A New Algorithm for Embedding Plane Graphs at Fixed Vertex Locations

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Abstract

We show that a plane graph can be embedded with its vertices at pre-assigned (fixed) locations in the plane and at most $2.5n + 1$ bends per edge. This improves and simplifies a classic result by Pach and Wenger [12]. The proof extends to grid embeddings, orthogonal embeddings, and minimum length embeddings.

Mathematics Subject Classifications: 05C10,05C62,68R10

1 Introduction

A *plane graph* is a crossing-free drawing of a graph, also called an *embedding*. Fáry's theorem famously shows that every plane graph has a crossing-free straight-line drawing in the plane. The straight-line drawing is even *isomorphic* to the original drawing of the graph; that is, there is a homeomorphism of the plane that maps the original drawing to the straight-line drawing.

We revisit a classic theorem by Pach and Wenger [12] on drawing plane graphs so that each vertex is drawn at a pre-assigned location in the plane; we refer to this as drawing a graph at *fixed vertex locations*. Without further restrictions this is easy: starting with the plane graph, simply apply a homeomorphism of the plane that moves each vertex to its fixed location, but we are interested in limiting the complexity of the resulting drawing. Our measure of complexity will be the number of bends per edge: Each edge is drawn as a chain of straight-line segments (a *polygonal chain*), with every two consecutive segments joined by a *bend*. A straight-line embedding, as in Fáry's theorem, then is a 0-bend embedding.

Pach and Wenger showed that every plane graph on n vertices has an isomorphic embedding at fixed vertex locations with at most $24(n + 2)$ bends per edge, and the drawing can be found in quadratic time.

They also proved that their upper bound is tight up to a constant factor by showing that a perfect matching whose vertices are randomly assigned to the corners of a convex polygon almost surely requires $n/(2 \cdot 40^3)$ bends per edge for a constant fraction of the edges.

The construction Pach and Wenger use in the proof of the upper bound is ingenious, but intricate. The original paper has been widely cited in the graph drawing and computational geometry literature, and has been applied in other results (e.g. [5]), so we think it is important to see whether the construction can be simplified or improved.

Badent, Di Giacomo, and Liotta [4] give an $O(n^2 \log n)$-time algorithm which achieves an upper bound of $6n - 1$ and they sketch an argument that this bound can be lowered to $3n + 2$ (see page 141 of their paper). Gordon [8] gives an $O(n^2)$-time algorithm achieving an upper bound of $3n + O(1)$.

We venture to present yet another proof, which we believe to be very simple and visual, and which also improves the constant in the upper bound[1]. The proof naturally extends to orthogonal drawings, with a slightly weaker bound.

Theorem 1. *Given a plane graph G on n vertices $v_1, \ldots, v_n$, and n points $p_1, \ldots, p_n$ we can find an isomorphic embedding of G in which v_i is located at p_i, for all $1 \leqslant i \leqslant n$, and every edge is a polygonal chain with at most $2.5n - 1$ bends. The embedding can be found in quadratic time.*

For comparison, the best known lower bound for arbitrary graphs is $n/6 - 3$ bends, as shown by Badent, Di Giacomo, and Liotta [4, Theorem 1].

Remark 2 (Other Models). If one does not care which vertex is assigned to which point location, much better upper bounds are known. Kaufmann and Wiese [10] showed that every plane graph on n vertices can be mapped to any n points in linear time with at most 3 bends per edge. They also show that the bound can be improved to 2 bends per edge; the running time goes up to quadratic, but that is the lesser issue; the area of the drawing may increase exponentially.

Remark 3 (History). Drawing plane graphs with vertices at assigned locations has a history in circuit design. Koppe's 1978 paper "Automatische Abbildung eines planaren Graphen in die Ebene mit beliebig vorgebbaren Örtern der Knotenbilder"[2] describes an algorithm to solve the problem [11]. Koppe implemented the algorithm (in Fortran), and the paper includes pictures of the resulting drawings. He does not claim any theoretical upper bounds.

