Jones' Conjecture in subcubic graphs*

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Abstract

We confirm Jones' Conjecture for subcubic graphs. Namely, if a subcubic planar graph does not contain $k+1$ vertex-disjoint cycles, then it suffices to delete $2k$ vertices to obtain a forest.

Mathematics Subject Classifications: 05C10, 05C38, 05C69

1 Introduction

We investigate the connection between the maximum number of vertex-disjoint cycles in a graph and the minimum number of vertices whose deletion results in a cycle-free graph, i.e. a forest. A *cycle packing* of a (multi)graph G is a set of vertex disjoint cycles that appear in G as subgraphs. We denote the maximum size of a cycle packing of G by $\mathrm{cp}(G)$.

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A *feedback vertex set* of a (multi)graph G is a set S of vertices such that $G - S$ is a forest. We denote an arbitrary minimum feedback vertex set of G as $FVS(G)$ and denote its size by $\mathrm{fvs}(G)$.

Erdős and Pósa [4] showed that there is a constant c such that for any graph G, $\mathrm{fvs}(G) \leqslant c \cdot \mathrm{cp}(G) \log \mathrm{cp}(G)$, and that this upper-bound is tight for some graphs. This seminal result led to rich developments. Two main directions: determining which structures can replace "cycles" there (free to increase the bounding function), and improving the bounding functions. Most notably, in the first line of research, Cames van Batenburg et al. [1] proved that "cycle" can directly be replaced with "minor of a given planar graph", which was earlier showed by Robertson and Seymour [9] with a worse bounding function.

We focus on the second line of research, and are interested in the best possible bound in the original theorem, when restricted to the specific case of planar graphs. A drastic improvement is then possible, and we were motivated by the following elusive conjecture.

Conjecture 1 ("Jones' Conjecture"[1], Kloks, Lee and Liu [5])**.** Every planar graph G satisfies $\mathrm{fvs}(G) \leqslant 2 \cdot \mathrm{cp}(G)$.

Note that Conjecture 1 is tight for wheels or for the dodecahedron. Currently, the best known bound is that every planar graph G satisfies $\mathrm{fvs}(G) \leqslant 3 \cdot \mathrm{cp}(G)$, as proved independently by Chappel et al. [2], Chen et al. [3], and Ma et al. [7].

In his PhD Thesis, Munaro [8] considered the case of subcubic graphs and made significant progress. Here we complete the case, and prove that Jones' Conjecture holds for subcubic graphs.

Theorem 2. *Every subcubic planar multigraph G satisfies* $\mathrm{fvs}(G) \leqslant 2 \cdot \mathrm{cp}(G)$.

2 Proof of Theorem 2

2.1 Notation

A multigraph is *simple* if it has no loops or multi-edges. In this case, we simply refer to it as a graph.

Let $G = (V, E)$ be a (multi)graph. For $W \subseteq V$, we denote by $G[W]$ the subgraph of G induced by W, and by $G - W$ the subgraph of G induced by $V \setminus W$. If $W = \{v\}$, then we denote $G - v = G - W$. For $F \subseteq E$, we denote $G - F = (V, E \setminus F)$. If $F = \{e\}$, then we denote $G - e = G - F$. For $W \cap V = \emptyset$, $G + W$ is the disjoint union of G and a set W of isolated vertices. If $W = \{v\}$, then we denote $G + v = G + W$. For F a set of pairs of edges with $F \cap E = \emptyset$, we denote $G + F = (V, E \cup F)$. If $F = \{e\}$, then we denote $G + e = G + F$. Graph is called *cubic* if all of its vertices have degree exactly 3. Graph is called *subcubic* all of its vertices have degree at most 3.

A (multi)graph is *k-connected* if it has at least $k + 1$ vertices and the removal of at most $k - 1$ vertices leaves the graph connected. A (multi)graph is *k-edge-connected*

[1] http://www.openproblemgarden.org/op/jones_conjecture

if the removal of at most $k-1$ edges leaves the graph connected. Note that a subcubic (multi)graph with at least $k+1$ vertices is k-connected if and only if it is k-edge-connected. In a connected (multi)graph, a *separating edge* or *bridge* is an edge the removal of which disconnects the graph.

A (multi)graph is *essentially* 4*-edge-connected* if the removal of at most three edges does not yield two components with at least two vertices each. A (multi)graph is *cyclically* 4*-edge-connected* if the removal of at most three edges does not yield two components that both contain a cycle. For a cubic (multi)graph, these last two notions are equivalent.

2.2 Proof

We proceed by contradiction. Let G be a counter-example to Theorem 2 with the minimum number of vertices. To obtain a contradiction, we first argue that G is a simple graph that is essentially 4-edge-connected, as follows.

Lemma 3. *The multigraph G is an essentially* 4*-edge-connected simple graph.*

Proof. While Claims 4, 5, 6 are known and easy properties of a minimum counter-example to Jones' conjecture on subcubic graphs (see e.g. [8]), we include their proofs because we believe they may constitute a useful warm-up. The uninterested reader may skip them.

