Independent sets in subgraphs of a Shift graph

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Abstract

Erdős, Hajnal and Szemerédi proved that any subset G of vertices of a shift graph Sh_n^k has the property that the independence number of the subgraph induced by G satisfies $\alpha(\mathrm{Sh}_n^k[G]) \geqslant \left(\frac{1}{2} - \varepsilon\right)|G|$, where $\varepsilon \to 0$ as $k \to \infty$. In this note we prove that for $k = 2$ and $n \to \infty$ there are graphs $G \subseteq \binom{[n]}{2}$ with $\alpha(\mathrm{Sh}_n^2[G]) \leqslant \left(\frac{1}{4} + o(1)\right)|G|$, and $\frac{1}{4}$ is best possible. We also consider a related problem for infinite shift graphs.

Mathematics Subject Classifications: 05C69, 05C63

1 Introduction

For $n > k \in \mathbb{N}$ the shift graph Sh_n^k with

$$V(\mathrm{Sh}_n^k) = \{(x_1, \ldots, x_k) \ : \ 1 \leqslant x_1 < \cdots < x_k \leqslant n\}$$

is a graph in which two vertices $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{y} = (y_1, \ldots, y_k)$ are adjacent if $x_i = y_{i+1}$ for all $i \in \{1, \ldots, k-1\}$ (or $y_i = x_{i+1}$ for all $i \in \{1, \ldots, k-1\}$). Shift graphs were introduced by Erdős and Hajnal [3],[4] and are standard examples of graphs with large chromatic number and large odd girth. More precisely, while the odd girth of Sh_n^k is

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$2k+1$, they proved* that Sh_n^k has chromatic number $(1+o(1))\log^{(k-1)} n$, where $\log^{(k-1)}$ stands for $k-1$ times iterated $\log_2$.

Shift graphs have another interesting property: For each finite set $G \subseteq V(\mathrm{Sh}_n^k)$ the induced subgraph $\mathrm{Sh}_n^k[G]$ has a relatively large independent set with respect to $|G|$. In other words, the property "having a large independent subset" is hereditary for Sh_n^k. Namely, for

$$\alpha_n^k = \min\left\{\frac{\alpha(\mathrm{Sh}_n^k[G])}{|G|} \;:\; \emptyset \neq G \subseteq V(\mathrm{Sh}_n^k)\right\}, \tag{1}$$

Erdős, Hajnal and Szemerédi [5, Theorem 1] proved the following.

Theorem 1 (Erdős, Hajnal, Szemerédi)**.** *For positive integers $k < n$*

$$\alpha_n^k \geqslant \frac{1}{2} - \frac{1}{k}.$$

As for the upper bound, for $n \geqslant 2k+1$ the shift graph Sh_n^k contains an odd cycle and so $\alpha_n^k < 1/2$. Therefore, Theorem 1 yields a lower bound which for large values of k is essentially optimal.

Nevertheless, determining the values of α_n^k for fixed k and large n seems to represent an interesting and non-trivial problem. We will concentrate our attention on the case $k = 2$. In this case the bound from Theorem 1 is not optimal, as we observe that $\alpha_n^2 \geqslant 1/4$ for all n, and prove a matching upper bound.

Theorem 2. $\lim\limits_{n\to\infty} \alpha_n^2 = \dfrac{1}{4}$.

In [2], Czipszer, Erdős and Hajnal proved that the densest independent set of the infinite graph $\mathrm{Sh}_{\mathbb{N}}^2$ has density $1/4$ (see Section 3 for precise formulation). We complement their result by showing that the infinite shift graph $\mathrm{Sh}_{\mathbb{N}}^2$ does not have a similar hereditary property, i.e., there exists $G \subseteq V(\mathrm{Sh}_{\mathbb{N}}^2)$ such that any independent set in $\mathrm{Sh}_{\mathbb{N}}^2[G]$ has density zero in G (see Theorem 7).

