *for all* $a \in M_0(x)$.

Proof. (i) It follows from the definitions of $M_r(x)$ and $\langle x \rangle$.

(ii) If $x^k \in M_1(x)$ then $-x^k = x^{n-k} \notin M_1(x)$, as $k \equiv 1 \pmod 4$ implies $n - k \equiv 3 \pmod 4$. Thus $M_1(x)$ is a skew-symmetric subset of Γ. Similarly, $M_3(x)$ is also a skew-symmetric subset of Γ.

(iii) As $k \equiv 1 \pmod 4$ if and only if $n - k \equiv 3 \pmod 4$, we get $-x^k = x^{n-k}$. Therefore $-M_1(x) = M_3(x)$ and $-M_3(x) = M_1(x)$.

(iv) Let $a \in M_2(x)$ and $y \in a + M_1(x)$. Then $a = x^{k_1}$ and $y = x^{k_1} + x^{k_2} = x^{k_1+k_2}$, where $k_1 \equiv 2 \pmod 4$ and $k_2 \equiv 1 \pmod 4$. Since $k_1 + k_2 \equiv 3 \pmod 4$, we have $y \in M_3(x)$ implying that $a + M_1(x) \subseteq M_3(x)$. Since size of both sets $M_1(x)$ and $M_3(x)$ are same, hence $a + M_1(x) = M_3(x)$. Similarly, $a + M_3(x) = M_1(x)$ for all $a \in M_2(x)$.

(v) The proof is similar to Part (iv). □

Lemma 13. *Let* $x \in \Gamma(4)$. *Then* $i\Bigg(\sum_{s \in M_1(x)} \psi_\alpha(s) - \sum_{s \in M_3(x)} \psi_\alpha(s)\Bigg) \in \mathbb{Z}$ *for all* $\alpha \in \Gamma$.

Proof. Let $x \in \Gamma(4)$, $\alpha \in \Gamma$ and

$$\mu_\alpha = i\Bigg(\sum_{s \in M_1(x)} \psi_\alpha(s) - \sum_{s \in M_3(x)} \psi_\alpha(s)\Bigg).$$

Case 1: There exists $a \in M_2(x)$ such that $\psi_\alpha(a) \neq -1$. Then

$$\begin{aligned}
\mu_\alpha &= -i\Bigg(\sum_{s \in M_3(x)} \psi_\alpha(s) - \sum_{s \in M_1(x)} \psi_\alpha(s)\Bigg) \\
&= -i\Bigg(\sum_{s \in a+M_1(x)} \psi_\alpha(s) - \sum_{s \in a+M_3(x)} \psi_\alpha(s)\Bigg) \\
&= -i\Bigg(\sum_{s \in M_1(x)} \psi_\alpha(a+s) - \sum_{s \in M_3(x)} \psi_\alpha(a+s)\Bigg) \\
&= -i\psi_\alpha(a)\Bigg(\sum_{s \in M_1(x)} \psi_\alpha(s) - \sum_{s \in M_3(x)} \psi_\alpha(s)\Bigg) \\
&= -\psi_\alpha(a)\mu_\alpha,
\end{aligned}$$

We have $(1+\psi_\alpha(a))\mu_\alpha = 0$. Since $\psi_\alpha(a) \neq -1$, so $\mu_\alpha = 0 \in \mathbb{Z}$.
Case 2: There exists $a \in M_0(x)$ such that $\psi_\alpha(a) \neq 1$. Applying the same process as in Case 1, we get $\mu_\alpha = 0 \in \mathbb{Z}$.
Case 3: Assume that $\psi_\alpha(a) = -1$ for all $a \in M_2(x)$ and $\psi_\alpha(a) = 1$ for all $a \in M_0(x)$. Then $\psi_\alpha(a) = -\psi_\alpha(x)$ for all $a \in M_3(x)$ and $\psi_\alpha(a) = \psi_\alpha(x)$ for all $a \in M_1(x)$. Therefore

$$\mu_\alpha = i\Bigg(\sum_{s\in M_1(x)} \psi_\alpha(s) - \sum_{s\in M_3(x)} \psi_\alpha(s)\Bigg) = 2i\psi_\alpha(x)|M_1(x)|.$$

Since $\psi_\alpha(x)^4 = 1$ and μ_α is real, we have $\psi_\alpha(x) = \pm i$. Thus $\mu_\alpha = \pm 2|M_1(x)| \in \mathbb{Z}$. □

