

Random Van der Waerden Theorem

Ohad Zohar

Tel Aviv University, Israel

ohadvb@gmail.com

Submitted: Jul 27, 2020; Accepted: Nov 12, 2021; Published: Jan 28, 2022

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Abstract

In this paper, we prove a sparse random analogue of the Van der Waerden Theorem. We show that, for all $r > 2$ and all $q_1 \geq q_2 \geq \dots \geq q_r \geq 3 \in \mathbb{N}$, $n^{-\frac{q_2}{q_1(q_2-1)}}$ is a threshold for the following property: For every r -coloring of the p -random subset of $\{1, \dots, n\}$, there exists a monochromatic q_i -term arithmetic progression colored i , for some i . This extends the results of Rödl and Ruciński for the symmetric case $q_1 = q_2 = \dots = q_r$. The proof of the 1-statement is based on the Hypergraph Container Method by Balogh, Morris and Samotij and Saxton and Thomason. The proof of the 0-statement is an extension of Rödl and Ruciński's argument for the symmetric case.

Mathematics Subject Classifications: 05C88, 05C89

1 Introduction

For $n, q \in \mathbb{N}$ and $r \geq 2$ let $[n] \rightarrow (q)_r$ denote the property that, for every r -coloring of $[n] = \{1, \dots, n\}$, there exists a monochromatic arithmetic progression of length q . More generally, we denote by $[n] \rightarrow (q_1, \dots, q_r)$ the property that, for every r -coloring of $[n]$, there exists i such that there is some arithmetic progression of length q_i colored i . A classical result in Ramsey theory due to Van der Waerden [22], states that, for every choice of $q_1, \dots, q_r \in \mathbb{N}$, there exists n_0 such that for every $n \geq n_0$ we have $[n] \rightarrow (q_1, \dots, q_r)$. One might think of Van der Waerden's theorem as the arithmetic analogue to the graph-theoretic Ramsey Theorem.

This work aims to determine necessary and sufficient conditions for the property $[n] \rightarrow (q_1, \dots, q_r)$ to hold when we replace $[n]$ with a typical set of a given density. We define A_p as the random subset of a set A , where every element $a \in A$ belongs to A_p with probability p , independently of all other elements of A . Many interesting questions in Ramsey theory deal with determining the thresholds for values of p for which it is no longer possible to color A_p without introducing specific monochromatic substructures.

Even though the use of probabilistic methods in Ramsey theory has a long history, the study of Ramsey properties of random structures was initiated more recently by Frankl and Rödl [5], who applied probabilistic methods to prove the existence of a graph G with no K_4 for which every 2-coloring must contain a monochromatic triangle.

In a series of papers, Rödl and Ruciński [17, 18, 19], determined the thresholds for the symmetric Van der Waerden property. Let $[n]_p \rightarrow (q)_r$ denote the event that, for every r -coloring of $[n]_p$, there exists a monochromatic arithmetic progression of length q . Similarly, let $[n]_p \rightarrow (q_1, \dots, q_r)$ denote the event that, for every r -coloring of $[n]_p$, there exists a color i such that there exists an arithmetic progression of length q_i colored i .

Theorem 1 (Rödl and Ruciński). *For $3 \leq q \in \mathbb{N}$ and every $r \geq 2$, there exist $c, C > 0$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}([n]_p \rightarrow (q)_r) = \begin{cases} 1 & \text{if } p \geq C \cdot n^{-\frac{1}{q-1}}, \\ 0 & \text{if } p \leq c \cdot n^{-\frac{1}{q-1}}. \end{cases}$$

Similarly to the notation for the Van der Waerden property, one lets $G \rightarrow (F_1, \dots, F_r)$ denote the property that, for every r -coloring of $E(G)$, there exists i such that there is a copy of F_i colored i . As well as Theorem 1, Rödl and Ruciński have also determined the threshold for the event $G(n, p) \rightarrow (F, \dots, F)$. In 1997, Kohayakawa and Kreuter [11] initiated the study of the asymmetric case for graphs; they determined the threshold for the event $G(n, p) \rightarrow (C_1, \dots, C_r)$ where C_1, \dots, C_r are all cycles, and conjectured the location of the threshold for general subgraphs (F_1, \dots, F_r) . Several papers have since extended Kohayakawa and Kreuter's result to other families of subgraphs. For instance, Marciniszyn, Skokan, Spöhel and Steger [13], showed that the conjecture holds when F_1, \dots, F_r are all cliques. More recently Mousset, Nenadov and Samotij [14] proved an upper bound for the threshold function in the Kohayakawa–Kreuter conjecture for general subgraphs, extending a result of Gugelmann, Nenadov, Person, Škorić, Steger and Thomas [7], and settling the 1-statement. Very recently, Liebenau, Mattos, Mendonça and Skokan [12], have shown that the 0-statement holds for $r = 2$ for any pair of cycles and cliques. However, the 0-statement for general subgraphs remains open.

In this paper we prove the following natural analogue of the Kohayakawa–Kreuter conjecture for the Van der Waerden theorem:

Theorem 2. *For every $r \geq 2$ and $q_1 \geq q_2 \geq \dots \geq q_r \in \mathbb{N}$, there exist $c, C > 0$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}([n]_p \rightarrow (q_1, \dots, q_r)) = \begin{cases} 1 & \text{if } p \geq C \cdot n^{-\frac{q_2}{q_1(q_2-1)}}, \\ 0 & \text{if } p \leq c \cdot n^{-\frac{q_2}{q_1(q_2-1)}}. \end{cases}$$

Whether the threshold in Theorem 2 is sharp remains an interesting open question; so far it has only been shown that this is the case in \mathbb{Z}_n when $r = 2$ and $q_1 = q_2$ by Friedgut, Hàn, Person and Schacht [6].

It is important to note that only the two largest lengths determine the threshold. Therefore, in the proof of the 1-statement, it suffices to assume $q_2 = \dots = q_r$. For the 0-statement, we will show that a proper coloring exists using only the first two colors, as

is necessary for the case $r = 2$, since that is also sufficient for all values of r . By that reasoning, it is natural to divide the proof into the symmetric ($q_1 = q_2$) and asymmetric ($q_1 > q_2$) cases.

A generalization of Van der Waerden's theorem is the classical Rado theorem [16], which characterizes the so-called partition-regular matrices. A matrix A is partition-regular if every finite coloring of the positive integers admits a monochromatic solution to the equation $Ax = 0$. Independently to this work, some progress has been made towards an asymmetric version of the random Rado theorem: given a sequence of partition-regular matrices A_1, \dots, A_r , for which values of p does every r -coloring of $[n]_p$ admit a solution to $A_i x = 0$ colored i ? Aigner-Horev and Person [1] obtained an upper bound for the threshold value, which implies the 1-statement in Theorem 2. Soon after, Hancock and Treglown [8] obtained a matching lower bound for the case where every A_i has rank one; this result implies the 0-statement in a special case of theorems 1 and 2, where $q_i = 3$ for all i .

The structure of this paper is as follows. In Section 2, we present several known results that are used in our proofs. In Sections 3 and 4, we present short proofs for the symmetric case, already proved by Rödl and Ruciński [18]. We also correct an error in their proof for the 0-statement, which was independently discovered by Hancock and Treglown. These sections are included here strictly for completeness; readers familiar with these results are encouraged to continue reading from Section 5 in which we prove the 1-statement for the asymmetric case. Finally, in Section 6, we complete the proof of Theorem 2 by proving the 0-statement for the asymmetric case.

2 Preliminary results

In this section, we state several known results that are used throughout this paper. The first result is the Hypergraph Container Lemma proved by Balogh, Morris and Samotij [3], and independently by Saxton and Thomason [21]. For an introduction to the various applications of this lemma, as well as the formulation used in this paper, we refer the reader to [4].

Definition 3. For a k -uniform hypergraph H and a set $A \subset V(H)$ we define

$$d(A) = |\{e \in E(H) : A \subset e\}|,$$

and for $\ell \in \{1, \dots, k\}$ we define

$$\Delta_\ell(H) = \max\{d(A) : A \subset V(H) \text{ and } |A| = \ell\}.$$

Definition 4. Let H be a hypergraph, we denote the set of independent subsets of $V(H)$ by

$$\mathcal{I}(H) = \{I \subset V(H) : \forall E \in E(H), E \not\subset I\}.$$

Theorem 5 (The Hypergraph Container Lemma). *Let $k \in \mathbb{N}$ and $\epsilon > 0$. Let H be a nonempty k -uniform hypergraph, and suppose that:*

$$\Delta_\ell(H) \leq K \cdot \left(\frac{b}{v(H)}\right)^{\ell-1} \cdot \frac{e(H)}{v(H)}$$

for some $b, K \in \mathbb{N}$ and every $\ell \in \{1, \dots, k\}$. Then, there exists a constant $D = D(\epsilon, k, K)$, a collection $\mathcal{C} \subset P(V(H))$ and a function $f: P(V(H)) \rightarrow \mathcal{C}$ such that:

1. for every $I \in \mathcal{I}(H)$, there exists $S \subset I$ with $|S| \leq Db$ and $I \subset f(S)$,
2. each $C \in \mathcal{C}$ contains fewer than $\epsilon \cdot e(H)$ edges.

Claim 6. Let H be the hypergraph encoding q -APs in $[n]$, where $V(H) = [n]$ and $E(H)$ is the set of arithmetic progressions of length q . Then,

$$\Delta(H) = \Delta_1(H) \leq n$$

Proof. We denote the number of arithmetic progressions of length q in $[n]$ such that k is the i th element by $d(k, i)$. One easily checks that $d(k, i)$ satisfies:

$$d(k, i) = \begin{cases} \left\lfloor \frac{n-k}{q-1} \right\rfloor & \text{if } i = 1, \\ \left\lfloor \frac{k-1}{q-1} \right\rfloor & \text{if } i = q, \\ \min\left\{ \left\lfloor \frac{k-1}{i-1} \right\rfloor, \left\lfloor \frac{n-k}{q-i} \right\rfloor \right\} & \text{otherwise.} \end{cases}$$

We obtain that that the degree of k in H satisfies

$$d(k) = \sum_{i=1}^q d(k, i) \leq \frac{n}{q-1} + \sum_{i=2}^{q-1} d(k, i).$$

Applying the bound $\min\left\{ \left\lfloor \frac{k-1}{i-1} \right\rfloor, \left\lfloor \frac{n-k}{q-i} \right\rfloor \right\} \leq \frac{k-1+n-k}{i-1+q-i} < \frac{n}{q-1}$ we conclude that

$$d(k) = \sum_{i=1}^q d(k, i) \leq \frac{n}{q-1} + \sum_{i=2}^{q-1} d(k, i) \leq \frac{n}{q-1} + (q-2) \cdot \frac{n}{q-1} \leq n. \quad \square$$

Remark 7. Let H be the hypergraph encoding q -APs in $[n]$, where $V(H) = [n]$ and $E(H)$ is the set of arithmetic progressions of length q . Then:

1. The number of edges of H satisfies

$$e(H) = \sum_{i=1}^{n-q+1} \left\lfloor \frac{n-i}{q-1} \right\rfloor = \Theta(n^2)$$

since there are $\left\lfloor \frac{n-a}{q-1} \right\rfloor$ arithmetic progressions of length q in $[n]$ with smallest element a .

2. For $\ell = 1$, we have $\Delta_1(H) \leq n$, by the previous claim.
3. For every $\ell \geq 2$, we have $\Delta_\ell(H) \leq q^2$ since choosing the indices of two elements determines the arithmetic progression.

Therefore we may apply the Hypergraph Container Lemma with $b = qn^{\frac{q-2}{q-1}}$, since for $\ell = 1$

$$\Delta_1(H) \leq n \leq K \cdot \frac{e(H)}{v(H)}$$

and for $2 \leq \ell \leq q$

$$\Delta_\ell(H) \leq q^2 \leq q^{q-1} \leq K \cdot \frac{q^{q-1}n^{q-2}}{n^{q-1}} \cdot \frac{e(H)}{n} \leq K \cdot \left(\frac{qn^{\frac{q-2}{q-1}}}{v(H)} \right)^{\ell-1} \cdot \frac{e(H)}{v(H)},$$

provided that K and n are sufficiently large.

Definition 8. For brevity, we say a set of integers is q -AP-free if it contains no arithmetic progression of length q .

We obtain the following container lemma for arithmetic progressions.

Theorem 9. For every integer $q \geq 3$ and $\epsilon > 0$, there exists a constant $D = D(\epsilon, q)$ such that for each $n \in \mathbb{N}$, there exists a collection $\mathcal{G} \subset P([n])$ and a function $f: P([n]) \rightarrow \mathcal{G}$ such that:

1. each $G \in \mathcal{G}$ contains fewer than ϵn^2 many q -APs,
2. for every q -AP-free subset $I \subset [n]$, there exists $S \subset I$ with $|S| \leq D \cdot n^{\frac{q-2}{q-1}}$ and $I \subset f(S)$.

We also require two classical results in probabilistic combinatorics, the first of which is Janson's inequality [10].

Theorem 10 (Janson's inequality). Let Γ be a finite set and let $\mathcal{S} \subset P(\Gamma)$. For every $A \in \mathcal{S}$, let $I_A = 1$ if $A \subset \Gamma_p$ and $I_A = 0$ otherwise.

