Cooperative conditions for the existence of rainbow matchings

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Abstract

Let $k > 1$, and let $\mathcal{F}$ be a family of $2n + k - 3$ non-empty sets of edges in a bipartite graph. If the union of every k members of $\mathcal{F}$ contains a matching of size n, then there exists an $\mathcal{F}$-rainbow matching of size n. Replacing $2n + k - 3$ by $2n + k - 2$, the result is true also for $k = 1$, and it can be proved (for all k) both topologically and by a relatively simple combinatorial argument. The main effort is in gaining the last 1, which makes the result sharp.

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1 Introduction

Throughout the paper, "family" means "multiset", meaning that elements may repeat. To differentiate the notation, we use round brackets for families, and (as usual) curly brackets for sets. For a family $\mathcal{F}$, we write $\mathcal{F} \setminus \{F\}$ and $\mathcal{F} \cup \{F\}$ in the family sense. That is, $\mathcal{F} \setminus \{F\}$ contains one less copy of F than $\mathcal{F}$ if $F \in \mathcal{F}$, and $\mathcal{F} \cup \{F\}$ contains one more copy of F than $\mathcal{F}$.

Given a family $\mathcal{S} = (S_1, \dots, S_m)$ of sets, an *$\mathcal{S}$-rainbow set* is the image of a partial choice function of $\mathcal{S}$. So, it is a set $\{x_{i_j} \mid j \leqslant k\}$, where $1 \leqslant i_1 < \cdots < i_k \leqslant m$ and $x_{i_j} \in S_{i_j}$.

A *complex* is a closed down hypergraph, meaning that any subset of any edge is an edge. The injectivity - at most one element from every set S_i - is a "smallness" condition, in the sense that the set of injective choices is a complex. Hence statements of interest are of the form "there exists a large rainbow set satisfying certain conditions (like being a matching)". The classical theorem of this type is Hall's marriage theorem.

Below, again, $\mathcal{S} = (S_1, \dots, S_m)$ is a family of sets. For a set $I \subseteq [m]$, let $\mathcal{S}_I = \bigcup_{i \in I} S_i$.

Theorem 1. *If $|\mathcal{S}_J| \geqslant |J|$ for every $J \subseteq [m]$ then there is a full rainbow set, that is, a rainbow set of size m.*

Another well-known rainbow result is Drisko's theorem, on rainbow matchings. The following slightly more general version of the original theorem was proved in [1]:

Theorem 2. *[7] $2n - 1$ matchings in a bipartite graph, of size n each, have a rainbow matching of size n.*

There is a conspicuous difference between the two theorems: in the first the condition is "cooperative", namely it is on subfamilies of $\mathcal{S}$, whereas in the second it is on singletons - each S_i is assumed to be large by itself. On the other hand, there is a condition on the number of the sets S_i.

1.1 A cooperative version of the Kalai-Meshulam theorem

A complex $\mathcal{C}$ is said to be *d-Leray* if $\tilde{H}_k(\mathcal{C}[S]) = 0$ for all $S \subseteq V$ and all $k \geqslant d$ ($\tilde{H}_k$ is the reduced k-th homology group). Let $\lambda(\mathcal{C})$ be the smallest number d such that $\mathcal{C}$ is d-Leray.

A basic result in this direction is a theorem of Kalai and Meshulam [11]:

Theorem 3. *Let $\mathcal{M}$ and $\mathcal{C}$ be a matroid and a complex, respectively, on the same ground set. If $\lambda(lk_{\mathcal{C}}(S)) < rank_{\mathcal{M}}(V \setminus S)$ for every $S \in \mathcal{C}$ then $\mathcal{M} \setminus \mathcal{C} \neq \emptyset$.*

Here $lk_{\mathcal{C}}(S) = \{T \subseteq V \setminus S \mid S \cup T \in \mathcal{C}\}$. The theorem above is a re-formulation of Theorem 1.6 in [11].

The following was proved in [12]:

Theorem 4. *For any complex $\mathcal{C}$ and set $S \in \mathcal{C}$, $\lambda(lk_{\mathcal{C}}(S)) \leqslant \lambda(\mathcal{C})$.*

Theorems 3 and 4, combined, yield the following:

Theorem 5. *If $\lambda(\mathcal{C}) \leqslant d$ and $\mathcal{S} = (S_1, \ldots, S_{d+k})$ is a family of subsets of $V(\mathcal{C})$ satisfying $\mathcal{S}_I \notin \mathcal{C}$ whenever $I \subseteq [d+k]$ is of size k, then there exists an $\mathcal{S}$-rainbow non-$\mathcal{C}$ set.*

Proof. By duplicating vertices, if necessary (a vertex having a distinct copy for every set S_i it belongs to), we may assume that the sets S_i are disjoint. Let $\mathcal{M}$ be the partition matroid defined by the sets S_i. By Theorems 4 and 3 it suffices to show that if $S \in \mathcal{C}$ then $rank_{\mathcal{M}}(V \setminus S) > d$. This follows from the condition $\mathcal{S}_I \notin \mathcal{C} \quad (|I| \geqslant k)$ and the fact that $rank_{\mathcal{M}}(A) = |\{i : A \cap S_i \neq \emptyset\}|$. □

This is a "cooperative" version of the Kalai-Meshulam theorem, namely many sets join forces to contain a set not belonging to $\mathcal{C}$.

1.2 A cooperative version of Theorem 2

For a set F of edges we denote by $\nu(F)$ the maximal size of a matching in F. For a family $\mathcal{F} = (F_1, \ldots, F_m)$ of sets of edges, we denote by $\nu_R(\mathcal{F})$ the maximal size of an $\mathcal{F}$-rainbow matching.

Let t be an integer, and let $n \leqslant t$. Let $\mathcal{C}$ be the complex consisting of all $F \subseteq E(K_{t,t})$, satisfying $\nu(F) < n$. In [3] it was shown that $\lambda(\mathcal{C}) \leqslant 2n-2$. Together with Theorem 5 this yields:

Theorem 6. $2n+k-2$ *sets of edges in a bipartite graph, the union of any k of which contains a matching of size n, have a rainbow matching of size n.*

Notation 7. We write $(m, k, n) \to_{\mathcal{B}} q$ for the statement "every m nonempty sets of edges in a bipartite graph, the union of every k of which contains a matching of size n, have a rainbow matching of size q".

In this notation, the theorem says that $(2n+k-2, k, n) \to_{\mathcal{B}} n$. The case $k=1$ is Theorem 2. The main result of this paper is that for $k > 1$ this can be improved by 1, thereby obtaining a sharp bound.

Theorem 8. $(2n+k-3, k, n) \to_{\mathcal{B}} n$ *whenever* $1 < k \leqslant n$.

