Periodic solutions of one-dimensional cellular automata with uniformly chosen random rules

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Abstract

We study cellular automata whose rules are selected uniformly at random. Our setting are two-neighbor one-dimensional rules with a large number n of states. The main quantity we analyze is the asymptotic distribution, as $n \to \infty$, of the number of different periodic solutions with given spatial and temporal periods. The main tool we use is the Chen-Stein method for Poisson approximation, which establishes that the number of periodic solutions, with their spatial and temporal periods confined to a finite range, converges to a Poisson random variable with an explicitly given parameter. The limiting probability distribution of the smallest temporal period for a given spatial period is deduced as a corollary and relevant empirical simulations are presented.

Mathematics Subject Classifications: 60K35, 37B15, 68Q80

1 Introduction

We investigate one-dimensional cellular automata (CA), a class of temporally and spatially discrete dynamical systems, in which the update rule is selected uniformly at random, and thereafter applied deterministically. Our focus is the asymptotic behavior of the probability that such randomly chosen CA has a periodic solution with fixed spatial and temporal periods, as n, the number of states, goes to infinity. This complements the work

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in [6], where the limiting behavior of the longest temporal period with a given spatial period is explored. We assume the simplest nontrivial setting of two-neighbor rules.

The (**spatial**) **configuration** at time t of a one-dimensional CA with **number of states** n is a function ξ_t that assigns to every site $x \in \mathbb{Z}$ its **state** $\xi_t(x) \in \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. The evolution of spatial configurations is given by a local 2-neighbor **rule** $f : \mathbb{Z}_n^2 \to \mathbb{Z}_n$ that updates ξ_t to ξ_{t+1} as follows:

$$\xi_{t+1}(x) = f(\xi_t(x-1), \xi_t(x)), \quad \text{for all } x \in \mathbb{Z}.$$

We abbreviate the rule assignment $f(a, b) = c$ as $a\underline{b} \mapsto c$. We give a rule by listing its values for all pairs in reverse alphabetical order, from $(n-1, n-1)$ to $(0, 0)$.

Given ξ_0, the update rule determines the **trajectory** ξ_t, $t \in \mathbb{Z}_+ = \{0, 1, \ldots\}$, or, equivalently, the **space-time configuration**, which is the map $(x, t) \mapsto \xi_t(x)$ from $\mathbb{Z} \times \mathbb{Z}_+$ to $\mathbb{Z}_n$. By convention, a picture of this map is a painted grid, in which the temporal axis is oriented downward, the spatial axis is oriented rightward, and each state is given as a different color. To give an example, a piece of the space-time configuration is presented in Figure 1. In this figure, we have three states, i.e., $n = 3$, and the rule is *021102022*, i.e., $2\underline{2} \mapsto 0, 2\underline{1} \mapsto 2, 2\underline{0} \mapsto 1, 1\underline{2} \mapsto 1, 1\underline{1} \mapsto 0, 1\underline{0} \mapsto 2, 0\underline{2} \mapsto 0, 0\underline{1} \mapsto 2$ and $0\underline{0} \mapsto 2$.

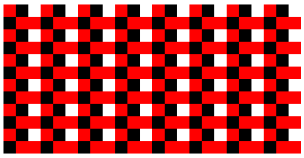


Figure 1: A piece of the space-time configuration of a 3-state rule. In the space-time configuration, 0, 1 and 2 are represented by white, red and black cells, respectively.

The space-time configuration in Figure 1 exhibits periodicity in both space and time. In the literature [3], such a configuration is called doubly or jointly periodic. Since these are the only objects we study, we simply refer to such a configuration as a **periodic solution** (PS). To be precise, start with a periodic spatial configuration ξ_0, such that there is a $\sigma > 0$ satisfying $\xi_0(x) = \xi_0(x + \sigma)$, for all $x \in \mathbb{Z}$. Run a CA rule f starting with ξ_0. If we have $\xi_\tau(x) = \xi_0(x)$, for all $x \in \mathbb{Z}$ and that σ and τ are both minimal, then we have found a periodic solution of **temporal period** τ and **spatial period** σ.

A **tile** is any rectangle with τ rows and σ columns within this space-time configuration. We interpret a tile as a configuration on a discrete torus; we will not distinguish between spatial and temporal translations of a PS, and therefore between either rotations of a tile. The tile of a PS is by definition unique and we will identify a PS with its tile. As an example, in Figure 1, we start with the initial configuration $\xi_0 = 120^\infty = \ldots 120120120 \ldots$ (we give a configuration as a bi-infinite sequence when the position of the origin is clear or unimportant). After 2 updates, we have $\xi_2(x) = \xi_0(x)$, for all $x \in \mathbb{Z}$, thus the PS has temporal period 2 and spatial period 3. Its tile is $\begin{matrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{matrix}$. A key role in our analysis is played by a special class of tiles, which are called simple tiles (see Section 3); in such a tile, the number of implied rule assignments is no larger than the number of states it contains, and it turns out that, in our setting, such tiles are the most likely.

CA that exhibit temporally periodic or jointly periodic behavior have been explored to some degree in the literature, and we give a brief review of some highlights. The foundational work is commonly considered to be [14]. That paper, together with its successors [10, 11], focuses on algebraic methods to investigate additive CA, but also lays the foundation for more general rules. More recent papers on temporal periodicity of additive binary rules include [4] and [15]. The literature on non-additive rules is more scarce, but includes notable works [2] and [3] on the density of periodic configurations, which use both rigorous and experimental methods. A method of finding temporally periodic trajectories is discussed in [21], which reiterates the utility of the relation between periodic configurations and cycles on graphs induced by the CA rules, introduced in [14]. This approach is useful in the present paper as well. Papers investigating long temporal periods of CA also include [18, 17], as well as our companion papers [6, 8]. To mention another take on periodicity, the paper [5] introduces *robust* PS, which are those that expand into any environment with positive speed, and investigates their existence in all range 2 (i.e., 3-neighbor) binary CA; see [7] for results on robust PS for uniformly random rules.

We now present a formal setting to investigate PS from uniformly selected random rules, which, to our knowledge, have not been explored before. Our rule space Ω_n consists of n^{n^2} rules and we assign a uniform probability $\mathbb{P}$ to each rule f, therefore $\mathbb{P}(\{f\}) = 1/n^{n^2}$. Let $\mathcal{P}_{\tau,\sigma,n}$ be the random set of PS with temporal period τ and spatial period σ of such a uniformly chosen CA rule. The main quantity we are interested in is the number of periodic solutions of fixed pair of periods τ and σ, $|\mathcal{P}_{\tau,\sigma,n}|$, as $n \to \infty$. In particular, we will determine $\lim \mathbb{P}\left(\mathcal{P}_{\tau,\sigma,n} \neq \varnothing\right)$, the limiting probability that a random CA rule has a PS with given temporal and spatial periods. In the following theorem, we prove that this limit is in $(0, 1)$ for any τ and σ. Define

$$\lambda_{\tau,\sigma} = \frac{1}{\tau\sigma} \sum_{d \,\middle|\, \gcd(\tau,\sigma)} \varphi(d)d, \tag{1}$$

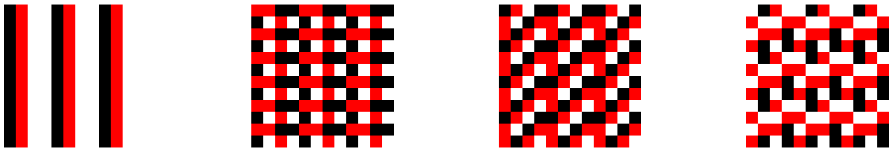
where φ, the Euler totient function, is given by

$$\varphi(d) = |\{k : 1 \leqslant k \leqslant d, \gcd(k, d) = 1\}|.$$

We denote by Poisson(λ) a Poisson random variable with expectation λ, and by d_{TV} the total variation distance.