[1]We originally found this proof without being aware of the previous work by Badent, Di Giacomo, Liotta [4] and Gordon [8]. Gordon's construction has some similarities with our construction, but we think we isolate a core geometric idea, which we call the synchronized skiers, which makes our proof almost a case of "just look at the picture".

[2]The title translates to "Automatic drawing of a planar graph in the plane with prescribed vertex locations.

2 The New Proof

As Pach and Wenger did, we first prove Theorem 1 for Hamiltonian graphs, with a slightly better bound, and then show how to make a graph Hamiltonian.

Lemma 4. *Any plane Hamiltonian graph G on n vertices has an isomorphic embedding at fixed vertex locations in which every edge is drawn as a polygonal chain with at most $2n-1$ bends. The embedding can be found in quadratic time given the Hamiltonian cycle.*

The main ingredient in the proof of Lemma 4 is a geometric pattern which we fit to the set of fixed vertex locations. Following this pattern allows us to reach each point from an initial placement of a vertex without requiring too many bends. Our simple geometric solution is shown in Figure 1. Imagine six skiers going down the mountain in parallel, picking up the points as needed by moving left and right.

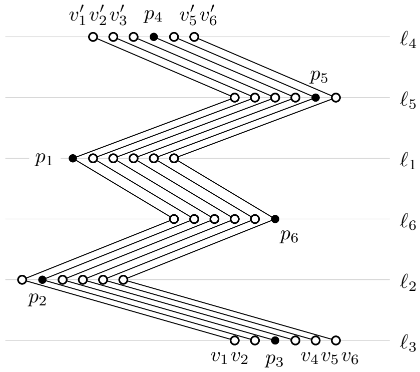

Figure 1: Six points $p_1, \ldots, p_6$ in the plane and corresponding polygonal chains P_i connecting v_i to v_i' for $1 \leqslant i \leqslant 6$ (with $v_3 = p_3$ and $v_4' = p_4$).

More formally, we say that polygonal chains $P_1, \ldots, P_n$ are a *synchronized covering* of the points $p_1, \ldots, p_n$, if the chains are pairwise disjoint, chain P_i contains p_i, for every $i \in [n]$, the i-th points of all chains lie on the same line ℓ_j (labeled so that $p_j \in \ell_j$) and called a *bend-line*, and polygonal chains do not cross bend-lines except at their vertices. For example, in Figure 1, the first points of the six chains all lie on the line containing p_4 so it is labeled ℓ_4.

By definition, each polygonal chain in a synchronized covering has $n-2$ bends (since it has n vertices), and there are two bend-lines so that all bend-points lie between those two bend-lines.

Lemma 5. *For every set of n distinct points we can find a synchronized covering such that all bend-lines are parallel.*

Proof. Suppose no two points lie on the same horizontal line, and p_b is the lowest (bottom) and p_t the highest (top) point. We can then let ℓ_i, the i-th bend-line, be the horizontal line through p_i. On each ℓ_i we place n points so that p_i is the i-th point on ℓ_i from the left. We then let P_i be the polygonal chain that connects all the i-th points on the bend-lines starting with ℓ_b at the bottom and ending with ℓ_t at the top. See Figure 1. By definition, P_i passes through p_i, it has $n-2$ bends (all at intermediate horizontal lines), no two chains intersect each other, and all bend-lines are parallel.

If there are two points that lie on the same horizontal line, we work with lines that are slightly angled instead of horizontal. Since there are only a finite number of slopes between the points, we can easily find a slope for which there is at most one point on each line with that slope. □

The construction in the proof gives us a lot of flexibility, e.g. we can choose the points on ℓ_i arbitrarily, as long as p_i is the i-th point. With Lemma 5 we can complete the proof of Lemma 4.