The sharpness of this result, namely the fact that $(2n+k-4, k, n) \not\to_{\mathcal{B}} n$ for any k, is given by the following example. In C_{2n}, take the odd edges matching repeated $n-1$ times, the even edges matching repeated $n-2$ times, and a singleton set, consisting of an even edge, repeated $k-1$ times. Explicitly:

Example 9. Consider a complete bipartite graph $K_{n,n}$ with sides $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$. Let

$$S_i = \begin{cases} \{a_1b_1, a_2b_2, \ldots, a_nb_n\} & \text{if } i \in [n-1], \\ \{a_1b_2, a_2b_3, \ldots, a_{n-1}b_n, a_nb_1\} & \text{if } i \in [2n-3] \setminus [n-1], \\ \{a_1b_2\} & \text{if } i \in [2n+k-4] \setminus [2n-3]. \end{cases}$$

Let $\mathcal{S} = (S_i \mid i = 1, \ldots, 2n+k-4)$. Then for any $I \subseteq [2n+k-4]$ with $|I| \geqslant k$, $\nu(\mathcal{S}_I) \geqslant n$, and $\nu_R(\mathcal{S}) < n$.

Remark 10. After our result was obtained, Holmsen and Lee [10] gave a topological proof of Theorem 8, using a strong version of Theorem 3. Their result is a somewhat stronger version of Theorem 8.

1.3 Cooperative versions of Colorful Caratheodory

Part of the motivation for Theorem 8 comes from the existence of cooperative versions of a famous rainbow result - Bárány's Colorful Caratheodory theorem [6]. In fact, as we shall see below (first proof of Theorem 25), the affinity is not merely formal. Theorem 6 follows from a cooperative version of Colorful Caratheodory.

Wegner [13] noted that the complex $\mathcal{C}$ of sets of vectors in $\mathbb{R}^d$ not containing a given vector v in their convex hull satisfies $\lambda(\mathcal{C}) = d$. Similarly, the complex $\mathcal{D}$ of sets not containing v in their cone (set of non-negative combinations) satisfies $\lambda(\mathcal{D}) = d-1$. This, together with Theorem 5, yields:

Theorem 11. *Let $v \in \mathbb{R}^d$.*

1. *If $\mathcal{S} = (S_1, \ldots, S_{d+k})$ is a family of subsets of $\mathbb{R}^d$ such that $v \in conv(\mathcal{S}_K)$ for every $K \subseteq [d+k]$ of size k, then there exists an $\mathcal{S}$-rainbow set S such that $v \in conv(S)$.*

2. *If $\mathcal{S} = (S_1, \ldots, S_{d+k-1})$ is a family of subsets of $\mathbb{R}^d$ such that $v \in cone(\mathcal{S}_K)$ for every $K \subseteq [d+k-1]$ of size k, then there exists an $\mathcal{S}$-rainbow set S such that $v \in cone(S)$.*

The case $k = 2$ of part (1) of the theorem was strengthened by Holmsen-Pach-Tverberg [9] and Arocha et.al. [5]:

Theorem 12. *If $S_1, \ldots, S_{d+1}$ are non-empty sets in $\mathbb{R}^d$, and $v \in conv(S_i \cup S_j)$ whenever $1 \leqslant i < j \leqslant d+1$, then there is a rainbow set S with $v \in conv(S)$.*

Holmsen [8] gave a topological proof of this result, using a notion he called "near d-Lerayness", which means that $lk_{\mathcal{C}}(S)$ is d-Leray for every non-empty $S \in \mathcal{C}$. The same argument can be used to prove the analogous strengthening for all $k > 1$:

Theorem 13. *Let $k > 1$, and let $\mathcal{S} = (S_1, \ldots, S_{d+k-1})$ be a family of non-empty sets in $\mathbb{R}^d$, such that every k of them contain v in the convex hull of their union. Then there is an $\mathcal{S}$-rainbow set containing v in its convex hull.*

The analogous strengthening of part (2) of Theorem 11 is false, as witnessed by simple counterexamples.

Example 14. Let $v_1, \ldots, v_{d+1}$ be the vertices of a d-dimensional simplex $\sigma \subseteq \mathbb{R}^d$ whose barycenter is the origin. Let v be the barycenter of face $\{v_1, \ldots, v_d\}$ of σ. Consider the family $\mathcal{S} = (S_1, \ldots, S_{d+k-2})$ of non-empty sets in $\mathbb{R}^d$, where $S_i = \{v_1, \ldots, v_d\}$ for $1 \leqslant i \leqslant d-1$ and $S_j = \{v_{d+1}\}$ for $d \leqslant j \leqslant d+k-2$. Among any k sets in $\mathcal{S}$, at least one is S_i for some $1 \leqslant i \leqslant d-1$, hence the convex cone spanned by their union contains v. However, there is no $\mathcal{S}$-rainbow set S such that $v \in cone(S)$.

2 Rainbow paths

The proof of Theorem 8 is based on a combinatorial proof of the result $(2n+k-2, k, n) \to_{\mathcal{B}} n$, and analysis of the extreme case. This proof, in turn, uses a lemma on rainbow paths in networks. To get the extra 1 we analyze the extreme cases of that lemma. The analysis uses ideas from an analogous lemma in [4], which is the case $k = 1$. But apart from a higher level of complexity, there is the difference that for $k > 1$ the analysis leads to an improvement of 1 in the theorem - which was not the case for $k = 1$.

A *network* is a triple $\mathcal{N} = (D, s, t)$, where D is a digraph, and s, t are two special vertices in it, called *source* and *target*. We assume that there are no edges going out of t or into s. We write $V(\mathcal{N})$ for $V(D)$. The set $V(\mathcal{N}) \setminus \{s, t\}$ is denoted by $V^\circ(\mathcal{N})$, and its elements are called "inner vertices". For an $s-t$ path P let $V^\circ(P) = V^\circ(\mathcal{N}) \cap V(P)$. Two $s-t$ paths P, Q are said to be *internally disjoint* if $V^\circ(P) \cap V^\circ(Q) = \emptyset$.

For an $s-t$ path Q let $B(Q)$ be the set of backward edges on Q, namely those directed edges pq where $p, q \in V(Q)$ and q precedes p on Q. Let s_Q be the vertex following s in Q, and t_Q the vertex preceding t in Q. Define $U(Q) = \{vs_Q \mid v \in V^\circ(\mathcal{N}) \setminus V(Q)\} \cup \{t_Q u \mid u \in V^\circ(\mathcal{N}) \setminus V(Q)\}$. ("$U$" stands for "useless", since such edges cannot be used as shortcuts - this will be clarified below).

We shall borrow a term - "regimented" - from [4], but its use is a bit different here.

Definition 15. Let $\mathcal{F}$ be a family of sets of edges in $\mathcal{N}$. A *regimentation* of $\mathcal{F}$ is a pair $\mathcal{R} = (\mathcal{Q} = \mathcal{Q}(\mathcal{R}), I = I(\mathcal{R}))$, where $\mathcal{Q}$ is a set of internally disjoint $s-t$ paths, and I is a function from a subset $\mathcal{E} = \mathcal{E}(\mathcal{R})$ of $\mathcal{F}$ (the "essential" sets) onto $\mathcal{Q}$, satisfying the following conditions:

1. $\bigcup_{Q \in \mathcal{Q}} V(Q) = V(\mathcal{N})$,
2. $E(I(F)) \subseteq F$ for every $F \in \mathcal{E}$, and
3. $|I^{-1}(Q)| = |E(Q)| - 1$ for every $Q \in \mathcal{Q}$.

