Cooperative conditions for the existence of rainbow matchings

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Abstract

Let $k > 1$, and let $\mathcal{F}$ be a family of $2n + k - 3$ non-empty sets of edges in a bipartite graph. If the union of every $k$ members of $\mathcal{F}$ contains a matching of size $n$, then there exists an $\mathcal{F}$-rainbow matching of size $n$. Replacing $2n + k - 3$ by $2n + k - 2$, the result is true also for $k = 1$, and it can be proved (for all $k$) both topologically and by a relatively simple combinatorial argument. The main effort is in gaining the last 1, which makes the result sharp.

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1 Introduction

Throughout the paper, “family” means “multiset”, meaning that elements may repeat. To differentiate the notation, we use round brackets for families, and (as usual) curly brackets for sets. For a family $\mathcal{F}$, we write $\mathcal{F} \setminus \{F\}$ and $\mathcal{F} \cup \{F\}$ in the family sense. That is, $\mathcal{F} \setminus \{F\}$ contains one less copy of $F$ than $\mathcal{F}$ if $F \in \mathcal{F}$, and $\mathcal{F} \cup \{F\}$ contains one more copy of $F$ than $\mathcal{F}$.

Given a family $S = (S_1, \ldots, S_m)$ of sets, an $S$-rainbow set is the image of a partial choice function of $S$. So, it is a set $\{x_{ij} \mid j \leq k\}$, where $1 \leq i_1 < \cdots < i_k \leq m$ and $x_{ij} \in S_{ij}$.

A complex is a closed down hypergraph, meaning that any subset of any edge is an edge. The injectivity - at most one element from every set $S_i$ - is a “smallness” condition, in the sense that the set of injective choices is a complex. Hence statements of interest are of the form “there exists a large rainbow set satisfying certain conditions (like being a matching)”.

Below, again, $S = (S_1, \ldots, S_m)$ is a family of sets. For a set $I \subseteq \mathcal{I}$, let $S_I = \bigcup_{i \in I} S_i$.

Theorem 1. If $|S_J| \geq |J|$ for every $J \subseteq \mathcal{I}$ then there is a full rainbow set, that is, a rainbow set of size $m$.

Another well-known rainbow result is Drisko’s theorem, on rainbow matchings. The following slightly more general version of the original theorem was proved in [1]:

Theorem 2. [7] $2n - 1$ matchings in a bipartite graph, of size $n$ each, have a rainbow matching of size $n$.

There is a conspicuous difference between the two theorems: in the first the condition is “cooperative”, namely it is on subfamilies of $S$, whereas in the second it is on singletons - each $S_i$ is assumed to be large by itself. On the other hand, there is a condition on the number of the sets $S_i$.

1.1 A cooperative version of the Kalai-Meshulam theorem

A complex $C$ is said to be $d$-Leray if $\bar{H}_k(C[S]) = 0$ for all $S \subseteq V$ and all $k \geq d$ ($\bar{H}_k$ is the reduced $k$-th homology group). Let $\lambda(C)$ be the smallest number $d$ such that $C$ is $d$-Leray.

A basic result in this direction is a theorem of Kalai and Meshulam [11]:

Theorem 3. Let $\mathcal{M}$ and $C$ be a matroid and a complex, respectively, on the same ground set. If $\lambda(\text{lk}_C(S)) < \text{rank}_{\mathcal{M}}(V \setminus S)$ for every $S \in C$ then $\mathcal{M} \setminus C \neq \emptyset$.

Here $\text{lk}_C(S) = \{T \subseteq V \setminus S \mid S \cup T \in C\}$. The theorem above is a re-formulation of Theorem 1.6 in [11].

The following was proved in [12]:

Theorem 4. For any complex $C$ and set $S \in C$, $\lambda(\text{lk}_C(S)) \leq \lambda(C)$.

Theorems 3 and 4, combined, yield the following:
Theorem 5. If \( \lambda(C) \leq d \) and \( S = (S_1, \ldots, S_{d+k}) \) is a family of subsets of \( V(C) \) satisfying \( S_i \notin C \) whenever \( I \subseteq [d+k] \) is of size \( k \), then there exists an \( S \)-rainbow non-

Proof. By duplicating vertices, if necessary (a vertex having a distinct copy for every set \( S_i \) it belongs to), we may assume that the sets \( S_i \) are disjoint. Let \( M \) be the partition matroid defined by the sets \( S_i \). By Theorems 4 and 3 it suffices to show that if \( S \in C \) then \( \text{rank}_M(V \setminus S) > d \). This follows from the condition \( S_i \notin C \ (|I| \geq k) \) and the fact that \( \text{rank}_M(A) = |\{i : A \cap S_i \neq \emptyset\}|. \)

This is a “cooperative” version of the Kalai-Meshulam theorem, namely many sets join forces to contain a set not belonging to \( C \).

1.2 A cooperative version of Theorem 2

For a set \( F \) of edges we denote by \( \nu(F) \) the maximal size of a matching in \( F \). For a family \( \mathcal{F} = (F_1, \ldots, F_m) \) of sets of edges, we denote by \( \nu_R(\mathcal{F}) \) the maximal size of an \( \mathcal{F} \)-rainbow matching.

Let \( t \) be an integer, and let \( n \leq t \). Let \( C \) be the complex consisting of all \( F \subseteq E(K_{t,n}) \), satisfying \( \nu(F) < n \). In [3] it was shown that \( \lambda(C) \leq 2n - 2 \). Together with Theorem 5 this yields:

Theorem 6. \( 2n + k - 2 \) sets of edges in a bipartite graph, the union of any \( k \) of which contains a matching of size \( n \), have a rainbow matching of size \( n \).

Notation 7. We write \( (m, k, n) \rightarrow_B q \) for the statement “every \( m \) nonempty sets of edges in a bipartite graph, the union of every \( k \) of which contains a matching of size \( n \), have a rainbow matching of size \( q \)”.

In this notation, the theorem says that \( (2n + k - 2, k, n) \rightarrow_B n \). The case \( k = 1 \) is Theorem 2. The main result of this paper is that for \( k > 1 \) this can be improved by 1, thereby obtaining a sharp bound.

Theorem 8. \( (2n + k - 3, k, n) \rightarrow_B n \) whenever \( 1 < k \leq n \).

