Quotients of uniform positroids

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Abstract

Two matroids M and N are said to be concordant if there is a strong map from N to M. This also can be stated by saying that each circuit of N is a union of circuits of M. In this paper, we consider a class of matroids called positroids, introduced by Postnikov, and utilize their combinatorics to determine concordance among some of them.

More precisely, given a uniform positroid, we give a purely combinatorial characterization of a family of positroids that is concordant with it. We do this by means of their associated decorated permutations. As a byproduct of our work, we describe completely the collection of circuits of this particular subset of positroids.

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1 Introduction

If two matroids M_1 and M_2 on the same ground set of ranks r_1, r_2, respectively, are such that every circuit of M_2 is a union of circuits of M_1, then we say that M_1 is a *quotient* of M_2, or that M_1 and M_2 are *concordant*. Moreover, such pair of matroids are said to form a *(two step) flag matroid*. The main contribution of this paper is to utilize the

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combinatorics of a special family of matroids, called positroids, in order to characterize when certain pairs of positroids form a quotient, and thus form a flag positroid.

Introduced in [12], positroids have proven to be a combinatorially exciting family of matroids. One may describe positroids via several combinatorial objects such as Grassmann necklaces, decorated permutations, and Le-diagrams [8, 12]. With such rich combinatorics, one can ask if certain matroidal properties may be better understood in the case of positroids through any of these objects.

Taking into account the fact that the uniform matroid $U_{k,n}$ is always a positroid, we combinatorially describe a family of positroids of rank $k-1$ that are quotients of $U_{k,n}$. Our characterization is a complete one for all possible quotients when $n < 6$. Our main result states that positroids of rank $k-1$ that are a quotient of $U_{k,n}$ can be obtained from the decorated permutation of $U_{k,n}$ after performing a cyclic shift on some of its values. We conjecture that all positroid quotients can be described in a similar way.

Our results provide a partial answer to the problem stated in [8], namely, determine combinatorially when two positroids are concordant. Additionally, our work includes a concrete description of the circuits of this family of positroids concordant to $U_{k,n}$.

A strong motivation for this work is the result of [2] which proves that positively oriented matroids are realizable. In this spirit, we believe our results provide a better understanding of certain flag positroids which can be useful in determining the realizability of positively oriented flag matroids.

In order to understand quotients of positroids, we introduce the *poset of positroid quotients*, whose elements are positroids on the same ground set and whose covering relation is given by $N \lessdot M$ if and only if N is a quotient of M and their ranks differ by one.

We conclude with a conjecture establishing a necessary condition for two arbitrary positroids to form a quotient. This conjecture is stated in terms of decorated permutations as well.

The paper is organized as follows. In Section 2 we provide the necessary background on positroids and quotients of matroids. In Section 3 we introduce the poset of quotients of positroids and explore some of its combinatorics. We also characterize families of positroids that are quotients of uniform positroids, and conjecture a general combinatorial rule for positroid quotients. In Section 4 we end with future work and some further questions.

2 Preliminaries

Matroids are combinatorial objects that generalize the notion of linear independence and can be defined through several equivalent ways. We suggest [9] for a wider view on matroid theory.

From here onwards we denote the set $\{1, 2, \ldots, n\}$ by $[n]$ and the k-subsets of $[n]$ by $\binom{[n]}{k}$. Whenever there is no room for confusion, we will denote the set $\{a_1, \ldots, a_n\}$ by $a_1 \ldots a_n$.

2.1 Matroids

Definition 1. A *matroid* M is an ordered pair $(E, \mathcal{B})$ that consists of a finite set E and a collection $\mathcal{B}$ of subsets of E that satisfies the following conditions:

(B1) $\mathcal{B} \neq \emptyset$,

(B2) If B_1, B_2 are distinct elements in $\mathcal{B}$ and $x \in B_1 \setminus B_2$, then there exists an element $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

The set E is the *ground set* of M and the collection $\mathcal{B} := \mathcal{B}(M)$ is called the *set of bases* of M.

It can be shown that every element of $\mathcal{B}$ has the same cardinality, denoted r_M, which is called the *rank* of M. We will say that a subset I of E is *independent* in the matroid $M = (E, \mathcal{B})$ if there exists $B \in \mathcal{B}$ such that $I \subseteq B$. In particular, notice that $\emptyset$ is always independent. By $\mathcal{I}(M)$ we denote the collection of independent sets of a matroid M. If I is not independent we call it *dependent*. In particular, a minimally dependent subset C of E is called a *circuit* of M. That is, C is dependent in M but every proper subset of C is independent. We denote by $\mathcal{C}(M)$ the collection of circuits of the matroid M.

A classic example of a matroid, and in fact our main object of study, is the uniform matroid.

Definition 2. Let n be a positive integer and $0 \leqslant k \leqslant n$. The *uniform matroid of rank k on* $[n]$, denoted $U_{k,n}$, is the ordered pair $U_{k,n} = \left([n], \binom{[n]}{k}\right)$. That is, the bases of $U_{k,n}$ are all the k-subsets of $[n]$.

Definition 3. The *dual* of a matroid $M = (E, \mathcal{B})$ is the ordered pair $M^* = (E, \mathcal{B}^*)$ where

$$B^* = \{E \setminus B \,:\, B \in \mathcal{B}\}.$$

The reader can see that the dual of a matroid of rank k is also a matroid and has rank $n - k$. More over, $U^*_{k,n} = U_{n-k,n}$.

2.2 Quotients and flag matroids

In this paper we are concerned with quotients of a particular class of matroids that will be defined in Section 2.3. Thus we now recall the notion of quotients in matroid theory.

Definition 4. Given two matroids M and N on the same ground set E, we say that M is a *quotient* of N if every circuit of N can be expressed as the union of circuits of M.

Example 5. The matroid $U_{1,3}$ is a quotient of $U_{2,3}$. This is clear since the only circuit of $U_{2,3}$ is the set $\{1, 2, 3\}$, which can be written as the union of $\{1, 2\}, \{2, 3\}, \{1, 3\}$. The latter are the circuits of $U_{1,3}$. In general, $U_{k,n}$ is a quotient of $U_{\ell,n}$ as long as $k \leqslant \ell$.

Definition 4 has been studied in Chapter 8 of [16] where, given two matroids M and N on the same ground set, the property that *M is a quotient of N* is equivalent to the identity being a strong map. In fact, quotients can be defined in many equivalent ways. We present the following proposition whose proof we omit (see [16, Prop. 8.1.6]).

Proposition 6. *Let M and N be matroids on the same ground set E. The following statements are equivalent:*

(a) M is a quotient of N.

(b) N^* is a quotient of M^*.

(c) For any pair of subsets A and B of E, such that $A \subset B$, it follows that

$$r_N(B) - r_N(A) \geqslant r_M(B) - r_M(A).$$

Notice that due to Proposition 6(c), if we take $A = \emptyset$ it follows that $r_N(B) \geqslant r_M(B)$ whenever M is a quotient of N. Moreover, equality holds in this case for all B at the same time if and only if $M = N$. In view of this, the following definition is in order.

Definition 7. Let $M_1, \ldots, M_k$ be a collection of distinct matroids on the ground set $[n]$, of respective ranks $1 \leqslant r_1, r_2, \ldots, r_k \leqslant n$. If for every $1 \leqslant i < j \leqslant k$ it holds that M_i is a quotient of M_j, we say that the collection $\{M_1, \ldots, M_k\}$ is a *flag matroid*. We denote this as $M_1 \subset \cdots \subset M_k$ and refer to the matroids M_i as the *constituents* of the flag matroid. When $k = n = |E|$, and thus $r_i = i$ for $i \in [n]$, the collection $\{M_1, \ldots, M_n\}$ is called a *full flag matroid*.