Let $\mathcal{IE}(\mathcal{R}) = \mathcal{F} \setminus \mathcal{E}(\mathcal{R})$ (the "inessential" sets) and $B(\mathcal{R}) = \bigcup_{Q \in \mathcal{Q}} B(Q)$.

If such a regimentation $\mathcal{R}$ exists, we say then that $\mathcal{F}$ is regimented by $\mathcal{R}$.

Conditions (1) and (3) imply:

Lemma 16. $|\mathcal{E}(\mathcal{R})| = |V^\circ(\mathcal{N})|$.

Proof. Since $\mathcal{E}(\mathcal{R}) = \bigcup_{Q \in \mathcal{Q}} I^{-1}(Q)$, we have $|\mathcal{E}(\mathcal{R})| = \sum_{Q \in \mathcal{Q}} |I^{-1}(Q)|$. Then by the condition (3) of a regimentation, we have

$$|\mathcal{E}(\mathcal{R})| = \sum_{Q \in \mathcal{Q}} |I^{-1}(Q)| = \sum_{Q \in \mathcal{Q}} (|E(Q)| - 1) = \sum_{Q \in \mathcal{Q}} |V^\circ(Q)|.$$

Since $\mathcal{Q}$ is a set of internally disjoint $s-t$ paths, the condition (1) of a regimentation implies $\sum_{Q \in \mathcal{Q}} |V^\circ(Q)| = |V^\circ(\mathcal{N})|$, and hence we obtain $|\mathcal{E}(\mathcal{R})| = |V^\circ(\mathcal{N})|$. $\square$

Notation 17 (Pruning and concatenation of paths)*.* If P is a directed path and $x \in V(P)$ then Px is the part of P up to and including x, and xP is the part of P starting at x. If two paths P and Q meet at a vertex x, then PxQ denotes the walk obtained by concatenating Px and xQ. If the endpoint of a path P coincides with the initial point in a path Q, we write PQ for the walk that is the concatenation of P and Q.

Lemma 18. *Suppose $\mathcal{F}$ is regimented by $\mathcal{R} = (\mathcal{Q}, I)$, and let $B = B(\mathcal{R}), \mathcal{IE} = \mathcal{IE}(\mathcal{R})$. If there is no $\mathcal{F}$-rainbow $s - t$ path, then $\bigcup \mathcal{IE} \subseteq B$ and $\bigcup I^{-1}(Q) \subseteq E(Q) \cup B \cup U(Q)$ for every $Q \in \mathcal{Q}$.*

(For a set $\mathcal{K}$ of sets $\bigcup \mathcal{K}$ is the union of all sets in $\mathcal{K}$.)

Proof. Let vu be an edge belonging to F for some $F \in \mathcal{F}$. Assume that $v \in V(Q_1)$, $u \in V(Q_2)$. Let $P = Q_1 vu Q_2$ (see Notation 17).

To obtain the conclusion of the lemma, we will show the following.

1. When $Q_1 = Q_2$, P is an $\mathcal{F}$-rainbow $s - t$ path unless $vu \in B(Q_1)$ or $vu \in E(Q_1)$ and $F \in I^{-1}(Q_1)$.

2. When $Q_1 \neq Q_2$, P is an $\mathcal{F}$-rainbow $s - t$ path unless $v = t_{Q_1}$ and $F \in I^{-1}(Q_1)$, or $u = s_{Q_2}$ and $F \in I^{-1}(Q_2)$.

First suppose that $Q_1 = Q_2$. If v precedes u on Q_1 and $vu \notin E(Q_1)$, then P is an $\mathcal{F}$-rainbow $s-t$ path, since by part (3) of Definition 15 it has enough represented sets for its length. If $vu \in E(Q_1)$, then P is an $\mathcal{F}$-rainbow $s - t$ path unless $F \in I^{-1}(Q_1)$. This proves (1).

Now assume $Q_1 \neq Q_2$. We may assume that $v \in V^{\circ}(Q_1)$ and $u \in V^{\circ}(Q_2)$ since if not the claim is a special case of (1). Then $Q_1 v$ and $u Q_2$ are rainbow, and they have enough represented sets in $I^{-1}(Q_1)$ and $I^{-1}(Q_2)$, respectively. If $F \notin I^{-1}(Q_1) \cup I^{-1}(Q_2)$, then P is rainbow. If $F \in I^{-1}(Q_1)$ and $v \neq t_{Q_1}$, then $Q_1 vu$ is rainbow since it has enough represented sets in $I^{-1}(Q_1)$, since it has length at most $|E(Q_1)| - 1$. Similarly if $F \in I^{-1}(Q_2)$ and $u \neq s_{Q_2}$, then $vu Q_2$ is rainbow since it has enough represented sets in $I^{-1}(Q_2)$. In both cases P is rainbow, which proves (2).

Since we assume there is no $\mathcal{F}$-rainbow $s-t$ path, if $F \in \mathcal{IE}$, then $vu \in B$ by (1) and (2). Thus $\bigcup \mathcal{IE} \subseteq B$. If $F \in I^{-1}(Q)$ for some $Q \in \mathcal{Q}$, then $vu \in E(Q) \cup B \cup U(Q)$ by (1) and (2). Thus $\bigcup I^{-1}(Q) \subseteq E(Q) \cup B \cup U(Q)$. □

Corollary 19. *Let $\mathcal{F}$ be regimented by $\mathcal{R}$, and assume that there is no $\mathcal{F}$-rainbow $s - t$ path. If $F \in \mathcal{IE}(\mathcal{R})$ then F does not contain an $s - t$ path.*

In fact, F does not even contain an edge sy.

Lemma 20. *Let P, Q be $s - t$ paths in a network (D, s, t). If $E(P) \subseteq E(Q) \cup B(Q) \cup \tilde{B} \cup U(Q)$ for some collection $\tilde{B}$ of edges that are vertex-disjoint from Q, then $P = Q$.*

Proof. The only edge leaving s in $E(Q) \cup B(Q) \cup \tilde{B} \cup U(Q)$ is $ss_Q \in E(Q)$, and the only edge to t is $t_Q t \in E(Q)$. So these are necessarily the first and last edges of P. Therefore P has no edges from $U(Q)$, since the in-degree of s_Q and the out-degree of t_Q in P are 1.