Theorem 1. *For any fixed integers $\tau \geqslant 1$ and $\sigma \geqslant 1$, $|\mathcal{P}_{\tau,\sigma,n}|$ converges weakly to* Poisson$(\lambda_{\tau,\sigma})$*. In particular, $\mathbb{P}\left(\mathcal{P}_{\tau,\sigma,n} \neq \varnothing\right) \to 1 - \exp\left(-\lambda_{\tau,\sigma}\right)$ as $n \to \infty$.*

In fact, our proof given in Section 4 provides an upper bound on the total variation distance: $d_{\mathrm{TV}}(|\mathcal{P}_{\tau,\sigma,n}|, \mathrm{Poisson}\left(\lambda_{\tau,\sigma}\right)) = \mathcal{O}(1/n)$.



(a) *012200210* (b) *021102120* (c) *100112122* (d) *101201021*

Figure 2: Pieces of PS for $\sigma = 4$ and 3-state rules, *012200210*, *021102120*, *100112122* and *101201021*, with temporal period $\tau = 1, 2, 3$ and 4, respectively. (See the discussion before Corollary 20.) These temporal periods are the smallest in each case, as verified by Algorithm 8 in Section 2.3. Algorithm 11 in Section 2.4 shows that $\sigma = 4$ is not the minimal spatial period of PS given the corresponding temporal period $\tau = 1, 2$ and 3 in the first three rules, while for the last rule $\sigma = 4$ is also the minimal spatial period of PS for temporal period $\tau = 4$.

We also prove a more general result that concerns the number of PS with a range of periods. Assume $\mathcal{S} \subset \mathbb{N} \times \mathbb{N}$ is finite, and define $\mathcal{P}_{\mathcal{S},n} = \mathcal{P}_{\mathcal{S},n}(f) = \bigcup_{(\tau,\sigma)\in\mathcal{S}} \mathcal{P}_{\tau,\sigma,n}$ and

$$\lambda_{\mathcal{S}} = \sum_{(\tau,\sigma)\in\mathcal{S}} \lambda_{\tau,\sigma}. \tag{2}$$

The following theorem shows that the random variables $|\mathcal{P}_{\mathcal{S}_1,n}|$ and $|\mathcal{P}_{\mathcal{S}_2,n}|$ are asymptotically independent for finite disjoint $\mathcal{S}_1$ and $\mathcal{S}_2$.

Theorem 2. *For any fixed finite set $\mathcal{S} \subset \mathbb{N} \times \mathbb{N}$, $|\mathcal{P}_{\mathcal{S},n}|$ converges weakly to* Poisson$(\lambda_{\mathcal{S}})$. *In particular, $\mathbb{P}(\mathcal{P}_{\mathcal{S},n} \neq \varnothing) \to 1 - \exp(-\lambda_{\mathcal{S}})$ as $n \to \infty$.*

We define the random variable

$$Y_{\sigma,n} = \min\{\tau : \mathcal{P}_{\tau,\sigma,n} \neq \varnothing\}$$

to be the smallest temporal period of a PS with spatial period σ of a randomly selected n-state rule. Figure 2 provides four examples of rules f, with $Y_{4,3}(f) = 1, 2, 3$ and 4. As a consequence of Theorem 2, for a given $\sigma > 0$, the random variable $Y_{\sigma,n}$ is stochastically bounded, in the sense of the following corollary. In fact, Theorem 2 provides an explicit formula for the limiting distribution referred to in the statement.

Corollary 3. *The random variable $Y_{\sigma,n}$ converges weakly to a nontrivial distribution (that is, one not concentrated on a single number) as $n \to \infty$.*

We now briefly discuss the relation between this corollary and the main results of [6] and [8]. In [6], we consider a more general setting of CA rules with r neighbors, that is, ξ_t updates to ξ_{t+1} according to the rule $f : \mathbb{Z}_n^r \to \mathbb{Z}_n$, so that

$$\xi_{t+1}(x) = f(\xi_t(x - r + 1), \ldots, \xi_t(x)), \quad \text{for all } x \in \mathbb{Z}.$$

We remark that results analogous to Theorems 1 and 2 can be proved for general $r \geqslant 2$ with the methods of the present paper, but we restrict to the case $r = 2$ to minimize inessential technical and notational issues. Fix a spatial period σ and an r. Let $X_{\sigma,n} = \max\{\tau : \mathcal{P}_{\tau,\sigma,n} \neq \varnothing\}$ be the largest temporal period of a PS with spatial period σ of a uniformly chosen random r-neighbor rule. In the case when $\sigma \leqslant r$, we prove that $X_{\sigma,n}/n^{\sigma/2}$ converges in distribution to a nontrivial limit, as $n \to \infty$. We also provide empirical evidence that the same result holds when $\sigma > r$, although in that case we do not have a rigorous proof even for $r = 2$. At least for $r = \sigma = 2$, therefore, the shortest temporal period is stochastically bounded while the longest is on the order of n. Moreover, it is not hard to see that the maxima of the random variables $Y_{2,n}$ and $X_{2,n}$ are both $n^2 - n$. More generally, in [8] and [13], we construct rules f with $Y_{\sigma,n}(f) \geqslant C(\sigma)n^\sigma$. That is, the maximum of the random variable $Y_{\sigma,n}$ is of the same order as its upper bound $n^\sigma - \mathcal{O}(n^{\sigma/2})$, guaranteed by the pigeonhole principle.

In the next section, we collect our main tools: tiles of PS; circular shifts; oriented graphs induced by a rule; and the Chen-Stein method. As already announced, we devote Section 3 to the key class of tiles, the simple tiles. We prove the main results in Section 4 and conclude with a discussion and several unsolved problems in Section 5.

2 Preliminaries

2.1 Tiles of a PS

We recall that the spatial and temporal periods σ and τ are assumed to be minimal, so a tile cannot be divided into smaller identical pieces. We now take a closer look into properties of tiles.

If we choose an element in a tile T to be placed at the position $(0,0)$, T may be expressed as a matrix $T = (a_{i,j})_{i=0,\dots,\tau-1,j=0,\dots,\sigma-1}$. We always interpret the two subscripts modulo τ and σ. The matrix is determined up to a space-time rotation, but note that two different rotations cannot produce the same matrix due to the minimality of σ and τ. We say that $a_{i,j}$ is an element in T, and write $a_{i,j} \in T$, when we want to refer to the element of the matrix at the position (i,j), and use the notation $\mathtt{row}_i$ and $\mathtt{col}_j$ to denote the ith row and jth column of a tile T, again after we fix $a_{0,0}$. All the properties we now introduce are independent of the chosen rotation (as they must be, to be meaningful).