Proof of Lemma 4. Let C be the Hamiltonian cycle in G. By relabeling vertices and points, if necessary, we can assume that the vertices $v_1, \ldots, v_n$ occur along C in that order. We split G into G_I and G_O, the parts of G drawn inside and outside of C, with C added to both. To distinguish the vertices of G_I and G_O we write v_i' for the vertex in G_O corresponding to v_i in G_I.

Apply the construction of Lemma 5 to points $p_1, \ldots, p_n$ to obtain polygonal chains P_i, $1 \leqslant i \leqslant n$; we will use these chains as guides for how to draw the edges of G_I and G_O. If more than one p_i lies on a horizontal line, the bend-lines will be at an angle; in the drawings, we will display the lines as being horizontal, even if they are not. Let p_b be the lowest and p_t the highest point (relative to the angle).

Bend-line ℓ_b contains the n starting points of the polygonal chains. Identify these points with $v_1, \ldots, v_n$ (in particular, v_b is located at p_b) and add a drawing of the outerplane G_I below ℓ_b on those n points. This requires at most one bend per edge of G_I (the edges in $E(C) - \{v_1 v_n\}$ require no bend).

Similarly, we draw G_O above ℓ_t, with its vertices $v_1', \ldots, v_n'$ located at the end-points of the n polygonal chains. Then v_t' is located at p_t.

Finally, we move each v_i and its incident edges along P_i to p_i, and each v_i' along P_i to p_i, adding bends as we pass a bend-line. See Figure 2 for an illustration. To do this, we erase the drawing in a small neighborhood of v_i (or v_i'); we then have several severed edge-ends, some to the left, and some to the right of v_i. We place the same number of points to the left and right of any bend-point we encounter along the polygonal chain before we reach p_i and route the edges through those points.

Moving v_i to p_i can add at most $n-1$ bends to each edge incident to v_i. Since an edge has at most two endpoints, the moves can add at most $2n-2$ bends to each edge. Since the edge had one initial bend, the total number of bends per edge is at most $2n-1$. We have obtained a drawing of G with vertices v_i located at p_i, but with two copies of C. We arbitrarily remove one of them to obtain the desired drawing of G. □

Remark 6. As suggested by a referee, we can reduce the number of bends in Lemma 4 by 1 by working with a slightly concave ℓ_b (and convex ℓ_t), allowing us to draw G_O and G_I without any bends. This modification makes it less obvious how to adapt the proof

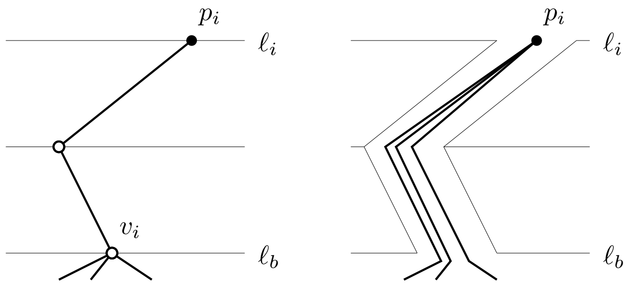

Figure 2: Moving edges incident to v_i to p_i by closely following P_i.

to other drawing models, so we have not adopted it. In practice, it may also lead to more congested drawings, since the edges of G_O and G_I will lie in very flat regions (the curvature of ℓ_b and ℓ_t is constrained by the p_i).

The second part of the proof consists in making the planar graph Hamiltonian. There are various approaches to that, but they all tend to rely on Whitney's theorem that a maximal planar graph (a triangulation) is Hamiltonian, if it does not contain a *separating triangle*, that is, a triangle that bounds a non-empty face [14]. We destroy these separating triangles following an idea by Kaufmann and Wiese [10, p.120-122].

Lemma 7. *Every plane graph on n vertices can be made Hamiltonian by subdividing each edge at most once, and adding some edges. Added vertices have degree at most* 4. *A Hamiltonian cycle can be found in linear time. Whenever the Hamiltonian cycle passes through a subdivision vertex, the two parts of the subdivided edge lie on opposite sides of the cycle.*

Proof. We add edges to the graph so it becomes maximally planar.[3] Suppose uv is the side of a separating triangle uvw. Then uv belongs to two facial triangles auv and buv, one inside uvw, and one outside. We subdivide uv by adding a vertex x, as well as edges xa and xb. Then x has degree 4, the graph remains maximally planar, and we reduced the number of separating triangles by at least one (no new separating triangles were created, and at least one was destroyed). See Figure 3 for an illustration (which also shows that x may later become incident to additional edges).