As $E(Q) \cup B(Q)$ and $\tilde{B}$ are disconnected, $E(P) \cap \tilde{B} = \emptyset$. It remains to show that $E(P) \cap B(Q) = \emptyset$, which follows from the fact that P does not repeat vertices. $\square$

Combining Lemmas 18 and 20 yields:

Corollary 21. *Let $\mathcal{F}$ be regimented by $\mathcal{R}$, and having no rainbow $s-t$ path. If $F \in \mathcal{E}(\mathcal{R})$ then $I(F)$ is the only $s-t$ path contained in F.*

By Corollaries 19 and 21, we can obtain the following corollary.

Corollary 22. *Let $\mathcal{F}$ be regimented by $\mathcal{R}$, and having no rainbow $s-t$ path. Then $F \in \mathcal{E}(\mathcal{R})$ if and only if F contains an $s-t$ path, and equivalently, $F \in \mathcal{IE}(\mathcal{R})$ if and only if F does not contain an $s-t$ path.*

The following argument will be used twice, and hence it receives separate mention:

Lemma 23. *Let $\mathcal{G}, \mathcal{H}$ be two families of sets of edges, none of which possesses a rainbow $s-t$ path. Suppose that $\mathcal{G}$ is regimented by $\mathcal{R} = (\mathcal{Q}, I)$ and $\mathcal{H}$ is regimented by $\mathcal{S} = (\mathcal{P}, J)$. Suppose that $\mathcal{G} \setminus \mathcal{H}$ consists of a single set of edges G, and $\mathcal{H} \setminus \mathcal{G}$ consists of single set of edges H. Then either $G \in \mathcal{IE}(\mathcal{R})$ and $H \in \mathcal{IE}(\mathcal{S})$, or $I(G) = J(H)$.*

Proof. Let $\mathcal{K} = \mathcal{G} \cap \mathcal{H}$. So $\mathcal{G} = \mathcal{K} \cup \{G\}$, $\mathcal{H} = \mathcal{K} \cup \{H\}$.

By Corollary 22, it is obvious that

$$\mathcal{K} \cap \mathcal{E}(\mathcal{R}) = \mathcal{K} \cap \mathcal{E}(\mathcal{S}). \tag{1}$$

By Corollary 21, $I(K) = J(K)$ for every $K \in \mathcal{K} \cap \mathcal{E}(\mathcal{R})$. Hence

$$\bigcup_{K \in \mathcal{E}(\mathcal{R}) \setminus \{G\}} V(I(K)) = \bigcup_{K \in \mathcal{E}(\mathcal{S}) \setminus \{H\}} V(J(K)) \tag{2}$$

Let us first show that $G \in \mathcal{IE}(\mathcal{R})$ if and only if $H \in \mathcal{IE}(\mathcal{S})$. Suppose that $G \in \mathcal{IE}(\mathcal{R})$. Then $\mathcal{E}(\mathcal{R}) \subseteq \mathcal{K}$. By (1) and Lemma 16, it follows that $\mathcal{E}(\mathcal{S}) = \mathcal{E}(\mathcal{R})$, so $H \in \mathcal{IE}(\mathcal{S})$. The converse implication is the same.

Assume next that $G \in \mathcal{E}(\mathcal{R})$ and $H \in \mathcal{E}(\mathcal{S})$. Let $Q_0 = I(G)$. Consider first the case that $V^\circ(Q_0)$ consists of a single vertex v. We have $\bigcup_{K \in \mathcal{E}(R) \setminus \{G\}} V(I(K)) = V^\circ \setminus \{v\}$, and hence by (2) we have also $\bigcup_{K \in \mathcal{E}(\mathcal{S}) \setminus \{H\}} V(J(K)) = V^\circ \setminus \{v\}$. Since the interiors of paths in $\mathcal{P}$ partition V°, it follows that $J(H)$ is the path svt, namely Q_0.

It remains to consider the case $|V^\circ(Q_0)| > 1$. Then, not counting multiplicities, $\mathcal{P} = \mathcal{Q}$, because every path of $\mathcal{Q}$ appears as $J(K)$ for some $K \in \mathcal{K}$. The only path in $\mathcal{P}$ not covered enough times by paths $J(K)$, $K \in \mathcal{E}(\mathcal{S}) \setminus \{H\}$, is Q_0. So, necessarily $J(H) = Q_0$. $\square$

The next theorem is the main step towards the proof of Theorem 8.

Theorem 24. *Let $\mathcal{N} = (D, s, t)$ be a network with n inner vertices. Let $\mathcal{F}$ be a family of $n + k - 1$ sets of edges in $\mathcal{N}$, satisfying the condition that $\bigcup \mathcal{K}$ contains an $s - t$ path, for every $\mathcal{K} \subseteq \mathcal{F}$ of size k. Then either there exists an $\mathcal{F}$-rainbow $s - t$ path, or $\mathcal{F}$ is regimented.*

The case $k = 1$ of the theorem is Theorem 3.3 in [4].

It is worth noting that the weaker result, with $\mathcal{F}$ being of size $n+k$, is not hard. First, the statement:

Theorem 25. *Let $\mathcal{N} = (D, s, t)$ be a network with n inner vertices. Let $\mathcal{F}$ be a family of $n + k$ sets of edges in $\mathcal{N}$, satisfying the condition that $\bigcup \mathcal{K}$ contains an $s - t$ path for every $\mathcal{K} \subseteq \mathcal{F}$ of size k. Then there exists an $\mathcal{F}$-rainbow $s - t$ path.*

Here are two proofs:

Proof 1. Observe that a set H of edges in $\mathcal{N}$ contains an $s - t$ path if and only if the cone of $\{\chi_b - \chi_a \mid ab \in H\}$ contains the vector $\chi_t - \chi_s$ (here χ_v is the function that is 1 on v and 0 on all other vertices). Also note that all these vectors reside in an $n + 1$-dimensional space (they are of length $n + 2$, but all are perpendicular to the all-1 vector). Apply now Theorem 11, part (2).

Proof 2. Take a maximal $\mathcal{F}$-rainbow tree T rooted at s. Assume, for contradiction, that it does not reach t. Then it represents at most n members of $\mathcal{F}$. Hence there are k sets $F \in \mathcal{F}$ not represented in T. By assumption, their union contains an $s - t$ path. The first edge leaving T can then be added to T to yield a larger rainbow tree, which contradicts the maximality of T.

Definition 26 (contracting an edge $\boldsymbol{sx}$)**.** Let sx be an edge of $\mathcal{N}$. We can contract sx to a newly defined vertex s', that will serve as the source of a new network $\mathcal{N}'$. Here is what this does to sets of edges, and to paths.

1. Let F be a set of edges in a network $\mathcal{N} = (D, s, t)$, and let sx be an edge, where x is an inner vertex. The contracted set of edges $F|_{sx \to s'}$ is obtained from F by replacing every edge sy or xy belonging to F by the edge $s'y$, and removing all edges yx.

2. An $s - t$ path P is transformed by the contraction of sx to an $s' - t$ path P', defined as follows. If $x \notin V(P)$ then $P' = P$ with s' replacing s. If $x \in V(P)$ then $P' = s'yP$ where y is the vertex following x in P (so, the vertices in Px disappear.) We also write $P' = P|_{sx \to s'}$. Note that in this definition $E(P')$ is not necessarily equal to $E(P)|_{sx \to s'}$.