If $M_1 \subset \cdots \subset M_k$ is a flag matroid, we will say that its constituents are *concordant*. That is, a collection of matroids $\{M_1 \ldots, M_k\}$ is concordant if M_i is a quotient of M_j, or vice versa, for all $i \neq j$.

It can be proved that if M is a quotient of N then every basis of M is contained in a basis of N and every basis of N contains a basis of M [3]. We point out that if M and N are two matroids on the same ground set, each basis of M being contained in a basis of N is equivalent to the identity map being a weak map [16, Prop. 9.1.2].

Example 8. Let M_1 be the matroid on the ground set [4] whose set of bases is $\mathcal{B}(M_1) = \{\{1\}, \{3\}, \{4\}\}$. Let $M_2 = U_{3,4}$. To see that $\{M_1, M_2\}$ is a flag matroid, notice M_1 is a quotient of M_2 since the only circuit $\{1, 2, 3, 4\}$ of M_2 can be written as the union $\{2\} \cup \{1, 3\} \cup \{1, 4\}$ of circuits of M_1. Moreover, the bases of M_1 and M_2 form the following 9 flags

$$\begin{array}{lll} \{1\} \subset \{1,2,3\} & \{1\} \subset \{1,2,4\} & \{1\} \subset \{1,3,4\} \\ \{3\} \subset \{1,2,3\} & \{3\} \subset \{1,3,4\} & \{3\} \subset \{2,3,4\} \\ \{4\} \subset \{1,2,4\} & \{4\} \subset \{1,3,4\} & \{4\} \subset \{2,3,4\} \end{array}.$$

2.3 Positroids

Consider a field $\mathbb{F}$ and let A be a $k \times n$ matrix with entries in $\mathbb{F}$. Let $I \subset [n]$ such that $|I| = k$. We think of the set $[n]$ as indexing the columns of A and thus the set I is a k-subset of the columns. Let $\Delta_I(A)$ denote the maximal minor given by the determinant of the $k \times k$ submatrix of A whose columns are those indexed by I in the order they appear in A.

Let $M = ([n], \mathcal{B})$ be a matroid of rank $r_M = k$. We say that M is *representable over* $\mathbb{F}$ if there exists a full rank $k \times n$ matrix A with entries in $\mathbb{F}$ such that $B \in \mathcal{B}(M)$ if and only if $\Delta_B(A) \neq 0$. In this way, we say that the matrix A *represents* the matroid M over $\mathbb{F}$ and we denote the matrix A by A_M. On the other hand, given a full rank $k \times n$ matrix A with entries in $\mathbb{F}$, we construct the matroid M_A with the set of bases $\mathcal{B}(M_A)$, where $\mathcal{B}_A = \left\{B \in \binom{[n]}{k} : \Delta_B(A) \neq 0\right\}$. We say that a matroid M is *realizable* if M is representable over $\mathbb{R}$.

Definition 9. A matroid $P = ([n], \mathcal{B})$ is called a *positroid* if P is realizable via a matrix A_P such that all the maximal minors of A are nonnegative.

Positroids are of particular interest as they have a strong connection to the positive Grassmannian. The *Grassmannian* $Gr_{k,n}(\mathbb{R})$ is the set of k-dimensional vector subspaces V in $\mathbb{R}^n$. Such a subspace V can be thought of as a $k \times n$ full dimensional matrix A by taking a basis of V as the rows of A. In this way, we can think of points in $Gr_{k\times n}(\mathbb{R})$ as full dimensional $k \times n$ matrices over $\mathbb{R}$ modulo left multiplication by full dimensional $k \times k$ matrices. Given a matroid $M = ([n], \mathcal{B})$, let

$$S_M = \left\{A \in Gr_{k\times n} : \Delta_I(A) \neq 0 \text{ iff } I \in \mathcal{B}(M), \text{ for all } I \in \binom{[n]}{k}\right\}.$$

Then the matroidal decomposition of the Grassmannian is

$$Gr_{k\times n}(\mathbb{R}) = \bigsqcup_M S_M,$$

where the union is over all matroids M on $[n]$ of rank k. Note this decomposition is not a stratification ([4]). However, the restriction to $Gr^{\geqslant 0}_{k\times n}$, the *nonnegative part of* $Gr_{k\times n}(\mathbb{R})$, that is,

$$Gr^{\geqslant 0}_{k\times n} = \left\{A \in Gr_{k\times n} : \Delta_I(A) \geqslant 0 \text{ for all } I \in \binom{[n]}{k}\right\},$$

does produce a matroidal decomposition. Postnikov proved that carrying the decomposition of the Grassmannian by matroids to the nonnegative Grassmannian with positroids provides a stratification for $Gr^{\geqslant 0}_{k\times n}$ [12, 13]. That is,

$$Gr^{\geqslant 0}_{k\times n} = \bigsqcup_P S^{\geqslant 0}_P,$$

where the union is over all positroids P on $[n]$ of rank k and $S^{\geqslant 0}_P$ consists of those $A \in Gr^{\geqslant 0}_{k\times n}$ such that $\Delta_I(A) > 0$ if and only if $I \in \mathcal{B}(P)$. In fact, Postnikov defined a realizable matroid M to be a positroid if the intersection $Gr^{\geqslant 0}_{k\times n} \cap S_M$ is non empty.

Example 10. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix}.$$

The reader can check that M_A is a positroid as each of the 10 maximal minors of A is nonnegative. In fact, M_A coincides with the matroid $U_{3,5}$.

A non-example is the matroid M on $[4]$ of rank 2 with bases $\{12, 14, 23, 34\}$. One can check that M is realizable but it is not a positroid.

Notice that unlike arbitrary representable matroids, positroids depend heavily on an ordering of the ground set. That is, changing the order of the columns of a matrix A can change the sign of its minors. However, being a positroid is closed under cyclic shift of the columns. That is, if the matrix $\begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{bmatrix}$ represents a positroid on $[n]$ of rank k, then $\begin{bmatrix} | & \cdots & | & | \\ v_2 & \cdots & v_n & (-1)^{k-1}v_1 \\ | & \cdots & | & | \end{bmatrix}$ also represents a positroid of rank k on [n]. In other words, although positroids are not closed under arbitrary permutation of the ground set, they are closed under cyclic permutations of $[n]$.

As a last note, we provide a proof that uniform matroids are positroids.

Lemma 11. *Let k, n be integers such that $0 \leqslant k \leqslant n$. The matroid $U_{k,n}$ is a positroid.*

Proof. Take $a_1, \ldots, a_n \in \mathbb{R}$ such that $0 < a_1 < \ldots < a_n$ and consider the $k \times n$ matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{k-1} & a_2^{k-1} & \cdots & a_n^{k-1} \end{bmatrix}.$$

Since any maximal minor of A is a Vandermonde matrix, we have that for all $I \in \binom{[n]}{k}$

$$\Delta_I(A) = \prod_{\substack{i_1 < i_2 \\ i_1, i_2 \in I}} (a_{i_2} - a_{i_1}).$$

As $i_1 < i_2$ implies that $a_{i_1} < a_{i_2}$, we have that $a_{i_2} - a_{i_1} > 0$ and this in turn shows that $\Delta_I(A)$ is a product of positive numbers. Therefore all maximal minors of A are positive and all collections of at most k columns of A form linearly independent sets. Thus, such a matrix A represents $U_{k,n}$ as a positroid. □

2.4 Grassmann necklaces and decorated permutations

The following objects appear in [12] as part of the family of combinatorial objects parametrizing positroids. In order to define them we introduce $<_i$, the *i-order on* $[n]$, which is the total order given by

$$i <_i i + 1 <_i \cdots <_i n <_i 1 <_i 2 <_i \cdots <_i i - 1.$$

Let $S = \{s_1 <_i \cdots <_i s_k\}$ and $T = \{t_1 <_i \cdots <_i t_k\}$ be k-subsets of $[n]$ that are totally ordered via $<_i$. We say that $S \leqslant_i T$ in the *i-Gale order* if and only if $s_j \leqslant_i t_j$ for all $j \in [k]$. With this in hand, we can now define Grassmann necklaces.