First the result is easily checked on small multigraphs, so we can assume that G has at least 4 vertices. Hence, for $k \leqslant 3$, G is k-connected if and only if it is k-edge connected.

Claim 4. *The multigraph G is cubic.*

Proof. Suppose G has a vertex v with degree at most 1. Then $G-v$ satisfies Jones' Conjecture by minimality of G. As no cycle of G contains v, $\mathrm{fvs}(G-v) = \mathrm{fvs}(G)$ and $\mathrm{cp}(G-v) = \mathrm{cp}(G)$, therefore G also satisfies Jones' Conjecture, a contradiction.

Suppose G has a vertex v with degree 2, and let u and w be the two neighbors of v. Then $G' = G - v + uw$ satisfies Jones' Conjecture, so $\mathrm{fvs}(G') \leqslant 2 \cdot \mathrm{cp}(G')$. The cycles of G are in bijection with the cycles of G', by exchanging the edges uv and vw and an edge uw when appropriate. Hence $\mathrm{cp}(G) = \mathrm{cp}(G')$. Moreover, if S is a feedback vertex set of G that does not contain v, then S is a feedback vertex set of G', and if S is a feedback vertex set of G that contains v, then $(S \setminus \{v\}) \cup \{u\}$ is a feedback vertex set of G'. Thus $\mathrm{fvs}(G) \leqslant \mathrm{fvs}(G') \leqslant 2 \cdot \mathrm{cp}(G') = \mathrm{cp}(G)$, and G satisfies Jones' Conjecture, a contradiction. Hence G is cubic. ⌟

Claim 5. *The multigraph G is* 2*-connected.*

Proof. Suppose that G is not 2-connected. As G is cubic, that means that G is not 2-edge-connected. Let e be a separating edge of G. Both components G_1 and G_2 of $G-e$ verify Jones' Conjecture by minimality of G. Since e is separating, it is not in any cycle of G. The union of any feedback vertex set of G_1 and any feedback vertex set of G_2 is a feedback vertex set of G, so $\mathrm{fvs}(G) \leqslant \mathrm{fvs}(G_1) + \mathrm{fvs}(G_2)$. The union of any cycle packing of G_1 and any cycle packing of G_2 is a cycle packing of G, so $\mathrm{cp}(G) \geqslant \mathrm{cp}(G_1) + \mathrm{cp}(G_2)$. Therefore G satisfies Jones' Conjecture, a contradiction. ⌟

Claim 6. *The multigraph G is 3-connected.*

Proof. Assume that it is not 3-connected, and thus not 3-edge-connected. Let $e = u_1u_2$ and $e' = v_1v_2$ be a 2-edge-cut, where u_1 and v_1 are in the same connected component of $G - \{e, e'\}$, which we denote G_1. Let G_2 be the other connected component of $G - \{e, e'\}$. We write $G'_1 = G_1 + \{u_1v_1\}$ and $G'_2 = G_2 + \{u_2v_2\}$. Note that this may lead to a double (or even triple) edge. See Figure 1 for an illustration. By minimality of G, we know that G_1, G_2, G'_1 and G'_2 all satisfy Jones' Conjecture.

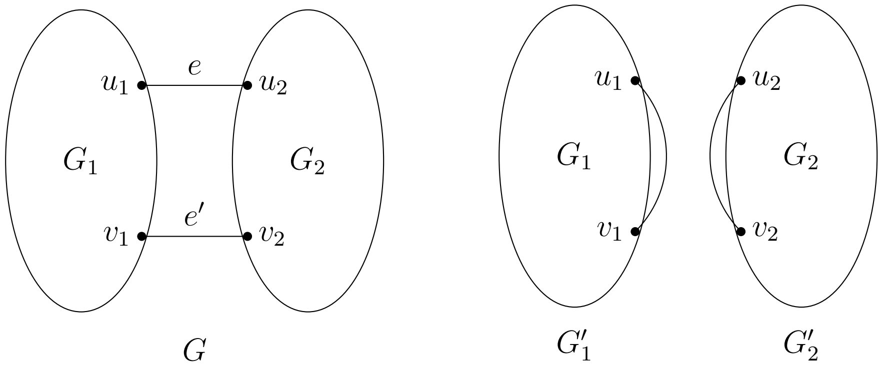

Figure 1: The graphs G, G'_1, and G'_2 in Claim 6.