Let $X = \sum_{A \in \mathcal{S}} I_A$ be the random variable counting the sets of \mathcal{S} which are entirely contained in Γ_p . Set

$$\mu = \mathbb{E}X \quad \text{and} \quad \Delta = \sum_{\substack{(A,B) \in \mathcal{S}^2 \\ A \neq B, A \cap B \neq \emptyset}} \mathbb{E}[I_A \cdot I_B].$$

Then,

$$\mathbb{P}(X = 0) \leq e^{-\mu + \frac{\Delta}{2}}.$$

Moreover, if $\Delta \geq \mu$, then

$$\mathbb{P}(X = 0) \leq e^{-\frac{\mu^2}{2\Delta}}.$$

The second inequality is also known as The Extended Janson Inequality. For an introduction, as well as proofs for both inequalities, we refer the reader to [2].

We will also require a special case of Harris's inequality [9].

Theorem 11 (Harris's inequality). *Let Γ be a finite set and let $f: P(\Gamma) \rightarrow \{0, 1\}$ be an indicator function for some family of sets \mathcal{A} . We say \mathcal{A} is increasing (equivalently decreasing), if $S_1 \subset S_2 \implies f(S_1) \leq f(S_2)$ (equivalently $f(S_1) \geq f(S_2)$). If \mathcal{A} is increasing and \mathcal{B} is decreasing, then*

$$\mathbb{P}(\Gamma_p \in \mathcal{A} \cap \mathcal{B}) \leq \mathbb{P}(\Gamma_p \in \mathcal{A}) \cdot \mathbb{P}(\Gamma_p \in \mathcal{B}).$$

Again, for a more detailed discussion, we refer the reader to [2].

The final result in this section is a well-known quantitative version of Van der Waerden's Theorem due to Varnavides [23].

Lemma 12. *For every $r \in \mathbb{N}$ and every $q \geq 2$, there exist $n_0 \in \mathbb{N}$ and $\epsilon > 0$ such that for all $n \geq n_0$, every $(r+1)$ -coloring of $[n]$ contains at least $(r+1) \cdot \epsilon \cdot n^2$ monochromatic arithmetic progressions of length q .*

Proof. From Van der Waerden's theorem we have $W = W(r+1, q)$ such that every $(r+1)$ -coloring of $[W]$ yields a monochromatic q -AP. Thus, in every coloring of $[n]$ every W -AP contains at least one monochromatic q -AP. We observe:

- (a) $\#W$ -APs in $[n] = \Theta\left(\frac{n(n-W)}{W-1}\right)$.
- (b) Every q -AP is contained in at most W^2 many W -APs (fixing the indices of two terms in an arithmetic progression determines the progression).

We obtain from (a) $\Theta\left(\frac{1}{W} \cdot n^2\right)$ many q -APs, however they may be contained in multiple W -APs. Since by (b) every q -AP is counted at most W^2 times, the lemma follows with $\epsilon = \Theta\left(\frac{W-3}{2(r+1)}\right)$. \square

3 The symmetric case 1-statement

In this section, we present a short proof to the following theorem (originally proved by Rödl and Ruciński [19]):

Theorem 13. *For every $r \in \mathbb{N}$ and $q \geq 3$, there exists $C > 0$ such that the following holds: If $p \geq Cn^{-\frac{1}{q-1}}$ then a.a.s. every r -coloring of $[n]_p$ contains a monochromatic q -AP.*

The general framework of the proof we present here is due to Nenadov and Steger [15] who applied a similar argument in the setting of graphs. We begin by describing a general outline of the proof.

From The Hypergraph Container Lemma, we obtain a set of containers \mathcal{G} for q -AP-free subsets of $[n]$. Assume for contradiction that $[n]_p$ has a coloring with no monochromatic

arithmetic progression of length q , and fix an arbitrary such coloring. Let $G_i \in \mathcal{G}$ be the container for the i -th color class. Each container contains strictly fewer than ϵn^2 many q -term arithmetic progressions. However, by Lemma 12 every coloring in $r + 1$ colors must contain some color class with at least ϵn^2 arithmetic progressions. In our case, we treat the remainder set $[n] \setminus \bigcup_i G_i$ as the final color class. Hence, it must have at least ϵn^2 arithmetic progressions and therefore has at least ϵn many elements. However, by definition, the remainder set (which depends only on G_1, \dots, G_r) and $[n]_p$ are disjoint. Thus, the probability that $[n]_p$ has a coloring with no monochromatic q -term arithmetic progression that obeys the coloring constraints set by (G_1, \dots, G_r) is at most $(1 - p)^{\epsilon n}$. Finally, we apply a union bound over all possible choices for the r -tuple of containers, obtaining that the probability that such $[n]_p$ has a proper r -coloring tends to zero.

Proof of Theorem 13. We say that a coloring is “proper” if it contains no monochromatic q -AP. We wish to prove that a.a.s. $[n]_p$ admits no such coloring. Applying Theorem 9 with $\epsilon = \epsilon(r)$ obtained from Lemma 12, we obtain a family of containers \mathcal{G} such that each $G \in \mathcal{G}$ contains fewer than ϵn^2 many q -APs. If we suppose that there exists a proper coloring for $[n]_p$, then there are q -AP-free subsets H_1, \dots, H_r , such that $\bigcup_{i=1}^r H_i = [n]_p$.

By Theorem 9, there exist a constant D and a function $f: P([n]) \rightarrow \mathcal{G}$ such that for every i there exists a set $S_i \subset H_i$, with $|S_i| \leq D \cdot n^{\frac{q-2}{q-1}}$ and $H_i \subset f(S_i) = G_i$. By Lemma 12, we obtain that for any coloring of $[n]$ with $r + 1$ colors there must be a color class with at least ϵn^2 arithmetic progressions.

Suppose we color the elements of $\bigcup G_i$ such that only elements of G_i are colored i and the elements of $[n] \setminus \bigcup_{i=1}^r G_i$ are colored $r + 1$. Since every G_i contains fewer than ϵn^2 arithmetic progressions, the set of elements colored $r + 1$ must contain ϵn^2 arithmetic progressions. Since, by Claim 6, each $k \in [n]$ belongs to at most n many q -APs we have

$$|[n] \setminus \bigcup_{i=1}^r G_i| \geq \frac{\epsilon n^2}{n} = \epsilon n.$$

In summary, the event “There exists a proper coloring of $[n]_p$ ” implies the following two events, for some $S_1, \dots, S_r \subset [n]$ with $|S_i| \leq D n^{\frac{q-2}{q-1}}$ for every $i \in \{1, \dots, r\}$:

1. $\bigcup_{i=1}^r S_i \subset [n]_p$,
2. $[n]_p \subset \bigcup_{i=1}^r f(S_i)$.

Note that (b) is equivalent to

$$[n]_p \cap \left([n] \setminus \bigcup_{i=1}^r f(S_i) \right) = \emptyset.$$

Since $S_i \subset f(S_i)$ for every i , (a) and (b) depend on disjoint subsets of $[n]$ and are independent events. Hence, the probability of both (a) and (b) occurring is:

$$\mathbb{P}\left(S_1, \dots, S_r \subset [n]_p \wedge [n]_p \subset \bigcup_{i=1}^r f(S_i)\right) \leq p^{|\bigcup_{i=1}^r S_i|} \cdot (1-p)^{\epsilon n}.$$

Given a set S with $|S| = s$, there are at most 2^{rs} sequences (S_1, \dots, S_r) such that $S = \bigcup S_i$. Taking the union bound over all choices of S_1, \dots, S_r (grouping by $s = |\bigcup_{i=1}^r S_i|$) we obtain

$$\begin{aligned} \mathbb{P}([n]_p \text{ admits a proper coloring}) &\leq \sum_{(S_1, \dots, S_r)} p^{|\bigcup_{i=1}^r S_i|} \cdot (1-p)^{\epsilon n} \\ &\leq (1-p)^{\epsilon n} \sum_{s=1}^{Drn^{\frac{q-2}{q-1}}} \binom{n}{s} 2^{rs} p^s \leq e^{-\epsilon pn} \sum_{s=1}^{Drn^{\frac{q-2}{q-1}}} \left(\frac{en2^r p}{s}\right)^s. \end{aligned}$$

Recall that $p = Cn^{-\frac{1}{q-1}}$. Since $x \mapsto \left(\frac{ea}{x}\right)^x$ is increasing for $x \leq a$, by choosing C sufficiently large we obtain that $Drn^{\frac{q-2}{q-1}} \leq \delta pn$ for some $\delta = \delta(C, D, r) > 0$ which can be made arbitrarily small by choosing C sufficiently large. Hence,

$$\begin{aligned} \sum_{s=1}^{Drn^{\frac{q-2}{q-1}}} \left(\frac{en2^r p}{s}\right)^s &\leq Drn^{\frac{q-2}{q-1}} \cdot \left(\frac{en2^r p}{Drn^{\frac{q-2}{q-1}}}\right)^{Drn^{\frac{q-2}{q-1}}} \\ &\leq \delta pn \cdot \left(\frac{e2^r}{\delta}\right)^{\delta pn} \leq e^{\frac{\epsilon pn}{2}} \end{aligned}$$

for sufficiently large C , since $\left(\frac{1}{\delta}\right)^\delta \rightarrow 1$ as $\delta \rightarrow 0$. Hence:

$$\mathbb{P}([n]_p \text{ admits a proper coloring}) \leq e^{-\frac{\epsilon pn}{2}} \rightarrow 0. \quad \square$$

4 The symmetric case 0-statement

In this section, we prove the following theorem (originally proved by Rödl and Ruciński [19]):

Theorem 14. *For any integer $q \geq 3$, there exists $c > 0$ such that for $p = c \cdot n^{-\frac{1}{q-1}}$, $[n]_p$ can a.a.s. be colored by two colors with no monochromatic q -term arithmetic progression.*

Note that this implies the 0-statement in Theorem 1 for any $r \geq 2$. The proof here is a specialization of the proof by Rödl and Ruciński [20] to the random Rado partition theorem, and is included here both for the sake of completeness and as an introduction to the techniques used for the asymmetric case in Section 6. The proof consists of two main lemmas. The deterministic lemma states that every non-2-colorable uniform hypergraph must contain one of a small family of hypergraphs which we refer to as 2-blocking hypergraphs (which will be defined next). Then, the probabilistic lemma states that in the random hypergraph of arithmetic progressions in $[n]_p$, the subhypergraphs mentioned in the deterministic lemma almost surely do not appear. We begin by defining several families of hypergraphs.

Definition 15. A *simple path* is a hypergraph consisting of edges E_1, \dots, E_ℓ , for $\ell \geq 1$, such that

$$|E_i \cap E_j| = \begin{cases} 1 & \text{if } |i - j| = 1, \\ 0 & \text{if otherwise.} \end{cases}$$

A *fairly simple cycle* is a hypergraph that consists of a simple path (E_1, \dots, E_ℓ) , with $\ell \geq 2$, and an edge E_0 such that

$$|E_0 \cap E_i| = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{if } i = 2, \dots, \ell - 1, \\ s & \text{if } i = \ell, \end{cases}$$

for some $s \geq 1$, and such that $E_0 \cap E_1 \cap E_\ell = \emptyset$. A fairly simple cycle is said to be *simple* if $s = 1$; otherwise, we say it is *special*. A path P in a hypergraph H is said to be *spoiled* if it is not an induced subhypergraph of H . We call an edge $E \subset V(P)$ such that $E \notin E(P)$ a *spoiling edge* for P .

The length of a path or a fairly simple cycle is the number of edges in it. A subhypergraph H_0 of H is said to have a *handle* if there is an edge E in H such that $|E| > |E \cap V(H_0)| \geq 2$.

Definition 16. We call a q -uniform hypergraph *2-blocking* if it is one of the following:

1. A special cycle.
2. A simple cycle with a handle.
3. A spoiled path.
4. For $q = 3$, the 3-uniform, 2-regular, 6-vertex, simple hypergraph, which we call “the reduced Fano plane” (See Figure 1a) ¹.

Lemma 17 (The deterministic lemma). *Let $q \geq 3$ be some integer, and let H be a q -uniform hypergraph which is not 2-colorable. Then H contains a 2-blocking hypergraph.*

Proof. Recall that we say a hypergraph is 3-edge-critical if it cannot be properly colored with two colors, but any proper subhypergraph of it is 2-colorable. We may assume that H is 3-edge-critical; otherwise, we may replace it with one of its 3-edge-critical subhypergraphs.

Claim 18. *If H is a 3-edge-critical hypergraph, then for every edge $E \in H$ and for every vertex $v \in E$ there is an edge $E' \in H$ such that $E \cap E' = \{v\}$.*

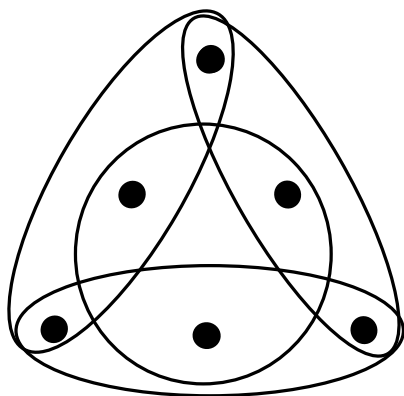
¹The name stems from the fact that this hypergraph is exactly the Fano plane with one vertex removed. It is also sometimes referred to as a Pasch configuration.