The sharpness of this result, namely the fact that \( (2n + k - 4, k, n) \not\rightarrow_B n \) for any \( k \), is given by the following example. In \( C_{2n} \), take the odd edges matching repeated \( n - 1 \) times, the even edges matching repeated \( n - 2 \) times, and a singleton set, consisting of an even edge, repeated \( k - 1 \) times. Explicitly:

Example 9. Consider a complete bipartite graph \( K_{n,n} \) with sides \( \{a_1, a_2, \ldots, a_n\} \) and \( \{b_1, b_2, \ldots, b_n\} \). Let

\[
S_i = \begin{cases} 
\{a_1b_1, a_2b_2, \ldots, a_nb_n\} & \text{if } i \in [n-1], \\
\{a_1b_2, a_2b_3, \ldots, a_{n-1}b_n, a_nb_1\} & \text{if } i \in [2n-3] \setminus [n-1], \\
\{a_ib_j\} & \text{if } i \in [2n+k-4] \setminus [2n-3]. 
\end{cases}
\]

Let \( S = (S_i \mid i = 1, \ldots, 2n + k - 4) \). Then for any \( I \subseteq [2n + k - 4] \) with \( |I| \geq k \), \( \nu(S_I) \geq n \), and \( \nu_R(S) < n \).
Remark 10. After our result was obtained, Holmsen and Lee [10] gave a topological proof of Theorem 8, using a strong version of Theorem 3. Their result is a somewhat stronger version of Theorem 8.

1.3 Cooperative versions of Colorful Carathéodory

Part of the motivation for Theorem 8 comes from the existence of cooperative versions of a famous rainbow result - Bárány’s Colorful Carathéodory theorem [6]. In fact, as we shall see below (first proof of Theorem 25), the affinity is not merely formal. Theorem 6 follows from a cooperative version of Colorful Carathéodory.

Wegner [13] noted that the complex $C$ of sets of vectors in $\mathbb{R}^d$ not containing a given vector $v$ in their convex hull satisfies $\lambda(C) = d$. Similarly, the complex $D$ of sets not containing $v$ in their cone (set of non-negative combinations) satisfies $\lambda(D) = d−1$. This, together with Theorem 5, yields:

**Theorem 11.** Let $v \in \mathbb{R}^d$.

1. If $S = (S_1, \ldots, S_{d+k})$ is a family of subsets of $\mathbb{R}^d$ such that $v \in \text{conv}(S_K)$ for every $K \subseteq [d+k]$ of size $k$, then there exists an $S$-rainbow set $S$ such that $v \in \text{conv}(S)$.

2. If $S = (S_1, \ldots, S_{d+k−1})$ is a family of subsets of $\mathbb{R}^d$ such that $v \in \text{cone}(S_K)$ for every $K \subseteq [d+k−1]$ of size $k$, then there exists an $S$-rainbow set $S$ such that $v \in \text{cone}(S)$.

The case $k = 2$ of part (1) of the theorem was strengthened by Holmsen-Pach-Tverberg [9] and Arocha et.al. [5]:

**Theorem 12.** If $S_1, \ldots, S_{d+1}$ are non-empty sets in $\mathbb{R}^d$, and $v \in \text{conv}(S_i \cup S_j)$ whenever $1 \leq i < j \leq d+1$, then there is a rainbow set $S$ with $v \in \text{conv}(S)$.

Holmsen [8] gave a topological proof of this result, using a notion he called “near $d$-Lerayness”, which means that $lk_C(S)$ is $d$-Leray for every non-empty $S \in C$. The same argument can be used to prove the analogous strengthening for all $k > 1$:

**Theorem 13.** Let $k > 1$, and let $S = (S_1, \ldots, S_{d+k−1})$ be a family of non-empty sets in $\mathbb{R}^d$, such that every $k$ of them contain $v$ in the convex hull of their union. Then there is an $S$-rainbow set containing $v$ in its convex hull.

The analogous strengthening of part (2) of Theorem 11 is false, as witnessed by simple counterexamples.

**Example 14.** Let $v_1, \ldots, v_{d+1}$ be the vertices of a $d$-dimensional simplex $\sigma \subseteq \mathbb{R}^d$ whose barycenter is the origin. Let $v$ be the barycenter of face $\{v_1, \ldots, v_d\}$ of $\sigma$. Consider the family $S = (S_1, \ldots, S_{d+k−2})$ of non-empty sets in $\mathbb{R}^d$, where $S_i = \{v_1, \ldots, v_d\}$ for $1 \leq i \leq d−1$ and $S_j = \{v_{d+k+1}\}$ for $d \leq j \leq d + k − 2$. Among any $k$ sets in $S$, at least one is $S_i$ for some $1 \leq i \leq d−1$, hence the convex cone spanned by their union contains $v$. However, there is no $S$-rainbow set $S$ such that $v \in \text{cone}(S)$.
2 Rainbow paths

The proof of Theorem 8 is based on a combinatorial proof of the result \((2n+k-2, k, n) \rightarrow B n\), and analysis of the extreme case. This proof, in turn, uses a lemma on rainbow paths in networks. To get the extra 1 we analyze the extreme cases of that lemma. The analysis uses ideas from an analogous lemma in [4], which is the case \(k = 1\). But apart from a higher level of complexity, there is the difference that for \(k > 1\) the analysis leads to an improvement of 1 in the theorem - which was not the case for \(k = 1\).

A network is a triple \(\mathcal{N} = (D, s, t)\), where \(D\) is a digraph, and \(s, t\) are two special vertices in it, called source and target. We assume that there are no edges going out of \(t\) or into \(s\). We write \(V(\mathcal{N})\) for \(V(D)\). The set \(V(\mathcal{N}) \setminus \{s, t\}\) is denoted by \(V^\circ(\mathcal{N})\), and its elements are called “inner vertices”. For an \(s - t\) path \(P\) let \(V^\circ(P) = V^\circ(\mathcal{N}) \cap V(P)\). Two \(s - t\) paths \(P, Q\) are said to be internally disjoint if \(V^\circ(P) \cap V^\circ(Q) = \emptyset\).

For an \(s - t\) path \(Q\) let \(B(Q)\) be the set of backward edges on \(Q\), namely those directed edges \(pq\) where \(p, q \in V(Q)\) and \(q\) precedes \(p\) on \(Q\). Let \(s_Q\) be the vertex following \(Q\), and \(t_Q\) the vertex preceding \(Q\) in \(Q\). Define \(U(Q) = \{v_{s_Q} \mid v \in V^\circ(\mathcal{N}) \setminus V(Q)\} \cup \{t_{Q} \mid u \in V^\circ(\mathcal{N}) \setminus V(Q)\\}. ("U" stands for “useless”, since such edges cannot be used as shortcuts - this will be clarified below).

We shall borrow a term - “regimented” - from [4], but its use is a bit different here.