Note that since $G'_1 = G_1 + \{u_1v_1\}$, $\mathrm{cp}(G_1) \leqslant \mathrm{cp}(G'_1) \leqslant \mathrm{cp}(G_1) + 1$. We first argue that $\mathrm{cp}(G'_1) = \mathrm{cp}(G_1) + 1$. Assume for a contradiction that $\mathrm{cp}(G'_1) = \mathrm{cp}(G_1)$. Note that for any feedback vertex set S_1 of G'_1, either $u_1 \in S_1$ or $v_1 \in S_1$ or u_1 and v_1 are in distinct components of $G_1 - S_1$, so $\mathrm{fvs}(G) \leqslant \mathrm{fvs}(G'_1) + \mathrm{fvs}(G_2)$. Thus, $\mathrm{fvs}(G) \leqslant \mathrm{fvs}(G'_1) + \mathrm{fvs}(G_2) \leqslant 2\,\mathrm{cp}(G'_1) + 2\,\mathrm{cp}(G_2) = 2\,\mathrm{cp}(G_1) + 2\,\mathrm{cp}(G_2) \leqslant 2\,\mathrm{cp}(G)$, a contradiction.

By symmetry, $\mathrm{cp}(G'_2) = \mathrm{cp}(G_2) + 1$. Therefore, every cycle packing of G'_1 contains the edge u_1v_1 and every cycle packing of G'_2 contains the edge u_2v_2. We can thus combine a cycle packing of G'_1 and a cycle packing of G'_2 by making a single cycle out of those two cycles. So $\mathrm{cp}(G) = \mathrm{cp}(G_1) + \mathrm{cp}(G_2) + 1$. However, if S_1 is a feedback vertex set of G_1 and S_2 is a feedback vertex set of G_2, then $S_1 \cup S_2 \cup \{u_1\}$ is a feedback vertex set of G. Therefore $\mathrm{fvs}(G) \leqslant \mathrm{fvs}(G_1) + \mathrm{fvs}(G_2) + 1 \leqslant 2\,\mathrm{cp}(G_1) + 2\,\mathrm{cp}(G_2) + 1 < 2\,\mathrm{cp}(G)$, a contradiction. Therefore G is 3-connected. ⌟

In particular since G is cubic, Claim 6 implies that G is a simple graph.

Claim 7. *The graph G is essentially 4-edge-connected.*

Proof. Assume that G is not essentially 4-edge-connected, and thus not cyclically 4-edge-connected. Consider a non-trivial 3-edge-cut $\{e_A, e_B, e_C\}$. Let G_1 and G_2 be the two components of $G \setminus \{e_A, e_B, e_C\}$. For $i \in \{1, 2\}$, we define G_i^{ABC} as the graph obtained from G by contracting G_{3-i} into a single vertex x. See Figure 2 for an illustration. We define G_i^{AB} (resp. G_i^{AC}, G_i^{BC}) as the graph obtained from G_i by connecting with an edge vertices from G_i incident to e_A and e_B (resp. to e_A and e_C for G_i^{AC} or to e_B and e_C for

G_i^{BC}). Again, this may lead to a double edge. Note that for both values of i, all of G_i, G_i^{AB}, G_i^{AC}, G_i^{BC} and G_i^{ABC} have fewer vertices than G, and thus satisfy Jones' Conjecture.

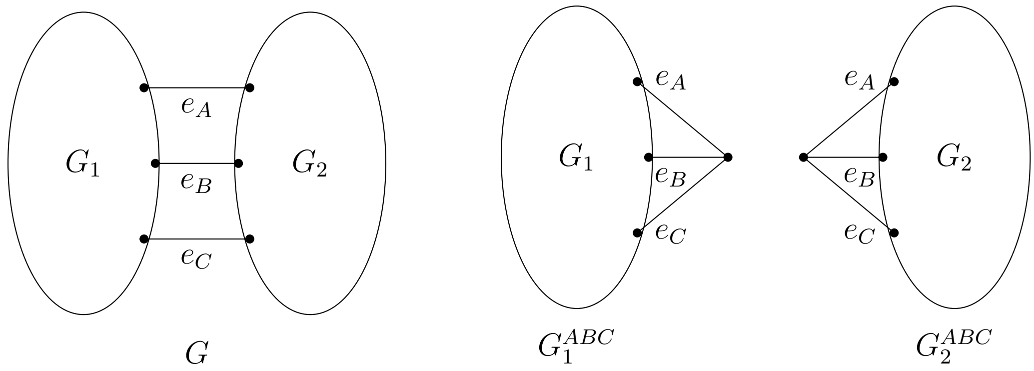

Figure 2: The graphs G, G_1^{ABC}, and G_2^{ABC}.