Proof. Let H be 3-edge-critical, and suppose that there are an edge $E \in H$ and a vertex $v \in E$ such that no other edge intersects E in exactly $\{v\}$; thus, every edge E' that contains v also contains another vertex of E . By the 3-edge-criticality, H can be colored red-blue in a way such that only E is monochromatic, say it is blue. Now, by changing the color of v to red, E is no longer monochromatic, and neither is any other edge that contains v , contradicting the fact that H is not 2-colorable. \square

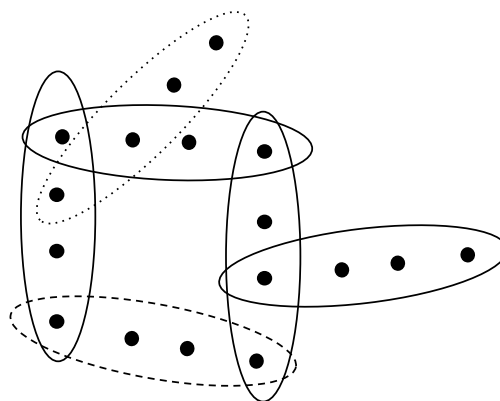
Let $P = (E_1, \dots, E_\ell)$ be a longest simple path in H . By the claim, $\ell \geq 2$. Let x, y be two of the vertices that belong only to E_1 , and let E_x, E_y be two edges of H that intersect E_1 only in x and y respectively. Since P is maximal, $h_z = |E_z \cap V(P)| \geq 2$, for $z = x, y$.

We may also assume that E_x, E_y only intersect one another and each edge of P in at most a single vertex (otherwise we obtain a special cycle). Let $i_z = \min\{i \geq 2 : E_z \cap E_i \neq \emptyset\}$, and without loss of generality assume $i_y \leq i_x$. If $h_z = q$ for some z then P is a spoiled simple path. Otherwise, either the edges $E_1, \dots, E_{i_y}, \dots, E_{i_x}, E_x$ form a fairly simple cycle, to which E_y is a handle (see Figure 1b) or $E_y \subset (E_1 \cup \dots \cup E_{i_x} \cup E_x)$.

Assuming the latter case, since $h_y < q$ and we assumed that $|E_x \cap E_y| \leq 1$, we must have $E_x \cap E_y = \{u\}$ for some vertex $u \notin V(P)$. We now split into several cases. First, if $i_y < i_x$ then the edges E_1, \dots, E_{i_y}, E_y form a fairly simple cycle to which E_x is a handle (since it only intersects the cycle at x and u). Finally, if $i_y = i_x = i$ then either E_x, E_y, E_i form a fairly simple cycle to which E_1 is a handle, or $i = 2, q = 3$ and E_1, E_2, E_x, E_y form the reduced Fano plane. \square



(a) The reduced Fano plane



(b) E_x (dashed) forms a cycle with P to which E_y (dotted) is a handle.

Lemma 19 (The probabilistic lemma). *Let H be the hypergraph with vertex set $V(H) = [n]$, whose edge set is the set of q -APs, and let H_p be its random subhypergraph induced by $[n]_p$. If $p = c \cdot n^{-\frac{1}{q-1}}$, then a.a.s. H_p contains no 2-blocking hypergraph, provided that c is sufficiently small.*

Proof. We apply a first-moment argument to several random variables. Let B be a large enough constant, we will show that a.a.s. no path of length $B \log n$ exists in H_p . We will then show that a.a.s. no 2-blocking graph with fewer than $B \log n$ edges exist.

The following calculations are used several times throughout the arguments. First, the number of choices for a q -term arithmetic progression in $[n]$ is $O(n^2)$, and the probability that all of its q elements belong to $[n]_p$ is $(cn^{-\frac{1}{q-1}})^q = O(n^{-1}p)$, so the expected number of q -term arithmetic progressions in $[n]_p$ is $O(np)$.

We say that an edge A extends a simple path $P = (E_1, \dots, E_\ell)$ in H , if $A \cap V(P) = \{v\}$ and $\{i : v \in E_i\}$ is exactly $\{1\}$ or $\{\ell\}$. Observe that the number of choices for an arithmetic progression A that extends a simple path P is bounded by $2 \cdot (q-1) \cdot q \cdot n$, and that the probability that $A \setminus V(P) \subset [n]_p$ is $(cn^{-\frac{1}{q-1}})^{q-1}$. Therefore, we may bound the expected number of possible edge choices for extending a simple path in H_p by some constant c_r , which may depend on q but not on n , and may be made arbitrarily small by changing c .

The number of arithmetic progressions containing a fixed set $\{v_1, \dots, v_s\}$ with $s \geq 2$, is bounded by a constant (for instance $\binom{q}{s}$ is a trivial bound). Hence the expected number of arithmetic progressions in H_p that contain $\{v_1, \dots, v_s\}$ is $O(p^{q-s})$, assuming $\{v_1, \dots, v_s\} \subset [n]_p$.

Since the number of simple paths of length t in H is at most $O(n^{1+t})$, and a simple path of length t must contain q vertices for the first edge and $q-1$ vertices for every subsequent edge, the expected number of simple paths of length t in H_p is

$$O(n^{1+t} \cdot p^{q+(t-1)(q-1)}) = O(np \cdot c_r^t).$$

We are now ready to proceed with the argument.

Let U be the random variable counting the number of simple paths in H_p of length at least $B \cdot \log n$. We bound the expected value of U by summing over different lengths of paths. By the above computations,

$$\mathbb{E}U \leq O\left(\sum_{t \geq B \log n} np \cdot c_r^t\right) = o(1),$$

provided that c_r is sufficiently small and B is sufficiently large.

Let W be the random variable counting the number of special cycles in H_p . For a given edge there are only a constant number of edges that intersect it in more than one vertex, therefore the number of such edge pairs in H is $O(n^2)$. We may bound the number of special cycles of length t in H by first fixing E_0 and E_{t-1} , then fixing a simple path of length $t-3$ starting from some vertex of E_0 and finally choosing E_{t-2} . Note that we have $O(1)$ many choices for E_{t-2} since it must include exactly one vertex from E_{t-3} and one vertex from E_{t-1} . Moreover, these vertices cannot coincide as $E_0 \cap E_1 \cap E_{t-1} = \emptyset$. In total we obtain that there are $O(n^2 \cdot n^{t-3}) = O(n^{t-1})$ such cycles in H . Since we must have at least $q+1$ vertices for E_0 and E_{t-1} , an additional $(t-3) \cdot (q-1)$ vertices for E_1, \dots, E_{t-3} , and exactly $q-2$ additional vertices for E_{t-2} we require a total of $2q-1 + (t-3) \cdot (q-1) = 1 + (t-1) \cdot (q-1)$ vertices in H_p . Hence,

$$\mathbb{E}W = O\left(\sum_{t > 2} n^{t-1} p^{1+(t-1)(q-1)}\right) = O\left(p \sum_{t > 2} c_r^{t-1}\right) = O(p) = o(1),$$

provided that c_r is sufficiently small.

Let X be the random variable counting the number of simple cycles with handles of length at most $B \log n$ in H_p . We denote the length of the cycle by t , and the size of the intersection between the handle and the cycle by k . Similarly to the previous argument, there are $O(n^t)$ many cycles of length t in H . Since the handle must attach in at least two vertices of the cycle we may bound the number of handles in H by $O(\log^2 n)$. For fixed values of t and k , such a configuration requires $t \cdot (q - 1)$ vertices for the cycle and an additional $q - k$ vertices for the handle. Summing over t and k ,

$$\mathbb{E}X = O\left(\sum_{t=3}^{B \log n} \sum_{2 \leq k \leq q-1} n^t \cdot p^{t \cdot (q-1) + q - k} \cdot \log^2 n\right) = O(p \log^3 n) = o(1).$$

Let Y be the random variable counting the number of spoiled simple paths of length less than $B \log n$ in H_p . Let E be an induced edge that is not one of the edges of a spoiled path. We now split into two cases. First, suppose that E intersects some edge in at least two vertices, we obtain a path (E_1, \dots, E_ℓ) and an edge $E_0 = E$, such that $|E_0 \cap E_1| = s \geq 2$, $|E_0 \cap (E_\ell \setminus E_1)| = t \geq 1$. If $t = 1$ and $E_0 \cap E_1 \cap E_\ell = \emptyset$ this yields a special cycle, and thus the expected number of such hypergraphs in H_p is $o(1)$. Otherwise, $t' = |E_\ell \cap (E_0 \cup E_1)| \geq 2$. As $|E_1 \cap E_\ell| \leq 1$ we have $s + t' \leq q + 1$. Hence, the expected number of choices for E_0, E_1, E_ℓ is

$$O\left(\sum_{s, t'} np \cdot p^{q-s} \cdot p^{q-t'}\right) = O\left(\sum_{s, t'} np^{2q-(s+t')+1}\right) = O(np^q) = o(1).$$

Assuming that E intersects every edge of a path (E_1, \dots, E_ℓ) in at most one vertex, we may define an ordering function $f: E \rightarrow [\ell]$ by $f(v) = \min\{i : v \in E_i\}$ and order the vertices of E by the values of f . Observe the shortest sub-path containing the first three vertices $\{v_1, v_2, v_3\} \subset E$, and denote $t_1 = f(v_2) - f(v_1)$ and $t_2 = f(v_3) - f(v_2)$. Let Y' count such configurations in H . We have an expected $O(np)$ many choices for E , and $O(c_r^{t_1-1} p^{q-2})$ many choices for a path of length t between two fixed vertices. Summing over t_1 and t_2 we obtain

$$\mathbb{E}Y' = O\left(\sum_{t_1 \geq 1} \sum_{t_2 \geq 1} np \cdot c_r^{t_1-1} p^{q-2} \cdot c_r^{t_2-1} p^{q-2}\right) = O(np^{2q-3}) = O(p^{q-2}) = o(1).$$

Thus,

$$\mathbb{E}Y = o(1).$$

Finally, let Z be the random variable counting the number of copies of the reduced Fano plane in H_p . First, we show that there are $O(n^2)$ copies of the reduced Fano plane in H . Suppose that $\{x_1, x_2, x_3\}$ is an edge in H , since there are $O(n^2)$ choices for an arithmetic progression of length three it suffices to show that there are at most $O(1)$ many choices for x_4, x_5, x_6 such that $\{x_1, \dots, x_6\}$ induce a copy of the reduced Fano plane. Denote $\vec{v} = (x_1, x_2, x_3)$ and $\vec{x} = (x_4, x_5, x_6)$. Since every pair of vertices in $\{x_4, x_5, x_6\}$ form a

3-AP with one vertex in $\{x_1, x_2, x_3\}$, we obtain the following family of equations:

$$A \cdot \vec{x} = \vec{v} \quad \text{for} \quad A = \begin{pmatrix} a & b & 0 \\ 0 & c & d \\ e & 0 & f \end{pmatrix} \quad \text{with} \quad a, b, c, d, e, f \in \{-1, \frac{1}{2}, 2\}.$$

Since $\det(A) = acf + bde \neq 0$ for all choices of A we obtain that A is invertible, and therefore fixing \vec{v} determines \vec{x} . Since the reduced Fano plane contains six vertices we obtain that the expected number of such configurations is $O(n^2 p^6) = O(n^{2-\frac{6}{2}}) = O(\frac{1}{n}) = o(1)$.

Thus, by Markov's inequality $\mathbb{P}(U = W = X = Y = Z = 0) \rightarrow 1$ as $n \rightarrow \infty$ completing the proof. \square

With the two lemmas in hand, the proof of Theorem 14 is immediate.

Proof of Theorem 14. Let H be the hypergraph with vertex set $V(H) = [n]_p$, whose edge set is the set of q -term arithmetic progressions. By Lemma 19, H a.a.s. contains no 2-blocking hypergraph. Therefore, by Lemma 17, H is 2-colorable. \square

5 The asymmetric case 1-statement

In this section, we prove the following result:

Theorem 20. *For every $r \geq 2$ and $q_1 > q_2 \geq \dots \geq q_r \in \mathbb{N}, q_r \geq 3$, there exists $C > 0$ such that the following holds:*

If $p \geq C n^{-\frac{q_2}{q_1(q_2-1)}}$ then a.a.s. for every r -coloring of $[n]_p$ there is some i such that there exists a monochromatic q_i -AP colored i .

We begin by describing a rough outline of the proof, which follows similar ideas to [7]. We say that a coloring is proper if it contains no monochromatic q_i -AP colored i ; we wish to prove that a.a.s. $[n]_p$ admits no such coloring. First, we reduce the problem by showing that every proper coloring may be modified to yield a ‘‘good’’ coloring where, additionally, every element not in a q_1 -AP is colored 1. Thus, it would suffice to show that no such good coloring exists.

Using The Hypergraph Container Lemma we obtain a set of containers \mathcal{G} for the q_2 -AP-free subsets of $[n]$. Supposing for contradiction that a good coloring exists, we fix one arbitrary such coloring along with containers $G_2, \dots, G_r \in \mathcal{G}$ for all the color classes but the first.

For $i \geq 2$, we denote the set of elements colored i by I_i . We denote the remainder set $I_1 = [n]_p \setminus (I_2 \cup \dots \cup I_r)$. Since, by Lemma 12, $[n] \setminus (G_2 \cup \dots \cup G_r)$ contains at least ϵn^2 many arithmetic progressions of length q_1 , the set of elements colored 1 is unlikely to be q_1 -AP-free. Indeed, in Lemma 21, we show that the probability that the set of elements colored 1 contains no q_1 -AP is exponentially small in $n^2 p^{q_1}$.

Note, that the existence of a good coloring implies two events: First, the signature sets for the containers G_2, \dots, G_r must all be covered by q_1 -APs in $[n]_p$, and second, the

remainder set I_1 must not contain a single q_1 -AP. Using Harris's inequality, we show that these two events are negatively correlated.

