Two results on Ramsey-Turán numbers

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Abstract

Let $f(n)$ be a positive function and H a graph. Denote by $\mathbf{RT}(n, H, f(n))$ the maximum number of edges of an H-free graph on n vertices with independence number less than $f(n)$. It is shown that $\mathbf{RT}(n, K_4 + mK_1, o(\sqrt{n \log n})) = o(n^2)$ for any fixed integer $m \geqslant 1$ and $\mathbf{RT}(n, C_{2m+1}, f(n)) = O(f^2(n))$ for any fixed integer $m \geqslant 2$ as $n \to \infty$.

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1 Introduction

For a graph G, let $v(G)$ and $e(G)$ be the number of vertices and edges of G, respectively, and let $\alpha(G)$ be the independence number of G. For graphs F, G and H, call G to be H-free if G does not contain H as a subgraph. Define $R(F, H)$ the minimum integer N such that any red/blue edge-coloring of K_N contains a red F or a blue H. The Turán number $ex(n; H)$ is defined as the maximum $e(G)$ of an H-free graph G with $v(G) = n$. The celebrated Erdős-Stone-Simonovits theorem shows that the asymptotic behavior of $ex(n; H)$ is determined by the chromatic number $\chi(H)$.

Theorem 1 ([9, 10]). *Let H be a graph with $\chi(H) = k \geqslant 2$. Then*

$$ex(n; H) = \left(\frac{k-2}{k-1} + o(1)\right)\binom{n}{2} \tag{1}$$

as $n \to \infty$.

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For a graph H and positive integers n and m, the Ramsey-Turán number $\mathbf{RT}(n, H, m)$ is defined as

$$\mathbf{RT}(n, H, m) = \max\{e(G) : G \text{ is } H\text{-free with } v(G) = n \text{ and } \alpha(G) < m\}. \tag{2}$$

Clearly, $\mathbf{RT}(n, H, m)$ is non-decreasing on m. The study of Ramsey-Turán numbers was introduced by Sós [21]. It was motivated by the classical theorems of Ramsey and Turán and their connections to geometry, analysis, and number theory. Ramsey-Turán theory has attracted a great deal of attention over the last 40 years, see, e.g, [3, 4, 6, 7, 8, 11, 12, 15, 18, 22, 23], and a survey by Simonovits and Sós [20].

Sometimes we want to consider the case when the bound $f(n)$ on $\alpha(G_n)$ is in form of $o(g(n))$. Namely, we shall consider $\mathbf{RT}(n, H, o(g(n)))$ or $\mathbf{RT}(n, H, g(n)/w_n)$, where the function $w_n \to \infty$ slowly and arbitrarily.

A further notation is as follows. If $\mathbf{RT}(n, H, f(n)) \leqslant cn^2 + o(n^2)$ for every $f(n) = o(g(n))$, then we write $\mathbf{RT}(n, H, o(g(n))) \leqslant cn^2 + o(n^2)$. If $\mathbf{RT}(n, H, f(n)) \geqslant cn^2 + o(n^2)$ for some $f(n) = o(g(n))$, then we write $\mathbf{RT}(n, H, o(g(n))) \geqslant cn^2 + o(n^2)$. When both inequalities become equalities, we write $\mathbf{RT}(n, H, o(g(n))) = cn^2 + o(n^2)$. Note that $\mathbf{RT}(n, H, o(g(n))) \leqslant o(n^2)$ is equivalent to $\mathbf{RT}(n, H, o(g(n))) = o(n^2)$.

Definition 2. Let H be a graph and f a positive function. Define

$$\overline{\rho\tau}(H, f) = \overline{\lim_{n\to\infty}} \frac{\mathbf{RT}(n, H, f(n))}{n^2}$$

and

$$\underline{\rho\tau}(H, f) = \underline{\lim}_{n\to\infty} \frac{\mathbf{RT}(n, H, f(n))}{n^2}.$$

If $\overline{\rho\tau}(H, f) = \underline{\rho\tau}(H, f)$, then we write the common value as $\rho\tau(H, f)$ and call it the Ramsey-Turán density of H with respect to f.

We try to understand that for a given graph H and large n, when we can observe crucial drops in the value of $\mathbf{RT}(n, H, m)$ while m is changing continuously from n to 2? In other words, we try to understand the asymptotic behavior of $\mathbf{RT}(n, H, f(n))$ when we replace $f(n)$ by a smaller $g(n)$.