First note that for both values of i, $\mathrm{fvs}(G) \leqslant \mathrm{fvs}(G_i^{ABC}) + \mathrm{fvs}(G_{3-i})$. In order to prove that let us assume without loss of generality that $i = 1$. Then remove $FVS(G_2)$ from G. What remains from G_2 after deleting $FVS(G_2)$ is a forest that could hypothetically create connections between vertices from G_1 incident to e_A, e_B, e_C. However if we are given any tree T and its three vertices $u, v, w \in V(T)$ then $P_{uv} \cap P_{vw} \cap P_{wu} \neq \emptyset$ (in fact it is always a single vertex), where P_{uv}, P_{vw}, P_{wu} are sets of vertices on unique paths between corresponding vertices, so it is possible to break the connections between all three pairs of these vertices by removing a single vertex of T. Because of that we see that $\mathrm{fvs}(G - FVS(G_2)) \leqslant \mathrm{fvs}(G_1^{ABC})$ what leads to $\mathrm{fvs}(G) \leqslant \mathrm{fvs}(G_1^{ABC}) + \mathrm{fvs}(G_2)$. Therefore we see that $\mathrm{fvs}(G) \leqslant \mathrm{fvs}(G_i^{ABC}) + \mathrm{fvs}(G_{3-i}) \leqslant \mathrm{fvs}(G_1) + \mathrm{fvs}(G_2) + 1 \leqslant 2\,\mathrm{cp}(G_1) + 2\,\mathrm{cp}(G_2) + 1$. We also have $\mathrm{cp}(G) \geqslant \mathrm{cp}(G_1) + \mathrm{cp}(G_2)$, yet $\mathrm{fvs}(G) > 2\,\mathrm{cp}(G)$.

It follows that for both values of i:

$$\mathrm{fvs}(G_i^{ABC}) = \mathrm{fvs}(G_i) + 1 \tag{1}$$

$$\mathrm{fvs}(G_i) = 2\,\mathrm{cp}(G_i) \tag{2}$$

And that:

$$\mathrm{fvs}(G) = \mathrm{fvs}(G_1) + \mathrm{fvs}(G_2) + 1 \tag{3}$$

$$\mathrm{cp}(G) = \mathrm{cp}(G_1) + \mathrm{cp}(G_2) \tag{4}$$

We are now ready for a closer analysis.

(i) For any i and for any $x \in \{A, B, C\}$,

$$\mathrm{fvs}(G) \leqslant \mathrm{fvs}(G_i^{ABC-x}) + \max_{y \neq x} \mathrm{fvs}(G_{3-i}^{ABC-y}).$$

Indeed, take without loss of generality $i = 1$ and $x = C$. Let us consider a minimum feedback vertex set S of G_1^{AB}. Note that in $G_1 \setminus S$, there is no path between the vertex incident to e_A and the vertex incident to e_B or at least one of them is in S. As a consequence, either vertex incident to e_C is in S or one of them, say the vertex incident to e_A, is either in S or is not in the same component as the vertex incident to e_C. For any minimum feedback vertex set S' of G_2^{BC}, we observe that $S \cup S'$ is a feedback vertex set of G, hence the conclusion. In particular, by combining with (3), if $\mathrm{fvs}(G_i^{ABC-x}) = \mathrm{fvs}(G_i)$ then $\mathrm{fvs}(G_{3-i}^{ABC-y}) = \mathrm{fvs}(G_{3-i}) + 1$ for some $y \neq x$.

(ii) For every $i \in \{1, 2\}$ and for every $x \in \{A, B, C\}$, if $\mathrm{fvs}(G_i^{ABC-x}) = \mathrm{fvs}(G_i) + 1$, then $\mathrm{cp}(G_i^{ABC-x}) = \mathrm{cp}(G_i) + 1$. That follows from (2), since G_i^{ABC-x} satisfies Jones' Conjecture.

(iii) For every $x \in \{A, B, C\}$,

$$\text{either } \mathrm{cp}(G_1^{ABC-x}) = \mathrm{cp}(G_1) \text{ or } \mathrm{cp}(G_2^{ABC-x}) = \mathrm{cp}(G_2).$$

Indeed, suppose not. Then both $\mathrm{cp}(G_1^{ABC-x}) = \mathrm{cp}(G_1) + 1$ and $\mathrm{cp}(G_2^{ABC-x}) = \mathrm{cp}(G_2) + 1$, for say $x = C$. Then for $i \in \{1, 2\}$, in every cycle packing of G_i^{AB} there is a cycle containing e_A and e_B. By taking a cycle packing of G_1^{AB} and a cycle packing of G_2^{AB}, we obtain a cycle packing of G (combining two cycles into one). So $\mathrm{cp}(G) \geqslant \mathrm{cp}(G_1) + \mathrm{cp}(G_2) + 1$, a contradiction with (4).

It follows from (1) and (2) that $\mathrm{cp}(G_i^{ABC}) = \mathrm{cp}(G_i) + 1$ for both values of i. Note that a maximum cycle packing of G_i^{ABC} uses two edges out of $\{e_A, e_B, e_C\}$. It follows that for some $z_i \in \{A, B, C\}$, it holds that $\mathrm{cp}(G_i^{ABC-z_i}) = \mathrm{cp}(G_i) + 1$.