Finally, we apply a union bound over all possible choices for the tuple of containers (G_2, \dots, G_r) , by iterating over their signature sets S_2, \dots, S_r . A critical part of the union bound argument is Lemma 23, which roughly states that if $p = \Theta(n^{-\frac{q_2}{q_1(q_2-1)}})$ then typically q_1 -APs rarely intersect, and therefore most of $S = \bigcup S_i$ is covered by isolated q_1 -APs in $[n]_p$.

We begin by proving the following lemmas:

Lemma 21. *Suppose \mathcal{A} is a collection of $\Omega(n^2)$ q -APs in $[n]$, and $np^{q-1} \ll 1$. Then,*

$$\mathbb{P}([n]_p \text{ does not contain any member of } \mathcal{A}) \leq \exp(-\Omega(n^2 p^q)).$$

Proof. Enumerate the elements of $\mathcal{A} = \{E_i : i \in I\}$. For each $i \in I$, let X_i be the indicator random variable for the event $E_i \subset [n]_p$ and let $X = \sum X_i$. Observe that

$$\begin{aligned} \mu &= \mathbb{E}X = \Omega(n^2 p^q), \\ \Delta &= \sum_{\substack{i \neq j \\ E_i \cap E_j \neq \emptyset}} \mathbb{P}(E_i \cup E_j \subset [n]_p) = \sum_{i \in I} \sum_{1 \leq k \leq q-1} \sum_{\substack{j \in I \\ |E_i \cap E_j|=k}} p^{2q-k}. \end{aligned}$$

Note that for a fixed i and $k > 1$ there are only at most $q^2 = O(1)$ many j s such that $|E_i \cap E_j| = k$, and for $k = 1$ there at most $O(n)$ such j s. For a fixed i this implies,

$$\sum_{1 \leq k \leq q-1} \sum_{\substack{j \in I \\ |E_i \cap E_j|=k}} p^{2q-k} = O(np^{2q-1} + \sum_{2 \leq k \leq q-1} p^{2q-k}) = O(np^{2q-1} + p^{q+1}),$$

and thus, as $|\mathcal{A}| = O(n^2)$,

$$\Delta = O(n^3 p^{2q-1} + n^2 p^{q+1}).$$

Moreover, by our assumption that $np^{q-1} \ll 1$ we have

$$n^3 p^{2q-1} + n^2 p^{q+1} \ll n^2 p^q.$$

Hence, by Janson's inequality,

$$\mathbb{P}(X = 0) \leq \exp(-\mu + \frac{\Delta}{2}) = \exp(-\Omega(\mu)) = \exp(-\Omega(n^2 p^q))$$

concluding the proof. □

Definition 22. We say that a q_1 -AP in $[n]_p$ is *isolated*, if it does not intersect any other q_1 -AP in $[n]_p$. We define the following random variables:

$$\begin{aligned} Q &= \bigcup \{A \subset [n]_p : A \text{ is a } q_1\text{-AP}\}, \\ Q_I &= \bigcup \{A \subset [n]_p : A \text{ is an isolated } q_1\text{-AP}\}. \end{aligned}$$

Let $\delta = \frac{\min\{q_2, q_1 - q_2\}}{2q_1(q_2 - 1)}$. If $|Q \setminus Q_I| < n^{1 - \frac{1}{q_2 - 1} - \delta}$ we say that Q is *mostly independent*.

Lemma 23. *If $p = O(n^{-\frac{q_2}{q_1(q_2-1)}})$ then Q is mostly independent a.a.s.*

Proof. First we compute the expected number of sets that are a union of two intersecting q_1 -APs. First, since $\frac{1}{q_2-1} - \frac{q_2}{q_1(q_2-1)} = \frac{q_1-q_2}{q_1(q_2-1)} \geq 2\delta$,

$$\begin{aligned} \mathbb{E}(\#\text{pairs of } q_1\text{-APs sharing exactly one element}) &= O(n^2 \cdot n \cdot p^{2q_1-1}) \\ &= O(n^{1-\frac{2}{q_2-1}+\frac{q_2}{q_1(q_2-1)}}) = O(n^{1-\frac{1}{q_2-1}-2\delta}). \end{aligned}$$

Second, for every $1 < m < q_1$

$$\begin{aligned} \mathbb{E}(\#\text{pairs of } q_1\text{-APs sharing exactly } m \text{ elements}) &= O(n^2 \cdot p^{2q_1-m}) \\ &= O(n^{-\frac{2}{q_2-1}+\frac{mq_2}{q_1(q_2-1)}}). \end{aligned}$$

Since $\frac{mq_2}{q_1(q_2-1)} \leq \frac{(q_1-1)q_2}{q_1(q_2-1)} = 1 + \frac{1}{q_2-1} - \frac{q_2}{q_1(q_2-1)} \leq 1 + \frac{1}{q_2-1} - 2\delta$,

$$\mathbb{E}(\#\text{pairs of } q_1\text{-APs sharing } m \text{ elements}) = O(n^{1-\frac{1}{q_2-1}-2\delta}).$$

In particular, since $|Q \setminus Q_I| \leq |\{(A, B) : A \cap B \neq \emptyset, A, B \in Q\}|$,

$$\mathbb{E}|Q \setminus Q_I| \leq O(n^{1-\frac{1}{q_2-1}-2\delta}).$$

Finally, we obtain from Markov's inequality,

$$\mathbb{P}(|Q \setminus Q_I| \geq n^{1-\frac{1}{q_2-1}-\delta}) \leq O\left(\frac{n^{1-\frac{1}{q_2-1}-2\delta}}{n^{1-\frac{1}{q_2-1}-\delta}}\right) = O(n^{-\delta}) = o(1).$$

Hence, a.a.s.

$$|Q \setminus Q_I| < n^{1-\frac{1}{q_2-1}-\delta}. \quad \square$$

We are now ready to prove Theorem 20.

Proof of Theorem 20. First, since p depends only on q_1 and q_2 , we may assume that $q_2 = q_3 = \dots = q_r$. We say that a coloring is proper if it contains no monochromatic q_i -AP colored i ; we wish to prove that a.a.s. $[n]_p$ admits no such coloring.

By our assumption, $p \geq Cn^{-\frac{q_2}{q_1(q_2-1)}}$ for some sufficiently large $C > 0$. Since not admitting a proper coloring is an increasing event, without loss of generality we may assume that $p = Cn^{-\frac{q_2}{q_1(q_2-1)}}$.

Note that the elements of the set $\{a \in [n]_p : a \text{ does not belong to a } q_1\text{-AP}\}$ may all be recolored 1 for any proper coloring of $[n]_p$, without creating a monochromatic q_1 -AP. We say that such a proper coloring is a "good" coloring. Since, by recoloring, the existence of a proper coloring implies the existence of a good coloring it suffices to show that no good coloring exists.

Suppose that there is such a coloring; then for each $i \in [r]$, the set I_i of elements colored i contains no q_i -APs. By The Hypergraph Container Lemma for every $\epsilon > 0$ and

every $i \geq 2$ there exist $S_i \subset I_i \subset G_i$ with $|S_i| \leq s_{\max} = O(n^{1-\frac{1}{q_2-1}})$ and such that G_i , which depends only on S_i , contains at most ϵn^2 many q_2 -APs (the implicit constant in the definition of s_{\max} may depend on ϵ and q_2). By our assumption,

$$I_1 = [n]_p \setminus (I_2 \cup \dots \cup I_r) \supset [n]_p \setminus (G_2 \cup \dots \cup G_r) = [n]_p \cap ([n] \setminus (G_2 \cup \dots \cup G_r)).$$

In particular,

$$I_1 \text{ contains no } q_1\text{-APs} \implies [n]_p \cap ([n] \setminus (G_2 \cup \dots \cup G_r)) \text{ contains no } q_1\text{-APs.}$$

For brevity, we write $A_{(S_2, \dots, S_r)} = [n]_p \cap ([n] \setminus (G_2 \cup \dots \cup G_r))$, as G_i depends only on S_i . Since $q_1 > q_2$, the number of q_1 -APs in any set of integers is at most as large as the number of q_2 -APs. Therefore, for every choice of (S_2, \dots, S_r) , Lemma 12 implies that $[n] \setminus (G_2 \cup \dots \cup G_r)$ contains at least ϵn^2 many q_1 -APs. Hence, by Lemma 21

$$\mathbb{P}(A_{(S_2, \dots, S_r)} \text{ contains no } q_1\text{-AP}) \leq e^{-D_1 n^2 p^{q_1}}$$

for some constant $D_1 > 0$ which depends only on ϵ and q_1 .

Let Q and Q_I be the variables defined in Definition 22. Suppose $S = \cup_{i=2}^r S_i$ is covered by elements of Q ; we fix a largest subset of S that is covered by pairwise-disjoint arithmetic progressions of length q_1 in $[n]_p$ and denote it S' . Since $S \setminus S' \subset S \cap (Q \setminus Q_I)$, if Q is mostly independent then $|S \setminus S'| < n^{1-\frac{1}{q_2-1}-\delta}$. Thus, if a good coloring exists and Q is mostly independent then there exists some choice of (S_2, \dots, S_r) such that

1. S is covered by q_1 -APs in $[n]_p$.
2. $|S \setminus S'| < n^{1-\frac{1}{q_2-1}-\delta}$.
3. $A_{(S_2, \dots, S_r)}$ contains no q_1 -AP.

For shorthand, we say that S is “well-covered” if it satisfies conditions 1 and 2.

Let $P = \mathbb{P}([n]_p \text{ admits a “good” coloring})$. We wish to show that $P = o(1)$. We first note P may be bounded by the sum of probabilities of two other events: either Q is not mostly independent or there exists a tuple (S_2, \dots, S_r) such that the above three events hold. Since by Lemma 23 the probability that Q is not mostly independent is $o(1)$ we obtain

$$P \leq o(1) + \sum_{(S_2, \dots, S_r)} \mathbb{P}(S \text{ is well-covered} \wedge A_{(S_2, \dots, S_r)} \text{ contains no } q_1\text{-AP}).$$

Note, that the event “ S is well-covered” is increasing, while the event “ $A_{(S_2, \dots, S_r)}$ contains no q_1 -AP” is decreasing. Therefore by Harris’s inequality we obtain

$$\begin{aligned} P &\leq o(1) + \sum_{(S_2, \dots, S_r)} \mathbb{P}(S \text{ is well-covered}) \cdot \mathbb{P}(A_{(S_2, \dots, S_r)} \text{ contains no } q_1\text{-AP}) \\ &\leq o(1) + \sum_{(S_2, \dots, S_r)} \mathbb{P}(S \text{ is well-covered}) \cdot e^{-D_1 n^2 p^{q_1}}. \end{aligned}$$

Our goal is now to obtain a bound on

$$\sum_{(S_2, \dots, S_r)} \mathbb{P}(S \text{ is well-covered}).$$

Let $\mathcal{C}(A)$ denote the event “ A is covered by pairwise-disjoint q_1 -APs”. Since there are at most $r^{|S \setminus S'|}$ many ways to distribute the elements of $S \setminus S'$ to S_2, \dots, S_r , we obtain

$$\sum_{(S_2, \dots, S_r)} \mathbb{P}(S \text{ is well-covered}) \leq \sum_{t=0}^{n^{1-\frac{1}{q_2-1}-\delta}} \binom{n}{t} r^t \cdot \sum_{(S'_2, \dots, S'_r)} \mathbb{P}(\mathcal{C}(S')),$$

where the second sum ranges over all $(r-1)$ -tuples of sets (S'_2, \dots, S'_r) satisfying $|S'_i| \leq s_{\max}$ for each i and $S' = \cup_{i=2}^r S'_i$; which we may bound from above by

$$O(e^{3 \log n \cdot n^{1-\frac{1}{q_2-1}-\delta}}) \cdot \sum_{(S'_2, \dots, S'_r)} \mathbb{P}(\mathcal{C}(S')).$$

We now move on to bound

$$\sum_{(S'_2, \dots, S'_r)} \mathbb{P}(\mathcal{C}(S')).$$

Suppose $S' = \cup_{i=2}^r S'_i$ is fixed, then there are $(r-1)^{|S'|}$ many ways to distribute its elements into $r-1$ different subsets; hence,

$$\sum_{(S'_2, \dots, S'_r)} \mathbb{P}(\mathcal{C}(S')) \leq O(2^{rs_{\max}}) \sum_{|S'| < rs_{\max}} \mathbb{P}(\mathcal{C}(S')).$$

We note that $\sum_{|S'| < rs_{\max}} \mathbb{P}(\mathcal{C}(S'))$ is simply the expected number of sets of size at most rs_{\max} that are covered by pairwise-disjoint q_1 -APs. This, in turn, may be bounded from above by

$$\sum_{N \leq rs_{\max}} \sum_{s < rs_{\max}} \binom{q_1 N}{s} \mathbb{E}(\# \text{collections of } N \text{ pairwise-disjoint } q_1\text{-APs in } [n]_p).$$

Since the expected number of choices for collections of N pairwise-disjoint q_1 -APs in $[n]_p$ is at most $\frac{(n^2 p^{q_1})^N}{N!}$, we conclude

$$\begin{aligned} \sum_{|S'| < rs_{\max}} \mathbb{P}(\mathcal{C}(S')) &\leq \sum_{N \leq rs_{\max}} \sum_{s < rs_{\max}} \frac{(n^2 p^{q_1})^N}{N!} \binom{q_1 N}{s} \\ &\leq \sum_{N \leq rs_{\max}} \frac{(n^2 p^{q_1})^N}{N!} 2^{q_1 N} \\ &\leq \sum_{N \leq rs_{\max}} \left(\frac{e 2^{q_1} n^2 p^{q_1}}{N} \right)^N. \end{aligned}$$

Note, that $N \leq r s_{\max} \leq D'_2 n^{1-\frac{1}{q_2-1}}$ for some $D'_2 > 0$ which does not depend on C . Since $x \rightarrow \left(\frac{ea}{x}\right)^x$ grows for $x \leq a$ and $n^2 p_1^q = C^{q_1} n^{1-\frac{1}{q_2-1}}$, for C large enough we have

$$\left(\frac{e2^{q_1} n^2 p^{q_1}}{N}\right)^N \leq \left(\frac{(e2C)^{q_1}}{D'_2}\right)^{D'_2 n^{1-\frac{1}{q_2-1}}} \leq e^{D'_2 q_1 \log(2eC) n^{1-\frac{1}{q_2-1}}}.$$

Hence,

$$\sum_{N < r s_{\max}} \left(\frac{e2^{q_1} n^2 p^{q_1}}{N}\right)^N \leq \sum_{N < r s_{\max}} e^{D'_2 q_1 \log(2eC) n^{1-\frac{1}{q_2-1}}} \leq e^{O(\log C n^{1-\frac{1}{q_2-1}})}.$$

Finally we obtain,

$$\begin{aligned} \sum_{(S_2, \dots, S_r)} \mathbb{P}(S \text{ is well-covered}) &\leq O(e^{O(n^{1-\frac{1}{q_2-1}} \log C + \log n \cdot n^{1-\frac{1}{q_2-1}-\delta})}) \\ &\leq e^{D_2 \log C n^{1-\frac{1}{q_2-1}}} \end{aligned}$$

for some constant $D_2 > 0$ which does not depend on C .