Definition 12. Let $0 \leqslant k \leqslant n$. A *Grassmann necklace* of type (k, n) is a sequence $I = (I_1, \ldots, I_n)$ of subsets $I_i \in \binom{[n]}{k}$ such that for any $i \in [n]$,

1. if $i \in I_i$, then $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$ for some $j \in [n]$,
2. if $i \notin I_i$, then $I_{i+1} = I_i$,

where $I_{n+1} := I_1$.

A Grassmann necklace is associated to every matroid in the following way.

Proposition 13. *[12, Lemma 16.3] Given a matroid $P = ([n], \mathcal{B})$ of rank k, let I_i be the minimal element of $\mathcal{B}$ with respect to the i-Gale order. Then $\mathcal{I}(P) = (I_1, \ldots, I_n)$ is a Grassmann necklace of type (k, n).*

Remark 14. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

We leave it to the reader to check that the respective bases of the matroids M_A and M_B are $\mathcal{B}(M_A) = \{12, 13, 14, 23, 24, 34\}$ and $\mathcal{B}(M_B) = \{12, 13, 14, 23, 34\}$, and that their corresponding Grassmann necklaces are $\mathcal{I}(M_A) = \mathcal{I}(M_B) = (12, 23, 34, 41)$. Although the matroids M_A and M_B have the same Grassmann necklace, one can show that M_A is a positroid but M_B is not. That is, although M_B is realizable there is no representation of M_B inside $Gr_{k,n}^{\geqslant 0}$.

This is a shadow of a nice property of positroids. That is, if M is a positroid then its Grassmann necklace allows us to recover all of its bases in the following way.

Proposition 15. *[8, 12] Given a Grassmann necklace $I = (I_1, \ldots, I_n)$ of type (k, n), the set*

$$\mathcal{B}(I) = \left\{ B \in \binom{[n]}{k} \,:\, B \geqslant_i I_i \text{ for all } i \in [n] \right\}$$

is the collection of bases of a positroid $\mathcal{P}(I) = ([n], \mathcal{B}(I))$ of rank k.

Remark 16. One can check from Propositions 13 and 15 that if M is a positroid and $I(M)$ its Grassmann necklace then $M = \mathcal{P}(\mathcal{I}(M))$ and $I = \mathcal{I}(\mathcal{P}(I))$. Moreover, if M is not a positroid then $M \subsetneq \mathcal{P}(\mathcal{I}(M))$ and therefore $P(\mathcal{I}(M))$ is the positroid that contains all matroids with the same Grassmann necklace. This is illustrated in Remark 14.

Making use of Grassmann necklaces we now define *decorated permutations*. These will be used as our main way to index positroids.

Definition 17. A *decorated permutation* is a bijection $\sigma : [n] \to [n]$ whose fixed points are decorated as $\sigma(\ell) = \underline{\ell}$ or $\sigma(\ell) = \overline{\ell}$. We denote the set of all decorated permutations on $[n]$ by $\mathcal{D}_n$.

Given $\sigma \in \mathcal{D}_n$ and $i \in [n]$ we say that $j \in [n]$ is a *weak i-excedance* of σ if $\sigma(j) = \overline{j}$ or $j <_i \sigma(j)$. We denote by $W_i(\sigma)$ the set of weak i-excedances of σ. It is shown in [14] that $|W_i(\sigma)| = |W_j(\sigma)|$ for all $i, j \in [n]$ and thus we denote the number of weak excedances of σ by $W(\sigma) := |W_1(\sigma)|$. For instance, if $\sigma = 5\underline{2}6134$ then $W_1(\sigma) = \{1, 3\}$ and $W(\sigma) = 2$.

Let $\mathcal{D}_{k,n}$ be the set of all decorated permutations on $[n]$ with k weak excedances. Then $\mathcal{D}_n = \bigsqcup_{k=0}^{n} \mathcal{D}_{k,n}$. The cardinality of $\mathcal{D}_{k,n}$ has been computed in [12, Proposition 23.1] and corresponds to the sequence A046802 [10]. The following proposition gives us a bijective way to go between Grassmann necklaces and decorated permutations. If σ is a decorated permutation we will denote by P_σ its corresponding positroid.

Proposition 18. *[1, Proposition 4.6] Let $I = (I_1, \dots, I_n)$ be a (k, n)-Grassmann necklace. Let $\pi(I)$ be the decorated permutation in $\mathcal{D}_{k,n}$ constructed as follows:*

- *if $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$ with $i \neq j$, then $\pi(I)(j) := i$.*
- *if $I_{i+1} = I_i$ where $i \in I_i$, then $\pi(I)(i) = \overline{i}$.*
- *if $I_{i+1} = I_i$ where $i \notin I_i$, then $\pi(I)(i) = \underline{i}$.*

Conversely, let σ be a decorated permutation on $[n]$ with k weak excedances, and let I_i be the set of weak i-excedances of σ. Then $\mathcal{I}(\sigma) = (I_1, \dots, I_n)$ is a Grassmann necklace of type (k, n). Moreover, these constructions are inverses of each other. Namely, $\pi(\mathcal{I}(\sigma)) = \sigma$, and $\mathcal{I}(\pi(I)) = I$.

Remark 19. The bijection between Grassmann necklaces and decorated permutations outlined here, and used throughout the paper, is due to Postnikov in [12]. Our results can also be expressed using a different bijection given in [8].

Example 20. Consider the Grassmann necklace $I = (13, 34, 34, 45, 56, 61)$. Using Proposition 18 we get the decorated permutation $\pi(I) = 5\underline{2}6134$. Conversely, the Grassmann necklace $(12, 23, 34, 41)$ that indexes the uniform positroid $M = U_{2,4}$ gives the decorated permutation $\pi_M = 3412$.

It follows from Proposition 18 that $\sigma(i) = \underline{i}$ implies $\{i\}$ is a *loop* (or 1-element circuit) of the associated positroid P_σ, and thus is never contained in an element of $\mathcal{I}(P_\sigma)$. Similarly, $\sigma(i) = \overline{i}$ implies $\{i\}$ is a *coloop* of P_σ and is in every element of $\mathcal{I}(P_\sigma)$ and thus in every basis of P_σ.

As part of our notation, we denote by $[a, b]$ any interval of $[n]$, including cyclic ones. Namely, the sets of form $\{a, a+1, \dots, b-1, b\}$ if $a \leqslant b$ and $\{a, a+1, \dots, n, 1, \dots, b\}$ if $b < a$. This allows us to describe some details in a more compact way. For example, the positroid $U_{k,n}$ is such that its Grassmann necklace is given by

$$\mathcal{I} = ([k], [2, k+1], \dots, [n, k-1])$$

and its decorated permutation is

$$\pi_{k,n} := (n-k+1)(n-k+2) \cdots n 1 2 \cdots (n-k).$$

That is, $\pi(i) = n - k + i \,(mod\, n)$ for $i \in [n]$.

As stated in the introduction, we are interested in a combinatorial characterization of quotients of positroids. Thus our main question is: Given two positroids P_1 and P_2 on the ground set $[n]$, can we determine combinatorially whether P_1 is a quotient of P_2, or vice-versa? We will make use of decorated permutations to give a partial answer to this question.

3 Poset of positroid quotients

For every $n \geqslant 1$, we denote by $\mathcal{P}_n$ the poset whose elements consist of the decorated permutations in $\mathcal{D}_n$ (i.e. positroids on $[n]$) and whose order relation is the transitive closure of the following covering relation: $\tau \lessdot \pi$ if and only if $\tau \in \mathcal{D}_{k-1,n}$, $\pi \in \mathcal{D}_{k,n}$ for some $k \in \{0, 1, \ldots, n\}$, and P_τ is a quotient of P_π. We call $\mathcal{P}_n$ *the poset of positroid quotients on* $[n]$. See Figure 1 for an illustration of $\mathcal{P}_3$.