Repeating this, we can eliminate all separating triangles.[4] The resulting graph is then Hamiltonian by Whitney's theorem [14], and we can find a Hamiltonian cycle C in linear time, using [3].[5]

Suppose C passes through a vertex x which subdivides edge uv in the original graph. Let xy and xz be the two edges of C incident to x. If $\{y, z\} = \{u, v\}$, then subdividing

[3]This is well-known to be achievable in linear time, see [9], for example.

[4]All (separating) triangles in a planar graph can be found in linear time, see [7].

[5]The authors of [3] remark that Whitney's original proof leads to a quadratic-time algorithm, which would be sufficient for us.

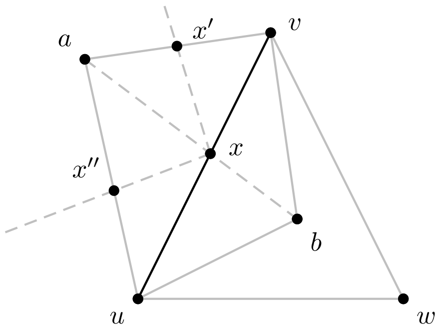

Figure 3: Edge uv was subdivided by x, and x connected to a and b to destroy a separating triangle, uvw; later au and av were also subdivided.

uv was unnecessary to make the graph Hamiltonian; we undo this by deleting x and all edges incident to x, drawing uv along uxv, and replacing yxz in C with $yz = uv$.

If $\{y, z\}$ and $\{u, v\}$ are disjoint, we are done if ux and xv lie on opposite sides of C. If they do not, then, as above, we undo the subdivision of uv as follows: delete x and all edges incident to it, add back uv along uxv and replace yxz in C with yz. If edge yz already exists we are done (it cannot have belonged to C, otherwise C would have been the cycle on yxz). So we need a drawing of yz which we get by following yxz closely (without intersecting x). For example, in Figure 3 we may have that yxz is $x'xx''$, in which case we need to add a drawing of $x'x''$, or yxz may be axx', in which case we already have ax'.

Otherwise, $\{y, z\}$ and $\{u, v\}$ have exactly one vertex in common, say $y = u$. Again, we delete x and all edges incident to it. We add back uv along uxv and draw $yz = uz$ along ux and xz, unless yz already exists in the graph, in which case we use the existing drawing. In C we replace yxz with yz. For example, in Figure 3 we may have that yxz is uxb, in which case we do not have to add a new edge ub, but yxz could also be uxx' in which case we need to add a drawing of ux' by closely following uxx' without intersecting it.

Finally, any subdivision vertex x is incident to at most two edges belonging to an original edge of G and at most two edges belonging to C. We delete all other edges incident to x ensuring that it has degree at most 4. □

The idea for the proof of the main result is now simple: Lemma 7 shows that every graph can be made Hamiltonian by subdividing each edge at most once; this doubles the upper bound of Lemma 4 to $4n + 1$ (with one extra bend at the subdivision vertex). For the sharper bound, we need to proceed more carefully.

Proof of Theorem 1. We are given a plane graph G on n vertices. Apply Lemma 7 to obtain a plane graph H with Hamiltonian cycle C. We use Lemma 4 with graph H and cycle C, but only consider the vertices in $V(G)$ as having a fixed location. For the vertices in $V(H) - V(G)$ we add polygonal chains that lie between the polygonal chains of their neighboring vertices (along C) and assign them locations on ℓ_b. We obtain an isomorphic embedding of H. In that embedding, the vertices in $V(G)$ are placed at their assigned locations, and every edge has at most $2n-1$ bends. In particular, any edge of G that was not subdivided has at most $2n-1$ bends.