We assume without loss of generality that $z_1 = C$. Note that from (iii), $\mathrm{cp}(G_2^{AB}) = \mathrm{cp}(G_2)$, hence $z_2 \neq C$, and $\mathrm{fvs}(G_2^{AB}) = \mathrm{fvs}(G_2)$ by (ii). We assume without loss of generality $z_2 = A$. By symmetry, $\mathrm{cp}(G_1^{BC}) = \mathrm{cp}(G_1)$ and $\mathrm{fvs}(G_1^{BC}) = \mathrm{fvs}(G_1)$. From (i) applied with $i = 1$ and $x = A$, there is $y \in \{B, C\}$ such that $\mathrm{fvs}(G_2^{ABC-y}) = \mathrm{fvs}(G_2) + 1$. Note that $y \neq C$, so $y = B$ and $\mathrm{fvs}(G_2^{AC}) = \mathrm{fvs}(G_2) + 1$. We derive from (ii) that $\mathrm{cp}(G_2^{AC}) = \mathrm{cp}(G_2) + 1$, hence $\mathrm{cp}(G_1^{AC}) = \mathrm{cp}(G_1)$ by (iii). Again from (ii), we obtain $\mathrm{fvs}(G_1^{AC}) = \mathrm{fvs}(G_1)$.

Therefore $\mathrm{fvs}(G_2^{AB}) = \mathrm{fvs}(G_2)$, $\mathrm{fvs}(G_1^{BC}) = \mathrm{fvs}(G_1)$, and $\mathrm{fvs}(G_1^{AC}) = \mathrm{fvs}(G_1)$. Now, (i) applied with $i = 2$ and $x = C$ yields a contradiction. ⌟

The conclusion follows directly from Claim 7. □

To obtain the desired contradiction, we combine Lemma 3 with the following very convenient theorem from Munaro [8]:

Theorem 8 (Theorem 3.4.10 in [8])**.** *If G is a simple subcubic graph which is a counter-example to Jones' Conjecture and which has the minimum number of vertices, then G is not cyclically 4-edge-connected.*

The main ingredient of the proof of Theorem 8 is an explicit formula for the minimum size of a feedback vertex set in a simple cubic cyclically 4-edge-connected graph. In [8], this formula is deduced using properties of the maximum genus of such a graph. In the next section, we include a more direct proof that, despite relying on the same combinatorial backbone — the matroid matching problem — is more self-contained and highlights the key properties of highly connected cubic graphs that allows us to compute the minimum size of a feedback vertex set.

3 An alternative proof of Theorem 8

As argued with Claim 4 in the proof of Lemma 3, to prove Theorem 8 it suffices to handle simple cubic graphs that are cyclically 4-edge-connected. That is, we henceforth assume G is a simple planar cyclically 4-edge-connected cubic graph and our goal is to prove that $\operatorname{fvs}(G) \leqslant 2\operatorname{cp}(G)$.

Instead of directly comparing $\operatorname{fvs}(G)$ and $\operatorname{cp}(G)$, we bound the value of each of them by a function of the number of vertices n. Note that the dual graph of G is also planar, hence it admits an independent set of size $\lceil \frac{f}{4} \rceil$, where f is the number of faces in the plane embedding of G. Note that an independent set in the dual of G is a cycle packing in G. (In fact, it is a face packing, that is, a cycle packing where every cycle is a face.) Therefore, G satisfies $\operatorname{cp}(G) \geqslant \lceil \frac{f}{4} \rceil$. By Euler's formula, $f = \frac{n}{2} + 2$, because G is cubic. So $\operatorname{cp}(G) \geqslant \lceil \frac{n}{8} + \frac{1}{2} \rceil$. That $2\operatorname{cp}(G) \geqslant \operatorname{fvs}(G)$ then follows from Theorem 9 below.

Theorem 9. *Every simple cyclically-4-edge-connected cubic graph G with n vertices satisfies* $\operatorname{fvs}(G) = \lceil \frac{n+1}{4} \rceil$.

Equivalently, we need to argue that in such a graph, the maximum number of vertices in an induced forest is $\lfloor \frac{3n-1}{4} \rfloor$. To look for a largest induced forest in G, we rephrase the question as a matroid matching problem in the line graph of G. To this end, we need a few definitions.

Let H be a multigraph. A *ν-pair* in H is a pair of edges of H with a common endpoint. A ν-graph is a pair $(H, \mathcal{V})$ where H is a multigraph and $\mathcal{V}$ is a partition of $E(H)$ into ν-pairs. A *cactus* in $(H, \mathcal{V})$ is a set $X \subseteq \mathcal{V}$ such that $\bigcup_{F \in X} F$ is a forest. Let $\beta(H, \mathcal{V})$ be the maximum size of a cactus in $(H, \mathcal{V})$.

Let G be a graph as in Theorem 9 and let $L(G)$ be the line graph of G. Construct a ν-graph $(H, \mathcal{V})$ as follows. Take $V(H) = V(L(G)) = E(G)$. For every $v \in V(G)$, let e_1, e_2, e_3 be the three edges of G incident with v; the line graph $L(G)$ features a triangle $e_1e_2e_3$. Pick arbitrarily two edges of this triangle, say e_1e_2 and e_2e_3, add them to $E(H)$ and add them as a ν-pair F_v to $\mathcal{V}$. The following observation is straightforward.