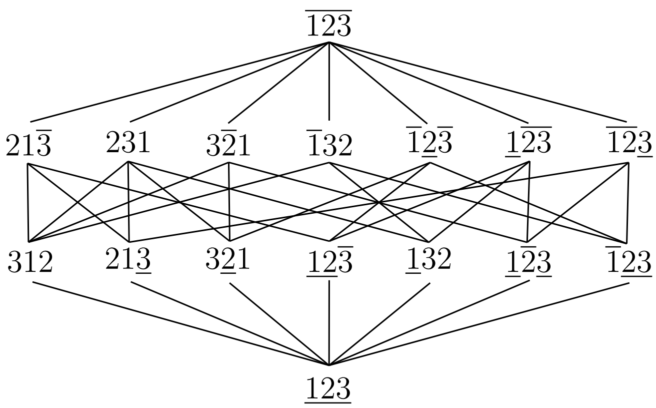

Figure 1: The poset of positroid quotients $\mathcal{P}_3$.

Recall that the decorated permutation corresponding to the uniform positroid $U_{k,n}$ is given by $\pi_{k,n} = (n-k+1)\cdots(n-1)\, n\, 1\, 2\, \cdots (n-k)$. We state the following properties of the poset $\mathcal{P}_n$, whose proof we leave to the reader:

1. It is a poset with $\hat{0}$ given by the decorated permutation $\pi_{0,n} = \underline{12\cdots n}$ and $\hat{1}$ given by the decorated permutation $\pi_{n,n} = \overline{12\cdots n}$.

2. It is graded and the rank of each decorated permutation is its number of weak excedances. Thus, its rank polynomial is symmetric and unimodal. The number of elements in each rank is recorded in the sequence A046802.

3. If $\sigma, \tau \in \mathcal{D}_n$ are decorated permutations such that $\tau \leqslant \sigma$ and $\sigma(i) = \underline{i}$, then $\tau(i) = \underline{i}$. On the other hand, if $\tau(i) = \overline{i}$, then $\sigma(i) = \overline{i}$.

Similarly, one can construct the poset $\mathcal{M}_n$ of matroid quotients whose elements are all matroids on the ground set $[n]$ and whose order relation is $M < N$ if and only if $M \neq N$

and M is a quotient of N. This is implicitly done in [16]. Moreover, it is shown in [16, Prop. 8.2.5] in the language of strong maps, that if $\{M, N\}$ form a flag matroid and $r_M < r_N$ then there exist matroids $M = M_0$, M_1, …, $M_k = N$ such that $r_{M_{i+1}} = r_{M_i} + 1$ for $i = 0, \dots, k-1$, and $\{M_0, \dots, M_k\}$ form a flag matroid. That is, any flag matroid $M \subset N$ can be extended to a saturated flag matroid $M = M_0 \subset M_1 \subset \cdots \subset M_k = N$ whose constituents M_i have each possible rank between r_M and r_N.

In other words, if $M < N$ in $\mathcal{M}_n$, then there is a saturated chain $M \lessdot M_1 \lessdot \cdots \lessdot N$ in $\mathcal{M}_n$. The existence of such saturated chains is made explicit via the Higgs lift (see [16, Prop. 8.2.5] for details). One may feel tempted to conclude the same in the poset $\mathcal{P}_n$. However, this is unclear as one needs to guarantee that the Higgs lift of a positroid is again a positroid.

Problem 21. Let $\sigma \leqslant \tau$ in $\mathcal{D}_n$. How can we construct positroids $P_\sigma = P_{\sigma_0}, \dots, P_{\sigma_m} = P_\tau$ such that $\sigma = \sigma_0 \lessdot \sigma_1 \lessdot \cdots \lessdot \sigma_m = \tau$ is a saturated chain in $\mathcal{D}_n$ where $m = r_\tau - r_\sigma$?

Our main theorem identifies a set of positroids that are quotients of $U_{k,n}$ for any $k \in [n]$. We do this by defining the following sequence of moves on decorated permutations.

Definition 22. Given a decorated permutation $\pi \in \mathcal{D}_n$ and a subset A of $[n]$, we denote by $\overleftarrow{\rho_A}(\pi)$ the element of $\mathcal{D}_n$ obtained from π by performing the following moves in order:

(F) **Freeze move:** Freeze the value i in π if and only if $i \in A$.

(S) **Shift move:** After freezing, cyclically shift the remaining values in π one place to the left, jumping over frozen elements when necessary.

(D) **Decoration move:** If any fixed point i appears after the shift move (S), decorate it as $\underline{i}$.

Analogously, we denote by $\overrightarrow{\rho_A}(\pi)$ the permutation obtained from π by performing the (F), (S), (D) moves, such that (S) shifts to the right instead of the left, and that (D) decorates any new fixed point i as $\overline{i}$. We call this sequence of moves a *FSD-shift* of π.

Example 23. Let $\pi = 34512$. Then

$$\begin{aligned} \overleftarrow{\rho_\emptyset}(\pi) &= 45123 & \overleftarrow{\rho_{12}}(\pi) &= 45\underline{3}12 \\ \overleftarrow{\rho_{25}}(\pi) &= 41532 & \overleftarrow{\rho_{4}}(\pi) &= 54123 \\ \overrightarrow{\rho}_{12}(\pi) &= 53412 & \overrightarrow{\rho_{4}}(\pi) &= 24\overline{3}51. \end{aligned}$$

Proposition 24. *Let $A \subseteq [n]$. Then $\overrightarrow{\rho_A}(\pi_{k,n})^{-1} = \overleftarrow{\rho_B}(\pi_{n-k,n})$, where B is the set given by $\pi_{k,n}^{-1}(A) = \pi_{n-k,n}(A)$.*

Proof. Let $A \subseteq [n]$, $\pi := \pi_{k,n}$, and recall that $\pi^{-1} = \pi_{n-k,n}$. With this in mind it is easy to see that freezing the elements $\pi(i)$ for every $\pi(i) \in A$ makes every element of $B = \{i \in [n] : i = \pi^{-1}(j) \text{ for } j \in A\}$ frozen in $\pi_{n-k,n}$.

Now consider the remaining values of π that are not frozen, in the order they appear. These values form a permutation ω over the set $[n] \setminus A$. The positions of ω are indexed by the ordered set $[n] \setminus B$. Thus a cyclic shift to the right of the values of ω is equivalent to a cyclic shift to the left of the indices of ω. Finally for $i \notin B$, the index $\omega^{-1}(i)$ is precisely the entry $\pi^{-1}(i) = \pi_{n-k,n}(i)$. Thus, $\overrightarrow{\rho_A}(\pi)^{-1} = \overleftarrow{\rho_B}(\pi^{-1})$. □

Example 25. Again, let $\pi = 34512$ and set $A = \{2\}$. Then $\overrightarrow{\rho_A}(\pi)^{-1} = (\overline{1}3452)^{-1} = \underline{1}5234$. On the other hand, $B = \pi^{-1}(A) = \{5\}$ and $\overleftarrow{\rho_B}(\pi_{n-k,n}) = \overleftarrow{\rho_5}(45123) = \underline{1}5234$.

3.1 Uniform quotients

Now we provide a characterization using decorated permutations for a family of positroids of rank $k-1$ that are quotients of the uniform positroid $U_{k,n}$. For the remainder of the paper we will consider $A \subseteq [n]$ as a union of disjoint cyclic intervals of $[n]$. That is, $A = [a_1, i_1] \cup \cdots \cup [a_m, i_m]$ while denoting singletons as $\{a_j\}$. We call each of these cyclic intervals a *cyclic component* of A. For example if $n = 13$ and $A = \{1, 2, 5, 8, 9, 12, 13\}$, then $A = [12, 2] \cup \{5\} \cup [8, 9]$.