Every edge of G was subdivided at most once, and the subdivision vertices lie on ℓ_b. Consider a subdivision vertex x on (original) edge $uv \in E(G)$. By Lemma 7 we know that xu and xv lie on opposite sides of C. Hence, while one of ux and xv may have $2n-1$ bends, the other edge has at most n bends, since x lies on ℓ_b, for a total of $3n$ bends. See Figure 4.

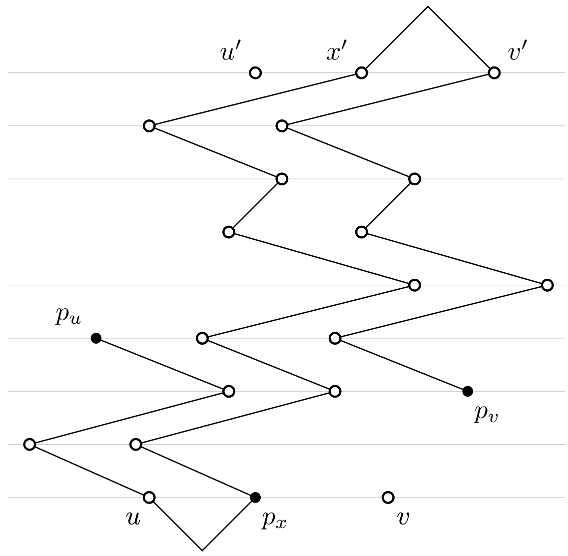

Figure 4: Pushing the vertices of the subdivided edge uxv into place, u to p_u, x' to p_x and v' to p_v.

To get the $2.5n$ bound, we revisit the way we destroy a separating triangle uvw in the proof of Lemma 7. Instead of picking uv arbitrarily, we work with the fixed locations of u, v, and w. Imagine a line parallel to the ℓ-lines so that at most half the vertices of G lie below that line, and at most half above (in Figure 4 that is simply the middle line). Then at least two of the three vertices, let us say u and v lie in the same half, and we destroy the separating triangle uvw by subdividing uv. If u and v both lie in the lower half (as illustrated in Figure 4), then one of the edges has at most $n/2+1$ bends, while the other has at most $n+(n-1)=2n-2$ bends (p_u and p_v cannot lie on ℓ_b), giving a total of $n/2+1+2n-2+1=2.5n$ bends. If u and v both lie in the upper half, then the two edges have at most n and $1.5n$ bends, for a total of $n+1.5n+1=2.5n+1$ bends. In either case, uv has at most $2.5n+1$ bends. □

The geometric construction at the root of Theorem 1—the synchronized covering created in Lemma 5—is quite versatile. In the following subsections we exhibit variants of Theorem 1 for grid, orthogonal, and minimum length drawings.

Grid Embeddings[6]

Suppose the fixed vertex locations for plane graph G in Theorem 1 lie on a $k \times k$, axis-parallel grid. The embedding produced by Theorem 1 can be made to lie on a somewhat larger grid. Let us assume that no two vertices of G lie on the same horizontal line. In this case the bend-lines in Lemma 5 can be chosen to be horizontal. We evenly space the n vertices of the polygonal chains on each bend-line at a distance which is $2n-1$ times the unit distance of the $k \times k$ grid. When pushing a vertex v along a polygonal chain in the proof of Lemma 4 we may have to create up to $n-1$ additional bend-points to the left or right of a vertex of the polygonal chain we pass to accommodate the at most $n-1$ edges v is incident to. Since we placed the vertices of the polygonal chains on each bend-line at distance $2n-1$, we can place the additional points (evenly spaced at unit distance), without interfering with bend-points added to another chain. We need $n(1+2(n-1)) < 2n^2$ points on each bend-line to place all vertices and bend-points, and those points may have to lie at the extreme left or right end of the grid, so we will be fine if we add $2n^2$ grid-points to the left and right of the existing k grid-points, for a total of $(k+4n^2)$ grid-points on each bend-line.