For $m \equiv 0 \pmod 4$ and $r \in \{1,3\}$, define

$$G_m^r(1) = \{k : k \equiv r \pmod 4, \gcd(k,m) = 1\}.$$

Define an equivalence relation $\approx$ on $\Gamma(4)$ such that $x \approx y$ if and only if $y = x^k$ for some $k \in G_m^1(1)$, where $m = \text{ord}(x)$. Observe that if $x, y \in \Gamma(4)$ and $x \approx y$ then $x \sim y$, but the converse need not be true. For example, consider $x = 5 \pmod{12}$, $y = 11 \pmod{12}$ in $\mathbb{Z}_{12}$. Here $x, y \in \mathbb{Z}_{12}(4)$ and $x \sim y$ but $x \not\approx y$. For $x \in \Gamma(4)$, let $[\![x]\!]$ denote the equivalence class of x with respect to the relation $\approx$.

Lemma 14. *Let Γ be an abelian group, $x \in \Gamma(4)$ and $m = ord(x)$. Then the following are true.*

(i) $[\![x]\!] = \{x^k : k \in G_m^1(1)\}$.

(ii) $[\![-x]\!] = \{x^k : k \in G_m^3(1)\}$.

(iii) $[\![x]\!] \cap [\![-x]\!] = \varnothing$.

(iv) $[x] = [\![x]\!] \cup [\![-x]\!]$.

Proof.

(i) Let $y \in [\![x]\!]$. Then $x \approx y$, and so $\text{ord}(x) = \text{ord}(y) = m$ and there exists $k \in G_m^1(x)$ such that $y = x^k$. Thus $[\![x]\!] \subseteq \{x^k : k \in G_m^1(1)\}$. On the other hand, let $z = x^k$ for some $k \in G_m^1(1)$. Then $\text{ord}(x) = \text{ord}(z)$ and so $x \approx z$. Thus $\{x^k : k \in G_m^1(1)\} \subseteq [\![x]\!]$.

(ii) Note that $-x = x^{m-1}$ and $m-1 \equiv 3 \pmod 4$. By Part (i),

$$\begin{aligned}[\![-x]\!] = \{(-x)^k : k \in G_m^1(1)\} &= \{x^{(m-1)k} : k \in G_m^1(1)\}\\ &= \{x^{-k} : k \in G_m^1(1)\}\\ &= \{x^k : k \in G_m^3(1)\}.\end{aligned}$$

(iii) Since $G_m^1(1) \cap G_m^3(1) = \varnothing$, so by Part (i) and Part (ii), $[\![x]\!] \cap [\![-x]\!] = \varnothing$ holds.

(iv) Since $[x] = \{x^k : k \in G_m(1)\}$ and $G_m(1)$ is a disjoint union of $G_m^1(1)$ and $G_m^3(1)$, by Part (i) and Part (ii), $[x] = [\![x]\!] \cup [\![-x]\!]$ holds. $\square$

Let D_g be the set of all odd divisors of g, and D_g^1 (resp. D_g^3) be the set of all odd divisors of g which are congruent to 1 (resp. 3) modulo 4. It is clear that $D_g = D_g^1 \cup D_g^3$.

Lemma 15. *Let Γ be an abelian group, $x \in \Gamma(4)$, $m = ord(x)$ and $g = \frac{m}{4}$. Then the following are true.*

(i) $M_1(x) \cup M_3(x) = \bigcup\limits_{h \in D_g} [x^h]$.

(ii) $M_1(x) = \bigcup\limits_{h \in D_g^1} [\![x^h]\!] \cup \bigcup\limits_{h \in D_g^3} [\![-x^h]\!]$.

(iii) $M_3(x) = \bigcup\limits_{h \in D_g^1} [\![-x^h]\!] \cup \bigcup\limits_{h \in D_g^3} [\![x^h]\!]$.

Proof. (i) Let $x^k \in M_1(x) \cup M_3(x)$, where $k \equiv 1 \text{ or } 3 \pmod 4$. To show $x^k \in \bigcup\limits_{h \in D_g} [x^h]$, it is enough to show $x^k \sim x^h$ for some $h \in D_g$. Let $h = \gcd(k, g) \in D_g$. Note that

$$\text{ord}(x^k) = \frac{m}{\gcd(m, k)} = \frac{m}{\gcd(g, k)} = \frac{m}{h} = \text{ord}(x^h).$$

Also, as $h = \gcd(k, m)$, we have $\langle x^k \rangle = \langle x^h \rangle$, and so $x^k = x^{hj}$ for some $j \in G_q(1)$, where $q = \text{ord}(x^h) = \frac{m}{h}$. Thus $x^k \sim x^h$ where $h = \gcd(k, g) \in D_g$. Conversely, let $z \in \bigcup\limits_{h \in D_g} [x^h]$. Then there exists $h \in D_g$ such that $z = x^{hj}$ where $j \in G_q(1)$ and $q = \frac{m}{\gcd(m,h)}$. Now $h \in D_g$ and $q \equiv 0 \pmod 4$ imply that both h and j are odd integers. Thus $hj \equiv 1 \text{ or } 3 \pmod 4$ and so $\bigcup\limits_{h \in D_g} [x^h] \subseteq M_1(x) \cup M_3(x)$. Hence $M_1(x) \cup M_3(x) = \bigcup\limits_{h \in D_g} [x^h]$.