Lemma 10. *For every $X \subseteq V(G)$, $G[X]$ is a forest if and only if $\bigcup_{v \in X} F_v$ induces a forest in H.*

Hence, the question of determining the maximum size of an induced forest in G is equivalent to the question of computing $\beta(H, \mathcal{V})$.

To achieve this goal, we rely on a min-max formula by Lovász [6], following an exposition of Szigeti [10]. Let $(H, \mathcal{V})$ be a ν-graph. Let $\mathcal{P} = \{V_1, \dots, V_\ell\}$ be a partition of $V(H)$ and let $\mathcal{Q} = \{U_1, \dots, U_k\}$ be a partition of $\mathcal{V}$. For every $1 \leqslant i \leqslant k$, let $P(U_i)$ be the set of those indices $1 \leqslant j \leqslant \ell$ for which there is a ν-pair in U_i where one or both of the edges have an endpoint in V_j. Define

$$\operatorname{val}(\mathcal{P}, \mathcal{Q}) = |V(H)| - \ell + \sum_{i=1}^{k} \left\lfloor \frac{|P(U_i)| - 1}{2} \right\rfloor .$$

It is not difficult to observe that, for fixed $\mathcal{P}$ and $\mathcal{Q}$, $\operatorname{val}(\mathcal{P}, \mathcal{Q})$ is an upper bound on $\beta(H, \mathcal{V})$. Lovász proved that some choice of $\mathcal{P}$ and $\mathcal{Q}$ actually yields a tight upper bound [6, 10].

Theorem 11. *For every ν-graph $(H, \mathcal{V})$ there exist partitions $\mathcal{P}$ and $\mathcal{Q}$ such that*

$$\operatorname{val}(\mathcal{P}, \mathcal{Q}) = \beta(H, \nu).$$

Recall that in our setting $V(H) = E(G)$, so $\mathcal{P} = \{V_1, V_2, \dots, V_\ell\}$ is actually a partition of $E(G)$. Furthermore, every ν-pair $F_v \in \mathcal{V}$ corresponds to a vertex $v \in V(G)$. By somewhat abusing the notation, we henceforth treat $\mathcal{Q} = \{U_1, U_2, \dots, U_k\}$ as a partition of $V(G)$. With this notation, $P(U_i)$ is the set of indices j such that V_j contains an edge (of G) incident with a vertex of U_i (which is a subset of $V(G)$).

Consider now the following pair of partitions $(\mathcal{P}_0, \mathcal{Q}_0)$:

- $\mathcal{P}_0$ is the finest partition of $E(G)$: each V_i is a singleton set and $\ell = |E(G)| = \frac{3n}{2}$.
- $\mathcal{Q}_0$ is the coarsest partition of $V(G)$: $k = 1$ and $U_1 = V(G)$.

Since n is even due to G being cubic,

$$\operatorname{val}(\mathcal{P}_0, \mathcal{Q}_0) = \left\lfloor \frac{\frac{3n}{2} - 1}{2} \right\rfloor = \left\lfloor \frac{3n-2}{4} \right\rfloor = \left\lfloor \frac{3n-1}{4} \right\rfloor .$$

We conclude that $\beta(H, \nu) \leqslant \left\lfloor \frac{3n-1}{4} \right\rfloor$ and it remains to argue that the minimum of $\operatorname{val}(\cdot, \cdot)$ is attained for the pair $(\mathcal{P}_0, \mathcal{Q}_0)$.

To this end, pick a pair $(\mathcal{P}, \mathcal{Q})$ of partitions that:

- minimizes $\operatorname{val}(\mathcal{P}, \mathcal{Q})$,
- subject to the above, maximizes ℓ, and
- subject to the above, minimizes k.

It remains to show that $\mathcal{P} = \mathcal{P}_0$ and $\mathcal{Q} = \mathcal{Q}_0$. We prove this in two steps.

Lemma 12. *$\mathcal{P} = \mathcal{P}_0$, that is, $\mathcal{P}$ partitions $E(G)$ into singletons.*

Proof. Assume the contrary. Let $\mathcal{P} = \{V_1, V_2, \ldots, V_\ell\}$ and w.l.o.g. assume $|V_\ell| > 1$. Pick $e \in V_\ell$. Consider a partition $\mathcal{P}' = \{V_1', \ldots, V_{\ell+1}'\}$ where $V_i' = V_i$ for $i < \ell$, $V_\ell' = V_\ell \setminus \{e\}$ and $V_{\ell+1}' = \{e\}$; that is, we split V_ℓ into $V_\ell' = V_\ell \setminus \{e\}$ and $V_{\ell+1}' = \{e\}$. By the choice of $\mathcal{P}$ and $\mathcal{Q}$,

$$\mathrm{val}(\mathcal{P}, \mathcal{Q}) < \mathrm{val}(\mathcal{P}', \mathcal{Q}). \tag{5}$$

Let $P(U_i)$ be the set of those indices $j \in [\ell]$ for which V_j contains an edge incident with a vertex in U_i. Similarly, let $P'(U_i)$ be the set of those indices $j \in [\ell+1]$ for which V_j' contains an edge incident with a vertex in U_i. Clearly, $P(U_i) \triangle P'(U_i) \subseteq \{\ell, \ell+1\}$.