Let T_1 and T_2 be two tiles and $a_{i,j}$, $b_{k,m}$ be elements in T_1 and T_2, respectively. We say that T_1 and T_2 are **orthogonal**, and denote this property by $T_1 \perp T_2$, if $(a_{i,j}, a_{i,j+1}) \neq (b_{k,m}, b_{k,m+1})$ for $i,j,k,m \in \mathbb{Z}_+$. It is important to observe that in this case the two assignments $\overline{a_{i,j}a_{i,j+1}} \mapsto a_{i+1,j+1}$ and $\overline{b_{k,m}b_{k,m+1}} \mapsto b_{k+1,m+1}$ occur independently.

We say that T_1 and T_2 are **disjoint**, and denote this property by $T_1 \cap T_2 = \varnothing$, if $a_{i,j} \neq b_{k,m}$, for $i,j,k,m \in \mathbb{Z}_+$. Clearly, every pair of disjoint tiles is orthogonal, but not vice versa.

Let $s(T) = |\{a_{i,j} : a_{i,j} \in T\}|$ be the number of different states in the tile. Furthermore, let $p(T) = |\{(a_{i,j}, a_{i,j+1}) : a_{i,j}, a_{i,j+1} \in T\}|$ be the **assignment number** of T; this is the number of assignments of the rule f specified by T. Clearly, $p(T) \geqslant s(T)$, so we define

$\ell = \ell(T) = p(T) - s(T)$ to be the **lag** of T. A tile is **simple** if its lag is 0. Simple tiles play a crucial role in our arguments and will be addressed further in Section 3. We record a few immediate properties of a tile in the following Lemma.

Lemma 4. *Let $T = (a_{i,j})_{i=0,\dots,\tau-1,j=0,\dots,\sigma-1}$ be the tile of a PS with periods τ and σ. Then T satisfies the following properties:*

1. *Uniqueness of assignment: if $(a_{i,j}, a_{i,j+1}) = (a_{k,m}, a_{k,m+1})$, then $a_{i+1,j+1} = a_{k+1,m+1}$.*

2. *Aperiodicity of rows: each row of T cannot be divided into smaller identical pieces.*

Proof. Part 1 is clear since T is generated by a CA rule. Part 2 follows from part 1 and the assumption that the spatial period of T is minimal. □

By contrast, we remark that there *may* exist periodic columns in a tile of a PS. For example, note that the first column in Figure 2(d) has period 2 rather than $4 = \tau$.

2.2 Circular shifts

In this section, we introduce circular shifts, operation on Z_n^σ (or Z_n^τ), the set of words of length σ (or τ) from the alphabet $\mathbb{Z}_n$. They will be useful in Section 3.

Definition 5. Let $\mathbb{Z}_n^\sigma$ consist of all length-σ words. A **circular shift** is a map $\pi : \mathbb{Z}_n^\sigma \to \mathbb{Z}_n^\sigma$, given by an $i \in \mathbb{Z}_+$ as follows: $\pi(a_0a_1 \dots a_{\tau-1}) = a_ia_{i+1} \dots a_{i+\sigma-1}$, where the subscripts are modulo σ. The **order** of a circular shift π is the smallest k such that $\pi^k(A) = A$ for all $A \in \mathbb{Z}_n^\sigma$, and is denoted by $\operatorname{ord}(\pi) = \sigma / \gcd(i, \sigma)$. Circular shifts on $\mathbb{Z}_n^\tau$ will also appear in the sequel and are defined in the same way.

Lemma 6. *Let π be a circular shift on $\mathbb{Z}_n^\sigma$ and let $A \in \mathbb{Z}_n^\sigma$ be an aperiodic length-σ word from alphabet $\mathbb{Z}_n$. Then: (1) $\operatorname{ord}(\pi) \mid \sigma$; and (2) for any $d \mid \sigma$,*

$$\left| \{B \in \mathbb{Z}_n^\sigma : A = \pi(B) \text{ for some } \pi \text{ with } \operatorname{ord}(\pi) = d\} \right| = \varphi(d).$$

Proof. Note that the σ circular shifts form a cyclic group of order σ. Moreover, $\operatorname{ord}(\pi)$ of a circular shift is its order in the group, thus (1) follows. To prove (2), observe that the circular shifts of order d generate a cyclic subgroup and the number of them is $\varphi(d)$. As A is aperiodic, the cardinality in the claim is the same. □

We say that two words A and B of length σ are **equal up to a circular shift** if $B = \pi(A)$ for some circular shift π. For example, words 0123 and 2301 are not equal, but are equal up to a circular shift.

2.3 Directed graph on configurations

Connections between directed graphs on periodic configurations and cycles are well-established [14, 20, 12, 21], as they are useful for analysis of PS with a fixed spatial period.

Definition 7. Let $A = a_0 \dots a_{\sigma-1}$ and $B = b_0 \dots b_{\sigma-1}$ be two words from alphabet $\mathbb{Z}_n$. We say that A **down-extends to** B, if $f(a_i, a_{i+1}) = b_{i+1}$, for all $i \geqslant 0$, where (as usual) the indices are modulo σ.

If A down-extends to B, then $\pi(A)$ also down-extends to $\pi(B)$, for any circular shift π on $\mathbb{Z}_n^\sigma$. Therefore, we can define, for a fixed σ, the **spatial digraph** on equivalence classes of words equal up to circular shifts, which has an arc from A to B if A down-extends to B (where we identify the equivalence class with any of its representatives). See Figure 3 for the spatial digraph of the 3-state rule *021102022*. For instance, there is an arc from 122 to 210 as $1\underline{2} \mapsto 1$, $2\underline{2} \mapsto 0$ and $2\underline{1} \mapsto 2$. The following algorithm and self-evident proposition determine the PS in Figure 1 from the length-2 cycle $120 \leftrightarrow 211$ in Figure 3.

Algorithm 8. *Input: Spatial digraph $D_{\sigma,f}$ of f and spatial period σ.*
Step 1: Find all the directed cycles in $D_{\sigma,f}$.
Step 2: For each cycle $A_0 \to A_1 \to \cdots \to A_{\tau-1} \to A_0$, form the tile T by placing configurations $A_0, A_1, \dots, A_{\tau-1}$ on successive rows.
Step 3: If the spatial period of T is minimal, output T.

Proposition 9. *All PS of spatial period σ of f are obtained by Algorithm 8.*

We remark that Step 3 in Algorithm 8 is necessary, as, for instance, the cycle $000 \leftrightarrow 222$ in Figure 3 results in a PS of spatial period 1 instead of 3. In the same vein, the periods of configurations are non-increasing, and may decrease, along any directed path on the spatial digraph. For example, in Figure 3, the configuration 100 down-extends to 222, thus the period is reduced from 3 to 1 and then remains 1. These period reductions play a crucial role in our companion paper [6].

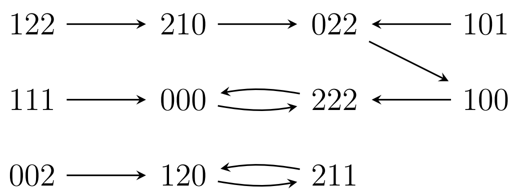


Figure 3: Spatial digraph of the 3-state rule *021102022* and spatial period $\sigma = 3$.

2.4 Directed graph on labels

In this subsection, we fix the temporal period τ, instead of the spatial period σ, and obtain another digraph induced by the rule. The construction below is an adaption of label trees from [5].