Therefore, for C large enough,

$$\begin{aligned} P &\leq o(1) + \mathbb{P}(S \text{ is well-covered}) \cdot e^{-D_1 n^2 p^{q_1}} \\ &= o(1) + O(e^{D_2 \log C n^{1-\frac{1}{q_2-1}}}) \cdot O(e^{-D_1 C^{q_1} n^{1-\frac{1}{q_2-1}}}) \rightarrow 0 \quad \square \end{aligned}$$

6 The asymmetric case 0-statement

In this section, we show that for any integers $q_1 > q_2$ there exists a sufficiently small positive $c > 0$ such that if $p = c \cdot n^{-\frac{q_2}{q_1(q_2-1)}}$ the elements of $[n]_p$ can a.a.s. be colored red/blue without a monochromatic q_1 -AP colored red or a monochromatic q_2 -AP colored blue. We note that this is sufficient for the 0-statement for any other number of colors. We begin by making several definitions that will assist us in stating the results of this section in the language of hypergraphs.

Definition 24. Throughout this section we will deal with hypergraphs with edges of two possible cardinalities q_1 and q_2 , we call such hypergraphs (q_1, q_2) -uniform. We will refer to edges as *long* or *short* edges, depending on their cardinalities. We say that a (q_1, q_2) -uniform hypergraph is *asymmetrically-2-colorable* if its vertices can be colored red/blue with no long edge colored red, and no short edge colored blue.

Definition 25. Let $H(n, q_1, q_2)$ be the hypergraph with vertex set $V(H) = [n]$, whose edge set is the set of arithmetic progressions of lengths q_1 and q_2 . We denote by $H(n, q_1, q_2, p)$ the random subhypergraph of $H(n, q_1, q_2)$ induced by $[n]_p$.

With these definitions in hand, we are ready to state this section's main result:

Theorem 26 (Asymmetric 0-statement). *For any integers $q_1 > q_2 \geq 3$, there exists $c > 0$ such that for $p \leq c \cdot n^{-\frac{q_2}{q_1(q_2-1)}}$, $H(n, q_1, q_2, p)$ is asymmetrically-2-colorable a.a.s.*

The proof we present here is similar in nature to the proof by Rödl and Ruciński [20] of the (symmetric) random Rado partition theorem and consists of two main lemmas. First, in Lemma 33, we show that a (q_1, q_2) -uniform hypergraph is asymmetrically-2-colorable unless it contains a member of a small family of hypergraphs which we call 2-blocking. Then, Lemma 34 states that those hypergraphs a.a.s. do not appear in $H(n, q_1, q_2, p)$. We begin by making several definitions.

Definition 27. We say that an edge $E = \{a_1, \dots, a_q\}$ has a *cover* if there are edges E_1, \dots, E_q such that $E \cap E_i = \{a_i\}$ and $|E_i| \neq |E|$ for all $i \in [q]$. We say that a cover for an edge is *simple* if $E_i \cap (\bigcup_{j \neq i} E_j) = \emptyset$ for all $i \in [q]$. If every edge in a hypergraph H has a cover, we say H is *covered*.

Definition 28. A *simple path* of length ℓ is a hypergraph consisting of short edges E_1, \dots, E_ℓ , and covering long edges $E_{1,1}, \dots, E_{1,q_2}, E_{2,1}, \dots, E_{\ell,q_2}$ such that $E_{i,1}, \dots, E_{i,q_2}$ cover E_i and such that:

1. $E_{i+1,1} = E_{i,q_2}$ for every $i < \ell$,
2. no two edges of the same cardinality intersect.

A simple path of length one is called a *block*; thus, a simple path consists of blocks, such that every pair of consecutive blocks share a long edge. For convenience, we refer to a single long edge as a simple path of length zero.

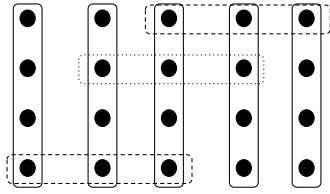
Definition 29. We say that a simple path P of length ℓ has a *saw* if for every $v_i \in E_{1,1} \setminus E_1$ there exists a short edge S_i such that $S_i \cap E_{1,1} = \{v_i\}$ and $|S_i \cap V(P)| = 2$. We call the edges S_i the *saw edges* for P .

Definition 30. We say that a simple path P of length ℓ is *spoiled* if there exists an edge $E \notin E(P)$ such that $|E \cap V(P)| \geq 3$ and $E \cap E_{\ell,q_2} = \{v\}$ for some $v \notin E_\ell$.

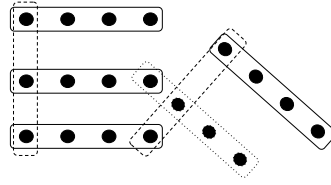
Definition 31. We say that a simple path P of length ℓ has a *spoiled extension*, if there exists a short edge $E_{\ell+1}$ along with a simple cover $E_{\ell+1,1}, \dots, E_{\ell+1,q_2}$, such that $E_{\ell+1} \cap V(P) = \{v\}$ for some $v \in E_{\ell,q_2} \setminus E_\ell$, $E_{\ell+1,1} = E_{\ell,q_2}$ and there exists $i \in \{2, \dots, q_2\}$ such that $E_{\ell+1,i} \cap V(P) \neq \emptyset$.

Definition 32. We say that a (q_1, q_2) -uniform hypergraph is *2-blocking* if it is one of the following:

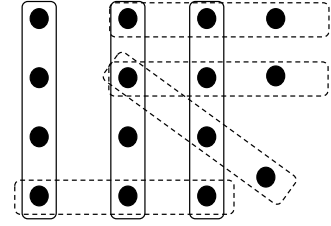
1. A short edge with a non-simple cover.
2. A spoiled simple path.
3. A simple path with a saw.



(a) A spoiled simple path of length 2, with the spoiling edge dotted.



(b) A block with a spoiled extension, with the spoiling long edge dotted.



(c) A path of length one with a saw.

Figure 2: Examples of 2-blocking hypergraphs for $q_1 = 4, q_2 = 3$.

4. A simple path with a spoiled extension.

Lemma 33 (The deterministic lemma). *Let $q_1 > q_2 \geq 3$ be some integers, and let H be a (q_1, q_2) -uniform hypergraph which is not asymmetrically-2-colorable. Then H contains a 2-blocking hypergraph.*

Lemma 34 (The probabilistic lemma). *Let $H = H(n, q_1, q_2, p)$, let c be a sufficiently small positive constant, and let $p = c \cdot n^{-\frac{q_2}{q_1(q_2-1)}}$. Then a.a.s. H contains no 2-blocking hypergraph.*

Theorem 26 follows immediately from these two lemmas. Most of this section deals with proving the probabilistic lemma, using a first-moment argument over several random variables. But first, we begin by proving the deterministic part of the theorem.

Proof of The deterministic lemma. We say that a hypergraph is edge-critical if it is not asymmetrically-2-colorable, but any proper subhypergraph is. We may assume that H is edge-critical; otherwise, we replace it with an edge-critical subhypergraph. We begin by showing that every edge critical hypergraph is covered.

Claim 35. *If H is an edge-critical hypergraph, then for every edge $E \in H$ and for every vertex $v \in E$ there is an edge $E' \in H$ such that $E \cap E' = \{v\}$ and $|E| \neq |E'|$; in other words, H is covered.*

Proof. Let H be edge-critical, and suppose that there are an edge $E \in H$ and a vertex $v \in E$, such that every edge E' of the other cardinality that contains v also contains another vertex of E . By the edge-criticality, H can be colored red/blue in such a way that only E violates the coloring condition. Without loss of generality, assume E is long (and colored red). Now, by changing the color of v to blue, E no longer violates the coloring condition, and neither does any short edge that contains v , contradicting the fact that H is not asymmetrically-2-colorable. \square

If there exists a short edge with a non-simple cover, then we are done, so we may assume all short edges have simple covers. Let P be a longest simple path, and let $\ell \geq 1$

be its length. We observe $E = E_{\ell, q_2}$; since E is covered by short edges, we have short edges S_i such that $S_i \cap E = \{v_i\}$ for every $v_i \in \{v_2, \dots, v_{q_1}\} = E \setminus E_\ell$.

Suppose first that there exists S_i such that $S_i \cap V(P) = \{v_i\}$ and observe $\{s_2, \dots, s_{q_2}\} = S_i \setminus E$. By the previous claim, we have a long covering edge for every s_i , and by the maximality of P we obtain that for every simple cover for S_i there exists a covering edge L that intersects P . Thus, we obtain that S_i forms a spoiled extension to P .

If, on the other hand, no S_i intersects P in exactly a single vertex, we obtain one of two cases: If there exists some i such that $|S_i \cap V(P)| \geq 3$, then since $|S_i \cap E| = 1$ we obtain that S_i is a spoiling edge for P . Otherwise, we have $|S_i \cap V(P)| = 2$ for all i and thus obtain a path with a saw, completing the proof. \square

The rest of this section deals with proving the probabilistic portion of Theorem 26. We will begin by proving upper bounds on the number of copies of several hypergraphs in $H(n, q_1, q_2)$ and showing that a.a.s. all short edges have simple covers. We will then prove an upper bound on the number of simple paths of arbitrary lengths. Finally, with the above results in hand, we will turn to prove Lemma 34.

Lemma 36. *Let $x, y \in [n]$ be distinct integers. Then, the number of choices for $a, b \in [n]$ such that there are q -APs that contain $\{x, a, b\}$ and $\{y, a, b\}$ is $O(1)$.*

Proof. Suppose that a, b are contained in a q -AP along with x , then there exists $t_1 \in \mathbb{Q}$ such that $x - a = t_1(b - a)$, thus, $x = (1 - t_1)a + t_1b$. Moreover, $t_1 = \frac{r_1}{r_2}$ for $r_1, r_2 \in \{-q, -q + 1, \dots, q\}$. The same also holds for y with another constant t_2 . We obtain the following system of linear equations:

$$\begin{pmatrix} 1 - t_1 & t_1 \\ 1 - t_2 & t_2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}. \quad (\star)$$

Since the determinant of the above matrix is $t_2 - t_1$, we obtain that as long as $t_1 \neq t_2$ there is only a single solution to (\star) . Since $x \neq y$ implies $t_1 \neq t_2$ and there are at most $O(q^2) = O(1)$ choices for t_1 and t_2 , there are only $O(1)$ many choices for a, b . \square

In Lemma 39 we show that a short edge with its q_2 covering long edges must contain almost $2q_1$ vertices. We will then use this lemma to show that non-simple covers are unlikely. We will require the following two elementary lemmas.

Lemma 37. *Let E_1, E_2 be two arithmetic progressions of length q_1 , with common differences d_1 and d_2 . If $d_1 < d_2$, then $|E_1 \cap E_2| \leq \lceil q_1 \cdot \frac{\gcd(d_1, d_2)}{d_2} \rceil$.*

Proof. Let A_1 and A_2 be the infinite arithmetic progressions containing E_1 and E_2 respectively. We obtain that $A = A_1 \cap A_2$ is either empty or an infinite arithmetic progression with common difference $\text{lcm}(d_1, d_2)$, thus $A \cap A_1$ contains every $\frac{\text{lcm}(d_1, d_2)}{d_1}$ -th element of A_1 . Therefore, a subsequence of length q_1 in A_1 contains at most $\lceil q_1 \cdot \frac{d_1}{\text{lcm}(d_1, d_2)} \rceil$ elements of A , and the result follows immediately. \square

Lemma 38. Let $n > m > 0$ be integers, and $A = (a_1, a_2, \dots, a_q)$ be an arithmetic progression of length $q > 3$ with common difference m . If we denote $t = \frac{n}{\gcd(n,m)}$, then $|\{a \in [n] : \exists i \in [q] \text{ such that } a \equiv a_i \pmod{n}\}| = \min\{t, q\}$. Moreover, $a_i \equiv a_{i+kt} \pmod{n}$ for all integers i and k such that $i, i + kt \in [q]$.