Definition 15. Let \(\mathcal{F}\) be a family of sets of edges in \(\mathcal{N}\). A regimentation of \(\mathcal{F}\) is a pair \(\mathcal{R} = (\mathcal{Q} = \mathcal{Q}(\mathcal{R}), I = I(\mathcal{R}))\), where \(\mathcal{Q}\) is a set of internally disjoint \(s - t\) paths, and \(I\) is a function from a subset \(\mathcal{E} = \mathcal{E}(\mathcal{R})\) of \(\mathcal{F}\) (the “essential” sets) onto \(\mathcal{Q}\), satisfying the following conditions:

1. \(\bigcup_{Q \in \mathcal{Q}} V(Q) = V(\mathcal{N})\),
2. \(E(I(F)) \subseteq F\) for every \(F \in \mathcal{E}\), and
3. \(|I^{-1}(Q)| = |E(Q)| - 1\) for every \(Q \in \mathcal{Q}\).

Let \(\mathcal{I}(\mathcal{R}) = \mathcal{F} \setminus \mathcal{E}(\mathcal{R})\) (the “inessential” sets) and \(B(\mathcal{R}) = \bigcup_{Q \in \mathcal{Q}} B(Q)\).

If such a regimentation \(\mathcal{R}\) exists, we say then that \(\mathcal{F}\) is regimented by \(\mathcal{R}\).

Conditions (1) and (3) imply:

Lemma 16. \(|\mathcal{E}(\mathcal{R})| = |V^\circ(\mathcal{N})|\).

Proof. Since \(\mathcal{E}(\mathcal{R}) = \bigcup_{Q \in \mathcal{Q}} I^{-1}(Q)\), we have \(|\mathcal{E}(\mathcal{R})| = \sum_{Q \in \mathcal{Q}} |I^{-1}(Q)|\). Then by the condition (3) of a regimentation, we have

\[|\mathcal{E}(\mathcal{R})| = \sum_{Q \in \mathcal{Q}} |I^{-1}(Q)| = \sum_{Q \in \mathcal{Q}} (|E(Q)| - 1) = \sum_{Q \in \mathcal{Q}} |V^\circ(Q)|.\]

Since \(\mathcal{Q}\) is a set of internally disjoint \(s - t\) paths, the condition (1) of a regimentation implies \(\sum_{Q \in \mathcal{Q}} |V^\circ(Q)| = |V^\circ(\mathcal{N})|\), and hence we obtain \(|\mathcal{E}(\mathcal{R})| = |V^\circ(\mathcal{N})|\). \(\Box\)
**Notation 17** (Pruning and concatenation of paths). If $P$ is a directed path and $x \in V(P)$ then $Px$ is the part of $P$ up to and including $x$, and $xP$ is the part of $P$ starting at $x$. If two paths $P$ and $Q$ meet at a vertex $x$, then $P\!x\!Q$ denotes the walk obtained by concatenating $Px$ and $xQ$. If the endpoint of a path $P$ coincides with the initial point in a path $Q$, we write $P\!Q$ for the walk that is the concatenation of $P$ and $Q$.

**Lemma 18.** Suppose $\mathcal{F}$ is regimented by $\mathcal{R} = (\mathcal{Q}, I)$, and let $B = B(\mathcal{R}), \mathcal{IE} = \mathcal{IE}(\mathcal{R})$. If there is no $\mathcal{F}$-rainbow $s - t$ path, then $\bigcup \mathcal{IE} \subseteq B$ and $\bigcup I^{-1}(Q) \subseteq E(Q) \cup B \cup U(Q)$ for every $Q \in \mathcal{Q}$.

(For a set $\mathcal{K}$ of sets $\bigcup \mathcal{K}$ is the union of all sets in $\mathcal{K}$.)

**Proof.** Let $vu$ be an edge belonging to $F$ for some $F \in \mathcal{F}$. Assume that $v \in V(Q_1)$, $u \in V(Q_2)$. Let $P = Q_1 vu Q_2$ (see Notation 17).

To obtain the conclusion of the lemma, we will show the following.

1. When $Q_1 = Q_2$, $P$ is an $\mathcal{F}$-rainbow $s - t$ path unless $vu \in B(Q_1)$ or $vu \in E(Q_1)$ and $F \in I^{-1}(Q_1)$.

2. When $Q_1 \neq Q_2$, $P$ is an $\mathcal{F}$-rainbow $s - t$ path unless $v = t_{Q_1}$ and $F \in I^{-1}(Q_1)$, or $u = s_{Q_2}$ and $F \in I^{-1}(Q_2)$.

First suppose that $Q_1 = Q_2$. If $v$ precedes $u$ on $Q_1$ and $vu \notin E(Q_1)$, then $P$ is an $\mathcal{F}$-rainbow $s - t$ path, since by part (3) of Definition 15 it has enough represented sets for its length. If $vu \in E(Q_1)$, then $P$ is an $\mathcal{F}$-rainbow $s - t$ path unless $F \in I^{-1}(Q_1)$. This proves (1).

Now assume $Q_1 \neq Q_2$. We may assume that $v \in V^\circ(Q_1)$ and $u \in V^\circ(Q_2)$ since if not the claim is a special case of (1). Then $Q_1 vu$ and $uQ_2$ are rainbow, and they have enough represented sets in $I^{-1}(Q_1)$ and $I^{-1}(Q_2)$, respectively. If $F \notin I^{-1}(Q_1) \cup I^{-1}(Q_2)$, then $P$ is rainbow. If $F \in I^{-1}(Q_1)$ and $v \neq t_{Q_1}$, then $Q_1 vu$ is rainbow since it has enough represented sets in $I^{-1}(Q_1)$, since it has length at most $|E(Q_1)| - 1$. Similarly if $F \in I^{-1}(Q_2)$ and $u \neq s_{Q_2}$, then $vuQ_2$ is rainbow since it has enough represented sets in $I^{-1}(Q_2)$. In both cases $P$ is rainbow, which proves (2).

Since we assume there is no $\mathcal{F}$-rainbow $s - t$ path, if $F \in \mathcal{IE}$, then $vu \notin B$ by (1) and (2). Thus $\bigcup \mathcal{IE} \subseteq B$. If $F \in I^{-1}(Q)$ for some $Q \in \mathcal{Q}$, then $vu \in E(Q) \cup B \cup U(Q)$ by (1) and (2). Thus $\bigcup I^{-1}(Q) \subseteq E(Q) \cup B \cup U(Q)$.

**Corollary 19.** Let $\mathcal{F}$ be regimented by $\mathcal{R}$, and assume that there is no $\mathcal{F}$-rainbow $s - t$ path. If $F \in \mathcal{IE}(\mathcal{R})$ then $F$ does not contain an $s - t$ path.

In fact, $F$ does not even contain an edge $sy$.

**Lemma 20.** Let $P, Q$ be $s - t$ paths in a network $(\mathcal{D}, s, t)$. If $E(P) \subseteq E(Q) \cup B(Q) \cup B \cup U(Q)$ for some collection $B$ of edges that are vertex-disjoint from $Q$, then $P = Q$. 