Definition 10. Let $A = a_0 \dots a_{\tau-1}$ and $B = b_0 \dots b_{\tau-1}$ be two words from alphabet $\mathbb{Z}_n$, which we call **labels** of length τ. (While it is best to view them as vertical columns, we write them horizontally for reasons of space, as in [5].) We say that A **right-extends to** B if $f(a_i, b_i) = b_{i+1}$, for all $i \in \mathbb{Z}_+$, where (as usual) the indices are modulo τ. We form the **temporal digraph** associated with a given τ by forming an arc from a label A to a label B if A right-extends to B.

A label $A = a_0 \dots a_{\tau-1}$ right-extends to B if and only if we preserve the temporal periodicity from a column A to the column B to its right. This fact is the basis for the Algorithm 11 below, which gives all the PS with temporal period τ. The temporal digraph of same rule as in Figure 3 and temporal period $\tau = 2$ is presented in Figure 4. For example, we have the arc from label 12 to 10 as $1\underline{1} \mapsto 0$, $2\underline{0} \mapsto 1$. Either of the two 3-cycles in the digraph generates the PS in Figure 1.

Algorithm 11. *Input: Temporal digraph $D_{\tau,f}$ of f with period τ.*
Step 1: Find all the directed cycles (in which we allow multiple visits to the same vertex) in $D_{\tau,f}$.
Step 2: For each cycle $A_0 \to A_1 \to \cdots \to A_{\sigma-1} \to A_0$, form the tile T by placing configurations $A_0, A_1, \dots, A_{\sigma-1}$ on successive columns.
Step 3: If both spatial and temporal periods of T are minimal, then output T.

Proposition 12. *All PS of temporal period τ of f can be obtained by the Algorithm 11.*

Again, Step 3 is necessary due to the same reason as Section 2.3. However, note the differences between the two graphs: the out-degrees in Figure 4 are between 0 and 3, and the temporal periods are not necessarily non-decreasing along a directed path. For example, 00 right-extends to 02. We also note that lifting the label digraph to one on equivalence classes, although possible, makes cycles more obscure and is thus less convenient.

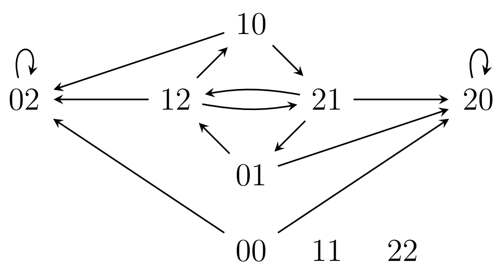


Figure 4: Temporal digraph of the 3-state rule *021102022* and temporal period $\tau = 2$. We remark again that the vertex labels are, in fact, columns, but are represented as rows in the text, so we keep that representation in the figure.

2.5 Chen-Stein method for Poisson approximation

The main tool we use to prove Poisson convergence is the Chen-Stein method [1]. We need the following setting for our purposes. Let I_i, for $i \in \Gamma$ be indicators of a finite family of events, which is indexed by Γ, $p_i = \mathbb{E}(I_i)$, $W = \sum_{i\in\Gamma} I_i$, $\lambda = \sum_{i\in\Gamma} p_i = \mathbb{E}W$, and $\Gamma_i = \{j \in \Gamma : j \neq i,\ I_i \text{ and } I_j \text{ are not independent}\}$. We quote Theorem 4.7 from [16].

Lemma 13. *We have*

$$d_{\mathrm{TV}}(W, \mathrm{Poisson}(\lambda)) \leqslant \min(1, \lambda^{-1}) \left[\sum_{i\in\Gamma} p_i^2 + \sum_{i\in\Gamma, j\in\Gamma_i} (p_i p_j + \mathbb{E}(I_i I_j))\right].$$

In our applications of the above lemma, all deterministic and random quantities depend on the number n of states, which we make explicit by the subscripts. In our setting, we prove that $d_{\mathrm{TV}}(W_n, \mathrm{Poisson}(\lambda_n)) = \mathcal{O}(1/n)$ and that $\lambda_n \to \lambda$ as $n \to \infty$, for an explicitly given λ, which implies that W_n converges to Poisson(λ) in distribution; see Theorems 1 and 2. In the proofs, we also need the following well-known result.

Lemma 14. *Let X_1, Y_1, X_2, Y_2 be integer valued random variables, such that X_1, Y_1 are defined on the same probability space, and so are X_2, Y_2. Then*

$$d_{\mathrm{TV}}(X_1 + Y_1, X_2 + Y_2) \leqslant d_{\mathrm{TV}}(X_1, X_2) + d_{\mathrm{TV}}(Y_1, Y_2).$$

3 Simple tiles

We recall that a tile T is **simple** if its lag vanishes: $\ell(T) = p(T) - s(T) = 0$. It turns out that simple tiles provide the dominant contribution to $|\mathcal{P}_{\tau,\sigma,n}|$, thus this class of tiles is of central importance. For example, consider the tiles

$$T_1 = \begin{matrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{matrix}, \qquad T_2 = \begin{matrix} 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 \end{matrix}.$$

Then T_1 is simple, as $s(T_1) = p(T_1) = 4$, but T_2 is not, as $s(T_2) = 3$ and $p(T_2) = 4$. Naturally, we call a PS simple if its tile is simple.

We denote by $\mathcal{P}^{(\ell)}_{\tau,\sigma,n}$ as the set of PS whose tile T has lag ℓ. Thus the set of simple PS is $\mathcal{P}^{(0)}_{\tau,\sigma,n}$. The following lemma addresses ramifications of repeated states in simple tiles.

Lemma 15. *Assume $T = (a_{i,j})_{i=0,\dots,\tau-1, j=0,\dots,\sigma-1}$ is a simple tile. Then*

1. *the states on each row of T are distinct;*
2. *if two rows of T share a state, then they are circular shifts of each other;*
3. *the states on each column of T are distinct; and*
4. *if two columns of T share a state, then they are circular shifts of each other.*

Proof. Part 1: When $\sigma = 1$, each row contains only one state, making the claim trivial. Now, assume that $\sigma \geqslant 2$ and that $a_{i,j} = a_{i,k}$ for some i and $j \neq k$. We must have $a_{i,j+1} = a_{i,k+1}$ in order to avoid $p(T) > s(T)$. Repeating this procedure for the remaining states on $\texttt{row}_i$ shows that this row is periodic, contradicting part 2 of Lemma 4.
Part 2: If $a_{i,j} = a_{k,m}$, for $i \neq k$, then the states to their right must agree, i.e., $a_{i,j+1} = a_{k,m+1}$, in order to avoid $p(T) > s(T)$. Repeating this observation for the remaining states on $\texttt{row}_i$ and $\texttt{row}_k$ gives the desired result.
Part 3: Assume a column contains repeated state, say $a_{i,j} = a_{k,j}$ for some i, j and k. By part 2, $\texttt{row}_i$ is exactly the same as $\texttt{row}_k$, so that the temporal period of this tile can be reduced, a contradiction.
Part 4: Assume that $a_{i,j} = a_{k,m}$, for $j \neq m$. Then $a_{i,j+1} = a_{k,m+1}$ by parts 1 and 2. So, $a_{i+1,j+1} = a_{k+1,m+1}$ by part 1 in Lemma 4. So, $a_{i+1,j} = a_{k+1,m}$, again by parts 1 and 2. Now, repeating the previous step for $a_{i+1,j} = a_{k+1,m}$ gives the desired result. □

We revisit the remark following Lemma 4: a tile may have periodic columns, but such a tile cannot be simple.