(ii) Let $x^k \in M_1(x)$, where $k \equiv 1 \pmod 4$. By Part (i), there exists $h \in D_g$ and $j \in G_q(1)$ such that $x^k = x^{hj}$, where $q = \frac{m}{\gcd(m,h)}$. Note that $k = jh$. If $h \equiv 1 \pmod 4$ then $j \in G_q^1(1)$, otherwise $j \in G_q^3(1)$. Thus using parts (i) and (ii) of Lemma 14, if $h \equiv 1 \pmod 4$ then $x^k \approx x^h$, otherwise $x^k \approx -x^h$. Hence $M_1(x) \subseteq \bigcup\limits_{h \in D_g^1} [\![x^h]\!] \cup \bigcup\limits_{h \in D_g^3} [\![-x^h]\!]$. Conversely, assume that $z \in \bigcup\limits_{h \in D_g^1} [\![x^h]\!] \cup \bigcup\limits_{h \in D_g^3} [\![-x^h]\!]$. This gives $z \in [\![x^h]\!]$ for an $h \in D_g^1$ or $z \in [\![-x^h]\!]$ for an $h \in D_g^3$. In the first case, by part (i) of Lemma 14, there exists $j \in G_q^1(1)$ with $q = \frac{m}{\gcd(m,h)}$ such that $z = x^{hj}$. Similarly, for the second case, by part (ii) of Lemma 14, there exists $j \in G_q^3(1)$ with $q = \frac{m}{\gcd(m,h)}$ such that $z = x^{hj}$. In both the cases, $hj \equiv 1 \pmod 4$. Thus $z \in M_1(x)$.

(iii) The proof is similar to Part (ii). $\square$

Lemma 16. *Let* $x \in \Gamma(4)$*. Then* $i\Big(\sum_{s\in[\![x]\!]} \psi_\alpha(s) - \sum_{s\in[\![-x]\!]} \psi_\alpha(s)\Big) \in \mathbb{Z}$ *for all* $\alpha \in \Gamma$.

Proof. Note that there exists $x \in \Gamma(4)$ with $\operatorname{ord}(x) = 4$. Apply induction on $\operatorname{ord}(x)$. If $\operatorname{ord}(x) = 4$, then $M_1(x) = [\![x]\!]$ and $M_3(x) = [\![-x]\!]$. Hence by Lemma 13,

$$i\Big(\sum_{s\in[\![x]\!]} \psi_\alpha(s) - \sum_{s\in[\![-x]\!]} \psi_\alpha(s)\Big) \in \mathbb{Z} \quad \text{for all } \alpha \in \Gamma.$$

Assume that the statement holds for all $x \in \Gamma(4)$ with $\operatorname{ord}(x) \in \{4, 8, \ldots, 4(g-1)\}$. We prove it for $\operatorname{ord}(x) = 4g$. Lemma 15 implies that

$$M_1(x) = \bigcup_{h\in D_g^1} [\![x^h]\!] \cup \bigcup_{h\in D_g^3} [\![-x^h]\!] \quad \text{and} \quad M_3(x) = \bigcup_{h\in D_g^1} [\![-x^h]\!] \cup \bigcup_{h\in D_g^3} [\![x^h]\!].$$

If $\operatorname{ord}(x) = 4g = m$ and $h > 1$ then $\operatorname{ord}(x^h), \operatorname{ord}(-x^h) \in \{4, 8, \ldots, 4(g-1)\}$. By induction hypothesis

$$i\Big(\sum_{s\in[\![x^h]\!]} \psi_\alpha(s) - \sum_{s\in[\![-x^h]\!]} \psi_\alpha(s)\Big) \in \mathbb{Z} \text{ for all } \alpha \in \Gamma.$$