Proof. The only edge leaving $s$ in $E(Q) \cup B(Q) \cup \tilde{B} \cup U(Q)$ is $ss_Q \in E(Q)$, and the only edge to $t$ is $tQt \in E(Q)$. So these are necessarily the first and last edges of $P$. Therefore $P$ has no edges from $U(Q)$, since the in-degree of $s_Q$ and the out-degree of $t_Q$ in $P$ are 1.

As $E(Q) \cup B(Q)$ and $\tilde{B}$ are disconnected, $E(P) \cap \tilde{B} = \emptyset$. It remains to show that $E(P) \cap B(Q) = \emptyset$, which follows from the fact that $P$ does not repeat vertices. \qed

Combining Lemmas 18 and 20 yields:

**Corollary 21.** Let $F$ be regimented by $R$, and having no rainbow $s-t$ path. If $F \in \mathcal{E}(R)$ then $I(F)$ is the only $s-t$ path contained in $F$.

By Corollaries 19 and 21, we can obtain the following corollary.

**Corollary 22.** Let $F$ be regimented by $R$, and having no rainbow $s-t$ path. Then $F \in \mathcal{E}(R)$ if and only if $F$ contains an $s-t$ path, and equivalently, $F \in \mathcal{IE}(R)$ if and only if $F$ does not contain an $s-t$ path.

The following argument will be used twice, and hence it receives separate mention:

**Lemma 23.** Let $G, H$ be two families of sets of edges, none of which possesses a rainbow $s-t$ path. Suppose that $G$ is regimented by $R = (Q, I)$ and $H$ is regimented by $S = (P, J)$. Suppose that $G \setminus H$ consists of a single set of edges $G$, and $H \setminus G$ consists of single set of edges $H$. Then either $G \in \mathcal{IE}(R)$ and $H \in \mathcal{IE}(S)$, or $I(G) = J(H)$.

**Proof.** Let $K = G \cap H$. So $G = K \cup \{G\}$, $H = K \cup \{H\}$.

By Corollary 22, it is obvious that

$$K \cap \mathcal{E}(R) = K \cap \mathcal{E}(S). \quad (1)$$

By Corollary 21, $I(K) = J(K)$ for every $K \in K \cap \mathcal{E}(R)$. Hence

$$\bigcup_{K \in \mathcal{E}(R) \setminus \{G\}} V(I(K)) = \bigcup_{K \in \mathcal{E}(S) \setminus \{H\}} V(J(K)) \quad (2)$$

Let us first show that $G \in \mathcal{IE}(R)$ if and only if $H \in \mathcal{IE}(S)$. Suppose that $G \in \mathcal{IE}(R)$. Then $\mathcal{E}(R) \subseteq K$. By (1) and Lemma 16, it follows that $\mathcal{E}(S) = \mathcal{E}(R)$, so $H \in \mathcal{IE}(S)$. The converse implication is the same.

Assume next that $G \in \mathcal{E}(R)$ and $H \in \mathcal{E}(S)$. Let $Q_0 = I(G)$. Consider first the case that $V^\circ(Q_0)$ consists of a single vertex $v$. We have $\bigcup_{K \in \mathcal{E}(R) \setminus \{G\}} V(I(K)) = V^\circ \setminus \{v\}$, and hence by (2) we have also $\bigcup_{K \in \mathcal{E}(S) \setminus \{H\}} V(J(K)) = V^\circ \setminus \{v\}$. Since the interiors of paths in $P$ partition $V^\circ$, it follows that $J(H)$ is the path $svt$, namely $Q_0$.

It remains to consider the case $|V^\circ(Q_0)| > 1$. Then, not counting multiplicities, $P = Q$, because every path of $Q$ appears as $J(K)$ for some $K \in K$. The only path in $P$ not covered enough times by paths $J(K)$, $K \in \mathcal{E}(S) \setminus \{H\}$, is $Q_0$. So, necessarily $J(H) = Q_0$. \qed

The next theorem is the main step towards the proof of Theorem 8.
Theorem 24. Let \( \mathcal{N} = (D, s, t) \) be a network with \( n \) inner vertices. Let \( \mathcal{F} \) be a family of \( n + k - 1 \) sets of edges in \( \mathcal{N} \), satisfying the condition that \( \bigcup \mathcal{K} \) contains an \( s - t \) path, for every \( \mathcal{K} \subseteq \mathcal{F} \) of size \( k \). Then either there exists an \( \mathcal{F} \)-rainbow \( s - t \) path, or \( \mathcal{F} \) is regimented.

The case \( k = 1 \) of the theorem is Theorem 3.3 in [4].

It is worth noting that the weaker result, with \( \mathcal{F} \) being of size \( n + k \), is not hard. First, the statement:

Theorem 25. Let \( \mathcal{N} = (D, s, t) \) be a network with \( n \) inner vertices. Let \( \mathcal{F} \) be a family of \( n + k \) sets of edges in \( \mathcal{N} \), satisfying the condition that \( \bigcup \mathcal{K} \) contains an \( s - t \) path for every \( \mathcal{K} \subseteq \mathcal{F} \) of size \( k \). Then there exists an \( \mathcal{F} \)-rainbow \( s - t \) path.

Here are two proofs:

Proof 1. Observe that a set \( H \) of edges in \( \mathcal{N} \) contains an \( s - t \) path if and only if the cone of \( \{ \chi_b - \chi_a \mid ab \in H \} \) contains the vector \( \chi_s - \chi_t \) (here \( \chi_a \) is the function that is 1 on \( v \) and 0 on all other vertices). Also note that all these vectors reside in an \( n + 1 \)-dimensional space (they are of length \( n + 2 \), but all are perpendicular to the all-1 vector). Apply now Theorem 11, part (2).

Proof 2. Take a maximal \( \mathcal{F} \)-rainbow tree \( T \) rooted at \( s \). Assume, for contradiction, that it does not reach \( t \). Then it represents at most \( n \) members of \( \mathcal{F} \). Hence there are \( k \) sets \( F \in \mathcal{F} \) not represented in \( T \). By assumption, their union contains an \( s - t \) path. The first edge leaving \( T \) can then be added to \( T \) to yield a larger rainbow tree, which contradicts the maximality of \( T \).

Definition 26 (contracting an edge \( sx \)). Let \( sx \) be an edge of \( \mathcal{N} \). We can contract \( sx \) to a newly defined vertex \( s' \), that will serve as the source of a new network \( \mathcal{N}' \). Here is what this does to sets of edges, and to paths.