Theorem 26. *Let $k \leqslant n$ and $A \subset [n]$ be a union of disjoint cyclic intervals of $[n]$ such that no interval in A has size greater than $k-1$. Then the positroid represented by $\overleftarrow{\rho_A}(\pi_{k,n})$ has rank $k-1$.*

Proof. Let $\pi = \pi_{k,n}$, $\sigma := \overleftarrow{\rho_A}(\pi)$, and P_σ be the positroid represented by σ. Since the number of weak excedances of a decorated permutation is equal to the rank of its corresponding positroid, it is enough to show that $|W_1(\sigma)| = k-1$. Recall that $W_1(\pi) = [k]$ and $\pi(x) = n-k+x$ for $x \in [1, k]$ and $\pi(x) = x-k$ for $x \in [k+1, n]$. We will show that $W_1(\sigma) = W_1(\pi) \setminus \{j\}$, where $j = \max\{i \in [k] : \pi(i) \notin A\}$.

Let $l \in [k] \setminus \{j\}$. If $\pi(l) \in A$, then $\sigma(l) = \pi(l)$ and $l \in W_1(\sigma)$. On the other hand, if $\pi(l) \notin A$ and $l \neq j$, there exists $l' = \min\{i \in [k] : l < i \text{ and } \pi(i) \notin A\}$. Since $\pi(l) = n-k+l$ and $\pi(l') = n-k+l'$ we have that $\sigma(l) = n-k+l'$. As $l' \in [k]$, it follows that $l < l' < n-k+l'$ and $l \in W_1(\sigma)$. This holds in particular if $l' = j$. Hence $W_1(\pi) \setminus \{j\} \subseteq W_1(\sigma)$. We will complete the proof by showing that neither $\{j\}$ nor $[k+1, n]$ are contained in $W_1(\sigma)$.

Let $j' = \min\{i \in [n] \setminus [k] : \pi(i) \notin A\}$ if it exists. Since $\pi(j) = n-k+j$ and $\pi(j') = j'-k$, we have that $\sigma(j) = j'-k$. Recall that the intervals of A have length less than k. Thus $j'-j \leqslant k$, implying that $j'-k \leqslant j$ and j is not a weak excedance in the inequality case. In the case of an equality, our construction tells us that fixed points are colored as $\sigma(i) = \underline{i}$ and we can assure that $j \notin W_1(\sigma)$.

If such a j' does not exist, then $\pi(i) \in A$ for all $i \in [k+1, n]$. This accounts for $n-k$ frozen elements. Due to our construction of j, the $k-j$ elements in $[k] \setminus [j]$ are also frozen. Now notice that in this case, for j to be a weak excedance of σ, all the first $j-(n-k)$ values of π would have to be frozen as well. This gives us a frozen interval of total size $n-k+k-j+j-(n-k) = k$ which contradicts our assumption that A has no interval of size greater than $k-1$. Thus we deduce that in both cases $j \notin W_1(\sigma)$.

Now let $l \in [k+1,n]$ and $\sigma(l) = l'$ for some l'. If $l <_1 l'$ then $\pi(l') = l' - k$ and $\sigma(l) = l' - k$. Since $l' - l \leqslant k$, we have that $l' - k \leqslant l$ and thus $l \notin W_1(\sigma)$.

On the other hand, if $l' \leqslant k$ then $\sigma(l) = n - k + l'$. Again, the cyclic interval $[l', l]$ contains at most $k-1$ elements. Thus $n + l' - l \leqslant k$ and $n - k + l' \leqslant l$ and we conclude that $l \notin W_1(\sigma)$. Therefore $W_1(\sigma) = W_1(\pi) \setminus j$ and P_σ has rank $k-1$ as desired. □

Corollary 27. *Let $k \leqslant n$ and let $A \in \binom{[n]}{\ell}$, for some $\ell \in \{0, \dots, k-1\}$. Then the positroid represented by the decorated permutation $\overleftarrow{\rho_A}(\pi_{k,n})$ has rank $k-1$.*

We are now ready to state our main result.

Theorem 28 (Main theorem)**.** *Let $\sigma \in D_{k-1,n}$ be a decorated permutation of rank $k-1$ and choose $A \in \binom{[n]}{\ell}$, for some $\ell \in \{0, \dots, k-1\}$. If $\sigma = \overleftarrow{\rho_A}(\pi_{k,n})$, then $\sigma \lessdot \pi_{k,n}$.*

The proof of Theorem 28 relies on showing that every circuit of $U_{k,n}$ is a union of circuits of the positroid represented by the decorated permutation $\overleftarrow{\rho_A}(\pi_{k,n})$ for the set $A \subset [n]$. To this end, we state the following results.

We first show that the set of circuits of the positroid P_σ represented by $\sigma = \overleftarrow{\rho_A}(\pi_{k,n})$ has a simple description. Then we show how to obtain the circuits of $\pi_{k,n}$ as the union of circuits of P_σ.

Theorem 29 (Circuit description of shifted uniform positroid)**.** *Let $A = [a_1, i_1] \cup \cdots \cup [a_m, i_m]$ be a subset of $[n]$ composed of disjoint cyclic intervals of lengths $l_1, \dots, l_m$ respectively. Then the circuits of the positroid P_σ represented by $\sigma = \overleftarrow{\rho_A}(\pi_{k,n})$ are given by the sets*

$$\mathcal{C}_A = \{[i_j+1, i_j+k-l_j] \,:\, j = 1, \dots, m\} \cup \left\{ C \in \binom{[n]}{k} \,:\, [i_j+1, i_j+k-l_j] \not\subset C \right\}. \quad (1)$$

Moreover, the circuits of size less than k can be read from the decorated permutation σ. They are precisely the intervals $[i_j+1, \sigma(i_j+1)]$.

Proof. Let $\pi := \pi_{k,n}$. Assume that $\sigma = \overleftarrow{\rho_A}(\pi_{k,n})$ for some $A \in \binom{[n]}{\ell}$ and $\ell \in \{0, \dots, k-1\}$. We begin by proving that each interval $[i_j+1, i_j+k-l_j]$ satisfies $\sigma(i_j+k-l_j) = i_j+1$ and that for all $r \in [n]$, the interval $[\sigma(j_r), j_r]$ is a circuit of P_σ. Let us suppose that within the frozen set A, there exist an interval $[a, b]$ which has length $b - a + 1$. Notice that the interval $[b+1, a+k-1]$ is the cyclic interval that would extend (clockwise) $[a, b]$ into a interval of length precisely k. In fact $[b+1, a+k-1]$ is the description of $[i_j+1, i_j+k-l_j]$ in terms of a and b. As we are freezing the values $[a, b]$ and $\pi_{k,n}(x) = x - k \pmod n$, we have that we are freezing the positions $[a+k, b+k] \pmod n$. Therefore we have that if

$$\pi = \begin{pmatrix} \cdots & a+k-1 & a+k & \dots & b+k & b+k+1 & \cdots \\ \cdots & a-1 & a & \dots & b & b+1 & \cdots \end{pmatrix},$$

then

$$\sigma = \begin{pmatrix} \cdots & a+k-1 & a+k & \dots & b+k & b+k+1 & \cdots \\ \cdots & b+1 & a & \dots & b & * & \cdots \end{pmatrix},$$

$\sigma(a+k-1) = b+1$, and $\sigma(i_j+k-l_j) = i_j+1$. For the rest of the proof we will denote such intervals as $[\sigma(j_r), j_r]$ for some $r \in [n]$.

We will now show the circuits described above and the k-subsets not containing these intervals are the only circuits of P_σ.

