Normal polytopes and ellipsoids

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Abstract

We show that: (1) unimodular simplices in a lattice 3-polytope cover a neighborhood of the boundary of the polytope if and only if the polytope is very ample, (2) the convex hull of lattice points in every ellipsoid in $\mathbb{R}^3$ has a unimodular cover, and (3) for every $d \geqslant 5$, there are ellipsoids in $\mathbb{R}^d$, such that the convex hulls of the lattice points in these ellipsoids are not even normal. Part (c) answers a question of Bruns, Michałek, and the author.

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1 Introduction

1.1 Main result

A convex polytope $P \subset \mathbb{R}^d$ is *normal* if it is *lattice*, i.e., has vertices in $\mathbb{Z}^d$, and satisfies the condition

$$\forall c \in \mathbb{N} \quad \forall x \in (cP) \cap \mathbb{Z}^d \quad \exists x_1, \dots, x_c \in P \cap \mathbb{Z}^d \qquad x_1 + \cdots + x_c = x.$$

A necessary condition for P to be normal is that the subgroup

$$\mathrm{gp}(P) := \sum_{x,y \in P \cap \mathbb{Z}^d} \mathbb{Z}(x - y) \subset \mathbb{Z}^d$$

must be a direct summand. Also, a face of a normal polytope is normal.

Normality is a central notion in toric geometry and combinatorial commutative algebra [7]. A weaker condition for lattice polytopes is *very ample*; see Section 1.2 for the definition. Normal polytopes define projectively normal embeddings of toric varieties whereas very ample polytopes correspond to normal projective varieties [3, Proposition 2.1].

A sufficient condition for a lattice polytope P to be normal is the existence of a *unimodular cover,* which means that P is a union of unimodular simplices. Unimodular covers play an important role in integer programming through their connection to the *Integral Carathéodory Property* [8, 12, 15].

There exist normal polytopes in dimensions $\geqslant 5$ without unimodular cover [6]. It is believed that all normal 3-polytopes have unimodular cover. But progress in this direction is scarce. Recent works [4, 11] show that all lattice 3-dimensional parallelepipeds and centrally symmetric 3-polytopes with unimodular corners have unimodular cover.

The normality of the convex hull of lattice points in an ellipsoid naturally comes up in [9]. We consider general ellipsoids, neither centered at 0 nor aligned with the coordinate axes. According to [9, Theorem 6.5(c)], the convex hull of the lattice points in any ellipsoid $E \subset \mathbb{R}^3$ is normal. [9, Question 7.2(b)] asks whether this result extends to higher dimensional ellipsoids.

Here we prove the following

Theorem. *Let $P \subset \mathbb{R}^3$ be a lattice polytope, $E \subset \mathbb{R}^d$ an ellipsoid, and $P(E)$ the convex hull of the lattice points in E.*

(a) *The unimodular simplices in P cover a neighborhood of the boundary ∂P in P if and only if P is very ample.*

(b) *If $d = 3$ then the polytope $P(E)$ is covered by unimodular simplices.*

(c) *For every $d \geqslant 6$, there exists E such that $\operatorname{gp}(P(E)) = \mathbb{Z}^d$ and $P(E)$ is not normal.*

If in (c) we drop the condition $\operatorname{gp}(P(E)) = \mathbb{Z}^d$, then ellipsoids $E \subset \mathbb{R}^d$ with $P(E)$ non-normal already exist for $d = 5$; see Remark 7.

1.2 Preliminaries

$\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the sets of non-negative integers and reals, respectively.

The convex hull of a set $X \subset \mathbb{R}^d$ is denoted by $\operatorname{conv}(X)$. The relative interior of a convex set $X \subset \mathbb{R}^d$ is denoted by $\operatorname{int} X$. The boundary of X is denoted by $\partial X = X \setminus \operatorname{int} X$.

Polytopes are assumed to be convex. For a polytope $P \subset \mathbb{R}^d$, its vertex set is denoted by $\operatorname{vert}(P)$.

A lattice n-simplex $\Delta = \operatorname{conv}(x_0, \dots, x_n) \subset \mathbb{R}^d$ is *unimodular* if $\{x_1 - x_0, \dots, x_n - x_0\}$ is a part of a basis of $\mathbb{Z}^d$.