Suppose a tile $T = (a_{i,j})_{i=0,\ldots,\tau-1, j=0,\ldots,\sigma-1}$ is simple. We will take a closer look at circular shifts of rows, so we fix a row, say the first row $\texttt{row}_0$. (We could start with any row, but we pick the first one for concreteness.) Let

$$i = \min\{k = 1, 2, \ldots, \tau - 1 : \texttt{row}_k = \pi(\texttt{row}_0),\ \text{for some circular shift } \pi : \mathbb{Z}^\sigma \to \mathbb{Z}^\sigma\}$$

be the smallest i such that $\texttt{row}_i$ is a circular shift of $\texttt{row}_0$, and let $i = 0$ if and only if T does not have circular shifts of $\texttt{row}_0$ other than this row itself. Then this circular shift satisfies $\texttt{row}_{(j+i) \bmod \tau} = \pi(\texttt{row}_j)$, for all $j = 0, \ldots, \tau - 1$ and i is determined by the tile T; we denote this circular shift by π^r_T. We denote by π^c_T the analogous circular shift for columns.

Lemma 16. *Assume the tile T of a PS is simple, and let $d_1 = ord\ (\pi^r_T)$ and $d_2 = ord\ (\pi^c_T)$. Then d_1 and d_2 are equal and divide* $\gcd(\tau, \sigma)$.

Proof. Fix an element as $a_{0,0}$. By Lemma 15, parts 1 and 2, $a_{0,0}$ appears in d_1 rows of T. It also appears in d_2 columns by Lemma 15, parts 3 and 4. As a consequence, $d_1 = d_2$. The divisibility follows from Lemma 6. □

For illustration, consider the following tile with $\tau = 4$ and $\sigma = 6$. Let

$$T_3 = \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 9 & 10 & 11 & 6 & 7 & 8 \end{matrix}.$$

Observe that $s(T_3) = p(T_3) = 12$, thus T_3 is a simple tile. Also note that $d = d_1 = ord(\pi^r_{T_3}) = d_2 = ord(\pi^c_{T_3}) = 2$, and that d divides $\gcd(\tau, \sigma)$ and $s(T_3) = \tau\sigma/d$. This is no coincidence, as proved in the next lemma.

Lemma 17. *An integer $s \leqslant n$ is the number of states in a simple tile*

$$T = (a_{i,j})_{i=0,\dots,\tau-1, j=0,\dots,\sigma-1}$$

of some PS if and only if there exists $d \mid \gcd(\tau, \sigma)$, such that $s = \tau\sigma/d$.

Proof. Let $T = (a_{i,j})_{i=0,\dots,\tau-1,j=0,\dots,\sigma-1}$. Assume that $s(T) = s$ and let $d = \text{ord } (\pi_T^r)$. Then by Lemma 15, parts 1 and 2, the first τ/d rows of T contain all states that are in T. As a result, $s = \tau\sigma/d$ and $d = \text{ord } (\pi_T^r) \mid \gcd(\tau, \sigma)$.

Now assume that $d \mid \gcd(\tau, \sigma)$. Then there exists a circular shift $\pi : \mathbb{Z}^\sigma \to \mathbb{Z}^\sigma$, such that ord $(\pi) = d$. To form a simple tile T with $s(T) = \tau\sigma/d$ states, construct a rectangle of τ/d rows and σ columns using $\tau\sigma/d$ different states in the first τ/d rows of T. Let $\texttt{row}_{\tau/d}$ be defined by $\pi(\texttt{row}_0)$ and the subsequent rows are all automatically defined by the maps that are assigned in the first τ/d rows, by Lemma 4, part 1. $\square$

The above lemma gives the possible values of $s(T)$ for a simple tile T and the next one enumerates the number of simple tiles of PS containing s different states.

Lemma 18. *The number of simple tiles of PS with temporal periods τ and spatial period σ containing s states is $\varphi(d)\binom{n}{s}(s-1)!$, where $d = \tau\sigma/s$.*

Proof. As in the proof of Lemma 17, if $s(T) = s = \tau\sigma/d$, then $d = \text{ord}(\pi_T^r)$. Moreover, there are $\binom{n}{s}(s-1)!$ ways to form the first τ/d rows of T. Then, to uniquely determine T, we need to select a circular shift $\pi : \mathbb{Z}^\sigma \to \mathbb{Z}^\sigma$ with ord $(\pi) = d$ and define $\texttt{row}_{\tau/d}$ to be $\pi(\texttt{row}_0)$. By Lemma 6, there are $\varphi(d)$ ways to do so. $\square$

Consider two different simple tiles T_1 and T_2 under the rule. As the final task of this section, we seek a lower bound on the combined number of values of the rule f assigned by T_1 *and* T_2, in terms of the number of states. If $s(T_1) = s_1$, then $p(T_1) \geqslant s_1$, i.e., there are at least s_1 values assigned by T_1. If there are s_2' states in T_2 that are not in T_1, then there are at least s_2' additional values to assign. Therefore, a lower bound of the number of values to be assigned in T_1 and T_2 is $s_1 + s_2'$. The next lemma states that we can increase this lower bound by at least 1 when $T_1 \cap T_2 \neq \varnothing$. This fact plays an important role in the proofs of Theorem 1 and Theorem 2.

Lemma 19. *Let T_1 and T_2 be two different simple tiles (for two different PS) for the same rule but possibly different periods. If T_1 and T_2 have at least one state in common, then there exist $a_{i,j} \in T_1$ and $b_{k,m} \in T_2$ such that $a_{i,j} = b_{k,m}$ and $a_{i,j+1} \neq b_{k,m+1}$.*

Proof. As T_1 and T_2 have at least one state in common, we may pick $a_{i,j} \in T_1$ and $b_{k,m} \in T_2$, such that $a_{i,j} = b_{k,m}$. If $a_{i,j+1} \neq b_{k,m+1}$, then we are done. Otherwise, we repeat this procedure for $a_{i,j+1}$ and $b_{k,m+1}$ and see if $a_{i,j+2} = b_{k,m+2}$. We repeat this procedure until we find two pairs such that $a_{i,j+q} = b_{k,m+q}$ and $a_{i,j+q+1} \neq b_{k,m+q+1}$. If we fail to do so, then $\texttt{row}_i$ in T_1 and $\texttt{row}_k$ in T_2 must be equal, up to a circular shift. This implies that T_1 and T_2 must be the same since they are tiles for same rule, a contradiction. $\square$

4 Proofs of main results

We will give a separate proof of Theorem 1 first, for transparency, and then we show how to adapt the argument to prove the stronger result,Theorem 2.