Proof. Let $G = \mathbb{Z}/n\mathbb{Z}$ be the additive cyclic group of order n . From elementary group theory we know that the order of m in G is $t = \frac{n}{\gcd(n,m)}$. Let G' be the cyclic subgroup generated by m . Then, the residues of A modulo n are contained in the coset $a_1 + G'$, which has $|a_1 + G'| = |G'| = t$, completing the proof. \square

Lemma 39. If E is a short edge and E_1, \dots, E_r are a subset of its covering edges, then

$$|E_1 \cup \dots \cup E_r| > 2q_1 \left(1 - \frac{1}{r}\right).$$

Proof. Denote $M = |E_1 \cup \dots \cup E_r|$ and assume for contradiction that $M \leq 2q_1(1 - \frac{1}{r})$. The case $r \in \{1, 2\}$ is trivial, therefore we may assume $r \geq 3$. We first show that no three covering edges share the same common difference.

Claim 40. Let E_1, E_2, E_3 be covering edges, and assume that all three q_1 -APs have the same common difference, i.e. $E_z = \{a_z + i \cdot d : i \in [q_1]\}$. Then,

$$|E_1 \cup E_2 \cup E_3| \geq 2q_1.$$

Proof. Suppose $\{v_z\} = E_z \cap E$ for $z \in \{1, 2, 3\}$. Without loss of generality assume that $v_1 < v_2 < v_3$, and that $E_z \cap E_2 \neq \emptyset$ for $z = 1, 3$, as otherwise $|E_z \cup E_2| = 2q_1$. Since each covering edge may only contain one vertex of E , and E_1 lies on the same infinite arithmetic progression of difference d as E_2 , we deduce that all the elements of E_1 must be strictly smaller than v_2 . Similarly, all elements of E_3 must be strictly larger than v_2 . Hence, $E_1 \cap E_3 = \emptyset$; and thus, $|E_1 \cup E_3| = 2q_1$. \square

We will now show that $r \geq 5$. Since $r \geq 3$, the previous claim implies that there exists a pair of edges with different common differences; without loss of generality we denote them E_1, E_2 . We note that $|E_1 \cap E_2| \leq \lfloor \frac{q_1}{2} \rfloor$, by Lemma 37. Let $E' = E \cap (E_1 \cup \dots \cup E_r)$. Since E_1, E_2 are covering edges of E , we have

$$\begin{aligned} |E' \cup E_1 \cup E_2| &= |E_1 \cup E_2| + |E'| - 2 = |E_1| + |E_2| - |E_1 \cap E_2| + |E'| - 2 \\ &\geq q_1 + \left\lfloor \frac{q_1}{2} \right\rfloor + r - 2. \end{aligned}$$

Thus, if $r \in \{3, 4\}$ we observe that (since $\lfloor \frac{q_1}{2} \rfloor \geq \frac{q_1}{2} - \frac{1}{2}$)

$$q_1 + \left\lfloor \frac{q_1}{2} \right\rfloor + r - 2 \geq \frac{3}{2}q_1 + r - \frac{5}{2} \geq \left(1 - \frac{1}{r}\right) 2q_1 + r - \frac{5}{2} > 2q_1 \left(1 - \frac{1}{r}\right)$$

for all $q_1 > r$. Finally, we will show that no three arithmetic progressions may have pairwise different common differences.

Claim 41. Let E_1, E_2, E_3 be covering edges, with distinct common differences d_1, d_2, d_3 respectively. Then

$$|E_1 \cup E_2 \cup E_3| \geq 2q_1 - 4,$$

and thus $M \geq |E_1 \cup E_2 \cup E_3| + (r - 3) \geq 2q_1 - 2 > 2q_1(1 - \frac{1}{r})$.

Proof. Assume for contradiction that $|E_1 \cup E_2 \cup E_3| < 2q_1 - 4$. Since

$$|E_1 \cup E_2 \cup E_3| \geq 3q_1 - |E_1 \cap E_2| - |E_1 \cap E_3| - |E_2 \cap E_3|,$$

at least one pair of edges intersects in more than $\frac{q_1}{3} + 1$ elements.

Recall that $q_1 > q_2 \geq r \geq 5$. Note that $\frac{q_1}{3} + 1 > \lceil \frac{q_1}{3} \rceil \geq \lceil \frac{q_1}{d} \rceil$ for all $d \geq 3$ and $q_1 \geq 6$. Thus, by Lemma 37, we have that for some $i, j \in [3]$ such that $d_i > d_j$ we have $\gcd(d_i, d_j) = \frac{d_i}{2}$ which can only occur if $d_i = 2d_j$. Let $k \notin \{i, j\}$. Since

$$|E_i \cup E_j \cup E_k| \geq 3q_1 - \left\lceil \frac{q_1}{2} \right\rceil - |E_i \cap E_k| - |E_j \cap E_k| \geq 3q_1 - \frac{q_1}{2} - 1 - |E_i \cap E_k| - |E_j \cap E_k|$$

we obtain that E_k must intersect one of the other edges in more than $\frac{q_1}{4} + 1$ elements. Note that $\frac{q_1}{4} + 1 > \lceil \frac{q_1}{4} \rceil \geq \lceil \frac{q_1}{d} \rceil$ for all $d \geq 4$ and $q_1 \geq 6$. Again, by Lemma 37, we have that $t \cdot \gcd(d_k, d_z) = \max(d_k, d_z)$ for some $z \in \{i, j\}$ and $t \in \{2, 3\}$.

If we set $d_j = d$, we get that $\frac{d_k}{d}$ may obtain one of six values: either $\frac{1}{2}$ or 4 for $t = 2$ or one of $\frac{1}{3}, \frac{2}{3}, 3, 6$ for $t = 3$. We note that by Lemma 38, if we observe the residues modulo d' of an arithmetic progression of length q_1 with common difference $d < d'$, we obtain that the progression cycles through $\frac{d'}{\gcd(d, d')}$ residues; and thus, it contains at least $\left\lfloor \frac{q_1 \cdot \gcd(d, d')}{d'} \right\rfloor$ elements of every residue class it encounters. By adjusting the constant d and permuting the indices we obtain that the triplet (d_1, d_2, d_3) must fall into one of five categories:

1. Assuming $(d_1, d_2, d_3) = (d, 2d, 4d)$, without loss of generality we assume $0 \in E_3$ and $d = 1$. Then, E_3 consists of q_1 elements which satisfy $a \equiv 0 \pmod{4}$. If E_2 contains elements which satisfy $a \equiv 1, 3 \pmod{4}$, we obtain that $|E_3 \cup E_2| = 2q_1$. Otherwise, we obtain that at least $\lfloor \frac{q_1}{2} \rfloor$ of the elements in E_2 satisfy $a \equiv 2 \pmod{4}$, and at least $2 \cdot \lfloor \frac{q_1}{4} \rfloor$ of the elements in E_1 satisfy $a \equiv 1, 3 \pmod{4}$. Thus,

$$|E_1 \cup E_2 \cup E_3| \geq q_1 + \lfloor \frac{q_1}{2} \rfloor + 2 \cdot \lfloor \frac{q_1}{4} \rfloor \geq 2q_1 - 2.$$

2. Assuming $(d_1, d_2, d_3) = (d, 2d, 3d)$, without loss of generality we assume $0 \in E_3$ and $d = 1$. Then, E_3 consists of q_1 elements which satisfy $a \equiv 0, 3 \pmod{6}$. Since at least $2 \lfloor \frac{q_1}{3} \rfloor$ of the elements in E_2 satisfy $a \equiv 2, 4 \pmod{6}$ (or equivalently $1, 5 \pmod{6}$), and at least $2 \lfloor \frac{q_1}{6} \rfloor$ of the elements in E_1 satisfy $a \equiv 1, 5 \pmod{6}$ (equivalently $2, 4 \pmod{6}$), we obtain

$$|E_1 \cup E_2 \cup E_3| \geq q_1 + 2 \lfloor \frac{q_1}{3} \rfloor + 2 \lfloor \frac{q_1}{6} \rfloor \geq 2q_1 - 3.$$

3. Assuming $(d_1, d_2, d_3) = (d, 2d, 6d)$, without loss of generality we assume $0 \in E_3$ and $d = 1$. Then, E_3 consists of q_1 elements which satisfy $a \equiv 0 \pmod{6}$. If E_2 contains elements which satisfy $a \equiv 1, 3, 5 \pmod{6}$ we obtain that $|E_3 \cup E_2| = 2q_1$. Otherwise, at least $2\lfloor \frac{q_1}{3} \rfloor$ of the elements in E_2 satisfy $a \equiv 2, 4 \pmod{6}$, and at least $3\lfloor \frac{q_1}{6} \rfloor$ of the elements in E_1 satisfy $a \equiv 1, 3, 5 \pmod{6}$. Hence,

$$|E_1 \cup E_2 \cup E_3| \geq q_1 + 2\lfloor \frac{q_1}{3} \rfloor + 3\lfloor \frac{q_1}{6} \rfloor \geq 2q_1 - 4.$$

4. Assuming $(d_1, d_2, d_3) = (d, 3d, 6d)$, without loss of generality we assume $0 \in E_3$ and $d = 1$. Then, E_3 consists of q_1 elements which satisfy $a \equiv 0 \pmod{6}$. If E_2 contains elements which satisfy $a \equiv 1, 4 \pmod{6}$ or $a \equiv 2, 5 \pmod{6}$ we obtain that $|E_3 \cup E_2| = 2q_1$. Otherwise, at least $\lfloor \frac{q_1}{2} \rfloor$ of the elements in E_2 satisfy $a \equiv 3 \pmod{6}$, and at least $4\lfloor \frac{q_1}{6} \rfloor$ of the elements in E_1 satisfy $a \equiv 1, 2, 4, 5 \pmod{6}$. Therefore,

$$|E_1 \cup E_2 \cup E_3| \geq q_1 + 4\lfloor \frac{q_1}{6} \rfloor + \lfloor \frac{q_1}{2} \rfloor \geq 2q_1 - 4.$$

5. Assuming $(d_1, d_2, d_3) = (2d, 3d, 6d)$, without loss of generality we assume $0 \in E_3$ and $d = 1$. Then, E_3 consists of q_1 elements which satisfy $a \equiv 0 \pmod{6}$. If E_2 contains elements which satisfy $a \equiv 1, 4 \pmod{6}$ or $a \equiv 2, 5 \pmod{6}$ we obtain that $|E_3 \cup E_2| = 2q_1$. If E_1 contains elements which satisfy $a \equiv 1, 3, 5 \pmod{6}$ we obtain that $|E_3 \cup E_1| = 2q_1$. Otherwise, at least $\lfloor \frac{q_1}{2} \rfloor$ of the elements in E_2 satisfy $a \equiv 3 \pmod{6}$, and at least $2\lfloor \frac{q_1}{3} \rfloor$ of the elements in E_1 satisfy $a \equiv 2, 4 \pmod{6}$. Thus,

$$|E_1 \cup E_2 \cup E_3| \geq q_1 + 2\lfloor \frac{q_1}{3} \rfloor + \lfloor \frac{q_1}{2} \rfloor \geq 2q_1 - 2. \quad \square$$

Finally we obtain that $r \geq 5$, no three edges may share a common difference, and there can be no more than two possible common differences which yields a contradiction. \square

Lemma 42. *Let H' be a hypergraph consisting of a short edge and its non-simple cover. If $p = O(n^{-\frac{q_2}{q_1(q_2-1)}})$, then $\mathbb{P}(H' \subset H(n, q_1, q_2, p)) = o(1)$.*

Proof. The following computations are used several times throughout the proof:

$$p^{q_1-1} = O(n^{-\frac{q_2}{q_2-1} + \frac{q_2}{q_1(q_2-1)}}) = O(n^{-1 - \frac{1}{q_2-1} + \frac{q_2}{q_1(q_2-1)}}) = O(n^{-1 - \frac{q_1 - q_2}{q_1(q_2-1)}}),$$

and

$$p^{q_1} = O(n^{-1 - \frac{1}{q_2-1}}).$$

Let E be a short edge and let E_1, \dots, E_{q_2} be its covering edges. We divide the covering edges into three categories: we say that a covering edge E_i is simple if $|E_i \cap (\bigcup_{j \neq i} E_j)| = 0$, we say a covering edge E_i is generic if $|E_i \cap (\bigcup_{j \neq i} E_j)| = 1$, otherwise we say this

edge is degenerate. Let $k_1 = |\{i : E_i \text{ is simple}\}|$, let $k_2 = |\{i : E_i \text{ is generic}\}|$ and let $k_3 = q_2 - k_1 - k_2$ be the number of degenerate edges. We are interested in counting configurations where $k_1 < q_2$. We will first show that non-simple covers with $k_3 = 0$ are unlikely to occur in $H(n, q_1, q_2, p)$.

We say that a vertex is generic if it belongs to more than one generic edge and denote $m = |\{v \in V(H') : v \text{ is generic}\}|$. Since every generic vertex belongs to at least two generic edges and every generic edge contains at most one generic vertex we obtain $1 \leq m \leq \frac{k_2}{2}$. There are $O(n^2)$ many possible choices for the short edge and $O(n^{k_1})$ many choices for the simple edges. Since $k_3 = 0$ every generic edge must contain a generic vertex, and thus, fixing all generic vertices determines the generic edges up to $O(1)$ many choices; thus, we have $O(n^m)$ many choices for the generic edges.