A *unimodular pyramid* over a lattice polytope Q is a lattice polytope $P = \operatorname{conv}(v, Q)$, where the point v is not in the affine hull of Q and the lattice height of v above Q inside the affine hull of P equals 1.

Cones C are assumed to be *pointed, rational,* and *finitely generated,* which means $C = \mathbb{R}_+x_1 + \dots + \mathbb{R}_+x_k$, where $x_1, \dots, x_k \in \mathbb{Z}^d$ and C does not contain a nonzero linear subspace. For a cone $C \subset \mathbb{R}^d$, the smallest generating set of the additive submonoid $C \cap \mathbb{Z}^d \subset \mathbb{Z}^d$ consists of the indecomposable elements of this monoid. This is a finite set, called the *Hilbert basis* of C and denoted by $\operatorname{Hilb}(C)$. See [7, Chapter 2] for a detailed discussion on Hilbert bases. For a lattice polytope $P \subset \mathbb{R}^d$, we have the inclusion of finite subsets of $\mathbb{Z}^{d+1}$:

$$\big(P \cap \mathbb{Z}^d, 1\big) \subset \mathrm{Hilb}(\mathbb{R}_+(P, 1)).$$

This inclusion is an equality if and only if P is normal.

A lattice polytope P is *very ample* if $\mathrm{Hilb}(\mathbb{R}_+(P - v)) \subset P - v$ for every vertex $v \in \mathrm{vert}(P)$. All normal polytopes are very ample, but already in dimension 3 there are very ample non-normal polytopes [7, Exercise 2.24]. For a detailed analysis of the discrepancy between the two properties see [3].

For a cone $C \subset \mathbb{R}^d$, we say that C has a *unimodular Hilbert triangulation (cover)* if C can be triangulated (resp., covered) by cones of the form $\mathbb{R}_+x_1 + \cdots + \mathbb{R}x_n$, where $\{x_1, \dots, x_n\}$ is a part of a basis of $\mathbb{Z}^d$ as well as of $\mathrm{Hilb}(C)$.

An *ellipsoid* $E \subset \mathbb{R}^d$ is a set of the form

$$\big\{x \in \mathbb{R}^d \mid (l_1(x) - a_1)^2 + \cdots + (l_d(x) - a_d)^2 = 1\big\} \subset \mathbb{R}^d,$$

where $l_1, \dots, l_d$ is a full-rank system of real linear forms and $a_1, \dots, a_d \in \mathbb{R}^d$.

For a lattice polytope P, the union of unimodular simplices in P will be denoted by $\mathrm{U}(P)$.

2 Unimodular covers close to the boundary

The following result of Sebő was later rediscovered in [1, 5] in a refined form in the context of toric varieties.

Theorem 1. *([16]) Every 3-dimensional cone C has a unimodular Hilbert triangulation.*

Notice. There exist 4-dimensional cones without unimodular Hilbert triangulation [5] and it is not known whether all 4- and 5-dimensional cones have unimodular Hilbert cover. According to [6], in all dimensions $\geqslant 6$ there are cones without unimodular Hilbert cover.

If $P \subset \mathbb{R}^3$ is very ample, then by Theorem 1, for every $v \in \mathrm{vert}(P)$, the cone $\mathbb{R}_+(P-v)$ has a unimodular Hilbert triangulation:

$$\mathbb{R}_+(P - v) = \bigcup_{T(v)} C_t,$$

where $T(v)$ is a finite index set, depending on v. In particular, the following unimodular simplices form a neighborhood of v in P:

$$\Delta_{v,t} = \mathrm{conv}(\mathrm{Hilb}(C_t), 0) + v, \quad t \in T(v).$$

Also, lattice polygons have unimodular triangulation [7, Corollary 2.54]. Therefore, the following lemma completes the proof of Theorem (a):

Lemma 2. *For a lattice polytope P of an arbitrary dimension, the following conditions are equivalent:*

(a) $\mathrm{U}(P)$ *is a neighborhood of ∂P within P;*

(b) $\mathrm{U}(P)$ *is a neighborhood within P of every vertex of P and $\partial P \subset \mathrm{U}(P)$.*

Proof. The implication (a)$\Longrightarrow$(b) is obvious.