Proof of Theorem 24. By induction on n. The case $n = 0$ is easy. So let $n \geqslant 1$ and assume that the theorem is valid when $n - 1$ replaces n.

Since $n+k-1 \geqslant k$, $\bigcup \mathcal{F}$ contains an $s-t$ path. So there exists at least one set $G \in \mathcal{F}$ containing an edge sx. If $x = t$ then the path st is rainbow, so we may assume that $x \neq t$. Now contract sx: for each $F \in \mathcal{F}$ let $F' = F|_{sx \to s'}$. Let $\mathcal{K}' = (F' \mid F \in \mathcal{F})$ for $\mathcal{K} \subseteq \mathcal{F}$. Let $\mathcal{N}'$ be the network with vertex set $V(\mathcal{N}) \setminus \{s, x\} \cup \{s'\}$, source s', target t, and edge set $\bigcup(\mathcal{F}' \setminus \{G'\})$.

Every $\mathcal{K} \subseteq \mathcal{F}$ of size k contains in its union the edge set of an $s-t$ path in $\mathcal{N}$, which is easily seen to imply the same, with s' replacing s, for $\mathcal{K}'$ in $\mathcal{N}'$. By the induction hypothesis, either there exists an $\mathcal{F}' \setminus \{G'\}$-rainbow $s'-t$ path P', or $\mathcal{F}' \setminus \{G'\}$ is regimented. In the first case, let y be the vertex following s' in P'. Then either syP' or $sxyP'$ is a rainbow $s-t$ path in $\mathcal{N}$, and we are done. So, we may assume the second possibility. Let $\mathcal{R}' = (\mathcal{Q}', I')$ be a regimentation of $\mathcal{F}' \setminus \{G'\}$, and let $\mathcal{E}' = \mathcal{E}(\mathcal{R}')$, $\mathcal{IE}' = \mathcal{IE}(\mathcal{R}')$.

Let $\tilde{\mathcal{IE}} = (F \in \mathcal{F} \setminus \{G\} \mid F' \in \mathcal{IE}')$ and $\tilde{\mathcal{E}} = (F \in \mathcal{F} \setminus \{G\} \mid F' \in \mathcal{E}')$.

By Lemma 16 $|\mathcal{E}'| = n - 1$, so

$$|\tilde{\mathcal{IE}}| = |\mathcal{IE}'| = k - 1. \tag{3}$$

In all claims below we assume that there is no $\mathcal{F}$-rainbow $s-t$ path.

Let $B' = \bigcup_{Q' \in \mathcal{Q}'} B(Q')$. By Lemma 18, $\bigcup \mathcal{IE}' \subseteq B'$ and $\bigcup I'^{-1}(Q') \subseteq E(Q') \cup B' \cup U(Q')$ for every $Q' \in \mathcal{Q}'$.

Notation 27 (two ways of un-contracting sx). Given an $s'-t$ path Q' in $\mathcal{N}'$, let $Q'^{(1)}$ be the path obtained from Q' by replacing s' with s and $Q'^{(2)}$ the path obtained from Q' by expanding its first edge $s'y$ to the path sxy.

Our aim is to glean from $\mathcal{R}'$ a regimentation $\mathcal{R} = (\mathcal{Q}, I)$ of $\mathcal{F}$. The set $\mathcal{E}(\mathcal{R})$ will contain G and $\mathcal{Q}$ will contain $s-t$ paths $f(Q')$, $Q' \in \mathcal{Q}'$, where f is an injective function defined as follows. Let $Q' \in \mathcal{Q}'$ and let $F \in \mathcal{F} \setminus \{G\}$ be such that $I'(F') = Q'$. By (3) and the condition of the theorem, the set $F \cup \bigcup \tilde{\mathcal{IE}}$ contains an $s-t$ path Q. Let $f(Q') = Q$.

Claim 28. *$Q' = Q|_{sx \to s'}$.*

Proof. By the choice of Q, we have $E(Q|_{sx\to s'}) \subseteq F' \cup \bigcup \mathcal{IE}'$. By Lemma 18, we have $F' \cup \bigcup \mathcal{IE}' \subseteq E(Q') \cup B' \cup U(Q') = E(Q') \cup B(Q') \cup \bigcup_{T' \in \mathcal{Q}' \setminus \{Q'\}} B(T') \cup U(Q')$. The claim now follows by Lemma 20. □

There are two possibilities:

(a) $x \notin V(Q)$. In this case $Q = Q'^{(1)}$.

(b) $x \in V(Q)$. Suppose, in this case, that Qx contains inner vertices. Let y be the first inner vertex of Qx. Then $y \in V^\circ(T')$ for some $T' \in \mathcal{Q}' \setminus \{Q'\}$, and then syT' is a rainbow $s-t$ path in $\mathcal{N}$ since it has enough represented sets in $I'^{-1}(T') \cup \{G\}$. So, we may assume that $V^\circ(Qx) = \emptyset$, meaning that the first edge on Q is sx, meaning in turn that $Q = Q'^{(2)}$.

Claim 29. *$sx \notin \bigcup \tilde{\mathcal{IE}}$.*

Proof. Let $F_0 \in \tilde{\mathcal{IE}}$ and suppose that $sx \in F_0$. Recall that $\mathcal{F}'$ is the family of sets of edges obtained, where, for every $F \in \mathcal{F}$, F' is the image of F under the contraction of sx. By the same argument as above, $\mathcal{F}' \setminus \{F_0'\}$ is regimented in $\mathcal{N}'$, by a regimentation $\mathcal{T} = (\mathcal{Q}(\mathcal{T}), J)$. Then $G' \in \mathcal{IE}(\mathcal{T})$ by Lemma 23, and hence G do not contain an edge yt. But this would imply that $G \bigcup \tilde{\mathcal{IE}}(\mathcal{R})$ does not contain such an edge, and hence that it does not contain an $s-t$ path, contrary to the assumption of the theorem. □

Since $E(Q) \subseteq F \cup \bigcup \tilde{\mathcal{IE}}$ and $\bigcup \mathcal{IE}' \subseteq B'$ by Lemma 18, a corollary of Claim 29 is:

$$E(Q) \subseteq F. \tag{4}$$

Claim 30. *The choice of $f(Q')$ is independent of the choice of F.*

Proof. We have to show that if $F_1, F_2 \in \mathcal{F} \setminus \{G\}$ satisfy $I'(F_i') = Q',\ \ i = 1,2$ and Q_i are $s-t$ paths whose edge sets are contained in $F_i \cup \tilde{\mathcal{IE}}$ $(i = 1,2)$ then $Q_1 = Q_2$. We know that Q_i are either $Q'^{(1)}$ or $Q'^{(2)}$. Assume, for contradiction, that $Q_1 \neq Q_2$, say $Q_1 = Q'^{(1)}$ and $Q_2 = Q'^{(2)}$. Then $sx \in E(Q_2)$ and hence $sx \in F_2$. The set $\mathcal{F}' \setminus \{F_2'\}$ lives in $\mathcal{N}'$, and repeating the previous argument we deduce that it has a regimentation $\mathcal{S} = (\mathcal{Q}(\mathcal{S}), J)$. By Lemma 23 $J(G') = I'(F_2') = Q'$. In particular $G' \supseteq E(Q')$. Since $Q_1 = Q'^{(1)}$, the edge $ss_{Q'}$ belongs to $E(Q_1) \subseteq F_1$. Then, using an edge from G and edges from the sets $F \in \mathcal{F}$ such that $F' \in I'^{-1}(Q')$ shows that $ss_{Q'}Q' = Q'^{(1)}$ is an $\mathcal{F}$-rainbow $s-t$ path (note that edges in $E(s_{Q'}Q')$ are also edges of F). This is the desired contradiction. □

Claim 31.