Now we have

$$\begin{aligned} i\Big(\sum_{s\in M_1(x)} \psi_\alpha(s) - \sum_{s\in M_3(x)} \psi_\alpha(s)\Big) &= i\Big(\sum_{s\in[\![x]\!]} \psi_\alpha(s) - \sum_{s\in[\![-x]\!]} \psi_\alpha(s)\Big) \\ &+ \sum_{h\in D_g^1, h>1} i\Big(\sum_{s\in[\![x^h]\!]} \psi_\alpha(s) - \sum_{s\in[\![-x^h]\!]} \psi_\alpha(s)\Big) \\ &+ \sum_{h\in D_g^3, h>1} i\Big(\sum_{s\in[\![-x^h]\!]} \psi_\alpha(s) - \sum_{s\in[\![x^h]\!]} \psi_\alpha(s)\Big). \end{aligned}$$

Hence

$$\begin{aligned} i\Big(\sum_{s\in[\![x]\!]} \psi_\alpha(s) - \sum_{s\in[\![-x]\!]} \psi_\alpha(s)\Big) &= i\Big(\sum_{s\in M_1(x)} \psi_\alpha(s) - \sum_{s\in M_3(x)} \psi_\alpha(s)\Big) \\ &- \sum_{h\in D_g^1, h>1} i\Big(\sum_{s\in[\![x^h]\!]} \psi_\alpha(s) - \sum_{s\in[\![-x^h]\!]} \psi_\alpha(s)\Big) \\ &+ \sum_{h\in D_g^3, h>1} i\Big(\sum_{s\in[\![x^h]\!]} \psi_\alpha(s) - \sum_{s\in[\![-x^h]\!]} \psi_\alpha(s)\Big) \end{aligned}$$

is also an integer for all $\alpha \in \Gamma$ because of Lemma 13 and induction hypothesis. □

For $exp(\Gamma) \equiv 0 \pmod 4$, define $\mathbb{D}(\Gamma)$ to be the set of all skew-symmetric subsets S of Γ such that $S = [\![x_1]\!] \cup \cdots \cup [\![x_k]\!]$ for some $x_1, \ldots, x_k \in \Gamma(4)$. For $exp(\Gamma) \not\equiv 0 \pmod 4$, define $\mathbb{D}(\Gamma) = \{\varnothing\}$.

Theorem 17. *Let* Γ *be an abelian group. If* $S \in \mathbb{D}(\Gamma)$ *then the oriented Cayley graph* $Cay(\Gamma, S)$ *is integral.*

Proof. Assume that $S \in \mathbb{D}(\Gamma)$. Then $S = [\![x_1]\!] \cup \cdots \cup [\![x_k]\!]$ for some $x_1, \ldots, x_k \in \Gamma(4)$. Let $Sp_H(Cay(\Gamma, S)) = \{\mu_\alpha : \alpha \in \Gamma\}$. We have

$$\mu_\alpha = i \sum_{s \in S} \Big(\psi_\alpha(s) - \psi_\alpha(-s) \Big) = \sum_{j=1}^{k} \sum_{s \in [\![x_j]\!]} i \Big(\psi_\alpha(s) - \psi_\alpha(-s) \Big).$$

Now by Lemma 16, $\mu_\alpha \in \mathbb{Z}$ for all $\alpha \in \Gamma$. Hence the oriented Cayley graph $Cay(\Gamma, S)$ is integral. □

Theorem 18. *Let* Γ *be an abelian group. If* $S \setminus \overline{S} \in \mathbb{B}(\Gamma)$ *and* $\overline{S} \in \mathbb{D}(\Gamma)$ *then the mixed Cayley graph* $Cay(\Gamma, S)$ *is integral.*

Proof. By Theorem 8, $Cay(\Gamma, S)$ is integral if and only if both $Cay(\Gamma, S \setminus \overline{S})$ and $Cay(\Gamma, \overline{S})$ are integral. Thus the result follows from Theorem 11 and Theorem 17. □

4 Characterization of integral mixed Cayley graphs over abelian groups

The *cyclotomic polynomial* $\Phi_n(x)$ is the monic polynomial whose zeros are the primitive n^{th} root of unity. That is

$$\Phi_n(x) = \prod_{a \in G_n(1)} (x - \omega_n^a),$$

where $\omega_n = \exp(\frac{2\pi i}{n})$. Clearly the degree of $\Phi_n(x)$ is $\varphi(n)$. See [11] for more details about cyclotomic polynomials.

Theorem 19. *[11] The cyclotomic polynomial* $\Phi_n(x)$ *is irreducible in* $\mathbb{Z}[x]$.