Definition 3. Given a graph H and two functions f and g with $f(n) > g(n)$, we shall say that H has a *phase transition* from f to g if $\underline{\rho\tau}(H, f) > \overline{\rho\tau}(H, g)$.

Trivially, $\mathbf{RT}(n, K_3, o(n)) = o(n^2)$ since a K_3-free graph G with $v(G) = n$ has maximum degree $\Delta(G) \leqslant \alpha(G)$. A celebrated result of Szemerédi [23], Bollobás and Erdős [4] is

$$\mathbf{RT}(n, K_4, o(n)) = \left(\frac{1}{8} + o(1)\right) n^2.$$

To clarify, the above result says that every K_4-free graph G with $v(G) = n$ and $\alpha(G) = o(n)$ has $e(G) \leqslant (1/8 + o(1))n^2$, and the bound is sharp. It is natural to ask whether or not $\mathbf{RT}(n, K_4, n^{1-\epsilon})$ is $\Theta(n^2)$ for some $\epsilon > 0$? A negative answer to this question was given by Sudakov [22]. Note that for any $\epsilon > 0$, the function $f(n) = e^{-\omega\sqrt{\log n}}n > n^{1-\epsilon}$ if $\omega = \omega_n < \epsilon\sqrt{\log n}$.

Theorem 4 ([22]). *Let $f(n) = e^{-\omega\sqrt{\log n}}n$, where $\sqrt{\log n} \geqslant \omega \to \infty$. Then*

$$\boldsymbol{RT}(n, K_4, f(n)) < e^{-\omega^2/2}n^2$$

for large n.

Let us define $q(3, n)$ as

$$q(3, n) = \min\{\alpha(G) : \ G \text{ is } K_3\text{-free and } v(G) = n\}. \tag{3}$$

Then $q(3, n) = \Theta(\sqrt{n \log n})$ from [1, 14, 19]. The function $\sqrt{n \log n}$ plays an important role in Ramsey-Turán theory. The orders of magnitude of $ex(n; H)$ with $\chi(H) \geqslant 3$ are all n^2, among which K_3 has the minimum $v(H)$. From definition of $\mathbf{RT}(n, H, m)$ in (2) and that of $q(3, n)$ in (3), and the fact $q(3, n) = \Theta(\sqrt{n \log n})$, we may say that an important phase transition of H with $\chi(H) \geqslant 3$ is that from $\sqrt{n \log n}$ to $o(\sqrt{n \log n})$.

It is known that [20]

$$\mathbf{RT}(n, K_5, \sqrt{n \log n}) = \mathbf{RT}(n, K_6, \sqrt{n \log n}) = \left(\frac{1}{4} + o(1)\right) n^2. \tag{4}$$

By answering a question of Erdős and Sós in [11], Balogh, Hu and Simonovits [3] proved the following result.

Theorem 5 ([3]). *As $n \to \infty$, it holds*

$$\boldsymbol{RT}(n, K_5, o(\sqrt{n \log n})) = o(n^2). \tag{5}$$

So by (4) and (5) we know that K_5 has a phase transition from $\sqrt{n \log n}$ to $o(\sqrt{n \log n})$. But the problem for K_6 on the same phase transition is still open.

Problem 6. Whether or not

$$\mathbf{RT}(n, K_6, o(\sqrt{n \log n})) = o(n^2)?$$

For vertex disjoint graphs G and H, let $G + H$ be the joint of G and H obtained from G and H by adding new edges connecting G and H completely. In this note, we shall make a small step to solve Problem 6. Our main result is as follows, in which $K_4 + K_1$ for $m = 1$ is K_5 in (5).

Theorem 7. *Let $m \geqslant 1$ be an integer. Then*

$$\boldsymbol{RT}(n, K_4 + mK_1, o(\sqrt{n \log n})) = o(n^2)$$

as $n \to \infty$.

Denote by $K_6^- = K_4 + 2K_1$, which is the graph obtained also from K_6 by dropping an edge. We shall list the results in [20] on Ramsey-Turán density of K_5, K_6^-, K_6 in the following table.

function\ graph	K_5	K_6^-	K_6
n	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{2}{5}$
$o(n)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{2}{7}$
$\sqrt{n\log n}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$o(\sqrt{n\log n})$	0	0	$\leqslant \frac{1}{6}$

Table 1: Ramsey-Turán density of K_5, K_6^-, K_6

Theorem 7 shows that $\mathbf{RT}(n, K_6^-, o(\sqrt{n\log n})) = o(n^2)$, which and a result in Table 1 tells us that K_6^- has a phase transition from $\sqrt{n\log n}$ to $o(\sqrt{n\log n})$.