1. *If $f(Q') = Q'^{(2)}$ then $G \supseteq E(f(Q'))$.*
2. *At most one $Q' \in \mathcal{Q}'$ satisfies $f(Q') = Q'^{(2)}$.*
3. *If $f(Q') = Q'^{(1)}$ for all $Q' \in \mathcal{Q}'$ then G contains the edges of the $s-t$ path sxt.*

Proof. To prove (1), let $f(Q') = Q'^{(2)}$ for some $Q' \in \mathcal{Q}'$.

Then, by Claim 30, $sx \in F$ for every $F' \in I'^{-1}(Q')$. We use the same trick as in the proof of Claim 30, interchanging the roles of F and G. Consider $\mathcal{F}' \setminus \{F'\}$. As above, we may assume that $\mathcal{F}' \setminus \{F'\}$ is regimented, by a regimentation $(\mathcal{P}', J')$. By Lemma 23, $J'(G') = I'(F') = Q'$, implying that $G' \supseteq E(Q')$. Then G contains either $E(Q'^{(1)})$ or $E(Q'^{(2)})$. If G contains $E(Q'^{(1)})$, then $ss_{Q'}Q'$ (which is just $Q'^{(1)}$) is an $\mathcal{F}$-rainbow $s-t$ path: the edge $ss_{Q'}$ represents G; since $|I'^{-1}(Q')| = |E(Q')| - 1$, the other edges have enough represented sets $F \in \mathcal{F}$ such that $F' \in I'^{-1}(Q')$ (remember that $G \notin I'^{-1}(Q')$). We have thus shown that G does not contain $E(Q'^{(1)})$, so it contains $E(Q'^{(2)})$, namely $G \supseteq E(f(Q'))$.

Next we prove (2). Let $f(Q') = Q'^{(2)}$ for some $Q' \in \mathcal{Q}'$. By the above argument and Corollary 21, $J'(G') = Q'$ is the only path contained in G'. This directly implies (2).

Finally, we prove (3). Assume that $f(Q') = Q'^{(1)}$ for all $Q' \in \mathcal{Q}'$. Let $\tilde{\mathcal{N}}$ be the network obtained from $\mathcal{N}$ by deleting the vertex x, and let $\tilde{F}$ be the set of edges of $\tilde{\mathcal{N}}$, obtained from F by deleting all edges incident with x. Let $\tilde{\mathcal{Q}} = \{Q'^{(1)} \mid Q' \in \mathcal{Q}'\}$, and $\tilde{I}(\tilde{F}) = f(I'(F'))$. By (4) and the assumption that $f(Q') = Q'^{(1)}$ for all $Q' \in \mathcal{Q}'$ the set $\tilde{\mathcal{F}} = (\tilde{F} \mid F \in \mathcal{F})$ is regimented by $(\tilde{\mathcal{Q}}, \tilde{I})$. The fact that there is no $\mathcal{F}$-rainbow $s-t$ path implies that there is also no $\tilde{\mathcal{F}}$-rainbow $s-t$ path. Therefore, by Lemma 18, we have $\tilde{G} \cup \bigcup_{F \in \tilde{\mathcal{IE}}} \tilde{F} \subseteq \bigcup_{Q \in \tilde{\mathcal{Q}}} B(Q)$. Thus

$$G \cup \bigcup \tilde{\mathcal{IE}} \subseteq \{sx, xt\} \cup \bigcup_{Q' \in \mathcal{Q}'} B(Q'^{(1)}) \cup U(sxt).$$

By the assumption of the theorem, $G \cup \bigcup \tilde{\mathcal{I}\mathcal{E}}$ contains an $s-t$ path, say Q_G. By Lemma 20 we have $Q_G = sxt$, and by Claim 29 we obtain $G \supseteq E(Q_G)$. This concludes the proof of the claim. $\square$

Remark 32. By the claim the paths $f(Q'),\ \ Q' \in \mathcal{Q}'$ are internally disjoint. In particular, there is at most one path $f(Q')$ containing x.

We can now complete the induction step in the proof of Theorem 24, by showing that $\mathcal{F}$ is regimented.

Case I: $f(Q') = Q'^{(1)}$ for all $Q' \in \mathcal{Q}'$.

Let $\mathcal{Q} = \{f(Q') \mid Q' \in \mathcal{Q}'\} \cup \{Q_0\}$ where $Q_0 = sxt$. Let $\mathcal{E} = (F \mid F' \in \mathcal{E}(\mathcal{R}')) \cup \{G\}$. Define $I : \mathcal{E} \to\ \mathcal{Q}$ by $I(F) = f(I'(F'))$ for $F \neq G$, and $I(G) = Q_0$.

Claim 33. $(\mathcal{Q}, I)$ *is a regimentation of* $\mathcal{F}$*.*

By Remark 32 and the fact that $x \notin \bigcup_{Q' \in \mathcal{Q}'} V(f(Q'))$, $\mathcal{Q}$ is a set of internally disjoint $s-t$ paths.

By (4) $E(I(F)) \subseteq F$ for all $F \in \mathcal{E} \setminus \{G\}$, and by part (3) of Claim 31 $E(I(G)) = E(Q_0) \subseteq G$. This implies condition (2) in Definition 15.

In addition,

$$|I^{-1}(Q)| = |I'^{-1}(f^{-1}(Q))| = |E(f^{-1}(Q))| - 1 = |E(f^{-1}(Q)^{(1)})| - 1 = |E(Q)| - 1$$

for all $Q \in \mathcal{Q} \setminus \{Q_0\}$, and

$$|I^{-1}(Q_0)| = 1 = |E(Q_0)| - 1.$$

This yields condition (3) of Definition 15.

Furthermore, since $\bigcup_{Q' \in \mathcal{Q}'} V^\circ(Q') = V^\circ(\mathcal{N}) \setminus \{x\}$ and $V^\circ(Q'^{(1)}) = V^\circ(Q')$, we have

$$\bigcup_{Q \in \mathcal{Q}} V^\circ(Q) = \bigcup_{Q' \in \mathcal{Q}'} V^\circ(Q'^{(1)}) \cup \{x\} = V^\circ(\mathcal{N}).$$

This implies condition (1) of Definition 15, thus completing the proof of the claim.