Proof of Theorem 1. Recalling the definition of $\mathcal{P}^{(\ell)}_{\tau,\sigma,n}$ before Lemma 15, we begin with the decomposition

$$|\mathcal{P}_{\tau,\sigma,n}| = |\mathcal{P}^{(0)}_{\tau,\sigma,n}| + \sum_{\ell=1}^{\tau\sigma} |\mathcal{P}^{(\ell)}_{\tau,\sigma,n}|. \tag{3}$$

For $\ell \geqslant 1$,

$$\mathbb{P}(|\mathcal{P}^{(\ell)}_{\tau,\sigma,n}| > 0) \leqslant \mathbb{E}(|\mathcal{P}^{(\ell)}_{\tau,\sigma,n}|) = \sum_{s=1}^{\tau\sigma} \binom{n}{s} g^{(\ell)}_{\tau,\sigma}(s) \frac{1}{n^{s+\ell}} = \mathcal{O}\left(\frac{1}{n^{\ell}}\right), \tag{4}$$

where $g^{(\ell)}_{\tau,\sigma}(s)$ counts the number of $\tau \times \sigma$ tiles that contain s different states and lag is ℓ. Note that the number of states s is fixed and does not depend on n. Here, $1/n^{s+\ell}$ is the probability of such a tile (determined by a PS), as there are $s+\ell$ assignments to make by a random map, and each assignment occurs independently with probability $1/n$. It follows that, with δ_0 the point-mass at 0,

$$d_{\mathrm{TV}}\left(\sum_{\ell=1}^{\tau\sigma} |\mathcal{P}^{(\ell)}_{\tau,\sigma,n}|,\, \delta_0\right) = \mathcal{O}\left(\frac{1}{n}\right). \tag{5}$$

To find the distributional limit of $|\mathcal{P}^{(0)}_{\tau,\sigma,n}|$ as $n \to \infty$, let $1 = d_1 < \ldots < d_u = \gcd(\sigma,\tau)$ be the common divisors of τ and σ and $s_j = \tau\sigma/d_j$, for $j = 1,\ldots,u$, be the possible numbers of states in simple tiles. We index the simple tiles that have s_j states in an arbitrary way, so that $T^{(j)}_k$ be the kth simple tile that has s_j states. Here $k = 1,\ldots,N_j$ and $N_j = \varphi(d_j)\binom{n}{s_j}(s_j-1)!$ is the number of simple tiles with s_j states (by Lemma 18). Let $I^{(j)}_k$ be the indicator random variable that $T^{(j)}_k$ is a tile determined by a PS. Let $W_n = \sum_{j=1}^{u}\sum_{k=1}^{N_j} I^{(j)}_k$. Then $W_n = |\mathcal{P}^{(0)}_{\tau,\sigma,n}|$ and

$$\begin{aligned}
\lambda_n = \mathbb{E}W_n = \sum_{j=1}^{u}\sum_{k=1}^{N_j} \mathbb{E}I^{(j)}_k &= \sum_{j=1}^{u} \varphi(d_j)\binom{n}{s_j}(s_j-1)!\frac{1}{n^{s_j}} \\
&\xrightarrow{n\to\infty} \sum_{j=1}^{u} \varphi(d_j)\frac{1}{s_j} \\
&= \sum_{j=1}^{u} \varphi(d_j)\frac{d_j}{\tau\sigma} = \frac{1}{\tau\sigma}\sum_{d \,\big|\, \gcd(\tau,\sigma)} \varphi(d)d = \lambda_{\tau,\sigma}.
\end{aligned}$$

We next show that $d_{\mathrm{TV}}(W_n, \mathrm{Poisson}(\lambda_n)) = \mathcal{O}(1/n)$. As orthogonal tiles have independent assignments, Lemma 13 implies that

$$\begin{aligned} &d_{\mathrm{TV}}(W_n, \mathrm{Poisson}(\lambda_n)) \\ &\leqslant \min(1, \lambda_n^{-1}) \left[\sum_{j,k} \left(\mathbb{E} I_k^{(j)} \right)^2 + \sum_{\substack{j,k,i,m \\ T_m^{(i)} \not\perp T_k^{(j)}}} \left(\mathbb{E} I_k^{(j)} \mathbb{E} I_m^{(i)} + \mathbb{E} I_k^{(j)} I_m^{(i)} \right) \right]. \end{aligned} \tag{6}$$

To bound $\sum_{j,k} \left(\mathbb{E} I_k^{(j)} \right)^2$, fix a $j \in \{1, \ldots, u\}$ and note that

$$\sum_{k=1}^{N_j} \left(\mathbb{E} I_k^{(j)} \right)^2 = \varphi(d_j) \binom{n}{s_j} (s_j - 1)! \left(\frac{1}{n^{s_j}} \right)^2 = \mathcal{O}\left(\frac{1}{n^{s_j}} \right). \tag{7}$$

It follows that $\sum_{j,k} \left(\mathbb{E} I_k^{(j)} \right)^2 = \mathcal{O}\left(1/n^{\mathrm{lcm}(\tau,\sigma)} \right) \to 0$, as $n \to \infty$. It remains to bound the sum over j, k, i, m in (6). For a fixed $i, j \in \{1, \ldots, u\}$,

$$\begin{aligned} &\sum_{k=1}^{N_j} \sum_{\substack{m=1 \\ T_m^{(i)} \not\perp T_k^{(j)}}}^{N_i} \left(\mathbb{E} I_k^{(j)} \mathbb{E} I_m^{(i)} + \mathbb{E} I_k^{(j)} I_m^{(i)} \right) \\ &\leqslant \sum_{k=1}^{N_j} \sum_{\substack{m=1 \\ T_m^{(i)} \cap T_k^{(j)} \neq \varnothing}}^{N_i} \left(\mathbb{E} I_k^{(j)} \mathbb{E} I_m^{(i)} + \mathbb{E} I_k^{(j)} I_m^{(i)} \right) \\ &= \sum_{k=1}^{N_j} \sum_{h=1}^{\min(s_i,s_j)} \sum_{\substack{m=1 \\ |T_m^{(i)} \cap T_k^{(j)}| = h}}^{N_i} \mathbb{E} I_k^{(j)} \mathbb{E} I_m^{(i)} + \sum_{k=1}^{N_j} \sum_{h=1}^{\min(s_i,s_j)} \sum_{\substack{m=1 \\ |T_m^{(i)} \cap T_k^{(j)}| = h}}^{N_i} \mathbb{E} I_k^{(j)} I_m^{(i)}, \end{aligned} \tag{8}$$

where the inequality holds because two tiles that share an assignment have to share at least one state. Label the two triple sums on the last line of (8) $S_{ij}^{(1)}$ and $S_{ij}^{(2)}$. Now, fix also an $h \in \{1, \ldots, \min(s_i, s_j)\}$. We first compute

$$\begin{aligned} \sum_{k=1}^{N_j} \sum_{\substack{m=1 \\ |T_m^{(i)} \cap T_k^{(j)}| = h}}^{N_i} \mathbb{E} I_k^{(j)} \mathbb{E} I_m^{(i)} &= \varphi(d_j) \binom{n}{s_j} (s_j - 1)! \varphi(d_i) \binom{s_j}{h} \binom{n - s_j}{s_i - h} (s_i - 1)! \frac{1}{n^{s_j}} \frac{1}{n^{s_i}} \\ &= \mathcal{O}\left(\frac{1}{n^h} \right), \end{aligned}$$

and therefore $S_{ij}^{(1)} = \mathcal{O}(1/n)$. Next, we estimate

$$\sum_{k=1}^{N_j} \sum_{\substack{m=1 \\ |T_m^{(i)} \cap T_k^{(j)}| = h}}^{N_i} \mathbb{E} I_k^{(j)} I_m^{(i)} \leqslant \varphi(d_j) \binom{n}{s_j} (s_j - 1)! \varphi(d_i) \binom{s_j}{h} \binom{n - s_j}{s_i - h} (s_i - 1)! \frac{1}{n^{s_j}} \frac{1}{n^{s_i - h}} \frac{1}{n}$$