The polynomial $\Phi_n(x)$ is irreducible over $\mathbb{Q}(i)$ if and only if $[\mathbb{Q}(i, \omega_n) : \mathbb{Q}(i)] = \varphi(n)$. Also $\mathbb{Q}(\omega_n)$ does not contain the number $i = \sqrt{-1}$ if and only if $n \not\equiv 0 \pmod 4$. Thus, if $n \not\equiv 0 \pmod 4$ then $[\mathbb{Q}(i, \omega_n) : \mathbb{Q}(\omega_n)] = 2 = [\mathbb{Q}(i), \mathbb{Q}]$, and therefore

$$[\mathbb{Q}(i, \omega_n) : \mathbb{Q}(i)] = \frac{[\mathbb{Q}(i, \omega_n) : \mathbb{Q}(\omega_n)].[\mathbb{Q}(\omega_n) : \mathbb{Q}]}{[\mathbb{Q}(i) : \mathbb{Q}]} = [\mathbb{Q}(\omega_n) : \mathbb{Q}] = \varphi(n).$$

Hence for $n \not\equiv 0 \pmod 4$, the polynomial $\Phi_n(x)$ is irreducible over $\mathbb{Q}(i)$.

Let $n \equiv 0 \pmod 4$. Then $\mathbb{Q}(i, \omega_n) = \mathbb{Q}(\omega_n)$, and so

$$[\mathbb{Q}(i, \omega_n) : \mathbb{Q}(i)] = \frac{[\mathbb{Q}(i, \omega_n) : \mathbb{Q}]}{[\mathbb{Q}(i) : \mathbb{Q}]} = \frac{\varphi(n)}{2}.$$

Hence the polynomial $\Phi_n(x)$ is reducible over $\mathbb{Q}(i)$.

We know that $G_n(1)$ is a disjoint union of $G_n^1(1)$ and $G_n^3(1)$. Define

$$\Phi_n^1(x) = \prod_{a \in G_n^1(1)} (x - \omega_n^a) \text{ and } \Phi_n^3(x) = \prod_{a \in G_n^3(1)} (x - \omega_n^a).$$

It is clear from the definition that $\Phi_n(x) = \Phi_n^1(x)\Phi_n^3(x)$.

Theorem 20. *[12] Let $n \equiv 0 \pmod 4$. The factors $\Phi_n^1(x)$ and $\Phi_n^3(x)$ of $\Phi_n(x)$ are irreducible monic polynomials in $\mathbb{Q}(i)[x]$ of degree $\frac{\varphi(n)}{2}$.*

In this section, first we prove that there is no integral oriented Cayley graph $Cay(\Gamma, S)$ for $exp(\Gamma) \not\equiv 0 \pmod 4$ and $S \neq \varnothing$. After that we find a necessary condition on the set S so that the mixed Cayley graph $Cay(\Gamma, S)$ is integral.

Theorem 21. *Let Γ be an abelian group and $exp(\Gamma) \not\equiv 0 \pmod 4$. Then the oriented Cayley graph $Cay(\Gamma, S)$ is integral if and only if $S = \varnothing$.*

Proof. Let $l = exp(\Gamma)$ and $Sp_H(Cay(\Gamma, S)) = \{\mu_\alpha : \alpha \in \Gamma\}$. Assume that $l \not\equiv 0 \pmod 4$ and $Cay(\Gamma, S)$ is integral. By Corollary 7, $\mu_\alpha = -\mu_{-\alpha} \in \mathbb{Q}$ and

$$\mu_\alpha = i \sum_{s \in S} \Big(\psi_\alpha(s) - \psi_\alpha(-s) \Big) \text{ for all } \alpha \in \Gamma.$$

Note that, $\psi_\alpha(s)$ and $\psi_\alpha(-s)$ are l^{th} roots of unity for all $\alpha \in \Gamma, s \in S$. Fix a primitive l^{th} root ω of unity and express $\psi_\alpha(s)$ in the form ω^j for some $j \in \{0, 1, \ldots, l-1\}$. Thus

$$\mu_\alpha = i \sum_{s \in S} \Big(\psi_\alpha(s) - \psi_\alpha(-s) \Big) = \sum_{j=0}^{l-1} a_j \omega^j,$$

where $a_j \in \mathbb{Q}(i)$. Since $\mu_\alpha \in \mathbb{Q}$, so $p(x) = \sum\limits_{j=0}^{l-1} a_j x^j - \mu_\alpha \in \mathbb{Q}(i)[x]$ and ω is a root of $p(x)$. Since $l \not\equiv 0(\mod 4)$, so $\Phi_l(x)$ is irreducible in $\mathbb{Q}(i)[x]$. Thus $p(\omega) = 0$ and $\Phi_l(x)$ is the monic irreducible polynomial over $\mathbb{Q}(i)$ having ω as a root. Therefore $\Phi_l(x)$ divides $p(x)$, and so $\omega^{-1} = \omega^{l-1}$ is also a root of $p(x)$. Note that, if $\psi_\alpha(s) = \omega^j$ for some $j \in \{0, 1, \ldots, l-1\}$ then $\psi_{-\alpha}(s) = \omega^{-j}$. We have

$$0 = p(\omega^{-1}) = \sum_{j=0}^{l-1} a_j \omega^{-j} - \mu_\alpha = \mu_{-\alpha} - \mu_\alpha \Rightarrow \mu_\alpha = \mu_{-\alpha}.$$