1. Let \( F \) be a set of edges in a network \( \mathcal{N} = (D, s, t) \), and let \( sx \) be an edge, where \( x \) is an inner vertex. The contracted set of edges \( F|_{sx \rightarrow s'} \) is obtained from \( F \) by replacing every edge \( sy \) or \( xy \) belonging to \( F \) by the edge \( s'y \), and removing all edges \( yx \).

2. An \( s - t \) path \( P \) is transformed by the contraction of \( sx \) to an \( s' - t \) path \( P' \), defined as follows. If \( x \not\in V(P) \) then \( P' = P \) with \( s' \) replacing \( s \). If \( x \in V(P) \) then \( P' = s'yP \) where \( y \) is the vertex following \( x \) in \( P \) (so, the vertices in \( Px \) disappear.) We also write \( P' = P|_{sx \rightarrow s'} \). Note that in this definition \( E(P') \) is not necessarily equal to \( E(P)|_{sx \rightarrow s'} \).

Proof of Theorem 24. By induction on \( n \). The case \( n = 0 \) is easy. So let \( n \geq 1 \) and assume that the theorem is valid when \( n - 1 \) replaces \( n \).

Since \( n + k - 1 \geq k \), \( \bigcup \mathcal{F} \) contains an \( s - t \) path. So there exists at least one set \( G \in \mathcal{F} \) containing an edge \( sx \). If \( x = t \) then the path \( st \) is rainbow, so we may assume that \( x \neq t \). Now contract \( sx \): for each \( F \in \mathcal{F} \) let \( F' = F|_{sx \rightarrow s'} \). Let \( \mathcal{K}' = (F' \mid F \in \mathcal{F}) \) for \( \mathcal{K} \subseteq \mathcal{F} \). Let \( \mathcal{N}' \) be the network with vertex set \( V(\mathcal{N}) \setminus \{s, x\} \cup \{s'\} \), source \( s' \), target \( t \), and edge set \( \bigcup(\mathcal{F}' \setminus \{G'\}) \).
Every $K \subseteq F$ of size $k$ contains in its union the edge set of an $s - t$ path in $\mathcal{N}$, which is easily seen to imply the same, with $s'$ replacing $s$, for $\mathcal{K}'$ in $\mathcal{N}'$. By the induction hypothesis, either there exists an $\mathcal{F}'\setminus\{G\}$-rainbow $s' - t$ path $P'$, or $\mathcal{F}'\setminus\{G\}$ is regimented. In the first case, let $y$ be the vertex following $s'$ in $P'$. Then either $syP'$ or $sxP'$ is a rainbow $s - t$ path in $\mathcal{N}$, and we are done. So, we may assume the second possibility. Let $\mathcal{R}' = (Q', I')$ be a regimentation of $\mathcal{F}' \setminus \{G\}$, and let $\mathcal{E}' = \mathcal{E}(\mathcal{R}')$, $\mathcal{E}^* = \mathcal{E}(\mathcal{R}')$.

Let $\mathcal{IE} = (F \in \mathcal{F} \setminus \{G\} \mid F' \in \mathcal{IE}')$ and $\mathcal{E} = (F \in \mathcal{F} \setminus \{G\} \mid F' \in \mathcal{E}')$.

By Lemma 16 $|\mathcal{E}'| = n - 1$, so

$$|\mathcal{IE}| = |\mathcal{IE}'| = k - 1. \quad (3)$$

In all claims below we assume that there is no $\mathcal{F}$-rainbow $s - t$ path.

Let $B' = \bigcup_{Q' \in \mathcal{Q}'} B(Q')$. By Lemma 18, $\bigcup \mathcal{IE} \subseteq B'$ and $\bigcup I'^{-1}(Q') \subseteq E(Q') \cup B' \cup U(Q')$ for every $Q' \in \mathcal{Q}'$.

Notation 27 (two ways of un-contracting sx). Given an $s' - t$ path $Q' \in \mathcal{Q}'$, let $Q'^{(1)}$ be the path obtained from $Q'$ by replacing $s'$ with $s$ and $Q'^{(2)}$ the path obtained from $Q'$ by expanding its first edge $s'y$ to the path $sx$.

Our aim is to glean from $\mathcal{R}'$ a regimentation $\mathcal{R} = (Q, I)$ of $\mathcal{F}$. The set $\mathcal{E}(\mathcal{R})$ will contain $G$ and $Q$ will contain $s - t$ paths $f(Q')$, $Q' \in \mathcal{Q}'$, where $f$ is an injective function defined as follows. Let $Q' \in \mathcal{Q}'$ and let $F \in \mathcal{F} \setminus \{G\}$ be such that $I'(F') = Q'$. By (3) and the condition of the theorem, the set $F \cup \bigcup \mathcal{IE}$ contains an $s - t$ path $Q$. Let $f(Q') = Q$.

Claim 28. $Q' = Q|_{sx \rightarrow s'}$.

**Proof.** By the choice of $Q$, we have $E(Q|_{sx \rightarrow s'}) \subseteq F' \cup \bigcup \mathcal{IE}$. By Lemma 18, we have $F' \cup \bigcup \mathcal{IE} \subseteq E(Q') \cup B' \cup U(Q') = E(Q') \cup B(Q') \cup \bigcup_{T' \in Q' \setminus \{G\}} B(T') \cup U(Q')$. The claim now follows by Lemma 20.

There are two possibilities:

(a) $x \notin V(Q)$. In this case $Q = Q'^{(1)}$.

(b) $x \in V(Q)$. Suppose, in this case, that $Qx$ contains inner vertices. Let $y$ be the first inner vertex of $Qx$. Then $y \in V^0(T')$ for some $T' \in Q' \setminus \{Q\}$, and then $syT'$ is a rainbow $s - t$ path in $\mathcal{N}$ since it has enough represented sets in $I'^{-1}(T') \cup \{G\}$. So, we may assume that $V^0(Qx) = \emptyset$, meaning that the first edge on $Q$ is $sx$, meaning in turn that $Q = Q'^{(2)}$.

Claim 29. $sx \notin \bigcup \mathcal{IE}$.

**Proof.** Let $F_0 \in \mathcal{IE}$ and suppose that $sx \in F_0$. Recall that $\mathcal{F}'$ is the family of sets of edges obtained, where, for every $F \in \mathcal{F}$, $F'$ is the image of $F$ under the contraction of $sx$. By the same argument as above, $\mathcal{F}' \setminus \{F_0\}$ is regimented in $\mathcal{N}'$, by a regimentation $\mathcal{Q} = (Q(T), J)$. Then $G' \in \mathcal{IE}(\mathcal{Q})$ by Lemma 23, and hence $G$ do not contain an edge $yt$. But this would imply that $G \cup \mathcal{IE}(\mathcal{R})$ does not contain such an edge, and hence that it does not contain an $s - t$ path, contrary to the assumption of the theorem. \[\square\]
Since \( E(Q) \subseteq F \cup \bigcup \mathcal{I} \mathcal{E} \) and \( \bigcup \mathcal{I} \mathcal{E}' \subseteq B' \) by Lemma 18, a corollary of Claim 29 is:

\[
E(Q) \subseteq F.
\]  

(4)

Claim 30. The choice of \( f(Q') \) is independent of the choice of \( F \).