For the opposite implication, let:

- $x \in \partial P$;
- F be the minimal face of P containing x;
- $v \in \mathrm{vert}(F)$;
- T_F be a unimodular cover of F with $\dim(F)$-simplices, contained in F;
- T_v be a unimodular cover of a neighbourhood of v in P;
- $T_{v,F}$ be the sub-family of T_v, consisting of simplices that have a $\dim(F)$- dimensional intersection with F;
- T_v/F be the collection of faces of simplices in $T_{v,F}$, opposite to F (that is, from each simplex in $T_{v,F}$ remove the $\dim(F)+1$ vertices that lie in F, so that one is left with a $(\dim(P)-\dim(F))$-simplex).

Then, the collection of $\mathrm{conv}(T_v/F, T_F)$ covers a neighbourhood of x in P and consists of unimodular simplices. □

3 Unimodular covers inside ellipsoids

3.1 Proof of Theorem (b)

The set of normal polytopes $P \subset \mathbb{R}^d$ carries a poset structure, where the order is generated by the elementary relation

$$P \leqslant Q \text{ if } P \subset Q \text{ and } \#(Q \cap \mathbb{Z}^d) = \#(P \cap \mathbb{Z}^d) + 1.$$

In [9] this poset is denoted by $\mathrm{NPol}(d)$. The trivial minimal elements of $\mathrm{NPol}(d)$ are the singletons from $\mathbb{Z}^d$. It is known that $\mathrm{NPol}(d)$ has nontrivial minimal elements for $d \geqslant 4$ [7, Exercise 2.27] and the first maximal elements for $d = 4, 5$ were found in [9]. It is possible that $\mathrm{NPol}(d)$ has isolated elements for some d.

Computer searches so far have found neither maximal nor nontrivial minimal elements in $\mathrm{NPol}(3)$ [9]. The next lemma is yet another evidence that all normal 3-polytopes have unimodular cover.

Lemma 3. *Let P be a normal 3-polytope. If $* \leqslant P$ in $\mathrm{NPol}(3)$ for a singleton $* \in \mathbb{Z}^3$ then $P = \mathrm{U}(P)$.*

Proof. If $Q \leqslant P$ is an elementary relation in $\mathrm{NPol}(d)$ and $\dim Q < \dim P$ then P is a unimodular pyramid over Q. In this case every full-dimensional unimodular simplex $\Delta \subset P$ is the unimodular pyramid over a unimodular simplex in Q and with the same apex as P. On the other hand, lattice segments and polygons are unimodularly triangulable. Therefore, it is enough to show that a polytope $P \in \mathrm{NPol}(3)$ has a unimodular cover if there is a 3-polytope $Q \in \mathrm{NPol}(3)$, such that Q has a unimodular cover and $Q \leqslant P$ is an elementary relation in $\mathrm{NPol}(3)$. Assume $\{v\} = \mathrm{vert}(P) \setminus Q$. By Theorem (a) we have the inclusion $P \setminus \mathrm{U}(P) \subset Q$. Since $Q = \mathrm{U}(Q)$ we have $P = \mathrm{U}(P)$. □

Call a subset $\mathcal{E} \subset \mathbb{Z}^d$ *ellipsoidal* and a point $v \in \mathcal{E}$ *extremal* if there is an ellipsoid $E \subset \mathbb{R}^d$, such that $\mathcal{E} = \mathrm{conv}(E) \cap \mathbb{Z}^d$ and $v \in E$.

Lemma 4. *Let $\mathcal{E} \subset \mathbb{R}^d$ be an ellipsoidal set. Then $\mathcal{E}$ has an extremal point and $\mathcal{E} \setminus \{v\}$ is also ellipsoidal for every extremal point $v \in \mathcal{E}$.*

Proof. Let $\mathcal{E} = \mathrm{conv}(E) \cap \mathbb{Z}^d$ for an ellipsoid $E \subset \mathbb{R}^d$. Applying an appropriate homothetic contraction, centered at the center of E, we can always achieve $\mathcal{E} \cap E \neq \emptyset$. In particular, $\mathcal{E}$ has an extremal point. For $v \in \mathcal{E} \cap E$, after changing E to its homothetic image with factor $(1+\varepsilon)$ and centered at v, where ε is a sufficiently small positive real number, we can further assume $\mathcal{E} \cap E = \{v\}$. Finally, applying a parallel translation to E by $\delta(z - v)$, where z is the center of E and $\delta > 0$ is a sufficiently small real number, we achieve $\mathrm{conv}(E) \cap \mathbb{Z}^d = \mathcal{E} \setminus \{v\}$. □