2 Proof of Theorem 2

Note that $\alpha_n^2 = \min\left\{\frac{\alpha(\mathrm{Sh}_n^2[G])}{|G|} \;:\; \emptyset \neq G \subseteq V(\mathrm{Sh}_n^2)\right\}$ is a nonincreasing positive sequence, so the sequence $\{\alpha_n^2\}$ has a limit. Additionally, we will often view $G \subseteq V(\mathrm{Sh}_n^2)$ as a graph with $V(G) = [n]$ and set of edges equal to G. Subsequently $|G|$ will denote both a size of G as a subset of $V(\mathrm{Sh}_n^2)$, and the number of edges in G when it is viewed as a graph.

*In [4] authors considered infinite graphs, however their proof can be adapted for finite case (see [1] and [6] for more detailed description).

2.1 Lower bound

We first show that the value of the limit in Theorem 2 is at least 1/4.

Claim 3. *For every set $G \subseteq V(\mathrm{Sh}_n^2)$ we have $\alpha(\mathrm{Sh}_n^2[G]) \geqslant \frac{1}{4}|G|$.*

Proof. Let $G \subseteq V(\mathrm{Sh}_n^2)$ be given. Consider a random colouring $c : [n] \to \{r, b\}$ such that every $i \in [n]$ is coloured red/blue with probability 1/2 independently of other elements of $[n]$.

Let G_c be a random subset of G defined by

$$G_c = \{(i, j) \in G \ : \ i < j,\ c(i) = b,\ c(j) = r\}.$$

Then such G_c is always an independent set in Sh_n^2. Moreover, $\mathbb{P}(e \in G_c) = \frac{1}{4}$ for every $e \in G$, and so $\mathbb{E}(|G_c|) = \frac{1}{4}|G|$. Therefore $\alpha(\mathrm{Sh}_n^2[G]) \geqslant \frac{1}{4}|G|$. □

2.2 Upper bound

We now proceed and prove the upper bound

$$\lim_{n\to\infty} \alpha_n^2 \leqslant \frac{1}{4}. \tag{2}$$

In what follows for every $\varepsilon > 0$, integer d satisfying $\frac{3+\ln d}{4d} \leqslant \frac{\varepsilon}{2}$, and for every integer $n \geqslant n_0(\varepsilon, d)$ that is a multiple of 2^d, we will construct a graph $G_\varepsilon(n, d) \subseteq V(\mathrm{Sh}_n^2)$ with

$$\alpha(\mathrm{Sh}_n^2[G_\varepsilon(n, d)]) \leqslant \left(\frac{1}{4} + \varepsilon\right) |G_\varepsilon(n, d)|.$$

To be more precise, for such ε and d we inductively build $G_\varepsilon(n, d)$ satisfying

$$\frac{\alpha(\mathrm{Sh}_n^2[G_\varepsilon(n, d)])}{|G_\varepsilon(n, d)|} \leqslant \left(\frac{1}{4} + \frac{3 + \ln d}{4d} + \frac{\varepsilon}{2}\right). \tag{3}$$

Since $\{\alpha_n^2\}$ is nonincreasing, (3) implies that $\lim_{n\to\infty} \alpha_n^2 \leqslant 1/4 + \varepsilon$, which subsequently implies (2) by letting $\varepsilon \to 0$.

While constructing $G_\varepsilon(n, d)$ we will use random bipartite graphs. Recall that if G is a graph and $X, Y \subseteq V(G)$ then $G[X, Y]$ is a graph consisting of edges of G with one vertex in X and another in Y. Finally let $e_G(X, Y) = |E(G[X, Y])|$ and we will omit subscript when G is obvious from the context. The following claim can be easily verified by considering a random graph and so the proof of Claim 4 is postponed to Appendix.

Claim 4. *For $\varepsilon > 0$ and $d \in \mathbb{N}$ there is $n_0 = n_0(\varepsilon, d)$ such that for all $n \geqslant n_0$ that are divisible by 2^d the following holds. Let $[n] = S \cup L$, where $S = \{1, \ldots, \frac{n}{2}\}$ and $L = [n] \setminus S$. There exists a bipartite graph $B_\varepsilon(n, d)$ with bipartition $V(B_\varepsilon(n, d)) = S \sqcup L$ such that*

(i) $|B_\varepsilon(n, d)| = \frac{n^2}{2^{d+1}}$.