$$= \mathcal{O}\left(\frac{1}{n}\right),$$

and therefore $S_{ij}^{(2)} = \mathcal{O}(1/n)$. The inequality and the three powers of n above are justified as follows: $1/n^{s_j}$ as there are s_j states in $T_k^{(j)}$, thus at least as many assignments; $1/n^{s_i-h}$ as there are $s_i - h$ states in $T_m^{(i)}$ that are not in $T_k^{(j)}$, thus at least as many assignments; and $1/n$ by Lemma 19, as $T_m^{(i)}$ and $T_k^{(j)}$ have $h \geqslant 1$ states in common and so there is at least one additional assignment. It follows that $d_{\mathrm{TV}}(W_n, \mathrm{Poisson}(\lambda_n))$ is bounded above by a constant times

$$\frac{1}{n^{\mathrm{lcm}(\tau,\sigma)}} + \sum_{i,j}\left(S_{ij}^{(1)} + S_{ij}^{(2)}\right) = \mathcal{O}\left(\frac{1}{n}\right),$$

which, together with (3), (5), and Lemma 14, gives

$$d_{\mathrm{TV}}(|\mathcal{P}_{\tau,\sigma,n}|, \mathrm{Poisson}(\lambda_n)) = \mathcal{O}\left(\frac{1}{n}\right),$$

and ends the proof. □

We now give the proof of Theorem 2, which mainly adds some notational complexity to the previous proof.

Proof of Theorem 2. Again, we begin with the decomposition

$$|\mathcal{P}_{\mathcal{S},n}| = |\mathcal{P}_{\mathcal{S},n}^{(0)}| + \sum_{\ell \geqslant 1} |\mathcal{P}_{\mathcal{S},n}^{(\ell)}|,$$

where $\mathcal{P}_{\mathcal{S},n}^{(\ell)}$ is the set of PS with periods $(\tau,\sigma) \in \mathcal{S}$ whose tile has lag ℓ. Note that the summation is finite since $\mathcal{S}$ is. For $\ell \geqslant 1$, as in (4),

$$\mathbb{P}(|\mathcal{P}_{\mathcal{S},n}^{(\ell)}| > 0) \leqslant \mathbb{E}(|\mathcal{P}_{\mathcal{S},n}^{(\ell)}|) = \mathcal{O}\left(\frac{1}{n^{\ell}}\right).$$

As a consequence,

$$d_{\mathrm{TV}}\left(\sum_{\ell \geqslant 1} |\mathcal{P}_{\mathcal{S},n}^{(\ell)}|,\, \delta_0\right) = \mathcal{O}\left(\frac{1}{n}\right).$$

This reduces the proof to finding the distributional limit of $|\mathcal{P}_{\mathcal{S},n}^{(0)}|$ as $n \to \infty$. We adopt the notation u, d_j, s_j, $T_k^{(j)}$ and $I_k^{(j)}$ from the proof of Theorem 1, for a fixed σ and τ. The dependence of these quantities on σ and τ will be suppressed from the notation, as the periods are taken from a finite range and thus do not affect the computation that follows. Now, $W_n = \sum_{(\tau,\sigma)} \sum_{j=1}^{u} \sum_{k=1}^{N_j} I_k^{(j)} = |\mathcal{P}_{\mathcal{S},n}^{(0)}|$ and

$$\Lambda_n = \sum_{(\tau,\sigma)} \sum_{j=1}^{u} \sum_{k=1}^{N_j} \mathbb{E} I_k^{(j)} \to \sum_{(\tau,\sigma)\in\mathcal{S}} \lambda_{\tau,\sigma} = \lambda_{\mathcal{S}},$$

as $n \to \infty$. It remains to show that $d_{\mathrm{TV}}(W_n, \mathrm{Poisson}(\Lambda_n)) = \mathcal{O}(1/n)$. From Lemma 13,

$$d_{\mathrm{TV}}(W_n, \mathrm{Poisson}(\Lambda_n)) \\ \leqslant \min(1, \Lambda_n^{-1}) \left[\sum_{(\tau,\sigma)} \sum_{j,k} \left(\mathbb{E} I_k^{(j)}\right)^2 + \sum_{(\tau,\sigma)} \sum_{j,k} \sum_{(\tau',\sigma')} \sum_{\substack{i,m \\ T_m^{(i)} \not\perp T_k^{(j)}}} \left(\mathbb{E} I_k^{(j)} \mathbb{E} I_m^{(i)} + \mathbb{E} I_k^{(j)} I_m^{(i)} \right) \right]. \tag{9}$$

To bound the double sum in (9), observe that, for a fixed (τ, σ), the sum over j, k is $\mathcal{O}\left(1/n^{\mathrm{lcm}(\tau,\sigma)}\right)$ by (7). As $\min_{\tau,\sigma} \mathrm{lcm}(\tau, \sigma) \geqslant 1$, the double sum in (9) is $\mathcal{O}(1/n)$.

To bound the quadruple sum in (9), fix a (τ, σ) for $I_k^{(j)}$, a (τ', σ') for $I_m^{(i)}$, and $i, j \in \{1, \ldots, u\}$. Then the sum over the remaining indices is bounded by $S_{ij}^{(1)} + S_{ij}^{(2)}$, exactly as in (6), except that now $S_{ij}^{(1)}$ and $S_{ij}^{(2)}$ also depend on the periods. The arguments that give $S_{ij}^{(1)} = \mathcal{O}(1/n)$ and $S_{ij}^{(2)} = \mathcal{O}(1/n)$ remain equally valid, and again imply $d_{\mathrm{TV}}(W_n, \mathrm{Poisson}(\Lambda_n)) = \mathcal{O}(1/n)$. The proof is now concluded in the same fashion as the proof of Theorem 1. □

The proof of Corollary 3 is now straightforward.

Proof of Corollary 3. Note that

$$\mathbb{P}(Y_{\sigma,n} \leqslant y) = \mathbb{P}(|\mathcal{P}_{[1,y]\times\{\sigma\},n}| > 0) \to 1 - \exp\left(-\lambda_{[1,y]\times\{\sigma\}}\right),$$

as $n \to \infty$, where $\lambda_{[1,y]\times\{\sigma\}} = \sum_{\tau=1}^{y} \lambda_{\tau,\sigma}$. □

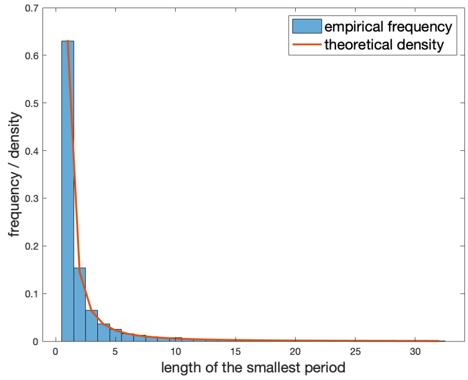
For $\sigma = 1, 2, 3$ and 4, the corresponding $\lambda_{\tau,\sigma}$ are

$$\lambda_{\tau,1} = \frac{1}{\tau}, \quad \lambda_{\tau,2} = \begin{cases} \frac{3}{2\tau}, & 2 \mid \tau \\ \frac{1}{2\tau}, & 2 \nmid \tau \end{cases}, \quad \lambda_{\tau,3} = \begin{cases} \frac{7}{3\tau}, & 3 \mid \tau \\ \frac{1}{3\tau}, & 3 \nmid \tau \end{cases}, \quad \lambda_{\tau,4} = \begin{cases} \frac{11}{4\tau}, & \tau = 0 \mod 4 \\ \frac{3}{4\tau}, & \tau = 2 \mod 4 \\ \frac{1}{4\tau}, & \tau = 1,3 \mod 4 \end{cases}.$$

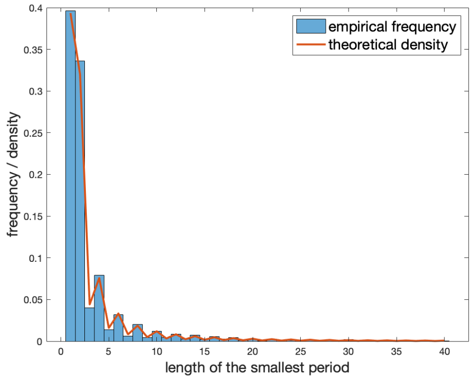
In Figure 5, we present computer simulations to test how close the distribution of $Y_{\sigma,n}$ is to its limit for moderately large n for the above four σ's. To compute $Y_{\sigma,n}(f)$, for every f in the samples, we apply Algorithm 8.