Next we complete the proof of Theorem (b). It follows from Lemma 3.2 that, for any natural number d and an ellipsoidal set $\mathcal{E} \subset \mathbb{Z}^d$, there is a descending sequence of ellipsoidal sets of the form

$$\mathcal{E} = \mathcal{E}_k \supset \mathcal{E}_{k-1} \supset \cdots \supset \mathcal{E}_1, \text{ with } \#\mathcal{E}_i = i \text{ for } i = 1, \ldots, k.$$

By [9, Theorem 6.5(c)], for $d = 3$, the $\mathrm{conv}(\mathcal{E}_i)$ are normal polytopes. Therefore, $* \leqslant \mathrm{conv}(\mathcal{E})$ in $\mathrm{NPol}(3)$ for some $* \in \mathbb{Z}^3$. Thus Lemma 3 applies. □

3.2 Alternative algorithmic proof in symmetric case

For the ellipsoids E with center in $\frac{1}{2}\mathbb{Z}^3$, there is a different proof of Theorem (b). It yields a simple algorithm for constructing a unimodular cover of $P(E)$.

Instead of Theorem 1 and [9, Theorem 6.5] this approach uses Johnson's 1916 *Circle Theorem* [13, 14]. We only need Johnson's theorem to derive the following fact, which does not extend to higher dimensions: for any lattice $\Lambda \subset \mathbb{R}^2$ and any ellipse $E' \subset \mathbb{R}^2$, such that $\mathrm{conv}(E')$ contains a triangle with vertices in Λ, every parallel translate $\mathrm{conv}(E') + v$, where $v \in \mathbb{R}^2$, meets Λ.

Assume an ellipsoid $E \subset \mathbb{R}^3$ has center in $\frac{1}{2}\mathbb{Z}^3$ and $\dim(P(E)) = 3$ (notation as in the theorem). Assume $\mathrm{U}(P(E)) \subsetneq P(E)$. Because $\partial P(E)$ is triangulated by unimodular triangles, there is a unimodular triangle $T \subset P(E)$, not necessarily in $\partial P(E)$, and a point $x \in \operatorname{int} T$, such that the points in $[0, x]$, sufficiently close to x, are not in $\mathrm{U}(P(E))$. For the plane, parallel to T on lattice height 1 above T and on the same side as 0, the intersection $E' = \operatorname{conv}(E) \cap H$ is at least as large as the intersection of $\operatorname{conv}(E)$ with the affine hull of T: a consequence of the fact that $P(E) \cap \mathbb{Z}^3$ is symmetric relative to the center of E. The mentioned consequence of Johnson's theorem implies that $\operatorname{conv}(E')$ contains a point $z \in \mathbb{Z}^3$. In particular, all points in $[x, 0]$, sufficiently close to x are in the unimodular simplex $\operatorname{conv}(T, z) \subset P(E)$, a contradiction.

4 High dimensional ellipsoids

For a lattice $\Lambda \subset \mathbb{R}^d$, define a Λ-*polytope* as a polytope $P \subset \mathbb{R}^d$ with $\operatorname{vert}(P) \subset \Lambda$. Using Λ as the lattice of reference instead of $\mathbb{Z}^d$, one similarly defines Λ-*normal polytopes* and Λ-*ellipsoidal sets*.

Consider the lattice $\Lambda(d) = \mathbb{Z}^d + \mathbb{Z}\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \subset \mathbb{R}^d$. We have $\left[\mathbb{Z}^d : \Lambda(d)\right] = 2$. Consider the $\Lambda(d)$-polytope $P(d) = \operatorname{conv}\left(\mathrm{B}(d) \cap \Lambda(d)\right)$, where $\mathrm{B}(d) = \left\{(\xi_1, \ldots, \xi_d) \mid \sum_{i=1}^d \left(\xi_i - \frac{1}{2}\right)^2 \leqslant \frac{d}{4}\right\} \subset \mathbb{R}^d$, i.e., $\partial(\mathrm{B}(d))$ is the circumscribed sphere for the cube $[0,1]^d$.