Corollary 8. *The graph K_6^- has a phase transition from $\sqrt{n\log n}$ to $o(\sqrt{n\log n})$.*

We also have a result on odd cycle C_{2m+1} of length $2m+1$.

Theorem 9. *Let $m \geqslant 2$ be an integer. If $n \to \infty$, then*

$$\boldsymbol{RT}(n, C_{2m+1}, f(n)) = O(f^2(n)).$$

Remark 1 It is easy to see that $\mathbf{RT}(n, C_{2m+1}, n) = ex(n; C_{2m+1}) = (\frac{1}{4} + o(1))n^2$ by Theorem 1. Thus Theorem 9 shows that C_{2m+1} has a phase transition from n to $o(n)$. Let us point out that $\mathbf{RT}(n, \{C_3, C_5, C_7\}, s) \leqslant s^2$ appeared in [18] (Lemma 7.1).

Remark 2 It should be remarked that m in Theorem 7 can be replaced by some ω_n by careful analysis.

2 Proofs of Main results

The Dependent Random Choice is a method developed by Füredi, Gowers, Kostochka, Rödl, Sudakov, and possibly others. The method is powerful for many problems, which is a "random double counting" in some sense. The next lemma is taken from Alon, Krivelevich and Sudakov [2]. Interested readers may check the survey on this method by Fox and Sudakov [13].

Lemma 10. *Let ℓ, r be positive integers. Let $G = (V, E)$ be a graph with n vertices and average degree $d = 2e(G)/n$. Then for any positive integer t, there exists a subset $U \subseteq V(G)$ with*

$$|U| \geqslant \frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{\ell}{n}\right)^t,$$

such that every r vertices in U have at least ℓ common neighbours.

Note that Lemma 10 makes sense only if

$$|U| \geqslant r, \quad \text{and} \quad \frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{\ell}{n}\right)^t > 0.$$

We need another lemma [16] for the proof of Theorem 7.

Lemma 11. *Let $B_m = K_2 + mK_1$. If $m \geqslant 1$ and $n \geqslant 3$, then*

$$R(B_m, K_n) \leqslant \frac{mn^2}{\log(n/e)}.$$

Proof of Theorem 7. Let $\omega_n \to \infty$ slowly and arbitrarily be a function and $\epsilon_n = \frac{\log \omega_n}{\omega_n}$. To show $\mathbf{RT}(n, K_4 + mK_1, \sqrt{n \log n}/\omega_n) \leqslant \epsilon_n n^2$, we shall show that if G is a $(K_4 + mK_1)$-free graph on n vertices with $\alpha(G) < \sqrt{n \log n}/\omega_n$, then $e(G) \leqslant \epsilon_n n^2$.

Suppose to the contrary, there is a $(K_4 + mK_1)$-free graph $G = (V, E)$ on n vertices with

$$e(G) \geqslant \epsilon_n n^2 \quad \text{and} \quad \alpha(G) < \sqrt{n \log n}/\omega_n,$$

and we shall find a contradiction.

Applying Lemma 10 to G with $d = 2\epsilon_n n,\ r = 2,\ t = \frac{\log n}{\omega_n},\ \ell = \frac{4mn}{\omega_n^2}$ and noting that

$$\log\left((2\epsilon_n)^t \cdot n\right) \sim \log n,$$

and

$$\log\left(\frac{n^2}{2} \cdot \left(\frac{4m}{\omega_n^2}\right)^t\right) = o(\log n),$$

and

$$\log\left(\sqrt{n \log n}/\omega_n\right) \sim \frac{1}{2}\log n,$$

we know that there exists a subset $U \subseteq V(G)$ with

$$\begin{aligned} |U| &\geqslant \frac{d^t}{n^{t-1}} - \binom{n}{r}\left(\frac{\ell}{n}\right)^t \\ &\geqslant (2\epsilon_n)^t \cdot n - \frac{n^2}{2} \cdot \left(\frac{4m}{\omega_n^2}\right)^t \\ &> \sqrt{n \log n}/\omega_n \end{aligned}$$