Case II: $f(Q'_0) = Q_0'^{(2)}$ for some $Q'_0 \in \mathcal{Q}$.

Let $\mathcal{Q} = \{f(Q') \mid Q' \in \mathcal{Q}'\}$ and $\mathcal{E} = (F \mid F' \in \mathcal{E}(\mathcal{R}')) \cup \{G\}$. Define $I : \mathcal{E} \to \mathcal{Q}$ by $I(F) = f(I'(F'))$ for all $F \in \mathcal{F} \setminus \{G\}$ and $I(G) = f(Q'_0)$.

Claim 34. $(\mathcal{Q}, I)$ *is (here, too) a regimentation of* $\mathcal{F}$*.*

By Remark 32, $\mathcal{Q}$ is a set of internally disjoint $s-t$ paths.

By (4) $E(I(F)) \subseteq F$ for $F \in \mathcal{E} \setminus \{G\}$, and by (1) of Claim 31 $E(I(G)) = E(f(Q'_0)) \subseteq G$, so condition (2) of Definition 15 is fulfilled.

In addition,

$$|I^{-1}(Q)| = |I'^{-1}(f^{-1}(Q))| = |E(f^{-1}(Q))| - 1 = |E(f^{-1}(Q)^{(1)})| - 1 = |E(Q)| - 1$$

for all $Q \neq f(Q'_0)$. On the other hand, for $Q = f(Q'_0)$,

$$|I^{-1}(Q)| = |I'^{-1}(f^{-1}(Q))| + 1 = |E(f^{-1}(Q))| = |E(f^{-1}(Q)^{(2)})| - 1 = |E(Q)| - 1.$$

This proves condition (3) in Definition 15.

Furthermore, since $\bigcup_{Q' \in \mathcal{Q}'} V^\circ(Q') = V^\circ(\mathcal{N}) \setminus \{x\}$, $V^\circ(Q'^{(1)}) = V^\circ(Q')$ and $V^\circ(Q'^{(2)}) = V^\circ(Q') \cup \{x\}$, we have

$$\bigcup_{Q \in \mathcal{Q}} V^\circ(Q) = \bigcup_{Q' \in \mathcal{Q}' \setminus \{Q'_0\}} V^\circ(Q'^{(1)}) \cup V^\circ(Q_0'^{(2)}) = V^\circ(\mathcal{N}).$$

So, condition (1) of Definition 15 is also valid, completing the proof of the theorem. □

3 Proof of Theorem 8

Let us first state the theorem in a slightly stronger form, that allows some of the edge sets to be empty.

Theorem 35. *Let $\mathcal{S}$ be a family of $2n+k-3$ sets of edges in a bipartite graph G, at most $k-2$ of them being empty. If $\nu(\bigcup \mathcal{K}) \geqslant n$ for every $\mathcal{K} \subseteq \mathcal{S}$ of size k then $\nu_R(\mathcal{S}) \geqslant n$.*

Before proving the theorem, we need the following definition.

Definition 36. For a matching N in a graph, a path is called N-*alternating* if every other edge in it belongs to N and it is called *augmenting* if its starting edge and ending edge are not in N.

Proof. Suppose, for contradiction, that $\nu_R(\mathcal{S}) =: m < n$. Let $M = \{f_S \mid S \in \mathcal{S}_0\}$ be a maximal size $\mathcal{S}$-rainbow matching, where $f_S \in S$. Let $\mathcal{S}_0^c = \mathcal{S} \setminus \mathcal{S}_0$.

Let A, B be the two sides of G. For every $h \in E(G)$ let h_A be the A-vertex of h, and h_B the B vertex.

We construct a network $\mathcal{N}$, having the property that its paths correspond to M-alternating paths, and its source-target paths correspond to augmenting M-alternating paths. Let $V(\mathcal{N}) = M \cup \{s, t\}$, where s represents $U_A := A \setminus \bigcup M$, and t represents $U_B := B \setminus \bigcup M$.

To every edge $h = ab \in E(G) \setminus M$ ($a \in A, b \in B$) we assign an edge $F(h)$ of $\mathcal{N}$, as follows.

1. If $a \in f \in M,\ b \in g \in M$ then $F(h) = fg$.
2. If $a \in U_A$ and $b \in g \in M$ then $F(h) = sg$.
3. If $b \in U_B$ and $a \in f \in M$ then $F(h) = ft$.
4. If $a \in U_A$ and $b \in U_B$ then $F(h) = st$.

For a set S of edges in G, let $F(S)$ be the set of edges in $\mathcal{N}$, defined by $F(S) = \{F(h) \mid h \in S \setminus M\}$. The function F is not one-to-one, because the inverse image of an edge sh ($h \in M$) can be any edge ah_B, $a \in U_A$.

Clearly, if $M \cup S$ contains an augmenting M-alternating path, then $F(S)$ contains an $s-t$ path in $\mathcal{N}$, and vice versa. Let $\mathcal{F} = \{F(S) \mid S \in \mathcal{S}_0^c\}$.

Since, by assumption, $m < n$, $|\mathcal{S}_0^c| = 2n-m+k-3 \geqslant m+k-1$. If N is a matching of size n, then $M \cup N$ contains an augmenting M-alternating path, and hence $F(N)$ contains an $s-t$ path. Hence, by Theorem 24 and Theorem 25, either

(i) there exists an $\mathcal{F}$-rainbow $s-t$ path P, or

(ii) $|\mathcal{S}_0^c| = m+k-1$ and $\mathcal{F}$ is regimented.

In case (i), as mentioned above, P yields an augmenting M-alternating path, whose application yields a larger rainbow matching. So we may assume (ii). Let $\mathcal{R} = (\mathcal{Q}, I)$ be the regimentation of $\mathcal{F}$. Let $F^{-1}(\mathcal{IE}(\mathcal{R})) = (S \in \mathcal{S}_0^c \mid F(S) \in \mathcal{IE}(\mathcal{R}))$. Since at most $k-2$ sets $S \in \mathcal{S}$ are empty and $|\mathcal{IE}(\mathcal{R})| = |\mathcal{S}_0^c| - |\mathcal{E}(\mathcal{R})| = k-1$ by Lemma 16, $\bigcup F^{-1}(\mathcal{IE}(\mathcal{R}))$ is non-empty.

Claim 37. *It is possible to choose M so that $\bigcup \mathcal{IE}(\mathcal{R}) \neq \emptyset$.*

This means that $\bigcup F^{-1}(\mathcal{IE}(\mathcal{R})) \setminus M \neq \emptyset$.