So far, we have argued that all vertices of the graph lie on a $(k+4n^2) \times k$-grid. This leaves us with placing the bend-points of edges in $G_I - E(C)$ and $G_O - E(C)$ below ℓ_b and above ℓ_t. We draw these edges at a $\pi/4$ angle at C, e.g. as shown in Figure 4. The corresponding bend-points can lie at most $k/2 + 2n^2$ units above or below the current grid, so they lie within the confines of a $(k+4n^2) \times (2k+4n^2)$-grid, but they may not lie on an actual grid-point. Since we chose angles of $\pi/4$, every bend-point either lies on a grid-line, or exactly in the middle between two grid-lines; hence it is sufficient to refine the grid by a factor of 2 to place the remaining bend-points. This proves the following result.

Corollary 8. *If the fixed vertex locations in Theorem 1 lie on a $k \times k$ grid, and no two locations lie on the same horizontal line, then the resulting embedding can be found on a $(2k+8n^2) \times (4k+8n^2)$-grid.*

If we cannot work with horizontal bend-lines, we need to rotate the grid. The number of distinct slopes r/s, with $1 \leqslant r, s \leqslant \ell$ is $\Omega(\ell^2)$ [2, Section 3.8], so we can choose $\ell \in O(n)$ sufficiently large such that r/s is none of the slopes determined by the given n fixed vertex locations. We can then work at slope r/s which requires an $O(n)$ refinement of the given grid, so in this case, we can construct an embedding on a grid of size $O(kn^2) \times O(kn^2)$.

Orthogonal Drawings[7]

In an *orthogonal drawing* edges may only consist of axis-parallel pieces, that is, all line-segments are horizontal or vertical. Since edges are usually not allowed to overlap, this

[6] I would like to thank one of the referees for suggesting grid embeddings.

[7] I am grateful to Hemanshu Kaul, who suggested to me that the proof of Theorem 1 extends to orthogonal drawings.

limits the vertex degrees of a representable graph to at most 4; graphs with this property are known as 4-*plane*.

Theorem 9. *Given a* 4*-plane graph* G *on* n *vertices* $v_1, \ldots, v_n$*, and* n *points* $p_1, \ldots, p_n$ *we can find an isomorphic orthogonal embedding of* G *in which* v_i *is located at* p_i*, for all* $1 \leqslant i \leqslant n$*, and every edge has at most* $10n + 6$ *bends. The embedding can be found in quadratic time.*

We sketch the modifications necessary in the proof of Theorem 1 to obtain Theorem 9. In the proof of Lemma 5 we replace the straight-line segments between consecutive lines with 2-bend orthogonal drawings of each edge looking like └┐ or ┌┘. Assuming no two vertices of G have to be located on the same horizontal line, each polygonal chain can be drawn with at most $2(n-1)$ bends. The initial drawing of an edge in Lemma 4 requires two bends in an orthogonal drawing, and we may need as many as two bends to attach an edge to its endpoint, so Lemma 4 requires at most $4(n-1)+2+2\cdot 2 = 4n+2$ bends per edge in an orthogonal drawing. This leads to a bound of at most $2(4n+2)+2 \leqslant 8n+6$ bends per edge in an orthogonal drawing of G. Adding edges to build a Hamiltonian cycle may result in vertices of degree more than 4, but we can remove the newly added edges of C after placing G_I and G_O and before pushing their vertices, so the degree at every vertex is at most 4 again.

We assumed that no two points lie on a horizontal line, but that may actually happen. In an orthogonal drawing that makes a difference, since we cannot arbitrarily change the angle of the underlying lines. Fortunately, the solution is simple: if we have a line containing multiple points, we bend all edges so they can traverse the line from left to right, then bend back to continue upwards, see Figure 5. This adds at most two bends per polygonal chain for every line containing more than one point. Since there can be at most $n/2$ such lines, this adds at most n bends per chain, increasing the overall upper bound to $10n + 6$ bends per edge in an orthogonal drawing.

