Enumeration of Gelfand–Cetlin type reduced words

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Abstract

The combinatorics of reduced words and their commutation classes plays an important role in geometric representation theory. For a semisimple complex Lie group G, a string polytope is a convex polytope associated with each reduced word of the longest element w_0 in the Weyl group of G encoding the character of a certain irreducible representation of G. In this paper, we deal with the case of type A, i.e., $G = \mathrm{SL}_{n+1}(\mathbb{C})$. A Gelfand–Cetlin polytope is one of the most famous examples of string polytopes of type A. We provide a recursive formula enumerating reduced words of w_0 such that the corresponding string polytopes are combinatorially equivalent to a Gelfand–Cetlin polytope. The recursive formula involves the number of standard Young tableaux of shifted shape. We also show that each commutation class is completely determined by a list of quantities called indices.

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1 Introduction

A *string polytope*, introduced by Littelmann [21], is a convex polytope $\Delta_{\mathbf{i}}(\lambda)$ determined by two data: a reduced word $\mathbf{i}$ of the longest element in the Weyl group of a complex semisimple Lie group and a dominant weight λ. Its lattice points parametrize the dual canonical basis elements of the irreducible representation with highest weight λ so that it can be regarded as a non-abelian generalization of a Newton polytope in toric geometry. The importance of string polytopes has been raised for the study of mirror symmetry of flag varieties (see, for example, [2]). We refer the reader to [22], [18], [21], [14], and [3] for various descriptions of string polytopes.

In this paper, we only deal with the case of type A, i.e., $G = \mathrm{SL}_{n+1}(\mathbb{C})$. A family of Gelfand–Cetlin polytopes provides one of the most famous examples of string polytopes. Similarly to string polytopes, Gelfand–Cetlin polytopes have been used to describe irreducible representations of $\mathrm{SL}_{n+1}(\mathbb{C})$. We recall the definition of Gelfand–Cetlin polytopes from [13] and [16]. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a sequence of nonnegative integers. For each λ, the *Gelfand–Cetlin polytope* $\mathrm{GC}(\lambda)$ is defined to be the closed convex polytope in $\mathbb{R}^{\bar{n}}$ consisting of the points $(x_{k,j})_{1 \leqslant j \leqslant k \leqslant n}$ satisfying the inequalities

$$x_{k+1,j} \geqslant x_{k,j} \geqslant x_{k+1,j+1}, \quad 1 \leqslant j \leqslant k \leqslant n,$$

where $\bar{n} = n(n+1)/2$, $x_{n+1,j} = \lambda_j + \cdots + \lambda_n$ for $1 \leqslant j \leqslant n$, and $x_{n+1,n+1} = 0$. Note that $\mathrm{GC}(\lambda)$ has the maximum dimension if each λ_i is positive, i.e., λ is regular. In this case, we say that $\mathrm{GC}(\lambda)$ is a full dimensional Gelfand–Cetlin polytope of rank n. It is shown in [21, Corollary 5 in Section 5] that the Gelfand–Cetlin polytope $\mathrm{GC}(\lambda)$ is an example of a string polytope of $\mathrm{SL}_{n+1}(\mathbb{C})$. More precisely, we have

$$\mathrm{GC}(\lambda) \simeq \Delta_{(1,2,1,3,2,1,\dots,n,n-1,\dots,1)}(\lambda),$$

where $\simeq$ means the unimodular equivalence[1]. In particular, these two polytopes are combinatorially equivalent. We refer the reader to [1] and references therein for more information on the combinatorics of Gelfand–Cetlin polytopes.

The string polytope $\Delta_{\mathbf{i}}(\lambda)$ has the maximum dimension if and only if the weight λ is regular. Once the weight is assumed to be regular, the combinatorial type of the string polytope is independent of the choice of the weight. In this paper, we consider string polytopes $\Delta_{\mathbf{i}}(\lambda)$ of type A for a fixed regular dominant weight λ so that each $\Delta_{\mathbf{i}}(\lambda)$ is determined by the reduced word $\mathbf{i}$.

The motivation of this paper is to enumerate the string polytope $\Delta_{\mathbf{i}}(\lambda)$ of type A which are combinatorially equivalent to the Gelfand–Cetlin polytope $\mathrm{GC}(\lambda)$. To this end, we study the reduced words $\mathbf{i}$ which give rise to such string polytopes $\Delta_{\mathbf{i}}(\lambda)$. To state our results, we introduce some terminologies.