If $A = \emptyset$, then $\mathcal{C} = \binom{[n]}{k}$, which are the circuits of the uniform positroid $U_{k-1,n}$ represented by σ. Now, let $\ell \in [k-1]$ and $A \in \binom{[n]}{\ell}$. Recall that $W_1(\pi) = [k]$. Moreover, for each $r \in [n]$, it holds that $W_r(\sigma) = [r, r+k-1] \setminus \{j_r\}$ where $j_r = \max\{i \in [r, r+k-1] : \pi(i) \notin A\}$. That is, j_r is the largest among the first k-positions in the $<_r$ order such that $\pi(j_r)$ is not frozen. Recall that the Grassmann necklace $\mathcal{I}_\sigma = (I_1, \ldots, I_n)$ corresponding to σ satisfies $I_r = W_r(\sigma)$ and that for each r this means that I_r is the minimal basis of P_σ under the r-Gale order.

We now show that for each $r \in [n]$, the interval $[\sigma(j_r), j_r]$ is a circuit of P_σ. We illustrate the proof with $r = 1$ as it is done analogously for each r. Set $j := j_1$ and notice that since $j \notin W_1(\sigma)$, then $\sigma(j) < j$ and $[\sigma(j), j] \subseteq [1, k]$. If $[\sigma(j), j]$ were independent, then $[\sigma(j), j]$ would be contained in $I_{\sigma(j)}$, but $I_{\sigma(j)}$ does not contain j since $j_{\sigma(j)} = j$. Therefore, $[\sigma(j), j]$ is dependent in P_σ. To show it is a circuit we will show that each of the sets $J_x = [\sigma(j), j] \setminus \{x\}$, for $x \in [\sigma(j), j]$, is independent by constructing a basis B_x of P_σ such that $J_x \subseteq B_x$. If $x = j$, then $B_x := I_1$ works. If $x \neq j$, we will prove the set $B_x := I_1 \cup \{j\} \setminus \{x\}$ is a basis of P_σ using Proposition 15.

Since $x < j$, $B_x = [1, x-1] \cup [x+1, k]$, and $I_1 = [1, j-1] \cup [j+1, k]$. A quick comparison of the sets shows that $B_x \geqslant_r I_1 \geqslant_r I_r$ for $r \in [x]$. For the other values of r, recall first that $I_r = [r, r+k-1] \setminus j_r$. Arranging the elements of I_r using the r-Gale order we get that $I_r = \{r, \ldots, j_r - 1, j_r + 1, \ldots, r+k-1\}$ taking mod n where needed. Arranging B_x we get $B_x = \{r, \ldots, k, 1, \ldots, x-1, x+1, \ldots, r-1\}$ if $r \in [k] \setminus [x]$, whereas for $r \in [n] \setminus [k]$ we get $B_x = \{1, \ldots, x-1, x+1, \ldots, k\}$. In either case one can see that $B_x \geqslant_r I_r$. This allows us to conclude that $[\sigma(j_r), j_r]$ is a minimally dependent set in P_σ for every $r \in [n]$ and therefore a circuit. The reader can verify that each of the sets $[i_j+1, i_j+k-l_j]$ from equation (1) are of the form $[\sigma(j_r), j_r]$ for some r.

We now proceed to show that any k-subset of $[n]$ that does not contain a $[\sigma(j_r), j_r]$ for some r is a circuit. Let D be a k-subset of $[n]$ such that D does not contain any of the sets $[\sigma(j_r), j_r]$ as given above. Since P_σ has rank $k-1$, D is automatically dependent. We only need now to show it is minimal. Consider a $(k-1)$-subset $F \subset D$ and let us see that F is a basis of P_σ. If F were not a basis then there would be a $r \in [n]$ such that $F \not\geqslant_r I_r$. Thus if $F = \{c_1 <_r \cdots <_r c_{k-1}\}$ and $I_r = \{b_1 <_r \cdots <_r b_{k-1}\}$ in r-Gale order, the fact that $F \not\geqslant_r I_r$ implies that for some l and $p \in [l-1]$, we have that $c_p \geqslant b_p$ and $c_l < b_l$. However, $I_r = [r, j_r - 1] \cup [j_r + 1, r+k-1]$ so l must be the position of j_r, which implies that $c_l = j_r$. Therefore $[r, j_r] \subset F$ and since $\sigma(j_r) = r$, we get that $[\sigma(j_r), j_r] \subset D$ which is a contradiction to our assumptions on D. We thus conclude that F is a basis of P_σ and D is a circuit.

So far we have shown that $\mathcal{C}_A$ is contained in $\mathcal{C}_\sigma$, the set of circuits of the positroid P_σ. To prove the reverse containment we will show that if $S \notin \mathcal{C}_A$ then $S \notin \mathcal{C}_\sigma$. First, notice that we only need to consider sets $S \subset [n]$ such that $|S| < k$ and that do not contain any interval $[\sigma(j_r), j_r]$. We will prove that such S can be extended to a set D of cardinality k in a way such that D does not contain any of the $[\sigma(j_r), j_r]$ and therefore, S will be independent. Suppose that the decomposition of the set A in cyclic intervals is $A = J_1 \cup \cdots \cup J_s$ and let L_i be the cyclic interval such that $J_i \cup L_i$ is an interval of size k for each i. Recall that all L_i are circuits of P_σ.

In order to prove that S can be extended to our desired D we will make use of $J_1 \cup L_1$. To this end, let $D' := J_1 \cup L_1 \cup S$ with cardinality at least k.

Case 1: Suppose D' contains no L_j except L_1. If $|S \setminus (J_1 \cup L_1)| \geqslant 1$ then S can be directly extended to a k-subset D of $D' \setminus \{a\}$ with an $a \in L_i \setminus S$ such that it does not contain any of the intervals in $\mathcal{C}_A$. On the other hand, if $S \subset J_1 \cup L_1$, let us take $D := ((J_1 \cup L_1) \setminus \{a\}) \cup \{b\}$ as an interval where $a \in L_1 \setminus S$ and $b := c + 1$ where c is the greatest element of L_1. Notice that such an a exists since S does not contain L_1 by hypothesis. If b does not exist, it means that $D' = [n]$ and thus $\sigma \lessdot \pi = U_{n,n}$ as any matroid is concordant to $U_{n,n}$. Meaning there is nothing to prove. Otherwise if such b exists, then D has size k, contains S, and does not contain any interval of $\mathcal{C}_A$. Meaning S is independent.

Case 2: Suppose D' contains L_1 and another different L_j. Without loss of generality call it L_2. If this is the case, then either $L_1 \cup L_2$ or $J_1 \cup L_2$ is a cyclic interval. In the former case, $|D'| \geqslant k + 1$ and D can be obtained by removing elements from D' of the form $a \in (L_1 \cup L_2) \setminus S$ until it has size k. These elements exist as $L_2 \not\subset S$ and $L_1 \cap L_2 \neq \emptyset$. In the latter case, D can be obtained as a subset of $D'' := (J_1 \setminus \{d\}) \cup (L_1 \setminus \{a\}) \cup \{b\} \cup S$ where $d \in (J_1 \cap L_2) \setminus S$, $a \in (L_1 \setminus S)$ and $d := c + 1$ where c is the clockwise greatest element of L_1. As $|D''| > k$ and does not contain any L_j but does contain S, we can extend S to a basis and make it an independent set.

Case 2.1: If simultaneously, D' contains L_2 and L_3 such that $L_1 \cup L_2$ and $J_1 \cup L_3$ are cyclic intervals, then either $J_3 \cap L_1 = \emptyset$ or $J_3 \cap L_1 \neq \emptyset$. In the former case, remove from D' any pair of elements $a \in (L_3 \cap J_1) \setminus S$ and $b \in (L_2 \cap L_1) \setminus S$. With this, $D' \setminus \{a, b\}$ will have still at least k elements as the cyclic components of S that intersect L_2 and L_3 have elements outside of D'.