Proof. We have to show that if \( F_1, F_2 \in \mathcal{F} \setminus \{G\} \) satisfy \( I'(F_i') = Q' \), \( i = 1, 2 \) and \( Q_i \) are \( s - t \) paths whose edge sets are contained in \( F_i \cup \mathcal{I} \mathcal{E} \) \( (i = 1, 2) \) then \( Q_1 = Q_2 \). We know that \( Q_i \) are either \( Q^{(1)} \) or \( Q^{(2)} \). Assume, for contradiction, that \( Q_1 \neq Q_2 \), say \( Q_1 = Q^{(1)} \) and \( Q_2 = Q^{(2)} \). Then \( sx \in E(Q_2) \) and hence \( sx \in F_2 \). The set \( \mathcal{F} \setminus \{F_2\} \) lives in \( \mathcal{N}' \), and repeating the previous argument we deduce that it has a regimentation \( \mathcal{S} = (Q(S), J) \). By Lemma 23 \( J(G') = I'(F_2') = Q' \). In particular \( G' \supseteq E(Q') \). Since \( Q_1 = Q^{(1)} \), the edge \( ss_Q' \) belongs to \( E(Q_1) \subseteq F_1 \). Then, using an edge from \( G \) and edges from the sets \( F \in \mathcal{F} \) such that \( F' \in I^{-1}(Q') \) shows that \( ss_Q'Q' = Q^{(1)} \) is an \( \mathcal{F} \)-rainbow \( s - t \) path (note that edges in \( E(ss_Q') \) are also edges of \( F \)). This is the desired contradiction. \( \square \)

Claim 31.

1. If \( f(Q') = Q^{(2)} \) then \( G \supseteq E(f(Q')) \).
2. At most one \( Q' \in Q' \) satisfies \( f(Q') = Q^{(2)} \).
3. If \( f(Q') = Q^{(1)} \) for all \( Q' \in Q' \) then \( G \) contains the edges of the \( s - t \) path \( sx \).

Proof. To prove (1), let \( f(Q') = Q^{(2)} \) for some \( Q' \in Q' \). Then, by Claim 30, \( sx \in F \) for every \( F' \in I^{-1}(Q') \). We use the same trick as in the proof of Claim 30, interchanging the roles of \( F \) and \( G \). Consider \( \mathcal{F} \setminus \{F'\} \). As above, we may assume that \( \mathcal{F} \setminus \{F'\} \) is regimented, by a regimentation \( (\mathcal{P}', J') \). By Lemma 23, \( J'(G') = I'(F'') = Q' \), implying that \( G' \supseteq E(Q') \). Then \( G \) contains either \( E(Q^{(1)}) \) or \( E(Q^{(2)}) \). If \( G \) contains \( E(Q^{(1)}) \), then \( ss_Q'Q' \) (which is just \( Q^{(1)} \)) is an \( \mathcal{F} \)-rainbow \( s - t \) path: the edge \( ss_Q' \) represents \( G \); since \( |I^{-1}(Q')| = |E(Q')| - 1 \), the other edges have enough represented sets \( F \in \mathcal{F} \) such that \( F' \in I^{-1}(Q') \) (remember that \( G \notin I^{-1}(Q') \)). We have thus shown that \( G \) does not contain \( E(Q^{(1)}) \), so it contains \( E(Q^{(2)}) \), namely \( G \supseteq E(f(Q')) \).

Next we prove (2). Let \( f(Q') = Q^{(2)} \) for some \( Q' \in Q' \). By the above argument and Corollary 21, \( J'(G') = Q' \) is the only path contained in \( G' \). This directly implies (2).

Finally, we prove (3). Assume that \( f(Q') = Q^{(1)} \) for all \( Q' \in Q' \). Let \( \mathcal{N} \) be the network obtained from \( \mathcal{N} \) by deleting the vertex \( x \), and let \( \bar{F} \) be the set of edges of \( \mathcal{N} \), obtained from \( F \) by deleting all edges incident with \( x \). Let \( \bar{Q} = \{Q^{(1)} \mid Q' \in Q' \} \), and \( \bar{I}(\bar{F}) = f(I'(F')) \). By (4) and the assumption that \( f(Q') = Q^{(1)} \) for all \( Q' \in Q' \) the set \( \bar{F} = (\bar{F} \mid F \in \mathcal{F}) \) is regimented by \( (\bar{Q}, \bar{I}) \). The fact that there is no \( \mathcal{F} \)-rainbow \( s - t \) path implies that there is also no \( \bar{F} \)-rainbow \( s - t \) path. Therefore, by Lemma 18, we have \( \bar{G} \cup \bigcup_{F \in \mathcal{I} \mathcal{E}} \bar{F} \subseteq \bigcup_{Q' \in Q'} B(Q) \). Thus

\[
G \cup \bigcup \mathcal{I} \mathcal{E} \subseteq \{sx, xt\} \cup \bigcup_{Q' \in Q'} B(Q^{(1)}) \cup U(sxt).
\]
By the assumption of the theorem, \( G \cup \bigcup \tilde{\mathcal{E}} \) contains an \( s - t \) path, say \( Q_G \). By Lemma 20 we have \( Q_G = sxt \), and by Claim 29 we obtain \( G \supseteq E(Q_G) \). This concludes the proof of the claim. \( \square \)

**Remark 32.** By the claim the paths \( f(Q') \), \( Q' \in Q' \) are internally disjoint. In particular, there is at most one path \( f(Q') \) containing \( x \).

We can now complete the induction step in the proof of Theorem 24, by showing that \( F \) is regimented.

**Claim 33.** \((Q, I)\) is a regimentation of \( F \).

By Remark 32 and the fact that \( x \notin \bigcup_{Q' \in \mathcal{Q}} V(f(Q')) \), \( Q \) is a set of internally disjoint \( s - t \) paths.

By (4) \( E(I(F)) \subseteq F \) for all \( F \in \mathcal{E} \setminus \{G\} \), and by part (3) of Claim 31 \( E(I(G)) = E(Q_0) \subseteq G \). This implies condition (2) in Definition 15.

In addition,

\[
|I^{-1}(Q)| = |I^{-1}(f^{-1}(Q))| = |E(f^{-1}(Q))| - 1 = |E(f^{-1}(Q))| - 1 = |E(Q)| - 1
\]

for all \( Q \in \mathcal{Q} \setminus \{Q_0\} \), and

\[
|I^{-1}(Q_0)| = 1 = |E(Q_0)| - 1.
\]

This yields condition (3) of Definition 15.