Since $\mu_{-\alpha} = -\mu_\alpha$, we get $\mu_\alpha = 0$, for all $\alpha \in \Gamma$. Hence $S = \varnothing$.

Conversely, if $S = \varnothing$ then all the eigenvalues of $Cay(\Gamma, S)$ are zero. Thus $Cay(\Gamma, S)$ is integral. □

Lemma 14 says that corresponding to each equivalence class of the relation $\sim$ we get two equivalence classes of the relation $\approx$. Define E to be the matrix of size $n \times n$, whose rows and columns are indexed by elements of Γ such that $E_{x,y} = i\psi_x(y)$. Note that each row of E corresponds to a character of Γ and $EE^* = nI_n$, where E^* is the conjugate transpose of E. Let $v_{[\![x]\!]}$ be the vector in $\mathbb{Q}^n$ whose coordinates are indexed by the elements of Γ, and the a^{th} coordinate of $v_{[\![x]\!]}$ is given by

$$v_{[\![x]\!]}(a) = \begin{cases} 1 & \text{if } a \in [\![x]\!], \\ -1 & \text{if } a \in [\![x^{-1}]\!], \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 16, we have $Ev_{[\![x]\!]} \in \mathbb{Q}^n$.

Lemma 22. *Let Γ be an abelian group, $v \in \mathbb{Q}^n$ and $Ev \in \mathbb{Q}^n$. Let the coordinates of v be indexed by elements of Γ. Then*

(i) $v_x = -v_{-x}$ *for all* $x \in \Gamma$.

(ii) $v_x = v_y$ *for all* $x, y \in \Gamma(4)$ *satisfying* $x \approx y$.

(iii) $v_x = 0$ *for all* $x \in \Gamma \setminus \Gamma(4)$.

Proof. Let E_x and E_y denote the column vectors of E indexed by x and y, respectively, and assume that $u = Ev \in \mathbb{Q}^n$. For $z \in \mathbb{C}$, let $\overline{z}$ denote the complex conjugate of z.

(i) We use the fact that $\overline{\psi_x(y)} = \psi_{-x}(y) = \psi_x(-y)$ for all $x, y \in \Gamma$. Again

$$u = Ev \Rightarrow E^*u = E^*Ev = (nI_n)v \Rightarrow \frac{1}{n}E^*u = v \in \mathbb{Q}^n.$$

Thus

$$\begin{aligned} v_x = \frac{1}{n}(E^*u)_x = \frac{1}{n}\sum_{a\in\Gamma} E^*_{x,a}u_a = \frac{1}{n}\sum_{a\in\Gamma} \overline{i\psi_a(x)}u_a &= -\frac{1}{n}\sum_{a\in\Gamma} i\psi_a(-x)u_a \\ &= -\overline{\frac{1}{n}\sum_{a\in\Gamma} \overline{i\psi_a(-x)}u_a} \\ &= -\overline{\frac{1}{n}\sum_{a\in\Gamma} E^*_{-x,a}u_a} \\ &= -\overline{\frac{1}{n}(E^*u)_{-x}} \\ &= -\overline{v_{-x}} = -v_{-x}. \end{aligned}$$

(ii) If $\Gamma(4) = \varnothing$ then there is nothing to prove. Now assume that $\Gamma(4) \neq \varnothing$, so that $exp(\Gamma) \equiv 0 \pmod 4$. Let $x, y \in \Gamma(4)$ and $x \approx y$. Then there exists $k \in G_m^1(1)$ such that $y = x^k$, where $m = \text{ord}(x) = \text{ord}(y)$. Assume that $x \neq y$, so that $k \geqslant 2$. Using