5 Discussion and open problems

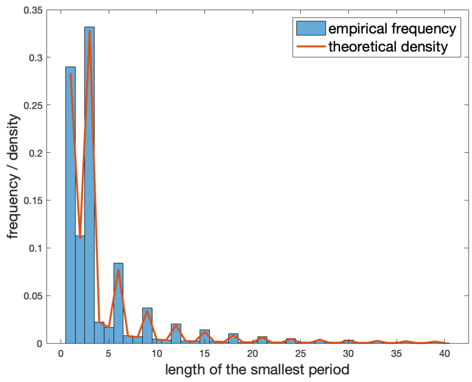
In this paper, we initiate the study of periodic solutions for one-dimensional CA with uniformly randomly selected rules. Our main focus is the limiting probability distribution of the number of PS when the number of states grows to infinity, and we show (Corollary 3) that the smallest temporal period of PS with a given spatial period σ is stochastically bounded.



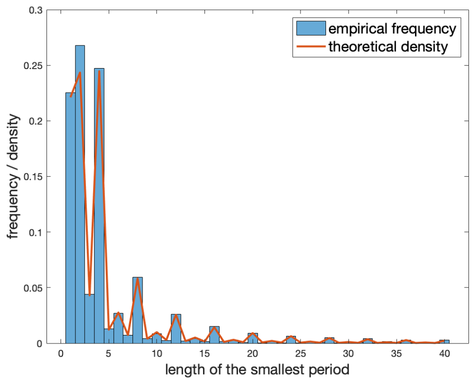
(a) $\sigma = 1$, $n = 100$



(b) $\sigma = 2$, $n = 100$



(c) $\sigma = 3$, $n = 60$



(d) $\sigma = 4$, $n = 20$

Figure 5: Lengths of the smallest temporal periods of PS with spatial periods $\sigma = 1$ to $\sigma = 4$ and various n. In each case, a histogram from a random sample from 10,000 rules is compared to the theoretical limiting distribution as $n \to \infty$, given by Corollary 3.

By a similar argument, we can also obtain an analogous result in which we fix the temporal period instead of the spatial period. Define another random variable

$$Y'_{\tau,n} = \min\{\sigma : \mathcal{P}_{\tau,\sigma,n} \neq \varnothing\},$$

which is the smallest spatial period of a PS given a temporal period τ. For example, for the four rules in Figure 2, we may verify that, by Algorithm 11, $Y'_{1,3}(\mathit{012200210}) = 1$ $(0 \to 0)$, $Y'_{2,3}(\mathit{021102120}) = 2$ $(12 \to 21 \to 12)$, $Y'_{3,3}(\mathit{100112122}) = 3$ $(102 \to 021 \to 210 \to 102)$ and $Y'_{4,3}(\mathit{101201021}) = 4$ $(0101 \to 2012 \to 1010 \to 0122 \to 0101)$, with one cycle that generates the minimal PS given parenthetically for each case.

Corollary 20. *The random variable $Y'_{\tau,n}$ converges to a nontrivial distribution as $n \to \infty$.*

Perhaps the most natural generalization of Theorem 2 would relax the condition that $\mathcal{S}$ is finite. The first case to consider surely is when $\mathcal{S}$ is a Cartesian product with one factor equal to $\mathbb{N}$. For example, it is clear that $\mathbb{P}(\mathcal{P}_{\mathbb{N}\times\mathbb{N},n} \neq \varnothing) = \mathbb{P}(\mathcal{P}_{\mathbb{N}\times\{1\},n} \neq \varnothing) = 1$, as any constant initial configuration eventually generates a PS with spatial period 1.

Now, consider a general $\sigma \geqslant 2$. Let ξ_0 be a periodic configuration of spatial period σ. Under any CA rule f, ξ_1 maintains the spatial periodicity, hence ξ_t eventually enters into a PS, whose spatial period is however a divisor of σ, not necessarily σ itself. For this reason, we cannot reach an immediate conclusion about $\lim \mathbb{P}(\mathcal{P}_{\mathbb{N}\times\{\sigma\},n} \neq \varnothing)$, as $n \to \infty$. We also refer the readers to [6], in which the reduction of temporal periods is explored in more detail.

For a fixed temporal period τ, the matter is even less clear as a rule may not have a PS with temporal period that divides τ. For a trivial example with τ odd and $n = 2$, consider the "toggle" rule that always changes the current state and thus $\xi_{t+1} = 1 - \xi_t$ and any initial state results in temporal period 2. Thus we formulate the following intriguing open problem.

Question 21. Let $\tau, \sigma \in \mathbb{N}$. What are the behaviors of $\mathbb{P}(\mathcal{P}_{\{\tau\}\times\mathbb{N},n} \neq \varnothing)$ and $\mathbb{P}(\mathcal{P}_{\mathbb{N}\times\{\sigma\},n} \neq \varnothing)$, as $n \to \infty$?

Another natural question addresses the case when σ and τ increase with n.

Question 22. For positive real numbers $a, b, c, d, \alpha, \beta, \gamma$ and δ, what is the asymptotic behavior of $\mathbb{P}\left(\mathcal{P}_{I_1\times I_2,n} \neq \varnothing\right)$, where $I_1 = [an^\alpha, bn^\beta]$ and $I_2 = [cn^\gamma, dn^\delta]$?

A wider topic for further research is to investigate how different the behavior of the shortest temporal period changes if we choose a random rule uniformly from a subset of the set of all rules. There are, of course, many possibilities for such a subset, and we selected two natural ones below. In each case, we keep the same notation $Y_{n,\sigma}$ for the resulting smallest temporal period of a PS with spatial period σ.

A rule is **left permutative** if the map $\psi_b : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $\psi_b(a) = f(a, b)$ is a permutation for every $b \in \mathbb{Z}_n$. Permutative rules, such as the famous *Rule 30* [19, 9], are good candidates for generation of long temporal periods.

Question 23. Let $\mathcal{L}$ be the set of all $(n!)^n$ permutative rules. Choosing one of these rules uniformly at random from $\mathcal{L}$, what is the asymptotic behavior of $Y_{n,\sigma}$?

Our final question concerns the most widely studied special class of CA, the additive rules [14]. Such a rule is given by $f(a, b) = \alpha a + \beta b$, for some $\alpha, \beta \in \mathbb{Z}_n$.

Question 24. Let $\mathcal{A}$ be the set of all n^2 additive rules. Again, what is the asymptotic behavior of $Y_{n,\sigma}$ if a rule from $\mathcal{A}$ is chosen uniformly at random?

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