Let $uv = e$, and let $a, b \in [k]$ be such that $u \in U_a$ and $v \in U_b$, where a and b are not required to be distinct. Observe that if $i \in [k] \setminus \{a, b\}$, then $P(U_i) = P'(U_i)$. For $i \in \{a, b\}$, it holds that $\ell + 1 \in P'(U_i)$, $\ell \in P(U_i)$ and, as $\mathcal{P}$ has ℓ sets V_j, $\ell + 1 \notin P(U_i)$. Hence $|P'(U_i)| = |P(U_i)| + 1$ if $\ell \in P'(U_i)$ and $|P'(U_i)| = |P(U_i)|$ if $\ell \notin P'(U_i)$.

From the above observations it in particular follows that

$$\mathrm{val}(\mathcal{P}', \mathcal{Q}) - \mathrm{val}(\mathcal{P}, \mathcal{Q}) = -1 + \sum_{i \in \{a,b\}} \left(\left\lfloor \frac{|P'(U_i)| - 1}{2} \right\rfloor - \left\lfloor \frac{|P(U_i)| - 1}{2} \right\rfloor \right) \tag{6}$$

Further, for $i \in \{a, b\}$ it holds that $|P'(U_i)| - |P(U_i)| \in \{0, 1\}$ and hence

$$\left\lfloor \frac{|P'(U_i)| - 1}{2} \right\rfloor - \left\lfloor \frac{|P(U_i)| - 1}{2} \right\rfloor \in \{0, 1\}. \tag{7}$$

Observe that (6) and (7) imply that the only possibility for (5) to hold is that $a \neq b$ and both of the following equalities hold:

$$\left\lfloor \frac{|P'(U_a)| - 1}{2} \right\rfloor - \left\lfloor \frac{|P(U_a)| - 1}{2} \right\rfloor = 1,$$
$$\left\lfloor \frac{|P'(U_b)| - 1}{2} \right\rfloor - \left\lfloor \frac{|P(U_b)| - 1}{2} \right\rfloor = 1.$$

The above implies that $|P(U_a)|$ is even and $|P'(U_a)| = |P(U_a)| + 1$, hence $\ell \in P'(U_a)$. Similarly we infer that $|P(U_b)|$ is even, $|P'(U_b)| = |P(U_b)| + 1$, and $\ell \in P'(U_b)$.

Consider now a partition $\mathcal{Q}'$ of $V(G)$ that is created from $\mathcal{Q}$ by merging U_a and U_b into one set U'. Let $P'(U')$ be the set of those indices $j \in [\ell+1]$ for which there is an edge of V_j' that is incident with a vertex in U'; that is, $P'(U') = P'(U_a) \cup P'(U_b)$. Recall that both ℓ and $\ell + 1$ are elements of $P'(U_a)$ and $P'(U_b)$, and thus

$$|P'(U')| \leqslant |P'(U_a)| + |P'(U_b)| - 2 = |P(U_a)| + |P(U_b)|.$$

Consequently, as both $|P(U_a)|$ and $|P(U_b)|$ are even,

$$\left\lfloor \frac{|P'(U')| - 1}{2} \right\rfloor \leqslant \left\lfloor \frac{|P(U_a)| + |P(U_b)| - 1}{2} \right\rfloor = \left\lfloor \frac{|P(U_a)| - 1}{2} \right\rfloor + \left\lfloor \frac{|P(U_b)| - 1}{2} \right\rfloor + 1.$$

We infer that $\mathrm{val}(\mathcal{P}', \mathcal{Q}') \leqslant \mathrm{val}(\mathcal{P}, \mathcal{Q})$ while $|\mathcal{P}'| > |\mathcal{P}|$, a contradiction to the choice of $\mathcal{P}$ and $\mathcal{Q}$. This finishes the proof of the lemma. □

Lemma 13. *If $\mathcal{P} = \mathcal{P}_0$, then $\mathcal{Q} = \mathcal{Q}_0$, that is, $\mathcal{Q}$ partitions $V(G)$ into one set.*

Proof. Assume the contrary, $k \geqslant 2$. By the assumption $\mathcal{P} = \mathcal{P}_0$, we can treat $P(U_i)$ as the set of edges of $E(G)$ incident with at least one vertex of U_i. By reordering the sets U_i if necessary, we can assume that for some $0 \leqslant r \leqslant k$ it holds that $|P(U_i)|$ is even if and only if $i \leqslant r$. Furthermore, let $\delta(U_i)$ be the set of edges of G having exactly one endpoint in U_i. Since G is cyclically-4-edge-connected, $\delta(U_i) \geqslant 3$ for every $i \in [k]$ and, furthermore, $\delta(U_i) \geqslant 4$ for every $i \in [r]$. Then, as G is cubic,