Lemma 3, entries of E_x and E_y are i times an m^{th} root of unity. Fix a primitive m^{th} root of unity ω, and express each entry of E_x and E_y in the form $i\omega^j$ for some $j \in \{0, 1, \ldots, m-1\}$. Thus

$$nv_x = (E^*u)_x = \sum_{j=0}^{m-1} a_j \omega^j,$$

where $a_j \in \mathbb{Q}(i)$ for all j. Thus ω is a root of $p(x) = \sum_{j=0}^{m-1} a_j x^j - nv_x \in \mathbb{Q}(i)[x]$. Therefore, $p(x)$ is a multiple of the irreducible polynomial $\Phi_m^1(x)$, and so ω^k is also a root of $p(x)$, because of $k \in G_m^1(1)$. As $y = x^k$ implies that $\psi_y(a) = \psi_x(a)^k$ for all $a \in \Gamma$, we have $(E^*u)_y = \sum_{j=0}^{m-1} a_j \omega^{kj}$. Hence

$$0 = p(\omega^k) = \sum_{j=0}^{m-1} a_j \omega^{kj} - nv_x = (E^*u)_y - nv_x = nv_y - nv_x \Rightarrow v_x = v_y.$$

(iii) Let $x \in \Gamma \setminus \Gamma(4)$ and $r = \mathrm{ord}(x) \not\equiv 0 \pmod 4$. Fix a primitive r^{th} root ω of unity, and express each entry of E_x in the form $i\omega^j$ for some $j \in \{0, 1, \ldots, r-1\}$. Thus

$$nv_x = (E^*u)_x = \sum_{j=0}^{r-1} a_j \omega^j,$$

where $a_j \in \mathbb{Q}(i)$ for all j. Thus ω is a root of $p(x) = \sum_{j=0}^{r-1} a_j x^j - nv_x \in \mathbb{Q}(i)[x]$. Therefore, $p(x)$ is a multiple of the irreducible polynomial $\Phi_r(x)$, and so ω^{-1} is also a root of $p(x)$. Since $\psi_{-x}(a) = \psi_x(a)^{-1}$ for all $a \in \Gamma$, therefore $(E^*u)_{-x} = \sum_{j=0}^{r-1} a_j \omega^{-j}$. Hence

$$0 = p(\omega^{-1}) = \sum_{j=0}^{r-1} a_j \omega^{-j} - nv_x = (E^*u)_{-x} - nv_x = nv_{-x} - nv_x,$$

implies that $v_x = v_{-x}$. This together with Part (i) imply that $v_x = 0$ for all $x \in \Gamma \setminus \Gamma(4)$. □

Theorem 23. *Let Γ be an abelian group. The oriented Cayley graph $Cay(\Gamma, S)$ is integral if and only if $S \in \mathbb{D}(\Gamma)$.*

Proof. Assume that the oriented Cayley graph $Cay(\Gamma, S)$ is integral. If $\Gamma(4) = \varnothing$ then by Theorem 21, we have $S = \varnothing$, and so $S \in \mathbb{D}(\Gamma)$. Now assume that $exp(\Gamma) \equiv 0 \pmod 4$ so that $\Gamma(4) \neq \varnothing$. Let v be the vector in $\mathbb{Q}^n$ whose coordinates are indexed by the elements of Γ, and the x^{th} coordinate of v is given by

$$v_x = \begin{cases} 1 & \text{if } x \in S, \\ -1 & \text{if } x \in S^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$(Ev)_a = \sum_{x\in\Gamma} E_{a,x} v_x = \sum_{x\in S} E_{a,x} - \sum_{x\in S^{-1}} E_{a,x} = i \sum_{x\in S} \left(\psi_a(x) - \psi_a(-x)\right).$$

Thus $(Ev)_a$ is an eigenvalue of the integral oriented Cayley graph $Cay(\Gamma, S)$. Therefore $Ev \in \mathbb{Q}^n$, and hence all the three conditions of Lemma 22 hold.

By the third condition of Lemma 22, $v_x = 0$ for all $x \in \Gamma \setminus \Gamma(4)$, and so we must have $S \cup S^{-1} \subseteq \Gamma(4)$. Again, let $x \in S$, $y \in \Gamma(4)$ and $x \approx y$. The second condition of Lemma 22 gives $v_x = v_y$, which implies that $y \in S$. Thus $x \in S$ implies $[\![x]\!] \subseteq S$. Hence $S \in \mathbb{D}(\Gamma)$. The converse part follows from Theorem 17. $\square$

The following example illustrates Theorem 23.