There are several variations of the orthogonal drawing model to accommodate graphs with vertices of degree larger than 4. Theorem 9 easily extends to such variants. In the *Kandinsky model*, vertices can be replaced by rectangles, with edges attaching to the outside of the rectangle; this makes it possible to create orthogonal drawings for graphs of arbitrary degrees. For the Kandinsky model, Angelini, Rutter, and T. P. [1] showed that if a graph G is given with a partial orthogonal embedding (so edges as well as vertices are fixed), and G has an orthogonal embedding extending the given embedding, then such an embedding needs at most $262|V(H)|$ bends per edge, where H is the vertex set of the partially embedded graph.

Minimal Length Embeddings

Chan, Hoffmann, Kiazyk, and Lubiw [6] introduced the *Minimum Length Planar Drawing [or Embedding] at Fixed Vertex Locations* problem. Their goal, different from ours, was to find a drawing of a given graph at fixed vertex location so that the total edge-length does not exceed the minimum possible length $L = L(G)$ by too much. Their paper

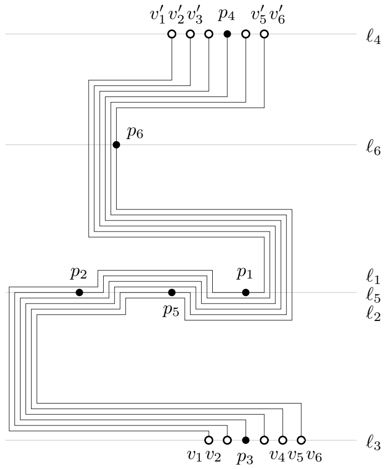

Figure 5: Six points $p_1, \ldots, p_6$ in the plane, with three points lying on the same horizontal line, and corresponding orthogonal drawings of polygonal chains P_i connecting v_i to v_i' for $1 \leqslant i \leqslant 6$.

showed (among other results) that one can always find a drawing of total edge-length at most $O(nL)$ in time $O(n^2)$. They do so by building a geometric spanning forest of total edge-length at most L, and then routing edges around that spanning forest adapting the original construction by Pach and Wenger. We will see that our construction can be modified to obtain the same upper bound, with an explicit constant factor (but slower running time).

At a first glance, our construction does not look too promising for achieving a minimal length embedding. Consider a regular n-gon of radius L. The polygonal chains we construct for this graph have length $\Omega(nL)$, so each edge could have length $\Omega(nL)$ for a total edge-length of $\Omega(n^2L)$. For a graph that has a straight-line planar embedding at the given vertex locations.

We conclude that we need to modify our construction of the polygonal chains. Fortunately that is not too difficult.

Lemma 10. *Given a plane graph G with fixed vertex locations, we can find a family of polygonal chains of total edge-length at most $L(G)$ in time $O(n^5)$, so that every connected component G' of G has all its vertex locations on the same polygonal chain. No two segments belonging to the polygonal chains cross.*

Proof. For each connected component of G we pick a minimum geometric spanning tree, and then use a depth-first traversal to turn it into a straight-line Hamiltonian cycle on the vertices of that component. This gives us a spanning set of cycles. There may be crossings both within and between cycles. The total edge length is at most $2L(G)$. If there are two edges uv and xy that cross, we remove these edges and add edges ux, vy, or edges uy, vx. We can always do so that afterwards all four vertices lie on the same cycle. This (standard) move (known as a 2-*opt*) will not increase the total edge-length of the cycles (though it may decrease the number of cycles, and it may increase the number of crossings). It is known that after at most n^3 steps, this process terminates, and each step can be performed in time $O(n^2)$.[8] We can now remove one edge from each cycle to obtain a family of polygonal chains of total edge-length at most $2L(G)$. By construction all vertices of a connected components lie on the same polygonal chain. □

Theorem 11. *Every plane graph G has an embedding at fixed vertex locations of total edge-length at most $18nK(G)$, and so that every edge has at most $2.5n + 1$ bends. Such an embedding can be found in time $O(n^5)$.*

The $O(n^5)$ (rather than quadratic) running time is caused by the crossing-removal operation in Lemma 10. Everything else can be done in time $O(n^2)$. A faster algorithm can be obtained by basing the polygonal chains on the geometric spanning forest constructed in [6, Lemma 5]. The number of bends would increase, but the total edge-length would improve, and the running time would be $O(n^2)$.