$$\begin{aligned}
2\mathrm{val}(\mathcal{P}, \mathcal{Q}) &= 2\sum_{i=1}^{k} \left\lfloor \frac{|P(U_i)| - 1}{2} \right\rfloor \\
&= 2\sum_{i=1}^{r} \left(\frac{|P(U_i)|}{2} - 1 \right) + 2\sum_{i=r+1}^{k} \left(\frac{|P(U_i)|}{2} - \frac{1}{2} \right) \\
&= -k - r + \sum_{i=1}^{k} |P(U_i)| \\
&= -k - r + |E(G)| + \frac{1}{2}\sum_{i=1}^{k} \delta(U_i) \\
&\geqslant -k - r + \frac{3n}{2} + \frac{1}{2}(3k + r) \\
&= \frac{3n}{2} + \frac{1}{2}(k - r) \geqslant \frac{3n}{2}.
\end{aligned}$$

Hence $\mathrm{val}(\mathcal{P}, \mathcal{Q}) \geqslant \lceil \frac{3n}{4} \rceil \geqslant \mathrm{val}(\mathcal{P}_0, \mathcal{Q}_0)$ while $|\mathcal{Q}| > |\mathcal{Q}_0|$, a contradiction. □

As we argued, Lemmas 12 and 13 finish the proof of Theorem 9, which in turn implies Theorem 8.

4 Conclusion

Through a non-trivial combination of elementary tricks and using a nice preliminary result of [8], we were able to close the case of Jones' Conjecture for subcubic graphs.

The obvious question is whether this can be at all used to solve the whole conjecture. The reduction we have for subcubic graphs extends easily to the general setting, in the sense that a smallest counter-example to Jones' Conjecture is essentially 4-edge-connected. It is not difficult to argue in a similar way that such a graph is 3-vertex-connected. However, a much harder question is whether it is essentially 4-vertex-connected. While it still seems possible, such a result using our approach would require additional tricks. Note that being in the general setting also gives us more leeway regarding possible reductions (no need to shy away from increasing the maximum degree, as long as there are fewer vertices).

A second obstacle to generalization is that even assuming that a smallest counter-example is essentially 4-vertex-connected, Theorem 8 only deals with the subcubic case. Another argument must then be devised.

A different approach would be not to aim for the conjectured bound of 2 but simply for any bound better than the existing one of 3. Unfortunately, this does not seem conceptually much easier. Let us emphasize this: a simple discharging argument yields $\mathrm{fvs}(G) \leqslant 3\,\mathrm{cp}(G)$ for every planar graph G, while even significant effort fails to grant a factor of $(3-\epsilon)$ instead of 3.

To highlight how little we understand around Jones' Conjecture, we conclude by posing the following stronger conjecture. Note that the example of many nested disjoint cycles shows that the embedding cannot be fixed. Also note that the simple discharging argument mentioned above does not imply the following conjecture with a factor of 3 instead of 2.

Conjecture 14. For every planar graph G,

$$\mathrm{fvs}(G) \leqslant 2 \cdot \mathrm{fp}(G),$$

where $\mathrm{fp}(G)$ is the maximum size of a face-packing of G, i.e., a cycle-packing where, for some embedding of G, every cycle bounds a face.

References

[1] Wouter Cames van Batenburg, Tony Huynh, Gwenaël Joret, and Jean-Florent Raymond. A tight Erdős–Pósa function for planar minors. *Advances in Combinatorics*, 33pp, 2019:2.

[2] Glenn G. Chappell, John Gimbel, and Chris Hartman. On cycle packings and feedback vertex sets. *Contributions to Discrete Mathematics*, 9(2), 2014.

[3] Hong-Bin Chen, Hung-Lin Fu, and Chie-Huai Shih. Feedback vertex set on planar graphs. *Taiwanese Journal of Mathematics*, 16(6):2077–2082, 2012.

[4] Paul Erdős and Lajos Pósa. On independent circuits contained in a graph. *Canadian Journal of Mathematics*, 17:347–352, 1965.

[5] Ton Kloks, Chuan-Min Lee, and Jiping Liu. New algorithms for k-face cover, k-feedback vertex set, and k-disjoint cycles on plane and planar graphs. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 282–295. Springer, 2002.

[6] László Lovász and M. D. Plummer. *Matching Theory.* North-Holland, 1986.

[7] Jie Ma, Xingxing Yu, and Wenan Zang. Approximate min-max relations on plane graphs. *Journal of Combinatorial Optimization*, 26(1):127–134, 2013.

[8] Andrea Munaro. *Sur quelques invariants classiques et nouveaux des hypergraphes.* PhD thesis, Grenoble Alpes, 2016.

[9] Neil Robertson and Paul D. Seymour. Graph minors. V. excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(1):92–114, 1986.

[10] Zoltán Szigeti. On a min-max theorem of cacti. In Robert E. Bixby, E. Andrew Boyd, and Roger Z. Ríos-Mercado, editors, *Integer Programming and Combinatorial Optimization*, pages 84–95. Springer, 1998.