Let $\mathfrak{S}_{n+1}$ be the symmetric group (i.e., the Weyl group of $\mathrm{SL}_{n+1}(\mathbb{C})$) on $[n+1] \coloneqq \{1, \dots, n+1\}$ and denote by $s_i \coloneqq (i, i+1)$ the simple transposition which swaps i and $i+1$

[1]Given integral polytopes $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^d$, we say that P and Q are *unimodularly equivalent* if there exist a matrix $U \in \mathrm{M}_{d\times d}(\mathbb{Z})$ and an integral vector $\mathbf{v} \in \mathbb{Z}^d$ such that $\det U = \pm 1$ and $Q = f_U(P) + \mathbf{v}$. Here, f_U is the linear transformation defined by U.

and fixes all other elements of $[n+1]$. The set $\{s_1, \dots, s_n\}$ of simple transpositions generates $\mathfrak{S}_{n+1}$, hence every element $w \in \mathfrak{S}_{n+1}$ can be written in the following form:

$$w = s_{i_1} \cdots s_{i_r}, \qquad i_1, \dots, i_r \in [n].$$

In this case, the sequence $\mathbf{i} := (i_1, \dots, i_r)$ is called a *word* of w. The *length* $\ell(w)$ of w is defined to be the smallest integer r for which $(i_1, \dots, i_r)$ is a word of w. A word $(i_1, \dots, i_r)$ of w is *reduced* if $r = \ell(w)$. We denote by $\mathcal{R}(w)$ the set of reduced words of w. There is a unique element $w_0^{(n+1)}$, called the *longest element*, in $\mathfrak{S}_{n+1}$ such that $\ell(w) \leqslant \ell(w_0^{(n+1)})$ for all $w \in \mathfrak{S}_{n+1}$.

For $i, j \in [n]$ satisfying $|i-j| > 1$, we have $s_i s_j = s_j s_i$. This induces an operation on the set $\mathcal{R}(w)$ defined by $(\dots, i, j, \dots) \mapsto (\dots, j, i, \dots)$, which is called a *commutation* (or a *2-move*). Define an equivalence relation $\sim$ on $\mathcal{R}(w)$ by

$$\mathbf{i} \sim \mathbf{i}' \quad \iff \quad \mathbf{i} \text{ is obtained from } \mathbf{i}' \text{ by a sequence of commutations.}$$

An element in $[\mathcal{R}(w)] := \mathcal{R}(w)/\sim$ is called a *commutation class* for w. For a recent account of the study of commutation classes, we refer the reader to [5, 26, 12, 17] and references therein.

One important fact about commutation classes for our purpose is that two string polytopes $\Delta_{\mathbf{i}}(\lambda)$ and $\Delta_{\mathbf{i}'}(\lambda)$ are combinatorially equivalent if (but not necessarily only if) $\mathbf{i}$ and $\mathbf{i}'$ are in the same commutation class (see [7, Lemma 3.1]). Accordingly, studying the elements in $[\mathcal{R}(w_0^{(n+1)})]$ is closely related to the classification problem of the combinatorial types of string polytopes.

We say that $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$ is a *Gelfand–Cetlin type* reduced word if the corresponding string polytope $\Delta_{\mathbf{i}}(\lambda)$ is combinatorially equivalent to a full dimensional Gelfand–Cetlin polytope of rank n. Let $\mathsf{gc}(n)$ be the number of Gelfand–Cetlin type reduced words in $\mathcal{R}(w_0^{(n+1)})$. By the definition of $\mathsf{gc}(n)$, it also counts the number of string polytopes $\Delta_{\mathbf{i}}(\lambda)$ with $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$ that are combinatorially equivalent to a full dimensional Gelfand–Cetlin polytope of rank n.

The first main result in this paper is the following recurrence relation for $\mathsf{gc}(n)$.

Theorem 1 (Theorem 37). *The number $\mathsf{gc}(n)$ of Gelfand–Cetlin type reduced words in $\mathcal{R}(w_0^{(n+1)})$ satisfies*

$$\mathsf{gc}(n) = \sum_{k=1}^{n} g^{(n,n-1,\dots,n-k+1)} \mathsf{gc}(n-k),$$

where g^μ is the number of standard Young tableaux of shifted shape $\mu = (\mu_1, \dots, \mu_t)$ and

$$g^\mu = \frac{|\mu|!}{\mu_1! \mu_2! \cdots \mu_t!} \prod_{i<j} \frac{\mu_i - \mu_j}{\mu_i + \mu_j}.$$

As a consequence of the proof of the above theorem, we obtain that the number of commutation classes consisting of Gelfand–Cetlin type reduced words in $\mathcal{R}(w_0^{(n+1)})$ is 2^{n-1}

(see Corollary 34). This result was also proved in a recent paper [17] using a different method.

We note that the number of string polytopes $\Delta_{\mathbf{i}}(\lambda)$ (for $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$) which are *unimodularly* equivalent to the Gelfand–Cetlin polytope $\mathrm{GC}(\lambda)$ is the same as the number $\mathsf{gc}(n)$. Accordingly, the above theorem also enumerates the number of string polytopes $\Delta_{\mathbf{i}}(\lambda)$ which are unimodularly equivalent to the Gelfand–Cetlin polytope $\mathrm{GC}(\lambda)$ (see Corollary 38).

A crucial object in the proof of Theorem 1 is a quantity, called the δ-index. For a sequence $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ of two letters A and D, the *δ-index* $\mathrm{ind}_\delta(\mathbf{i})$ of a reduced word $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$ is an element in $\mathbb{Z}^{n-1}$ which measures how far a given word is from the standard reduced word

$$(1, 2, 1, 3, 2, 1, \dots, n, n-1, \dots, 1)$$

of $w_0^{(n+1)}$. See Section 3 for the precise definition.