(ii) for all $X \subseteq S$ *and* $Y \subseteq L$

$$e(X,Y) = \frac{1}{2^{d-1}}|X||Y| \pm \frac{\varepsilon n^2}{2^{d+2}}.$$

Construction of $G_\varepsilon(n,d)$.

Definition 5. For every even n let $G_\varepsilon(n,1)$ be such that

$$G_\varepsilon(n,1) = \{(i,j) \ : \ 1 \leqslant i \leqslant \frac{n}{2} < j \leqslant n\},$$

i.e., $G_\varepsilon(n,1)$ is a complete balanced bipartite graph.

For $d \in \mathbb{N}$ define graph $G_\varepsilon(n,d)$ recursively for all sufficiently large† n such that $2^d|n$. Let $[n] = S \cup L$, where $S = \{1, \dots, \frac{n}{2}\}$ and $L = [n] \setminus S$. Then define

$$G_\varepsilon(n,d) = G_\varepsilon(S,d-1) \cup G_\varepsilon(L,d-1) \cup B_\varepsilon(n,d),$$

where $G_\varepsilon(S,d-1) = G_\varepsilon(\frac{n}{2},d-1)$, $V(G_\varepsilon(L,d-1)) = L$ and $G_\varepsilon(L,d-1) \cong G_\varepsilon(\frac{n}{2},d-1)$, and $B_\varepsilon(n,d)$ is a graph guaranteed by Claim 4.

To summarize, every $G_\varepsilon(n,d) = G$ satisfies the following properties (with $S_n = \{1, \dots, \frac{n}{2}\}$ and $L_n = [n] \setminus S_n$):

(i) $e_G(S_n, L_n) = \dfrac{n^2}{2^{d+1}}$.

(ii) for all $X \subseteq S_n$ and $Y \subseteq L_n$

$$e(X,Y) = \frac{1}{2^{d-1}}|X||Y| \pm \frac{\varepsilon n^2}{2^{d+2}}.$$

(iii) $G[S_n] \cong G[L_n] = G_\varepsilon(\frac{n}{2}, d-1)$

Using properties (i) and (iii) and induction on d it is easy to verify that for all $d \in \mathbb{N}$ and n divisible by 2^d

$$|G_\varepsilon(n,d)| = d\frac{n^2}{2^{d+1}}. \tag{4}$$

We will now proceed with proving (3). First let $G \subseteq V(\mathrm{Sh}_n^2)$ and let $I \subseteq G$ be an independent set in Sh_n^2. In other words there is no $1 \leqslant i < j < k \leqslant n$ with both (i,j) and (j,k) in I. One can observe that for each such $I \subseteq G$ there exists a 2-colouring $c : [n] \to \{r, b\}$ with $c(i) = r$ and $c(j) = b$ whenever $(i,j) \in I$, and then

$$I \subseteq G_c = \{(x,y) \in G \ : \ x < y, \ c(x) = b, \ c(y) = r\}. \tag{5}$$

† $n \geqslant 2^i n_0(\varepsilon, d-i)$ for all $i \in \{0, 1, \dots, d-2\}$, where $n_0(\varepsilon, d-i)$ is the number provided by Claim 4.

Therefore, in order to prove (3) we will show that for $G = G_\varepsilon(n,d)$ and any $c : [n] \to \{r,b\}$

$$\frac{|G_c|}{|G|} \leqslant \frac{1}{4} + \frac{3+\ln d}{4d} + \frac{\varepsilon}{2}. \tag{6}$$

For the rest of our calculation let ε be fixed. We will now prove (6) by induction on d. In order to make use of recursive structure of $G_\varepsilon(n,d)$ we will prove a version of (6) with an additional assumption that $|\{i : c(i) = b\}| = \alpha n$.