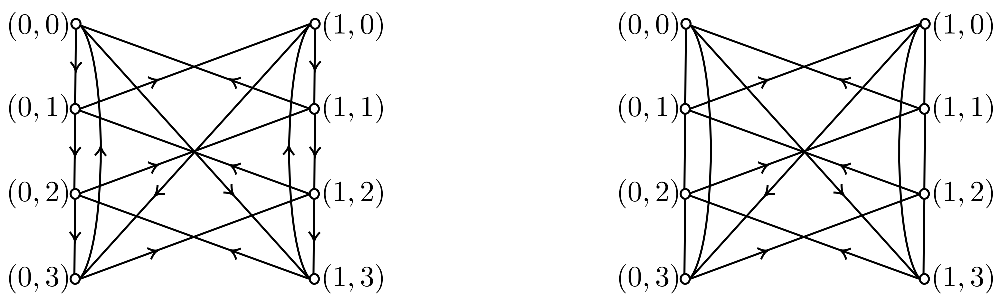
Example 24. Consider $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$ and $S = \{(0,1), (1,3)\}$. The graph $Cay(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$ is shown in Figure 1a. We see that $[\![(0,1)]\!] = \{(0,1)\}$ and $[\![(1,3)]\!] = \{(1,3)\}$. Therefore $S \in \mathbb{D}(\Gamma)$. Further, using Corollary 7 and Equation 1, the eigenvalues of $Cay(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$ are obtained as

$$\mu_\alpha = i[\psi_\alpha(0,1) - \psi_\alpha(0,3)] + i[\psi_\alpha(1,3) - \psi_\alpha(1,1)] \text{ for each } \alpha \in \mathbb{Z}_2 \times \mathbb{Z}_4,$$

where

$$\psi_\alpha(x) = (-1)^{\alpha_1 x_1} i^{\alpha_2 x_2} \text{ for all } \alpha = (\alpha_1, \alpha_2), x = (x_1, x_2) \in \mathbb{Z}_2 \times \mathbb{Z}_4.$$

It can be seen that $\mu_{(0,0)} = \mu_{(0,1)} = \mu_{(0,2)} = \mu_{(0,3)} = \mu_{(1,0)} = \mu_{(1,2)} = 0$, $\mu_{(1,1)} = -4$ and $\mu_{(1,3)} = 4$. Thus $Cay(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$ is integral.



(a) $S = \{(0,1), (1,3)\}$ (b) $S = \{(0,1), (0,3), (1,3)\}$

Figure 1: The graph $Cay(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$

Theorem 25. *Let* Γ *be an abelian group. The mixed Cayley graph* $Cay(\Gamma, S)$ *is integral if and only if* $S \setminus \overline{S} \in \mathbb{B}(\Gamma)$ *and* $\overline{S} \in \mathbb{D}(\Gamma)$*.*

Proof. By Theorem 8, the mixed Cayley graph $Cay(\Gamma, S)$ is integral if and only if both $Cay(\Gamma, S \setminus \overline{S})$ and $Cay(\Gamma, \overline{S})$ are integral. Note that $S \setminus \overline{S}$ is a symmetric set and $\overline{S}$ is a skew-symmetric set. Thus by Theorem 11, $Cay(\Gamma, S \setminus \overline{S})$ is integral if and only if $S \setminus \overline{S} \in \mathbb{B}(\Gamma)$. By Theorem 23, $Cay(\Gamma, \overline{S})$ is integral if and only if $\overline{S} \in \mathbb{D}(\Gamma)$. Hence the result follows. □

The following example illustrates Theorem 25.

Example 26. Consider $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_4$ and $S = \{(0,1),(0,3),(1,3)\}$. The graph $Cay(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$ is shown in Figure 1b. Observe that $\overline{S} = \{(1,3)\} = [\![(1,3)]\!] \in \mathbb{D}(\Gamma)$ and $S \setminus \overline{S} = \{(0,1),(0,3)\} = [(0,1)] \in \mathbb{B}(\Gamma)$. Further, using Lemma 5 and Equation 1, the eigenvalues of $Cay(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$ are obtained as

$$\mu_\alpha = [\psi_\alpha(0,1) + \psi_\alpha(0,3)] + i[\psi_\alpha(1,3) - \psi_\alpha(1,1)] \text{ for each } \alpha \in \mathbb{Z}_2 \times \mathbb{Z}_4,$$

where

$$\psi_\alpha(x) = (-1)^{\alpha_1 x_1} i^{\alpha_2 x_2} \text{ for all } \alpha = (\alpha_1, \alpha_2), x = (x_1, x_2) \in \mathbb{Z}_2 \times \mathbb{Z}_4.$$

One can see $\mu_{(0,0)} = \mu_{(0,1)} = \mu_{(1,0)} = \mu_{(1,3)} = 2$ and $\mu_{(0,2)} = \mu_{(0,3)} = \mu_{(1,1)} = \mu_{(1,2)} = -2$. Thus $Cay(\mathbb{Z}_2 \times \mathbb{Z}_4, S)$ is integral.

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