Consider the d-dimensional $\Lambda(d)$-polytope and the $(d-1)$-dimensional $\Lambda(d)$-simplex:

$$\begin{aligned} Q(d) &= \operatorname{conv}\left((P(d) \cap \Lambda(d)) \setminus \{\mathbf{e}_1 + \cdots + \mathbf{e}_d\}\right), \\ \Delta(d-1) &= \operatorname{conv}\left(\mathbf{e}_1 + \cdots + \mathbf{e}_{i-1} + \mathbf{e}_{i+1} + \cdots + \mathbf{e}_d \mid i = 1, \ldots, d\right), \end{aligned}$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_d \in \mathbb{R}^d$ are the standard basic vectors.

Notice. Although $P(d) \cap \mathbb{Z}^d = \{0,1\}^d$ for all d, yet $[0,1]^d \subsetneq P(d)$ for all $d \geqslant 4$. In fact, $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right) + k\mathbf{e}_i \in P(d) \cap \Lambda(d)$ for $1 \leqslant i \leqslant d$ and $-\left\lceil \frac{\sqrt{d}}{2} \right\rceil \leqslant k \leqslant \left\lfloor \frac{\sqrt{d}}{2} \right\rfloor$.

Lemma 5. *If $d \geqslant 5$ then $\Delta(d-1)$ is a facet of $Q(d)$ and $\Delta(d-1) \cap \Lambda(d) = \operatorname{vert}(\Delta(d-1))$.*

Proof. Assume $x = (\xi_1, \ldots, \xi_d) \in P(d) \cap \Lambda(d)$ satisfies $\xi_1 + \cdots + \xi_d \geqslant d-1$. We claim that there are only two possibilities: either $x = \mathbf{e}_1 + \cdots + \mathbf{e}_d$ or $x = \mathbf{e}_1 + \cdots + \mathbf{e}_{i-1} + \mathbf{e}_{i+1} + \cdots + \mathbf{e}_d$ for some index i. Since $P(d) \cap \mathbb{Z}^d = \{0,1\}^d$, only the case $x \in \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) + \mathbb{Z}^d$ needs to be ruled out. Assume $\xi_i = \frac{1}{2} + a_i$ for some integers a_i, where $i = 1, \ldots, d$. Then we have the inequalities

$$\sum_{i=1}^d a_i^2 \leqslant \frac{d}{4} \quad \text{and} \quad \sum_{i=1}^d a_i \geqslant \frac{d}{2} - 1.$$

Since the a_i are integers we have $\frac{d}{4} \geqslant \frac{d}{2} - 1$, a contradiction because $d \geqslant 5$. □

Lemma 6. *For every even natural number $d \geqslant 6$, there exists a point in $\left(\frac{d}{2} \cdot Q(d)\right) \cap \Lambda(d)$ which does not have a representation of the form $x_1 + \cdots + x_{\frac{d}{2}}$ with $x_1, \ldots, x_{\frac{d}{2}} \in Q(d) \cap \Lambda(d)$. In particular, $Q(d)$ is not $\Lambda(d)$-normal.*

Proof. Consider the baricenter $\beta(d) = \frac{d-1}{d} \cdot (\mathbf{e}_1 + \cdots + \mathbf{e}_d)$ of $\Delta(d-1)$. The point $\frac{d}{2} \cdot \beta(d)$ is the baricenter of the dilated simplex $\frac{d}{2} \cdot \Delta(d-1)$ and, simultaneously, a point in $\Lambda(d)$. Assume $\frac{d}{2} \cdot \beta = x_1 + \cdots + x_{\frac{d}{2}}$ for some $x_1, \ldots, x_{\frac{d}{2}} \in Q(d) \cap \Lambda(d)$. Lemma 5 implies $x_1, \ldots, x_{\frac{d}{2}} \in \Delta(d-1) \cap \Lambda(d) = \operatorname{vert}(\Delta(d-1))$. But this is not possible because the dilated $(d-1)$-simplex $c\Delta(d-1)$ has an interior point of the form $z_1 + \cdots + z_c$ with $z_1, \ldots, z_c \in \operatorname{vert}(\Delta(d-1))$ only if $c \geqslant d$. □