To that end for $d \in \mathbb{N}$, $\alpha \in [0,1]$ and $n \geqslant n_0(\varepsilon, d)$ let

$$f_d^\alpha(n) = d \cdot \max_c \left\{ \frac{|G_c|}{|G|} \;:\; G = G_\varepsilon(n,d),\ |\{i \;:\; c(i) = b\}| = \alpha n \right\}. \tag{7}$$

We will prove the following estimate on $f_d^\alpha(n)$.

Claim 6. *For every $d \in \mathbb{N}$, $\alpha \in [0,1]$ and $n \geqslant n_0(\varepsilon, d)$*

$$f_d^\alpha(n) \leqslant (d+3)(\alpha - \alpha^2) + \frac{1}{4}\ln d + \frac{d\varepsilon}{2}.$$

From (7) it follows that for $G = G_\varepsilon(n,d)$ and any colouring c we have

$$\frac{|G_c|}{|G|} \leqslant \max_{\alpha \in [0,1]} \frac{f_d^\alpha(n)}{d}.$$

Then by Claim 6 we get

$$\frac{|G_c|}{|G|} \leqslant \frac{1}{4}\frac{d+3}{d} + \frac{\ln d}{4d} + \frac{\varepsilon}{2},$$

establishing (6) and (3). Hence it remains to prove Claim 6 in order to finish the proof of the upper bound.

Proof of Claim 6. We prove a slightly stronger inequality for all $n \geqslant n_0(\varepsilon, d)$

$$f_d^\alpha(n) \leqslant (d+3)(\alpha - \alpha^2) + \frac{1}{4}\sum_{i=3}^{d+1}\frac{1}{i} + \frac{d\varepsilon}{2}. \tag{8}$$

The proof is by induction on d. For $d = 1$ recall that $G = G_\varepsilon(n,1)$ is a complete bipartite graph between S_n and L_n. Let $c : [n] \to \{r,b\}$ be such that for $B = \{i \;:\; c(i) = b\}$ we have $|B| = \alpha n$. Then in view of (5) the maximum value of $|G_c|$ is achieved when $B = [\alpha n]$ and so

$$f_1^\alpha(n) = \begin{cases} 2\alpha, & \alpha \in [0, \frac{1}{2}] \\ 2 - 2\alpha, & \alpha \in [\frac{1}{2}, 1]. \end{cases}$$

Now it is easy to verify that $f_1^\alpha \leqslant 4(\alpha - \alpha^2)$ for all $\alpha \in [0,1]$, establishing (8) in the case $d = 1$.

To prove inductive step let $G = G_\varepsilon(n,d)$ and let $c : [n] \to \{r,b\}$ be such that for $B = \{i \;:\; c(i) = b\}$ we have $|B| = \alpha n$. As before, let $S = \{1, \ldots, \frac{n}{2}\}$ and $L = [n] \setminus S$. Let

B_S, B_L, R_S and R_L denote the set of blue and red vertices in S and L respectively. We will further refine our analysis by assuming that $|B_S| = x\frac{n}{2}$ with some $x \in [0, 2\alpha]$. Since $|B_S| + |R_S| = \frac{n}{2}$, $|B_S| + |B_L| = |B| = \alpha n$, and $|B_L| + |R_L| = \frac{n}{2}$, we have $|R_S| = (1-x)\frac{n}{2}$, $|B_L| = (2\alpha - x)\frac{n}{2}$, and consequently $|R_L| = (1 - 2\alpha + x)\frac{n}{2}$ (see Figure 1). Then

$$|G_c| = |G_c[S]| + |G_c[L]| + e_G(B_S, R_L). \tag{9}$$

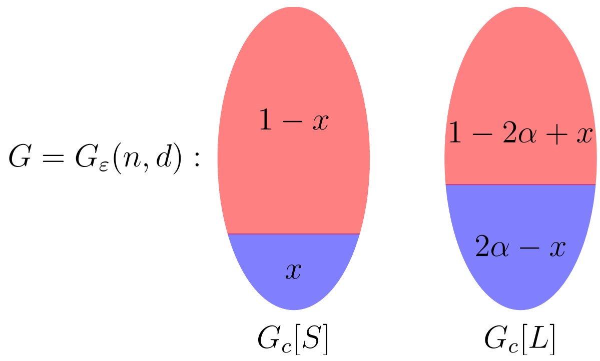

Figure 1: Proportions of red and blue vertices in $G_c[S]$ and $G_c[L]$.