We require q_2 vertices for the short edge, $q_1 - 1$ vertices for every simple edge and $k_2(q_1 - 1) - (k_2 - m)$ vertices for the generic edges. Hence, the expected number of such configurations is

$$\begin{aligned} O(n^{2+k_1+m} p^{q_2+(k_1+k_2)(q_1-1)-(k_2-m)}) &= O(n^{2+k_1+m} p^{q_2 q_1 - (k_2-m)}) \\ &= O(n^{2+k_1+m - (1 + \frac{1}{q_2-1})q_2 + \frac{(k_2-m)q_2}{q_1(q_2-1)}}) = O(n^{2-(k_2-m) - \frac{q_1 q_2 - (k_2-m)q_2}{q_1(q_2-1)}}). \end{aligned}$$

If $k_2 - m \geq 2$, as $k_2 - m < q_1$ this is clearly $o(1)$. If $k_2 - m \leq 1$ then since $k_2 \geq 2$ we must have $k_2 = 2$ and $m = 1$ and therefore

$$O(n^{2-(k_2-m) - \frac{q_1 q_2 - (k_2-m)q_2}{q_1(q_2-1)}}) = O(n^{1 - \frac{q_2(q_1-1)}{q_1(q_2-1)}}) = O(n^{1-1 - \frac{q_1-q_2}{q_1(q_2-1)}}) = o(1).$$

Therefore, we may assume that $k_3 > 0$. Assume first that there exists a degenerate edge E_1 along with two other covering edges E_2, E_3 such that $|E_1 \cap E_i| = 1$ for $i \in \{2, 3\}$ and $|E_2 \cap E_3| < 2$. Then fixing E_1 determines all three edges up to $O(1)$ many choices. Thus, we have at most $O(n)$ many choices for those three edges. Since we require q_2 vertices for the short edge and $|E_1 \cup E_2 \cup E_3| \geq 3q_1 - |E_1 \cap E_2| - |E_1 \cap E_3| - |E_2 \cap E_3| > 3q_1 - 3$, the expected number of such configurations is

$$O(n^3 p^{3q_1-6+q_2}) = O(n^3 p^{3(q_1-1)+q_2-3}) = O(n^{-3 \cdot \frac{q_1-q_2}{q_1(q_2-1)} p^{q_2-3}}) = o(1).$$

We thus deduce that every degenerate edge must either intersect some other covering edge in at least two vertices or intersect two other degenerate edges that intersect each other in at least two vertices. Therefore, $k_3 \geq 2$, and by Lemma 36 we have at most $O(1)$ many choices for the degenerate edges.

We now divide the generic edges into two categories: We say a generic edge is weakly-generic if it is a generic edge that lies in the same connected component of $H' \setminus \{E\}$ as a degenerate edge; otherwise, we say it is strongly-generic. We denote $k'_2 = |\{i : E_i \text{ is strongly-generic}\}|$. We say that a vertex is strongly-generic if it belongs to more than one strongly-generic edge, and denote $m' = |\{v \in V(H') : v \text{ is strongly-generic}\}|$. Since every strongly-generic vertex belongs to at least two strongly-generic edges and every strongly-generic edge contains at most one strongly-generic vertex we obtain $0 \leq m' \leq \frac{k'_2}{2} \leq \frac{k_2}{2}$. Again, we note that fixing the strongly-generic vertices determines the

strongly-generic edges up to $O(1)$ many choices. We also note that fixing E determines the degenerate and weakly-generic edges up to $O(1)$ many choices. Since, by Lemma 39, we require at least $2q_1(1 - \frac{1}{k_3})$ vertices for the degenerate edges, at least $k_2(q_1 - 1)$ additional vertices for the generic edges and k_1q_1 vertices for the simple edges, we get that the expected number of such configurations is (since $k_1 + k_2 + k_3 = q_2$, $k_3 \geq 2$ and $m' \leq \frac{k_2}{2}$)

$$\begin{aligned} & O(n^{2+k_1+m'} p^{k_1q_1+k_2(q_1-1)+2q_1(1-\frac{1}{k_3})}) \\ &= O(n^{2+k_1+m'-k_1(1+\frac{1}{q_2-1})-k_2(1+\frac{q_1-q_2}{q_1(q_2-1)})} p^{2q_1(1-\frac{1}{k_3})}) \\ &= O(n^{2-k_1\frac{1}{q_2-1}+m'-k_2-k_2\frac{q_1-q_2}{q_1(q_2-1)}} p^{2q_1(1-\frac{1}{k_3})}), \end{aligned}$$

since $m' - k_2 \leq -\frac{k_2}{2}$ and $p^{2q_1(1-\frac{1}{k_3})} = O(n^{-2-\frac{2}{q_2-1}+\frac{2q_2}{k_3(q_2-1)}})$,

$$\begin{aligned} &= O(n^{2-\frac{k_1}{q_2-1}-\frac{k_2}{2}-k_2\frac{q_1-q_2}{q_1(q_2-1)}-2-\frac{2}{q_2-1}+\frac{2q_2}{k_3(q_2-1)}}) \\ &= O(n^{-k_2\frac{q_1-q_2}{q_1(q_2-1)}-\frac{(2+k_1+\frac{k_2}{2}(q_2-1))k_3-2q_2}{k_3(q_2-1)}}) \\ &= O(n^{-k_2\frac{q_1-q_2}{q_1(q_2-1)}-\frac{2k_3+k_1k_3+k_2k_3-2q_2}{k_3(q_2-1)}}) = o(1). \end{aligned}$$

Finally, we conclude by Markov's inequality,

$$\mathbb{P}[\#\text{short edges with non-simple covers} > 0] = \sum_{k_1 < q_2} \sum_{k_2 < q_2 - k_1} o(1) = o(1). \quad \square$$

Lemma 43. *Let $H \subset H(n, q_1, q_2)$ be a simple path of length $\ell = O(\log n)$. Define $Y = |\{T \subset H(n, q_1, q_2) : H \cup T \text{ is a simple path of length } \ell + 1\}|$. Then for every $c_r > 0$, there exists $c > 0$ such that for $p \leq c \cdot n^{-\frac{q_2}{q_1(q_2-1)}}$ we have*

$$\mathbb{E}[Y \mid V(H) \subset [n]_p] < c_r.$$

Proof. Without loss of generality we may assume that the short edge in T intersects H only at E_{ℓ, q_2} . We denote $T = \{E = E_{\ell+1}, E_{\ell+1, 2}, \dots, E_{\ell+1, q_2}\}$. Since E intersects H in a single vertex we have $O(n)$ many choices for it. By fixing E we limit the number of choices for each covering edge to $O(n)$, giving us a total of $O(n^{q_2})$ many choices for T in $H(n, q_1, q_2)$.

Since T requires $q_2 - 1 + (q_2 - 1)(q_1 - 1) = (q_2 - 1)q_1$ new vertices, we obtain

$$\mathbb{E}[Y \mid V(H) \subset [n]_p] = O(n^{q_2} p^{(q_2-1)q_1}) = O(n^{q_2-q_2} c^{(q_2-1)q_1}) = O(c^{(q_2-1)q_1}). \quad \square$$

Corollary 44. *Let $\ell = O(\log n)$, and let Y be the random variable counting simple paths of length ℓ in $H(n, q_1, q_2, p)$. Then,*

$$\mathbb{E}Y = O(c_r^\ell n^{1-\frac{1}{q_2-1}}).$$

Proof. We may construct a simple path of length ℓ by first choosing a single long edge, and then choosing ℓ extensions.

Since two integers belong to only a constant number of q_1 -term arithmetic progressions, we obtain that the number of choices for the initial long edge is $O(n^2)$. Since we require q_1 vertices for the long edge, we obtain that the expected number of long edges is $O(n^2 p^{q_1}) = O(n^{1-\frac{1}{q_2-1}})$.

By the previous lemma, we may extend the path step by step, each time adding a factor of c_r to the expectation, and the corollary immediately follows. \square

We are now ready to prove the probabilistic lemma, thus completing the proof of Theorem 26.

Proof of The probabilistic lemma. Let $H = H(n, q_1, q_2, p)$, let B be a sufficiently large constant and set $\ell' = B \log n$. The proof of this lemma can be summarized as follows:

First, we recall that, by Lemma 42, a.a.s no non-simply covered short edges exist, and show that all simple paths terminate at lengths smaller than ℓ' , using Corollary 44. We then apply a first-moment argument to several random variables, showing that every small 2-blocking hypergraph contains some sub-hypergraph with $o(1)$ many expected copies in H ; thus, the probability that the 2-blocking hypergraph appears in our random hypergraph is $o(1)$.

Specifically, we will show that the existence of a 2-blocking hypergraph implies either the existence of additional vertices such that the expected number of choices for them is $o(n^{-1+\frac{1}{q_2-1}})$, or the existence of a simple path that obeys an additional constraint that causes us to lose a degree of freedom in the path construction. Since, by Corollary 44, we have only an expected $O(n^{1-\frac{1}{q_2-1}})$ many choices for a simple path, we obtain that replacing a factor of n with a factor of $O(\log^k n)$ causes the expectation to tend to zero as n grows.

As seen in the proof for Corollary 44, we may construct a path by selecting a long edge and then iteratively extending the path; therefore, in some cases, we refer to a single long edge as a simple path of length zero, allowing us to treat a single block as an extension to an existing path.

We will also sometimes assume that the edges of a simple path are ordered $E_{1,1}, E_{1,2}, \dots, E_{1,q_2}, E_{2,1}, \dots, E_{\ell,q_2}$. In such an ordering, if there are no other constraints on the path, we have $O(n^2)$ many choices for the first edge and $O(n)$ many choices for every other edge. For convenience, we say a vertex $v \in V(P)$ precedes another vertex $u \in V(P)$ if v belongs to an edge that precedes all edges that contain u .

Let U be the random variable counting short edges with non-simple covers. By Lemma 42 we have

$$\mathbb{P}(U > 0) = o(1).$$

Let W be the random variable counting simple paths of length ℓ' . By Corollary 44 and Markov's inequality we obtain

$$\mathbb{P}(W > 0) \leq \mathbb{E}W = O(n^{1-\frac{1}{q_2-1}} c_r^{B \log n}) = o(1).$$

Let X be the random variable counting spoiled simple paths of length $\ell < \ell'$. Let P be a simple path and let E be a spoiling edge for it. By Corollary 44, there are at most $O(n^{1-\frac{1}{q_2-1}})$ many choices for the path up to the final block. For the final block we have $O(n)$ many choices for the short edge, and for all the long edges except E_{ℓ, q_2} . Since $|E \cap (V(P) \setminus E_{\ell, q_2})| \geq 2$ we have at most $O(\log^2 n)$ many choices for E , and therefore $O(\log^2 n)$ many choices for E_{ℓ, q_2} . Since we require $q_2 - 1$ vertices for the short edge, along with $(q_2 - 1)(q_1 - 1)$ vertices for the long edges we conclude that

$$\mathbb{E}X = \sum_{\ell < \ell'} O(n^{1-\frac{1}{q_2-1}} n^{q_2-1} p^{q_1(q_2-1)} \log^2 n) = O(n^{q_2-q_2-\frac{1}{q_2-1}} \log^2 n) = o(1),$$

and therefore by Markov's inequality,

$$\mathbb{P}(X > 0) = o(1).$$

Let Y be the random variable counting simple paths of length $\ell < \ell'$ with a saw. We denote the path by P . Since the saw edges are not entirely contained in P , each one must contain a vertex $v \notin V(P)$. Let S_2, \dots, S_{q_1} be the saw edges. For each $k \in \{2, \dots, q_1\}$, let $\{s_k\} = (V(P) \cap S_k) \setminus E_{1,1}$ (there is only one such vertex, since $|S_k \cap V(P)| = 2$), and let S'_k be the first edge in P that contains s_k . We split into several cases:

Case 1. There exist two edges S_i, S_j such that $|S_i \cap S_j| \geq 2$.

By Lemma 36, we obtain that fixing $E_{1,1}$ determines S_i and S_j up to $O(1)$ many choices. Thus, if we iteratively extend a path from $E_{1,1}$ we obtain that when we select S'_i it must intersect one (or both) of S_i, S_j , giving us only $O(1)$ many choices for it, and thus the expected number of such paths with S_i and S_j in H is $O(n^{-\frac{1}{q_2-1}} \log n) = o(1)$. Therefore, the probability that such saw edges exist is $o(1)$ and we may assume all saw edges S_i and S_j intersect each other in at most one vertex.

Case 2a. There exist i, j such that $(S_i \cap S_j) \setminus V(P) \neq \emptyset$ and $S'_i \neq S'_j$.

Without loss of generality we assume S'_i follows S'_j in the edge ordering of P . If we fix the subpath up to (but not including) S'_i , then we have at most $O(\log n)$ many choices for S_j since it intersects both $E_{1,1}$ and S'_j which have already been chosen; this implies that there are further $O(1)$ many choices for S_i , as it must intersect both S_j and $E_{1,1}$. Therefore, there are further $O(1)$ many choices for S'_i as it must intersect both S_i and either a short edge or the final long edge of the previous block. We now have $O(n)$ many choices for each remaining edge in the block that contains S'_i .

We obtain that for the final block, we have $O(n)$ many choices for each edge except for S'_i for which we have only $O(\log n)$ many choices. Hence the expected number of such configurations is

$$\sum_{\ell < \ell'} O(n^{1-\frac{1}{q_2-1}} n^{q_2-1} p^{q_1(q_2-1)} \log n) = O(n^{q_2-q_2-\frac{1}{q_2-1}} \log^2 n) = o(1).$$

Case 2b. There exist i, j such that $(S_i \cap S_j) \setminus V(P) \neq \emptyset$ and $S'_i = S'_j = S' \neq E_1$.