On the other hand, if $J_3 \cap L_1 \neq \emptyset$, in order to maintain the cardinality of $D' \setminus \{a, b\}$ above or equal to k with a, b as above, we would need to guarantee that such elements can be substituted. This can be achieved directly if $[n] \setminus (J_1 \cup L_1)$ has at least 2 elements. If $|[n] \setminus (J_1 \cup L_1)| = 0$, we land again in the $U_{n,n}$ case. If instead $|[n] \setminus (J_1 \cup L_1)| = 1$, then $J_3 \cap J_1 \neq \emptyset$ (as the reader can check) which is a contradiction since all J_i are disjoint. This exhausts all the possibilities and the proof is complete. □

Proposition 30. *Every circuit of $\pi_{k,n}$ can be obtained as a union of elements in $\mathcal{C}_A$.*

Proof. As $\pi_{k,n}$ corresponds to the uniform positroid $U_{k,n}$, all of its circuits are all the $(k + 1)$-subsets of $[n]$. Let O be any such circuit and let

$$\mathcal{C}_O = \left\{ C \in \binom{[n]}{k} : C \in \mathcal{C}_A, C \subset O \right\}.$$

If $|\mathcal{C}_O| \geqslant 2$ then any two elements in $\mathcal{C}_O$ cover O. If $|\mathcal{C}_O| = 1$, then $\mathcal{C}_O = \{C\}$ for some

$C \in \binom{[n]}{k}$ and there is one remaining element x in O we have yet to cover. Let us take the k-subset $D := (C \setminus \{y\}) \cup \{x\}$ for any $y \in C$. As $D \neq C$ and $|D| = k$, then $D \notin \mathcal{C}_O$. Thus there is a cyclic interval $L \in \mathcal{C}_A$ such that $x \in L$ and $L \subset D$. This means that, $O = L \cup C$ and the claim is proved in this case.

Finally, if $\mathcal{C}_O = \emptyset$ then every k-subset C of $[n]$ contained in O properly contains at least one of the cyclic intervals $[i_j + 1, i_j + k - l_j]$ of A. Moreover, O contains at least two distinct intervals L_1 and L_2. To see this, take any k-subset $C_1 = O \setminus \{x\}$ where $x \in O$ and let $L_1 \subset C_1$ be an interval in $\mathcal{C}_A$. Now let $y \in L_1$ and set $C_2 = O \setminus \{y\}$. Since $C_2 \subset O$, there is an interval $L_2 \in \mathcal{C}_A$ such that $L_2 \subset C_2$. As $y \in L_1 \subset C_1$ and $x \notin C_1$, then $L_1 \neq L_2$.

Now assume $L_1, \ldots, L_m$ are the intervals in $\mathcal{C}_A$ contained in O. Denote $L = L_1 \cup \cdots \cup L_m$. Clearly $L \subseteq O$. We will prove the reverse containment. Suppose that the intervals in L are not pairwise disjoint. Without loss of generality let $L_1 \cap L_2 \neq \emptyset$. As $L_1, L_2 \in \mathcal{C}_A$, there are two disjoint cyclic intervals $J_1, J_2 \subseteq A$ that give rise to L_1, L_2, respectively. That is, $J_r \cup L_r$ is a cyclic interval of length k for $r = 1, 2$. Now suppose that the least element in L_1 is smaller than the one in L_2. Then $J_2 \cup L_2 \subset L_1 \cup L_2$ (otherwise J_1 overlaps J_2 which cannot happen) and this implies that $k < |L_1 \cup L_2| \leqslant k + 1$. The first inequality follows from $J_2 \cup L_2 = k$ and the second since $L_1 \cup L_2 \subseteq L \subseteq O$. Thus in this case, $|L_1 \cup L_2| = k + 1$ and we conclude that $O = L_1 \cup L_2$.

Now suppose that the intervals in L are pairwise disjoint. Denote by $J_i \subseteq A$ the frozen interval that gives rise to L_i. We know that $\sum_{i=1}^m |J_i| \leqslant k - 1$ and $|J_i| + |L_i| = k$ for all $i \in [m]$. Thus we get that $\sum_{i=1}^m |J_i| + \sum_{i=1}^m |L_i| = mk$ and

$$\sum_{i=1}^{m} |L_i| = mk - \sum_{i=1}^{m} |J_i| \geqslant mk - (k-1) = (m-1)k + 1 \geqslant k + 1,$$

for $m \geqslant 2$. But since $L \subseteq O$ and $|O| = k + 1$, we get that $L = O$. This finishes the proof. □

Theorem 28 follows immediately as a consequence of Theorem 29 and Proposition 30.

Example 31 (Illustrating Theorem 28.)**.** Let $n = 9$, $k = 5$ and $A = [9, 1] \cup \{6\}$. Then $\sigma = 768291345$ and the circuits $[\sigma(j_r), j_r]$ and the components I_r of the Grassmann necklace I_σ are given in the following table.

r	1	2	3	4	5
$[\sigma(j_r), j_r]$	[2,4]	[2,4]	[3,7]	[4,8]	[5,9]
I_r	$[1,3] \cup \{5\}$	$[2,3] \cup [5,6]$	[3,6]	[4,7]	[5,8]
r	6	7	8	9	
$[\sigma(j_r), j_r]$	[7,1]	[7,1]	[8,3]	[2,4]	
I_r	[6,9]	$[7,9] \cup \{2\}$	[8,2]	[9,3]	

Remark 32. The converse of Theorem 28 holds up to $n = 5$. That is, $\sigma \lessdot \pi_{k,5}$ if and only if $\sigma = \overleftarrow{\rho_A}(\pi_{k,n})$ for some $A \in \binom{[n]}{\ell}$ where $\ell \in \{0, \ldots, k-1\}$. In Table 1, for various values of k and n, we show the number of positroids of rank $k-1$ on $[n]$ that are quotients of $\pi_{k,n}$ but not characterized via Theorem 28. We want to point out that the only instance in which it fails for $n = 6$ is when $k = 3$. In particular, with the positroid $\pi = \pi_{3,6} = 456123$ that covers 24 positroids in the poset P_6 while there are $22 = 1 + 6 + 15$ positroids of the form $\overleftarrow{\rho_A}(\pi)$, where $A \in \binom{[6]}{\ell}$ and $\ell \in \{0, 1, 2\}$. The two extra positroids not obtained from this characterization correspond to the permutations $\sigma = 652143$ and $\tau = 416325$. Notice, however, that $\sigma = \overleftarrow{\rho_{135}}(\pi)$ and $\tau = \overleftarrow{\rho_{246}}(\pi)$.

n	k	# of quotients	Characterized by Theorem 28	Missing
6	3	24	22	2
7	3	36	29	7
	4	71	64	7
8	3	55	37	18
	4	119	93	26
	5	179	163	16
9	3	85	46	39
	4	202	130	72
	5	322	256	66
	6	412	382	30
10	3	133	56	77
	4	343	176	167
	5	583	386	197
	6	773	638	135
	7	898	848	50

Table 1: Number of quotient positroids of $U_{n,k}$.

We highlight that although there are quotients of $U_{k,n}$ that do not satisfy all the conditions in Theorem 28, these quotients also seem to have the same circuit structure that we have presented through FSD-shifts. Examples 33 and 34 illustrate this in the case of uncharacterized uniform positroid quotients and certain general positroid quotients respectively.

Example 33. Consider the uniform positroid $U_{4,9}$ indexed by its decorated permutation $\pi_{4,9} = 678912345$ and let $A = \{1, 3, 5, 6, 8\} = \{1\} \cup \{3\} \cup [5, 6] \cup \{8\}$. We obtain that

$\sigma := \rho_A(\pi) = 698214375$. On the other hand, calculating the circuits of P_σ we get that

$$\mathcal{C}_A = \{2,3,4\} \cup \{4,5,6\} \cup \{7,8\} \cup \{1,2,9\} \cup \left\{ C \in \binom{[9]}{4} \,:\, [2,4],[4,6],[7,8],[9,1] \not\subset C \right\}.$$

Making use of SageMath we conclude that $\sigma \lessdot \pi_{4,9}$.