Now, by (iii) $G[S] = G_\varepsilon(\frac{n}{2}, d-1)$ and we assumed $|B_S| = x\frac{n}{2}$, so

$$|G_c[S]| \overset{(7)}{\leqslant} \frac{f^x_{d-1}(\frac{n}{2})}{d-1}|G[S]| \overset{(4)}{=} \frac{n^2}{2^{d+2}} f^x_{d-1}\left(\frac{n}{2}\right). \tag{10}$$

Similarly, since $|B_L| = (2\alpha - x)\frac{n}{2}$ we have

$$|G_c[L]| \leqslant \frac{n^2}{2^{d+2}} f^{2\alpha - x}_{d-1}\left(\frac{n}{2}\right). \tag{11}$$

And finally, since $G = G_\varepsilon(n, d)$,

$$e_G(B_S, R_L) \overset{(ii)}{\leqslant} \frac{1}{2^{d-1}}|B_S||R_L| + \frac{\varepsilon n^2}{2^{d+2}} = \frac{n^2}{2^{d+1}}\left(x(1 - 2\alpha + x) + \frac{\varepsilon}{2}\right). \tag{12}$$

Combining (9) with (10), (11), and (12) we obtain

$$|G_c| \leqslant \frac{n^2}{2^{d+1}}\left(\frac{1}{2}\left(f^x_{d-1}\left(\frac{n}{2}\right) + f^{2\alpha-x}_{d-1}\left(\frac{n}{2}\right)\right) + x(1 - 2\alpha + x) + \frac{\varepsilon}{2}\right).$$

Finally, $|G| = |G_\varepsilon(n, d)| \overset{(4)}{=} d\frac{n^2}{2^{d+1}}$ and so by (7) we deduce that

$$f^\alpha_d(n) \leqslant \max_{x \in \mathbb{R}}\left\{\frac{1}{2}\left(f^x_{d-1}\left(\frac{n}{2}\right) + f^{2\alpha-x}_{d-1}\left(\frac{n}{2}\right)\right) + x(1 - 2\alpha + x) + \frac{\varepsilon}{2}\right\}. \tag{13}$$

The last inequality allows us to incorporate induction hypothesis. In particular, by induction hypothesis we have

$$f_{d-1}^{x}(\frac{n}{2}) \leqslant (d+2)(x-x^2) + \frac{1}{4}\sum_{i=3}^{d}\frac{1}{i} + \frac{(d-1)\varepsilon}{2},$$
$$f_{d-1}^{2\alpha-x}(\frac{n}{2}) \leqslant (d+2)(2\alpha-x)(1-2\alpha+x) + \frac{1}{4}\sum_{i=3}^{d}\frac{1}{i} + \frac{(d-1)\varepsilon}{2},$$

and these two inequalities together with (13), after some simple but tedious algebraic manipulations yield

$$f_d^{\alpha}(n) \leqslant \max_{x\in\mathbb{R}}\left\{-(d+1)x^2 + (1+2\alpha(d+1))x + (d+2)(\alpha-2\alpha^2) + \frac{1}{4}\sum_{i=3}^{d}\frac{1}{i} + \frac{d\varepsilon}{2}\right\}.$$