Proof of Theorem (c). Since $\mathbf{e}_1, \ldots, \mathbf{e}_d, (\frac{1}{2}, \ldots, \frac{1}{2}) \in Q(d)$ we have the equality $\operatorname{gp}(Q(d)) = \Lambda(d)$. By Lemmas 4 and 5, the set $Q(d) \cap \Lambda(d)$ is $\Lambda(d)$-ellipsoidal for $d \geqslant 5$. By applying a linear transformation, mapping $\Lambda(d)$ isomorphically to $\mathbb{Z}^d$, Lemma 6 already implies Theorem (c) for d even.

One involves all dimensions $d \geqslant 6$ by observing that (i) if $\mathcal{E} \subset \mathbb{R}^d$ is an ellipsoidal set then $\mathcal{E} \times \{0,1\} \subset \mathbb{R}^{d+1}$ is also ellipsoidal and (ii) the normality of $\operatorname{conv}(\mathcal{E} \times \{0,1\})$ implies that of $\operatorname{conv}(\mathcal{E})$. While (ii) is straightforward, for (i) one applies an appropriate affine transformation to achieve $\mathcal{E} = \operatorname{conv}(S^{d-1}) \cap \Lambda$, where $S^{d-1} \subset \mathbb{R}^d$ is the unit sphere, and $\Lambda \subset \mathbb{R}^d$ is a shifted lattice. In this case the ellipsoid $E = \left\{(\xi_1, \ldots, \xi_d) \;\middle|\; \frac{\xi_1^2}{a^2} + \cdots + \frac{\xi_{d-1}^2}{a^2} + \frac{\xi_d^2}{a^2} + \frac{(\xi_{d+1} - \frac{1}{2})^2}{b^2} = 1\right\} \subset \mathbb{R}^{d+1}$ with $b > \frac{1}{2}$ and $a = \frac{2b}{\sqrt{4b^2-1}}$, is within the $(b - \frac{1}{2})$-neighborhood of the region of $\mathbb{R}^{d+1}$ between the hyperplanes $(\mathbb{R}^d, 0)$ and $(\mathbb{R}^d, 1)$ and satisfies the following conditions: $E \cap (\mathbb{R}^d, 0) = (S^{d-1}, 0)$ and $E \cap (\mathbb{R}^d, 1) = (S^{d-1}, 1)$. In particular, when $\frac{1}{2} < b < \frac{3}{2}$ we have $\operatorname{conv}(E) \cap (\Lambda \times \mathbb{Z}) = \mathcal{E} \times \{0,1\}$. □

Remark 7. The definition of a normal polytope in the introduction is stronger than the one in [7, Definition 2.59]: the former is equivalent to the notion of an *integrally closed* polytope, whereas 'normal' in the sense of [7] is equivalent to $\operatorname{gp}(P)$-normal. Examples of $\operatorname{gp}(P)$-normal polytopes, which are not normal, are lattice non-unimodular simplices, whose only lattice points are the vertices. Lemma 5 and the proof of Lemma 6 show that the 5-simplex $\Delta(5)$ is not $\Lambda(6)$-unimodular. Applying an appropriate affine transformation we obtain a lattice non-unimodular simplices $\Delta' \subset \mathbb{R}^5$ with $\operatorname{vert}(\Delta')$ ellipsoidal. Such examples in $\mathbb{R}^5$ have been known sine the 1970s: a construction of Voronoi [2] yields a lattice $\Lambda \subset \mathbb{R}^5$ and a 5-simplex $\Delta \subset \mathbb{R}^5$ of Λ-multiplicity 2, whose circumscribed sphere does not contain points of Λ inside except $\operatorname{vert}(\Delta)$.

We do not know whether there are ellipsoidal subsets $\mathcal{E} \subset \mathbb{R}^5$ with $\operatorname{conv}(\mathcal{E})$ non-normal and $\operatorname{gp}(\operatorname{conv}(\mathcal{E})) = \mathbb{Z}^5$. For instance, $Q(5)$ is $\Lambda(5)$-normal, as checked by Normaliz [10].

Acknowledgments

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