Proof. Assume, for contradiction, that $\bigcup F^{-1}(\mathcal{IE}(\mathcal{R})) \subseteq M$. Since $\bigcup F^{-1}(\mathcal{IE}(\mathcal{R}))$ is non-empty, there is an element $S_0 \in \mathcal{S}_0$ such that $f_{S_0} \in M \cap \bigcup F^{-1}(\mathcal{IE}(\mathcal{R}))$. Let S_1 be a set in $F^{-1}(\mathcal{IE}(\mathcal{R}))$ containing f_{S_0}. By the condition of the theorem, $\bigcup F^{-1}(\mathcal{IE}(\mathcal{R})) \cup S_0$ contains a matching of size n. This, in turn, means that there exists an edge $f \in \bigcup F^{-1}(\mathcal{IE}(\mathcal{R})) \cup S_0 \setminus M$. Since by assumption $\bigcup F^{-1}(\mathcal{IE}(\mathcal{R})) \subseteq M$, we have $f \in S_0$. Now we can consider $\mathcal{S}_1 = (\mathcal{S}_0 \setminus \{S_0\}) \cup \{S_1\}$ as a represented set of M by changing the roles of S_0 and S_1. Let $\tilde{\mathcal{F}} = (F(S) \mid S \in \mathcal{S}_1^c)$. Then by the same reasoning as above, we may assume that $\tilde{\mathcal{F}}$ is regimented by $\tilde{\mathcal{R}} = (\tilde{\mathcal{Q}}, \tilde{I})$. By Lemma 23, we have $F(S_0) \in \mathcal{IE}(\tilde{\mathcal{R}})$ and $f \in S_0 \setminus M$, which implies $\bigcup \mathcal{IE}(\tilde{\mathcal{R}}) \neq \emptyset$. $\square$

So, we assume $\bigcup \mathcal{IE}(\mathcal{R}) \neq \emptyset$. Let pq be an edge in $F(S)$ for some $F(S) \in \mathcal{IE}(\mathcal{R})$. By Lemma 18, pq is a backward edge on some path $Q \in \mathcal{Q}$. Let $Q = sy_1y_2 \ldots y_ct$. For each $1 \leqslant i < c$ let e_i be the edge connecting the $(y_i)_A$ with $(y_{i+1})_B$, in G (these are the F^{-1} images of the edges of Q).

Let ℓ be such that $p = y_\ell$. As p is an edge in M, p is contained in a set $S_p \in \mathcal{S}_0$. By the condition of the theorem, the set $S_p \cup \bigcup F^{-1}(\mathcal{IE}(\mathcal{R}))$ contains a matching N of size n. Since $|M| < n$, N contains an edge ax, where $a \in U_A$ (recall that $U_A = A \setminus \bigcup M$). Suppose $x \in U_B$. If $ax \in \bigcup F^{-1}(\mathcal{IE}(\mathcal{R}))$, then $M \cup \{ax\}$ is a rainbow matching, contradicting the maximality of M. Thus we have $ax \in S_p$. Let $q = y_{\ell'}$ for some $\ell' < \ell$. Now consider

$$N = (M \cup \{ax, p_Aq_B\} \cup \{(y_i)_A(y_{i+1})_B \mid \ell' \leqslant i \leqslant \ell - 1\}) \setminus \{y_{\ell'}, y_{\ell'+1}, \ldots, y_\ell\}.$$

Since $p_Aq_B \in S$ and $\{(y_i)_A(y_{i+1})_B \mid \ell' \leqslant i \leqslant \ell-1\}$ has enough represented sets in $I^{-1}(Q)$, then N is a rainbow matching. However, it is a contradiction to the maximality of M since N has size $|M|+1$.

Hence, we may assume that x lies on an edge h of M, meaning that sh is an edge in $F(S_p) \cup \bigcup \mathcal{IE}(\mathcal{R})$. Since all edges in $\bigcup \mathcal{IE}(\mathcal{R})$ are backwards, and sh is not a backward edge on any path, sh belongs to $F(S_p)$.

Let $h \in V(Q_h)$ for $Q_h \in \mathcal{Q}$, and let P be the $s-t$ path shQ_h. Let $\tilde{P}$ be a path in $F^{-1}(P)$, whose first vertex is a, meaning that its first edge belongs to S_p. Let $X \triangle Y$ be the symmetric difference of X and Y, that is, $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. Let $N = M \triangle E(\tilde{P})$.

Consider two possibilities:

Possibility I: $h = y_d$ for $d \leqslant \ell$.

In this case N is an $\mathcal{S}$-rainbow matching of size $m+1$: we let the first edge, ah_B, represents S_p, and the other edges in $E(\tilde{P}) \setminus M$ has a represented sets in $I^{-1}(Q)$ and keep all other representations as they are. Since the edge in M representing S_p is removed by the symmetric difference, this assignment of representation yields an $\mathcal{S}$-rainbow matching.

Possibility II: Either $h \notin V(Q)$ or $h = y_d$ for $d > \ell$.

In this case, N is not $\mathcal{S}$-rainbow, since there are two edges representing S_p, namely p and ah_B. But this is rectifiable, using the edge pq. Suppose that $q = y_b$, where $b < \ell$. Let C be the cycle whose edges are $p_A q_B, q, e_b, y_{b+1}, e_{b+1}, \ldots, e_{\ell-1}, p = y_\ell$. Let $N' = N \triangle E(C)$. Then N' is a matching of size $m+1$, and it is $\mathcal{S}$-rainbow, since S_p is represented in it just once - by the edge ah_B. □

4 Somewhere over the rainbow - two possible strengthenings

It is possible that Theorem 8 can be given a strong cooperation generalisation.

Conjecture 38. Let $\mathcal{F}$ be a family of $2k-1$ sets of edges in a bipartite graph. If $\nu(\bigcup \mathcal{K}) \geqslant \min(|\mathcal{K}|, k)$ for every $\mathcal{K} \subseteq \mathcal{F}$ then $\nu_R(\mathcal{F}) \geqslant k$.

This generalises the following theorem from [2]:

Theorem 39. *If $\mathcal{F} = (F_1, \ldots, F_{2k-1})$ is a family of matchings in a bipartite graph, and $|F_i| = \min(i, k)$ for all i, then there exists an $\mathcal{F}$-rainbow matching of size k.*

Here is another possible strong version of Theorem 8.

Conjecture 40. Let $\mathcal{F} = (F_1, \ldots, F_{2k-1})$ be a system of bipartite sets of edges, sharing the same bipartition, and suppose that $\nu(F_i) \geqslant k$ for all $i \leqslant 2k-1$. Let V' be a copy of V disjoint from V, let F_i' be a copy of F_i on V' ($i \leqslant 2k-1$) and let $\tilde{F}_i = F_i \cup F_i'$ for $i \leqslant 2k-1$. Then the system $(\tilde{F}_i \mid i \leqslant 2k-1)$ has a full rainbow matching.

This implies Theorem 2, since by the pigeonhole principle either V or V' contains a rainbow matching of size k. Conjecture 40 would follow from the following conjecture of the first author and Eli Berger [1].

Conjecture 41. n matchings of size n in any graph have a rainbow matching of size $n-1$.

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