for all large n, such that all subsets of U of size 2 have at least $\frac{4mn}{\omega_n^2}$ common neighbours. Noting that $|U| > \sqrt{n \log n}/\omega_n$ and $\alpha(G) < \sqrt{n \log n}/\omega_n$, we know that $G[U]$ must have an edge. Assume $u_0 v_0 \in G[U]$. Now we shall construct a $K_4 + mK_1$ as follows. As we know $|N(u_0) \cap N(v_0)| \geqslant \frac{4mn}{\omega_n^2}$, by Lemma 11, we know either $G[N(u_0) \cap N(v_0)]$ contains an independent set of size at least

$$(1 - o(1))\sqrt{\frac{1}{2m} \cdot \frac{4mn}{\omega_n^2} \log\left(\frac{4mn}{\omega_n^2}\right)} > \frac{\sqrt{n \log n}}{\omega_n} > \alpha(G),$$

or $B_m \subset G[N(u_0, v_0)]$, which yields $K_4 + mK_1$ in G, a contradiction. □

Proof of Theorem 9 In order to prove Theorem 9, we need the following well-known result, which was proved by Chvatál [5].

Theorem 12 ([5]). *Let T_m be a tree of order m. Then the Ramsey number*

$$R(T_m, K_n) = (m-1)(n-1) + 1.$$

We also need the following lemma, which plays the key role in the proof.

Lemma 13 ([17]). *Let $m \geqslant 2$ be an integer and let a graph $G = (V, E)$ be C_{2m+1}-free. Then*

$$\alpha(G) \geqslant \frac{1}{(2m-1)2^{(m-1)/m}} \left(\sum_{v \in V} d(v)^{1/(m-1)} \right)^{(m-1)/m},$$

where $d(v)$ is the degree of v in G.

Proof To avoid the triviality we may assume that $f(n) \leqslant n$. To show Theorem 9, we shall show that any graph G on n vertices which is C_{2m+1}-free and $\alpha(G) < f(n)$ has at most $O(f^2(n))$ edges.

For $m = 2$, the assertion is clear since

$$f(n) > \alpha(G) \geqslant \frac{1}{3\sqrt{2}} \left(\sum_{v \in V} d(v) \right)^{1/2} = \left(\frac{nd}{18} \right)^{1/2},$$

where d is the average degree of G. It follows that $e(G) = \frac{nd}{2} \leqslant 9f^2(n)$.

In the following, we shall suppose $m \geqslant 3$ and separate the proof into two cases.

Case 1. The maximum degree $\Delta(G)$ of the graph G satisfies $\Delta(G) > \sqrt{nd}$, i.e., there is some vertex v such that $d(v) > \sqrt{nd}$. As the neighborhood $N(v)$ of v contains no path P_{2m} of order $2m$, it follows from Theorem 12 that

$$f(n) > \alpha(G) \geqslant \frac{d(v)}{2m} > \frac{\sqrt{nd}}{2m},$$

and thus $e(G) = \frac{nd}{2} \leqslant 2m^2 f^2(n)$.

Case 2. $\Delta(G) \leqslant \sqrt{nd}$. For every vertex v we have

$$d(v)^{1/(m-1)} \geqslant \frac{d(v)}{\Delta(G)^{(m-2)/(m-1)}}.$$

Together with Lemma 13 and $\Delta(G) \leqslant \sqrt{nd} = \sqrt{2e(G)}$, this yields

$$\begin{aligned} \alpha(G) &\geqslant \frac{1}{2m-1} \left(\frac{1}{2} \sum_{v \in V} \frac{d(v)}{\Delta(G)^{(m-2)/(m-1)}} \right)^{(m-1)/m} \\ &\geqslant \frac{1}{2m-1} \left(\frac{e(G)}{(2e(G))^{(m-2)/(2m-2)}} \right)^{(m-1)/m} \geqslant \frac{\sqrt{e(G)}}{\sqrt{8} \cdot m}, \end{aligned}$$

whence $e(G) \leqslant 8m^2 \alpha(G)^2$. Hence $e(G) = O(f^2(n))$ that completes the proof. □

3 Conclusions

It was shown [16] that $R(K_1+T_m, K_n) \leqslant \frac{(2m-3)n^2}{\log(n/e)}$ for all $m \geqslant 2$ and $n \geqslant 3$, thus Theorem 7 can be generalized to $\mathbf{RT}(n, K_3 + T_m, o(\sqrt{n \log n})) = o(n^2)$.

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