By fixing $E_{1,1}$ and $\{v\} = S_i \cap S_j$ we determine S_i and S_j up to $O(1)$ many choices. Since $v \notin V(P)$, we also determine S' up to $O(1)$ many choices as it must intersect $S_i \cup S_j$ in two vertices other than v . If we now extend a path from $E_{1,1}$ towards S' , we obtain that we have only $O(1)$ many choices for the edge connecting the path to S' (whether it is a short edge, or the final long edge in a block). Note that this connecting edge cannot be $E_{1,1}$ by our assumption that $S' \neq E_1$. Thus, the number of choices for both S' and the previous edge is $O(n)$ - the number of choices for v .

If we now add the remaining edges in the final block, we obtain that the expected number of such paths is

$$O(n^{-\frac{1}{q_2-1}} \log n) = o(1).$$

Case 2c. There exist i, j such that $(S_i \cap S_j) \setminus V(P) \neq \emptyset$ and $S'_i = S'_j = S' = E_1$.

From the previous cases we may assume that all saw edges intersect each other in at most one vertex, and any saw edge that intersects another saw edge outside of $V(P)$ must intersect $V(P) \setminus E_{1,1}$ in one vertex that lies on E_1 .

Let $A = \bigcup_{i=2}^{q_1} (S_i \setminus V(P))$ and for each $v \in A$ define $S(v) = \{i : v \in S_i\}$. Suppose that $|S(v)| \geq 2$ for some $v \in A$. Then,

$$\left| \bigcup_{i \in S(v)} (S_i \cap (E_1 \setminus E_{1,1})) \right| = |S(v)|,$$

since no two saw edges share more than one vertex and any saw edge that intersects another has a vertex in $E_1 \setminus E_{1,1}$. Therefore, $|S(v)| \leq |E_1 \setminus E_{1,1}| = q_2 - 1$ for all $v \in A$. Since,

$$(q_2 - 2)(q_1 - 1) = \sum_{i=2}^{q_1} |S_i \setminus V(P)| = \sum_{v \in A} |S(v)| \leq |A|(q_2 - 1),$$

we deduce that we require $|A| \geq \frac{(q_2-2)(q_1-1)}{q_2-1} = q_1 - 1 - \frac{q_1-1}{q_2-1}$ additional vertices for the saw edges.

If we assume P is fixed, we have at most $O(\log n)$ many choices for each saw edge. Thus, the expected number of choices for the additional vertices is

$$\begin{aligned} O(\log^{q_1-1} n \cdot p^{q_1-1-\frac{q_1-1}{q_2-1}}) &= O(\log^{q_1-1} n \cdot n^{-1-\frac{1}{q_2-1}+\frac{q_2}{q_1(q_2-1)}+\frac{q_1 q_2}{q_1(q_2-1)^2}-\frac{q_2}{q_1(q_2-1)^2}}) \\ &= O(\log^{q_1-1} n \cdot n^{-1-\frac{q_1-q_2}{q_1(q_2-1)}+\frac{1}{q_2-1}+\frac{q_1-q_2}{q_1(q_2-1)^2}}) \\ &= O(\log^{q_1-1} n \cdot n^{-1+\frac{1}{q_2-1}-\frac{(q_1-q_2)(q_2-2)}{q_1(q_2-1)^2}}). \end{aligned}$$

Thus by Corollary 44, the expected number of choices for the path and the saw edges is

$$O(\log^{q_1} n \cdot n^{-\frac{(q_1-q_2)(q_2-2)}{q_2(q_2-1)^2}}) = o(1).$$

Case 3. $(S_i \cap S_j) \setminus V(P) = \emptyset$ for all $i \neq j \in \{2, \dots, q_1\}$.

Since each saw edge contains at least one vertex not in $V(P)$, there are at least $q_1 - 1$ additional vertices introduced by the saw edges. We have $O(\log n)$ many choices for each

saw edge as they must intersect $V(P)$ in two vertices each, one of which lies on $E_{1,1}$. Thus, the expected number of paths with saws such as above is

$$\begin{aligned} \sum_{\ell < \ell'} O(n^{1-\frac{1}{q_2-1}} (\log n \cdot p)^{q_1-1}) &= O(\log^{q_1} n \cdot n^{1-\frac{1}{q_2-1}-\frac{q_2}{q_2-1}+\frac{q_2}{q_1(q_2-1)}}) \\ &= O(\log^{q_1} n \cdot n^{-\frac{2q_1-q_2}{q_1(q_2-1)}}) = o(1). \end{aligned}$$

Thus, by Markov's inequality,

$$\mathbb{P}(Y > 0) = o(1).$$

Finally, let Z be the random variable counting simple paths of length $\ell < \ell'$ with a spoiled extension. Once more, we divide the argument into several cases. First, let Z_1 be the random variable counting simple paths with spoiled extensions such that no long edge in the extension intersects the path P in more than one vertex.

We denote the number of long edges in the extension that intersect $V(P)$ by k . We have an expected $O(n^{1-\frac{1}{q_2-1}})$ many choices for P , $O(n)$ many choices for the short edge and each of the q_2-1-k long edges that are disjoint from $V(P)$, and $O(\log n)$ many choices for each of the k long edges that intersect P . We also require $k(q_1-1) + (q_2-1-k)q_1$ new vertices. Thus, the expected number of such configurations is

$$\begin{aligned} \mathbb{E}Z_1 &= \sum_{\ell < \ell'} \sum_{k=1}^{q_2} O(n^{1-\frac{1}{q_2-1}+q_2-k} p^{(q_2-1-k)q_1+k(q_1-1)} \log^k n) \\ &= \sum_{k=1}^{q_2} O(n^{1-\frac{1}{q_2-1}+q_2-k} p^{(q_2-1)q_1-k} \log^{q_2+1} n) \\ &= \sum_{k=1}^{q_2} O(n^{1-\frac{1}{q_2-1}-k(1-\frac{q_2}{q_1(q_2-1)})} \log^{q_2+1} n) \\ &= O(n^{-\frac{q_1-q_2}{q_1(q_2-1)}} \log^{q_2+1} n) = o(1). \end{aligned}$$

Next, let Z_2 be the random variable counting paths with spoiled extensions that contain exactly one long edge L that intersects the path P in at least two vertices. Fixing P , we have only $O(\log^2 n)$ many choices for L and $O(1)$ further choices for $E_{\ell+1}$, as it must intersect both L and E_{ℓ,q_2} . Once again, we sum over k , the number of long edges in the extension that intersect $V(P)$ in exactly one vertex. We have $O(n^{q_2-2-k})$ choices for all edges of the extension, and we require at least $(q_2-2)q_1-k+1$ new vertices. Thus, since $k \leq q_2-2$ and

$$p^{q_1(q_2-2)} = O(n^{-\frac{q_2(q_2-2)}{q_2-1}}) = O(n^{-q_2+1+\frac{1}{q_2-1}}),$$

the expected number of such configurations is

$$\begin{aligned} \mathbb{E}Z_2 &= \sum_{\ell < \ell'} \sum_{k=0}^{q_2-2} O(n^{q_2-k-1-\frac{1}{q_2-1}} p^{(q_2-2)q_1-k+1} \log^2 n) \\ &= \sum_{k=0}^{q_2-2} O(n^{-k} p^{1-k} \log^3 n) = O(\log^3 n \cdot p) = o(1). \end{aligned}$$

Finally, let Z_3 be the random variable counting paths with spoiled extensions such that there exist distinct $i, j \in [q_2]$ such that $|V(P) \cap E_{\ell+1, z}| \geq 2$ for $z \in \{i, j\}$. We denote $L_z = E_{\ell+1, z}$.

Since $E_{\ell+1}$ must have a simple cover we obtain that $L_z \cap E_{\ell, q_2} = \emptyset$ for $z \in \{i, j\}$. Therefore, if we fix the path P up to (but not including) E_{ℓ, q_2} , we have at most $O(\log^2 n)$ many choices for each of L_i and L_j , since both edges intersect $V(P)$ in at least two vertices. Therefore, we have at most $O(\log^4 n)$ many choices for $E_{\ell+1}$ as it must intersect both L_i and L_j .

Finally, we obtain that we have $O(\log^4 n)$ many choices for E_{ℓ, q_2} as it must intersect both $E_{\ell+1}$ and E_ℓ . Since we have an expected $O(n^{1-\frac{1}{q_2-2}})$ many choices for the path up to the final block, and we have $O(n^{q_2-1} \log^4 n)$ many choices for the final block (but still require $q_1(q_2 - 1)$ vertices), the expected number of choices for P is

$$\sum_{\ell < \ell'} O(n^{-\frac{1}{q_2-1}} \log^4 n) = o(1),$$

hence

$$\mathbb{P}(Z_3 > 0) = o(1),$$

and therefore

$$\mathbb{P}(Z > 0) \leq \mathbb{P}(Z_1 + Z_2 + Z_3 > 0) = o(1).$$

Finally,

$$\mathbb{P}(U = W = X = Y = Z = 0) \rightarrow 1 \text{ as } n \rightarrow \infty$$

completing the proof. □

Acknowledgements

I would like to thank my advisor Wojciech Samotij for his introduction to this problem and for his guidance towards its solution. I would also like to express my deepest appreciation for his kindness, patience and meticulousness during the research and writing process.

References

- [1] Elad Aigner-Horev and Yury Person, *An asymmetric random Rado theorem: 1-statement*, [arXiv:1906.05614](https://arxiv.org/abs/1906.05614).
- [2] Noga Alon and Joel H. Spencer, *The probabilistic method*, fourth ed., Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2016. MR 3524748
- [3] József Balogh, Robert Morris, and Wojciech Samotij, *Independent sets in hypergraphs*, *J. Amer. Math. Soc.* **28** (2015), no. 3, 669–709. MR 3327533
- [4] ———, *The method of hypergraph containers*, *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018*. Vol. 3, 2018, pp. 3045–3078.

- [5] Péter Frankl and Vojtěch Rödl, *Large triangle-free subgraphs in graphs without K_4* , Graphs Combin. **2** (1986), no. 2, 135–144. MR 932121
- [6] Ehud Friedgut, Hiệp Hàn, Yury Person, and Mathias Schacht, *A sharp threshold for van der Waerden’s theorem in random subsets*, Discrete Anal. (2016), Paper No. 7, 20. MR 3533306
- [7] Luca Gugelmann, Rajko Nenadov, Yury Person, Nemanja Škorić, Angelika Steger, and Henning Thomas, *Symmetric and asymmetric Ramsey properties in random hypergraphs*, Forum Math. Sigma **5** (2017), e28, 47. MR 3725732
- [8] Robert Hancock and Andrew Treglown, *An asymmetric random Rado theorem for single equations: the 0-statement*, arXiv:2004.14076v1.
- [9] T. E. Harris, *A lower bound for the critical probability in a certain percolation process*, Proc. Cambridge Philos. Soc. **56** (1960), 13–20. MR 0115221
- [10] Svante Janson, *Poisson approximation for large deviations*, Random Structures Algorithms **1** (1990), no. 2, 221–229. MR 1138428
- [11] Yoshiharu Kohayakawa and Bernd Kreuter, *Threshold functions for asymmetric Ramsey properties involving cycles*, Random Structures Algorithms **11** (1997), no. 3, 245–276. MR 1609513
- [12] Anita Liebenau, Letícia Mattos, Walner Mendonça, and Jozef Skokan, *Asymmetric ramsey properties of random graphs for cliques and cycles*, 2020.
- [13] Martin Marciniszyn, Jozef Skokan, Reto Spöhel, and Angelika Steger, *Asymmetric Ramsey properties of random graphs involving cliques*, Random Structures Algorithms **34** (2009), no. 4, 419–453. MR 2531778
- [14] Frank Mousset, Rajko Nenadov, and Wojciech Samotij, *Towards the Kohayakawa–Kreuter conjecture on asymmetric Ramsey properties*, Combinatorics, Probability and Computing **29** (2020), no. 6, 943–955.
- [15] Rajko Nenadov and Angelika Steger, *A short proof of the random Ramsey theorem*, Combin. Probab. Comput. **25** (2016), no. 1, 130–144. MR 3438289
- [16] Richard Rado, *Studien zur Kombinatorik*, Mathematische Zeitschrift (1933), 424–470.
- [17] Vojtěch Rödl and Andrzej Ruciński, *Lower bounds on probability thresholds for Ramsey properties*, Combinatorics, Paul Erdős is eighty, Vol. 1, Bolyai Soc. Math. Stud., János Bolyai Math. Soc., Budapest, 1993, pp. 317–346. MR 1249720
- [18] ———, *Random graphs with monochromatic triangles in every edge coloring*, Random Structures Algorithms **5** (1994), no. 2, 253–270. MR 1262978
- [19] ———, *Threshold functions for Ramsey properties*, J. Amer. Math. Soc. **8** (1995), no. 4, 917–942. MR 1276825
- [20] ———, *Rado partition theorem for random subsets of integers.*, Proceedings of the London Mathematical Society **74** (1997), no. 3, N.PAG.
- [21] David Saxton and Andrew Thomason, *Hypergraph containers*, Invent. Math. **201** (2015), no. 3, 925–992. MR 3385638

- [22] Bartel Leendert Van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw. Arch. Wisk. (1927).
- [23] Panayiotis Varnavides, *On certain sets of positive density*, J. London Math. Soc. **34** (1959), 358–360. MR 106865