Example 34. Let $A = \{2,4,7,8,9\} = [2] \cup [4] \cup [7,8,9]$ and consider the positroid P_σ indexed by the decorated permutation $\sigma = 698214375$. Then $\tau := \rho_A(\sigma) = \underline{1}98234576$ and the circuits of P_τ are

$$\mathcal{C}_A = \{1\} \cup \{7,8\} \cup \{2,9\} \cup \left\{ C \in \binom{[9]}{3} \,:\, [1],[7,8],\{2,9\} \not\subset C \right\}.$$

Again, using SageMath we see that $\tau \lessdot \sigma$.

Following Example 34, we would like to conjecture that Theorem 28 is an if and only if but this does not hold. Take $\sigma = 4321$ and $\tau = 3\underline{2}1\underline{4}$. In this case, $\tau = \rho_\emptyset(\sigma)$ but P_τ is not a quotient of P_σ as their respective circuits are $\{\{2\},\{4\},\{1,3\}\}$ and $\{\{2,3\},\{1,4\}\}$.

The dual version of Theorem 28 can be stated as follows.

Corollary 35. *Let $\tau \in D_{n-k+1,n}$ be a decorated permutation such that $\tau = \overrightarrow{\rho_B}(\pi_{n-k,n})$ for some $B \in \binom{[n]}{\ell}$ where $\ell = 0,\dots,k-1$. Then $\pi_{n-k,n} \lessdot \tau$.*

Proof. Consider the decorated permutation $\pi_{n-k,n}$, $B \in \binom{[n]}{\ell}$ where $\ell \in \{0,\dots,k-1\}$ and $\tau = \overrightarrow{\rho_B}(\pi_{k,n})$. Taking the corresponding positroids of $\pi_{n-k,n}$ and τ and using Proposition 6, we get that $U_{n-k,n}$ is a quotient of P_τ if and only if P_τ^* is a quotient of $U_{n-k,n}^*$. As $U_{n-k,n}^* = U_{k,n}$ and $P_\tau^* = P_{\tau^{-1}}$ (see [7]), we know that $U_{n-k,n}$ is a quotient of P_τ if and only if $P_{\tau^{-1}}$ is a quotient of $U_{k,n}$. Now $\tau = \overrightarrow{\rho_B}(\pi_{k,n})$ implies that $\tau^{-1} = \overleftarrow{\rho_A}(\pi_{n-k,n})$ where $A = \tau^{-1}(B)$ because of Proposition 24. This with the fact that $A \in \binom{[n]}{\ell}$ where $\ell \in \{0,\dots,k-1\}$ and Theorem 28 gives us that $\tau^{-1} \leqslant \pi_{k,n}$. Following back our trail of if and only ifs, this implies that $U_{n-k,n}$ is a quotient of P_τ and $\pi_{n-k,n} \lessdot \tau$ as desired. □

We point out that if N is a paving matroid then its simple truncation is a uniform matroid (see [11]). Thus, the positroids τ in Corollary 35 correspond to paving matroids. Hence, via our work we have characterized a family of paving positroids.

In view of Corollary 35, we see that $\pi_{n,n}$ covers r positroids if and only if $\pi_{0,n}$ is covered by r positroids. However, since $\pi_{0,n}$ and $\pi_{n,n}$ are the bottom and top elements of P_n, respectively, then $r = |D_{1,n}| = |D_{n-1,n}| = 2^n - 1$ and $|D_{1,n}| = \sum_{\ell=0}^{n-1}\binom{n}{\ell}$. This allows us to conclude that the converse of Theorem 28 also holds for $\pi_{n,n}$.

Remark 36. Let $\tau_1, \tau_2 \in P_n$ be such that $\tau_1 \lessdot \tau_2$, and let $I^i = (I_1^i, \cdots, I_n^i)$ be the Grassmann necklace of τ_i, for $i = 1,2$ respectively. Then $I_j^1 \subseteq I_j^2$ for each $j \in [n]$. This follows from [3, Corollary 1.7.2]. However, the converse is not true as can be seen by taking the necklace $I = (1,3,3,1)$ which is componentwise contained in $J = (12,23,34,41)$. The positroid corresponding to J has $\{2,3,4\}$ as a circuit which can not be written as a union of elements in $\mathcal{C}_I = \{\{2\},\{4\},\{1,3\}\}$.

We end by providing a conjecture that summarizes the findings detailed in this paper. This conjecture is based on evidence generated using SageMath for positroids on the ground set up to size 12.

Conjecture 37. Let σ and π be positroids on $[n]$ of respective ranks $k-1$ and k. If $\sigma \lessdot \pi$ then there is a subset $A \subset [k]$ such that $\sigma = \overleftarrow{\rho_A}(\pi)$.

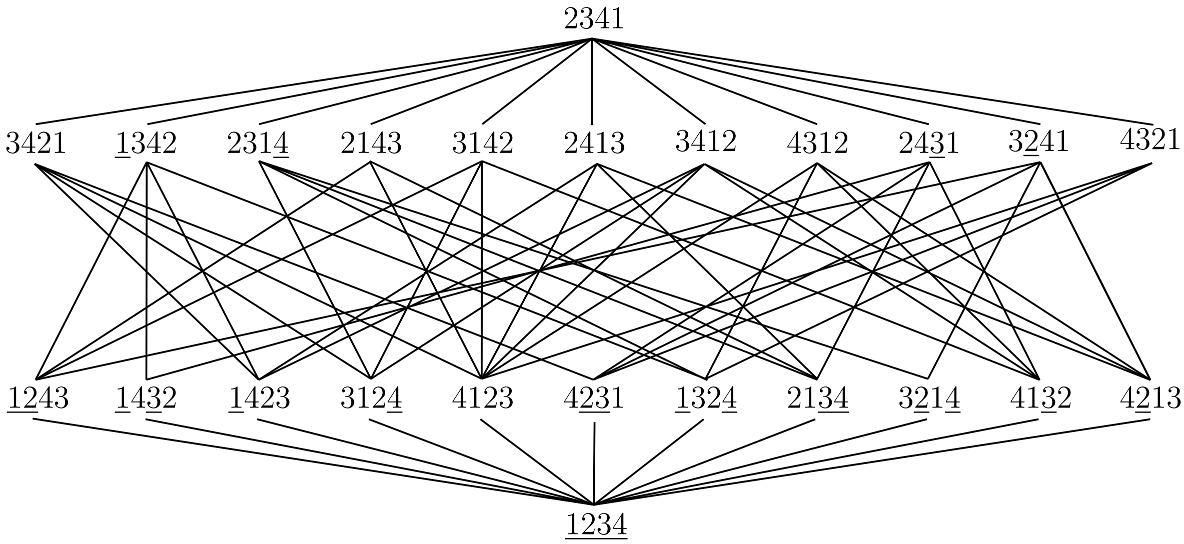

Figure 2: The interval $[\underline{1234}, 2341]$.

4 Future work

A parallel approach to the matroid quotient problem can be taken via positroid polytopes. That is, one may study chains of positroids that form a flag via the inequalities of their corresponding flag polytope. Thus, we pose the problem of characterizing flag positroids via flag matroid polytopes. That is, what conditions must a polytope have to guarantee that it corresponds to a flag of positroids?

Finally, the path we took in this paper has unveiled the poset of positroid quotients which deserves to be explored further. Some interesting questions in this direction are:

(a) *What is the Möbius function of the poset P_n?* Up to $n = 4$ the first values of $\mu(P_n)$ are $1, -1, 2, -9, 92$.

(b) *Is there an ER-labelling of P_n?* A candidate is labelling the edge of the covering $\tau \lessdot \sigma$ by the set that is frozen when passing from σ to τ. Answering this question may give a Whitney dual for this poset, in the sense of [6].

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