Furthermore, since \( \bigcup_{Q' \in \mathcal{Q}} V^\circ(Q') = V^\circ(\mathcal{N}) \setminus \{x\} \) and \( V^\circ(Q'^{(1)}) = V^\circ(Q') \), we have

\[
\bigcup_{Q' \in \mathcal{Q}} V^\circ(Q) = \bigcup_{Q' \in \mathcal{Q}} V^\circ(Q'^{(1)}) \cup \{x\} = V^\circ(\mathcal{N}).
\]

This implies condition (1) of Definition 15, thus completing the proof of the claim.

**Case II:** \( f(Q_0') = Q_0'^{(2)} \) for some \( Q_0' \in \mathcal{Q} \).

Let \( Q = \{f(Q') \mid Q' \in \mathcal{Q}'\} \) and \( \mathcal{E} = (F \mid F' \in \mathcal{E}(\mathcal{R}')) \cup \{G\} \). Define \( I : \mathcal{E} \rightarrow \mathcal{Q} \) by \( I(F) = f(I'(F')) \) for all \( F \in \mathcal{F} \setminus \{G\} \) and \( I(G) = f(Q_0') \).

**Claim 34.** \((Q, I)\) is (here, too) a regimentation of \( F \).

By Remark 32, \( Q \) is a set of internally disjoint \( s - t \) paths.

By (4) \( E(I(F)) \subseteq F \) for \( F \in \mathcal{E} \setminus \{G\} \), and by (1) of Claim 31 \( E(I(G)) = E(f(Q_0')) \subseteq G \), so condition (2) of Definition 15 is fulfilled.

In addition,

\[
|I^{-1}(Q)| = |I^{-1}(f^{-1}(Q))| = |E(f^{-1}(Q))| - 1 = |E(f^{-1}(Q))| - 1 = |E(Q)| - 1
\]
for all $Q \neq f(Q'_0)$. On the other hand, for $Q = f(Q'_0)$,

$$|I^{-1}(Q)| = |I^{-1}(f^{-1}(Q))| + 1 = |E(f^{-1}(Q))| = |E(f^{-1}(Q)(2))| - 1 = |E(Q)| - 1.$$  

This proves condition (3) in Definition 15.

Furthermore, since $\bigcup_{Q' \in Q} V^o(Q') = V^o(N) \setminus \{x\}$, $V^o(Q'(1)) = V^o(Q')$ and $V^o(Q'(2)) = V^o(Q') \cup \{x\}$, we have

$$\bigcup_{Q \in Q} V^o(Q) = \bigcup_{Q' \in Q \setminus \{Q'_0\}} V^o(Q'(1)) \cup V^o(Q'(2)) = V^o(N).$$

So, condition (1) of Definition 15 is also valid, completing the proof of the theorem. \qed

3 Proof of Theorem 8

Let us first state the theorem in a slightly stronger form, that allows some of the edge sets to be empty.

**Theorem 35.** Let $S$ be a family of $2n + k - 3$ sets of edges in a bipartite graph $G$, at most $k - 2$ of them being empty. If $\nu(\bigcup K) \geq n$ for every $K \subseteq S$ of size $k$ then $\nu_R(S) \geq n$.

Before proving the theorem, we need the following definition.

**Definition 36.** For a matching $N$ in a graph, a path is called $N$-alternating if every other edge in it belongs to $N$ and it is called augmenting if its starting edge and ending edge are not in $N$.

**Proof.** Suppose, for contradiction, that $\nu_R(S) = m < n$. Let $M = \{f_S \mid S \in S_0\}$ be a maximal size $S$-rainbow matching, where $f_S \in S$. Let $S'_0 = S \setminus S_0$.

Let $A, B$ be the two sides of $G$. For every $h \in E(G)$ let $h_A$ be the $A$-vertex of $h$, and $h_B$ the $B$ vertex.

We construct a network $N$, having the property that its paths correspond to $M$-alternating paths, and its source-target paths correspond to augmenting $M$-alternating paths. Let $V(N) = M \cup \{s, t\}$, where $s$ represents $U_A := A \setminus \bigcup M$, and $t$ represents $U_B := B \setminus \bigcup M$.

To every edge $h = ab \in E(G) \setminus M$ ($a \in A, b \in B$) we assign an edge $F(h)$ of $N$, as follows.

1. If $a \in f \in M$, $b \in g \in M$ then $F(h) = fg$.
2. If $a \in U_A$ and $b \in g \in M$ then $F(h) = sg$.
3. If $b \in U_B$ and $a \in f \in M$ then $F(h) = ft$.
4. If $a \in U_A$ and $b \in U_B$ then $F(h) = st$. 

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For a set $S$ of edges in $G$, let $F(S)$ be the set of edges in $N$, defined by $F(S) = \{ F(h) \mid h \in S \setminus M \}$. The function $F$ is not one-to-one, because the inverse image of an edge $sh$ ($h \in M$) can be any edge $ah_B$, $a \in U_A$.

Clearly, if $M \cup S$ contains an augmenting $M$-alternating path, then $F(S)$ contains an $s-t$ path in $N$, and vice versa. Let $\mathcal{F} = \{ F(S) \mid S \in S_0 \}$.

Since, by assumption, $m < n$, $|S_0'| = 2n - m + k - 3 \geq n + m - 1$. If $N$ is a matching of size $n$, then $M \cup N$ contains an augmenting $M$-alternating path, and hence $F(N)$ contains an $s-t$ path. Hence, by Theorem 24 and Theorem 25, either

(i) there exists an $\mathcal{F}$-rainbow $s-t$ path, or

(ii) $|S_0'| = m + k - 1$ and $\mathcal{F}$ is regimented.

In case (i), as mentioned above, $P$ yields an augmenting $M$-alternating path, whose application yields a larger rainbow matching. So we may assume (ii). Let $R = (Q, I)$ be the regimentation of $\mathcal{F}$. Let $F^{-1}(IE(R)) = (S \in S_0 \mid F(S) \in IE(R))$. Since at most $k - 2$ sets $S \in S$ are empty and $|IE(R)| = |S_0'| - |E| = k - 1$ by Lemma 16, $\bigcup F^{-1}(IE(R))$ is non-empty.

Claim 37. It is possible to choose $M$ so that $\bigcup IE(R) \neq \emptyset$.

This means that $\bigcup F^{-1}(IE(R)) \setminus M \neq \emptyset$.