In other words $f_d^{\alpha}(n) \leqslant \max_{x\in\mathbb{R}}\{g(x)\}$, where $g(x) = ax^2+bx+c$ with $a = -(d+1)$. Since $a < 0$ we have $\max_{x\in\mathbb{R}} g(x) = g(\frac{-b}{2a}) = c - \frac{b^2}{4a}$. Therefore after another set of algebraic manipulations we obtain

$$f_d^{\alpha}(n) \leqslant \max_{x\in\mathbb{R}}\{g(x)\} \leqslant (d+3)(\alpha-\alpha^2) + \frac{1}{4}\sum_{i=3}^{d+1}\frac{1}{i} + \frac{d\varepsilon}{2},$$

finishing the proof of the inductive step and Claim 6. □

3 Infinite graphs

Recall that Theorem 2 states

$$\lim_{n\to\infty} \min\left\{\frac{\alpha(\mathrm{Sh}_n^2[G])}{|G|} \;:\; \emptyset \neq G \subseteq V(\mathrm{Sh}_n^2)\right\} = \frac{1}{4}. \tag{14}$$

On the other hand, considering $I = \{(i,j) \;:\; 1 \leqslant i \leqslant \frac{n}{2} < j \leqslant n\}$ we clearly have $\alpha(\mathrm{Sh}_n^2) \geqslant \lfloor \frac{n^2}{4} \rfloor$. Moreover $\lfloor \frac{n^2}{4} \rfloor$ is optimal, since any graph $G \subseteq V(\mathrm{Sh}_n^2)$ with $|G| \geqslant \lfloor \frac{n^2}{4} \rfloor + 1$ contains a triangle and hence such G is not an independent set in Sh_n^2. Therefore,

$$\lim_{n\to\infty} \frac{\alpha(\mathrm{Sh}_n^2)}{|\mathrm{Sh}_n^2|} = \frac{1}{2}. \tag{15}$$

It may be interesting to note that infinite version of (15) was considered by Czipszer, Erdős and Hajnal [2] who proved that if I is independent set in countable shift graph $\mathrm{Sh}_{\mathbb{N}}^2$, then the density of I does not exceed 1/4, i.e.

$$\liminf_{n\to\infty} \frac{\left|I \cap \binom{[n]}{2}\right|}{\binom{n}{2}} \leqslant \frac{1}{4}. \tag{16}$$

(Here $\frac{1}{4}$ is clearly optimal, since $I = \{(i,j) \ : \ i < j, \ i \text{ odd}, \ j \text{ even}\}$ is independent in $\mathrm{Sh}^2_{\mathbb{N}}$.)

To complete this discussion we provide an infinite variant of (14).

Theorem 7. *There is* $G \subseteq V(\mathrm{Sh}^2_{\mathbb{N}})$ *such that if* I *is an independent set in* $\mathrm{Sh}^2_{\mathbb{N}}[G]$*, then*

$$\liminf_{n\to\infty} \frac{\left|I \cap \binom{[n]}{2}\right|}{\left|G \cap \binom{[n]}{2}\right|} = 0.$$

Proof. Consider an infinite ordered tree G with $V(G) = \mathbb{N}$, and with vertices labeled v_i^j, where j denotes the "level" L_j that vertex v_i^j belongs to and i denotes the order in which vertices are listed on the level.

Consider a labeling of vertices of G by integers satisfying $v_i^j < v_i^{j'}$ if $j < j'$ and $v_i^j < v_{i'}^j$ if $i < i'$ such that for all v_i^j the finite set $N^+(v_i^j)$ of all children of v_i^j forms an interval (and these intervals on the level L_{j+1} follow the order of their parents on L_j, see Figure 2). Finally we will assume that for all v_i^j

$$|N^+(v_i^j)| \geqslant 2^j \sum_{v<v_i^j} |N^+(v)|. \tag{17}$$

Now, let $I \subseteq G$ be an infinite independent set in $\mathrm{Sh}^2_{\mathbb{N}}$ and let $(v_k^{j-1}, v_i^j) \in I$, where v_k^{j-1} and v_i^j are parent and child respectively. Let $w = \max\{N^+(v_i^j)\}$ be the largest son of v_i^j and let $W = \{1, \dots, w\}$ (see Figure 2). Then

$$G[W] = \bigcup_{v \leqslant v_i^j} \left\{(v,u) \ : \ u \in N^+(v)\right\}. \tag{18}$$