Proof. Using Lemma 10 we construct a family of polygonal chains, so that each connected component is covered by a polygonal chain. Let P be one of the polygonal chains, and let G' be the union of all connected components of G covered by P. We can now perform a construction similar to the one pictured in Figure 1 for G', by having the polygonal chains for the vertices of G' follow P closely (in parallel). We still need at most one bend per inner vertex, so the same analysis as above gives us a bound of $2.5n + 1$ bends per edge. And the length of each edge of G' is at most three times the length of P, which is at most $2L(G)$. Hence, every edge has length at most $6L(G)$. Since every graph has at most $3n - 6$ edges, the total edge-length of the resulting embedding of G is at most $18nL(G)$. □

3 Conclusion

We presented a simplified and strengthened proof of a result first shown by Pach and Wenger [12]. How does the new bound and construction impact existing results in the literature? One example are the minimum length embeddings discussed earlier. Another example (one I was involved in) is Theorem 1 from [5] which generalizes Pach and Wenger's result to the case that edges as well as vertices may be fixed. More precisely: If a plane graph G contains a straight-line drawing of a subgraph H of G, it is possible to draw the edges of $E(G) - E(H)$ with at most $72|V(H)|$ bends each. Can the constant factor be improved? Can the construction be simplified? We also mentioned that Angelini, Rutter, and T. P. [1] showed that a similar result is true for orthogonal drawings in the Kandinsky model, with a bound of $192|V(H)|$. Again we may ask whether the factor can be improved and the construction simplified.

Reducing the Number of Bends. In practice the number of bends can be reduced. In Figure 1 we moved the n points on each bend-line in the same way, making the pieces of the

[8]This result was proved by van Leeuwen and Schoone [13] for Hamiltonian cycles, but it also works for a spanning set of cycles. The untangling step is the bottle-neck in our running time, but it appears there have been no improvements on the original bound since it was published.

polygonal chain parallel. This is an easy way to visualize the construction (synchronized skiers), but the proof does not require it, the vertices can be placed anywhere along the line as long as they are in the right order. Figure 6 shows an improved placement that only requires a single bend (in the polygonal chain from v_5 to p_5); we do not count the bends at the p_i, since we do not push through such a vertex.

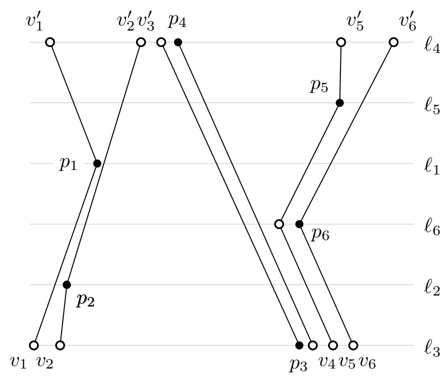

Figure 6: The six points $p_1, \ldots, p_6$ from Figure 1 with polygonal chains P_i only requiring a single bend (not counting bends at the p_i).

Given points $p_1, \ldots, p_n$ and vertices $v_1, \ldots, v_n$ along a line one may ask for the fewest number of bends that is achievable in a synchronized covering. And one may want to optimize that over all angles at which the line can be placed. Can this number be calculated efficiently based on the set of points and the permutation? Is it related to some natural intrinsic parameter of the point-set? Gordon [8] considers the possibility of choosing different directions of traversing the point-set, but does not seem to make use of that possibility.

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