Proof. Assume, for contradiction, that $\bigcup F^{-1}(IE(R)) \subseteq M$. Since $\bigcup F^{-1}(IE(R))$ is non-empty, there is an element $S_0 \in S_0$ such that $f_{S_0} \in M \cap \bigcup F^{-1}(IE(R))$. Let $S_1$ be a set in $F^{-1}(IE(R))$ containing $f_{S_0}$. By the condition of the theorem, $\bigcup F^{-1}(IE(R)) \cup S_0$ contains a matching of size $n$. This, in turn, means that there exists an edge $f \in \bigcup F^{-1}(IE(R)) \cup S_0 \setminus M$. Since by assumption $\bigcup F^{-1}(IE(R)) \subseteq M$, we have $f \in S_0$. Now we can consider $S_1 = (S_0 \setminus \{ S_0 \}) \cup \{ S_1 \}$ as a represented set of $M$ by changing the roles of $S_0$ and $S_1$. Let $\mathcal{F} = (F(S) \mid S \in S_1)$. Then by the same reasoning as above, we may assume that $\mathcal{F}$ is regimented by $R = (\hat{Q}, \hat{I})$. By Lemma 23, we have $F(S_0) \in IE(\hat{R})$ and $f \in S_0 \setminus M$, which implies $\bigcup IE(\hat{R}) \neq \emptyset$.

So, we assume $\bigcup IE(R) \neq \emptyset$. Let $pq$ be an edge in $F(S)$ for some $F(S) \in IE(R)$. By Lemma 18, $pq$ is a backward edge on some path $Q \in Q$. Let $Q = sy_1y_2 \ldots y_ct$. For each $1 \leq i < c$ let $e_i$ be the edge connecting the $(y_i)_A$ with $(y_{i+1})_B$, in $G$ (these are the $F^{-1}$ images of the edges of $Q$).

Let $\ell$ be such that $p = ye$. As $p$ is an edge in $M$, $p$ is contained in a set $S_p \in S_0$. By the condition of the theorem, the set $S_p \cup \bigcup F^{-1}(IE(R))$ contains a matching $N$ of size $n$. Since $|M| < n$, $N$ contains an edge $ax$, where $a \in U_A$ (recall that $U_A = A \setminus M$). Suppose $x \in U_B$. If $ax \in \bigcup F^{-1}(IE(R))$, then $M \cup \{ ax \}$ is a rainbow matching, contradicting the maximality of $M$. Thus we have $ax \in S_p$. Let $q = y_{\ell'}$ for some $\ell' < \ell$. Now consider

$$N = (M \cup \{ ax, pAq_B \} \cup \{(y_i)_A(y_{i+1})_B \mid \ell' \leq i \leq \ell - 1\}) \setminus \{ ye, ye_{\ell+1}, \ldots, ye_{\ell}\}.$$

Since $pAq_B \in S$ and $\{(y_i)_A(y_{i+1})_B \mid \ell' \leq i \leq \ell - 1\}$ has enough represented sets in $I^{-1}(Q)$, then $N$ is a rainbow matching. However, it is a contradiction to the maximality of $M$ since $N$ has size $|M| + 1$. 

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Hence, we may assume that $x$ lies on an edge $h$ of $M$, meaning that $sh$ is an edge in $F(S_p) \cup \bigcup I\mathcal{E}(\mathcal{R})$. Since all edges in $\bigcup I\mathcal{E}(\mathcal{R})$ are backwards, and $sh$ is not a backward edge on any path, $sh$ belongs to $F(S_p)$.

Let $h \in V(Q_h)$ for $Q_h \in \mathcal{Q}$, and let $P$ be the $s-t$ path $shQ_h$. Let $\hat{P}$ be a path in $F^{-1}(P)$, whose first vertex is $a$, meaning that its first edge belongs to $S_p$. Let $X \triangle Y$ be the symmetric difference of $X$ and $Y$, that is, $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. Let $N = M \triangle E(\hat{P})$.

Consider two possibilities:

**Possibility I:** $h = y_d$ for $d \leq \ell$.

In this case, $N$ is an $\mathcal{S}$-rainbow matching of size $m + 1$: we let the first edge, $ah_B$, represents $S_p$, and the other edges in $E(\hat{P}) \setminus M$ has a represented sets in $I^{-1}(Q)$ and keep all other representations as they are. Since the edge in $M$ representing $S_p$ is removed by the symmetric difference, this assignment of representation yields an $\mathcal{S}$-rainbow matching.

**Possibility II:** Either $h \notin V(Q)$ or $h = y_d$ for $d > \ell$.

In this case, $N$ is not $\mathcal{S}$-rainbow, since there are two edges representing $S_p$, namely $p$ and $ah_B$. But this is rectifiable, using the edge $pq$. Suppose that $q = y_b$, where $b < \ell$. Let $C$ be the cycle whose edges are $p, q, e_b, y_{b+1}, e_{b+1}, \ldots, e_{\ell-1}, p = y_f$.

Then $N'$ is a matching of size $m + 1$, and it is $\mathcal{S}$-rainbow, since $S_p$ is represented in it just once - by the edge $ah_B$.

\[\square\]

4 Somewhere over the rainbow - two possible strengthenings

It is possible that Theorem 8 can be given a strong cooperation generalisation.

**Conjecture 38.** Let $\mathcal{F}$ be a family of $2k - 1$ sets of edges in a bipartite graph. If $\nu(\bigcup \mathcal{K}) \geq \min(|\mathcal{K}|, k)$ for every $\mathcal{K} \subseteq \mathcal{F}$ then $\nu_R(\mathcal{F}) \geq k$.

This generalises the following theorem from [2]:

**Theorem 39.** If $\mathcal{F} = (F_1, \ldots, F_{2k-1})$ is a family of matchings in a bipartite graph, and $|F_i| = \min(i, k)$ for all $i$, then there exists an $\mathcal{F}$-rainbow matching of size $k$.

Here is another possible strong version of Theorem 8.

**Conjecture 40.** Let $\mathcal{F} = (F_1, \ldots, F_{2k-1})$ be a system of bipartite sets of edges, sharing the same bipartition, and suppose that $\nu(F_i) \geq k$ for all $i \leq 2k - 1$. Let $V'$ be a copy of $V$ disjoint from $V$, let $F'_i$ be a copy of $F_i$ on $V'$ ($i \leq 2k - 1$) and let $\tilde{F}_i = F_i \cup F'_i$ for $i \leq 2k - 1$. Then the system $(\tilde{F}_i \mid i \leq 2k - 1)$ has a full rainbow matching.

This implies Theorem 2, since by the pigeonhole principle either $V$ or $V'$ contains a rainbow matching of size $k$. Conjecture 40 would follow from the following conjecture of the first author and Eli Berger [1].

**Conjecture 41.** $n$ matchings of size $n$ in any graph have a rainbow matching of size $n - 1$. 

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