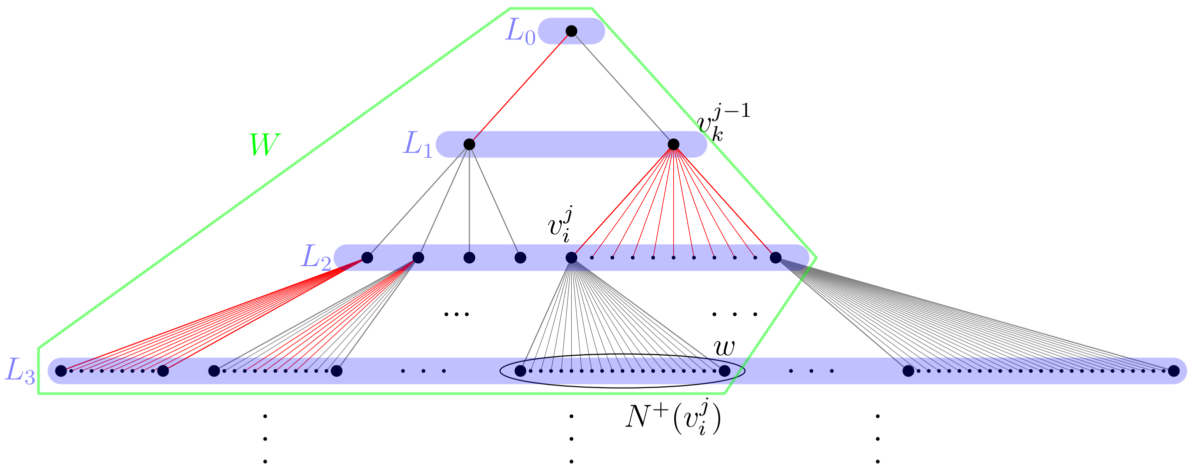

Figure 2: Infinite tree G, vertices are ordered top to bottom, left to right. Edges of I are labeled with red.

In particular in view of (17)

$$|G[W]| \geqslant \left| \left\{ (v_i^j, u) \ : \ u \in N^+(v) \right\} \right| = |N^+(v_i^j)|. \tag{19}$$

On other hand, since $(v_k^{j-1}, v_i^j) \in I$ and I is independent set in $\mathrm{Sh}^2_{\mathbb{N}}$, the set I does not contain any other edge incident to v_i^j. Consequently,

$$|I[W]| \leqslant \left| \bigcup_{v < v_i^j} \left\{ (v,u) \ : \ u \in N^+(v) \right\} \right| \overset{(17)}{\leqslant} 2^{-j} |N^+(v_i^j)|. \tag{20}$$

In view of (19) and (20) we have $|I[W]|/|G[W]| \leqslant \frac{1}{2^j}$. Now, since I is infinite there are edges $(v_k^{j-1}, v_i^j) \in I$ with sufficiently large j, hence the ratio $|I[W]|/|G[W]|$ can be made arbitrary small, finishing the proof. □

4 Concluding remarks

In [5] it was proved[‡] that for any n, k

$$\alpha_n^k \geqslant \begin{cases} \frac{1}{2} - \frac{1}{k}, & \text{if } k \text{ is even,} \\ \frac{1}{2} - \frac{1}{2k}, & \text{if } k \text{ is odd.} \end{cases} \tag{21}$$

It remains an open problem to determine for any $k \geqslant 3$ the exact value of $\lim_{n\to\infty} \alpha_n^k$. For $k = 4$ we were able to improve the constant in the lower bound (21) from $\frac{1}{4}$ to $\frac{3}{8}$[§] and for $k = 3$ we believe that estimate in (21) is sharp.

Problem 8. Show that $\lim_{n\to\infty} \alpha_n^3 = \frac{1}{3}$.

Finally, all of the results in this paper can be reformulated in terms of subgraphs with no increasing paths of length two. For instance, Theorem 2 implies that for any $\varepsilon > 0$ there exists an vertex-ordered graph G such that if $G' \subseteq G$ with $|G'| \geqslant \left(\frac{1}{4} + \varepsilon\right)|G|$, then G' contains an increasing path of length two, i.e. there are $i < j < k$ with $(i,j), (j,k) \in G'$. One can ask similar questions for longer increasing paths.

Problem 9. For any $\varepsilon > 0$ does there exist an ordered graph G such that if $G' \subseteq G$ with $|G'| \geqslant \left(\frac{1}{3} + \varepsilon\right)|G|$, then G' contains an increasing path of length three?

Note that in regards to Problem 9, one can consider a random coloring c of $V(G)$ with colors $\{0,1,2\}$ and define G' to be the collection of all $(i,j) \in E(G)$ with $i < j$ and $c(i) < c(j)$. Then such G' on average contains $\frac{1}{3}|G|$ edges and has no increasing paths of length three, motivating the constant $\frac{1}{3}$ in the problem.

[‡]the result follows from the proof of Theorem 1 in [5]

[§]$\alpha(\mathrm{Sh}_n^4[G]) \geqslant \frac{3}{8}|G|$ can be proved by considering a random colouring $c : [n] \to \{0,1\}$ and forming an independent set in Sh_n^4 by taking hyperedges of G of form 1000, 1110, or $x01y$ for some $x, y \in \{0,1\}$.

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Appendix

Proof of Claim 4. Let $B_\varepsilon(n,d) = G$, where G is a random graph between S and L obtained by selecting a random subset of size $\frac{n^2}{2^{d+1}}$ without replacement from $K_{S,L}$ (complete bipartite graph between S and L). Then G satisfies (i) and we will show that G satisfies (ii) almost surely.

For every $X \subseteq S$ and $Y \subseteq L$, $e(X,Y) = e_G(X,Y)$ is distributed as a hypergeometric random variable $\mathrm{H}\left(\frac{n^2}{4}, \frac{n^2}{2^{d+1}}, |X||Y|\right)$ with expectation $\frac{1}{2^{d-1}}|X||Y|$. Let $B_{X,Y}$ be the event that

$$\left| e_G(X,Y) - \frac{1}{2^{d-1}}|X||Y| \right| > \frac{\varepsilon n^2}{2^{d+2}},$$

i.e., $B_{X,Y}$ is the event that (ii) fails for given X and Y.

We will use a concentration inequality for hypergeometric random variables (this version is a corollary of Theorem 2.10 and inequalities (2.5),(2.6) of Janson, Luczak, Rucinski [7]).

Theorem 10. *Let $Z \sim H(N,m,k)$ be a hypergeometric random variable with the expectation $\mu = \frac{mk}{N}$, then for $t \geqslant 0$*

$$\mathbb{P}(|Z-\mu| > t) \leqslant 2\exp\left(\frac{-t^2}{2(\mu + t/3)}\right).$$

For a given $X \subseteq L$ and $Y \subseteq R$, as a consequence of Theorem 10 with $Z = e_G(X,Y)$, $t = \frac{\varepsilon n^2}{2^{d+2}}$ and $\mu = \frac{1}{2^{d-1}}|X||Y| \leqslant \frac{n^2}{2^{d+1}}$ we get

$$\mathbb{P}(B_{X,Y}) = e^{-\Omega(n^2)},$$

where constant in $\Omega()$ term depends on ε and d only. Therefore,

$$\mathbb{P}\left(\bigcup_{X,Y} B_{X,Y}\right) \leqslant \sum_{X,Y} \mathbb{P}\left(B_{X,Y}\right) \leqslant 2^n e^{-\Omega(n^2)} = o(1).$$

In particular, $\mathbb{P}(G \text{ satisfies (ii)}) = \mathbb{P}\left(\bigcap_{X,Y} \overline{B_{X,Y}}\right) = 1 - o(1)$. Hence, G almost surely satisfies (ii). □