Recently, the first and the third authors together with Kim and Park [7, Theorem A] classified all Gelfand–Cetlin type reduced words in $\mathcal{R}(w_0^{(n+1)})$ in terms of δ-indices. More precisely, they showed that $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$ is a Gelfand–Cetlin type reduced word if and only if there is a sequence $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ such that $\mathrm{ind}_\delta(\mathbf{i}) = (0, \dots, 0) \in \mathbb{Z}^{n-1}$.

It turns out that for $\mathbf{i}, \mathbf{j} \in \mathcal{R}(w_0^{(n+1)})$, if $\mathrm{ind}_\delta(\mathbf{i}) = \mathrm{ind}_\delta(\mathbf{j}) = (0, \dots, 0) \in \mathbb{Z}^{n-1}$ for some $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$, then $\mathbf{i} \sim \mathbf{j}$ (see Proposition 32). However, the condition $\mathrm{ind}_\delta(\mathbf{i}) = \mathrm{ind}_\delta(\mathbf{j})$ for some $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ does not always imply $\mathbf{i} \sim \mathbf{j}$ (see Example 33).

The second main result in this paper shows that the δ-indices $\mathrm{ind}_\delta(\mathbf{i})$ for all $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ completely determine the commutation class of $\mathbf{i}$.

Theorem 2 (Theorem 39)**.** *Let* $\mathbf{i}, \mathbf{j} \in \mathcal{R}(w_0^{(n+1)})$. *Then,* $\mathrm{ind}_\delta(\mathbf{i}) = \mathrm{ind}_\delta(\mathbf{j})$ *for all* $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ *if and only if* $\mathbf{i} \sim \mathbf{j}$.

In order to prove our main results, we consider word posets, which are similar to wiring diagrams. A word poset is a poset P together with a function $f_P : P \to \mathbb{Z}_{>0}$. Each commutation class in $[\mathcal{R}(w_0^{(n+1)})]$ corresponds to a word poset and the cardinality of the commutation class is equal to the number of linear extensions of the corresponding word poset. See Section 2.3 for the precise description.

This paper is organized as follows. In Section 2, we give basic definitions. In Section 3, we recall the operations on $\mathcal{R}(w_0^{(n+1)})$ called contractions and extensions. Moreover, we provide the definition of indices and prove several properties of them. In Section 4, we study Gelfand–Cetlin type reduced words and provide a proof of Theorem 1. In Section 5, we give a proof of Theorem 2.

2 Basic definitions

In this section, we give basic definitions and properties of commutation classes, wiring diagrams, and word posets which will be used throughout this paper.

2.1 Commutation classes

Let w be an element in $\mathfrak{S}_{n+1}$ and denote by $\mathcal{R}(w)$ the set of reduced words of w, i.e.,

$$\mathcal{R}(w) = \{(i_1, \dots, i_\ell) \in [n]^\ell \mid s_{i_1} s_{i_2} \cdots s_{i_\ell} = w\},$$

where $\ell = \ell(w)$ is the length of w. It is well known that

$$\ell(w) = \#\{1 \leqslant i < j \leqslant n \mid w(i) > w(j)\}.$$

We denote by $w_0^{(n+1)}$ the longest element $n+1\ n\ \cdots\ 1$ of $\mathfrak{S}_{n+1}$ (using the one-line notation). The length $\bar{n}$ of the longest element $w_0^{(n+1)}$ is given by

$$\bar{n} = \ell(w_0^{(n+1)}) = \frac{n(n+1)}{2} \qquad \text{for } n \in \mathbb{Z}_{>0} \coloneqq \{1, 2, \dots\}. \tag{1}$$

Recall that for a given $\mathbf{i} \in \mathcal{R}(w)$, one can produce new reduced words using *braid moves*. There are two types of braid moves as follows:

- A *2-move* replaces two consecutive elements i, j in $\mathbf{i}$ by j, i for some integers i and j with $|i - j| > 1$.
- A *3-move* replaces three consecutive elements i, j, i in $\mathbf{i}$ by j, i, j for some integers i and j with $|i - j| = 1$.

Note that braid moves do not change the product of the simple transpositions for $\mathbf{i}$ since $s_i s_j = s_j s_i$ for $|i - j| > 1$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. According to Tits' Theorem [30], any two reduced words in $\mathcal{R}(w)$ are connected by a sequence of braid moves. The braid moves and Tits' theorem can be generalized to other Coxeter systems (see [4, §3.3]).

Define an equivalence relation '$\sim$' on $\mathcal{R}(w)$ by

$$\mathbf{i} \sim \mathbf{i}' \qquad \Longleftrightarrow \qquad \mathbf{i} \text{ is obtained from } \mathbf{i}' \text{ by a sequence of 2-moves.}$$

We denote by $[\mathcal{R}(w)] \coloneqq \mathcal{R}(w)/\sim$ the set of equivalence classes and call an element $[\mathbf{i}] \in [\mathcal{R}(w)]$ a *commutation class*.

Remark 3. There is no known exact formula for the number $\mathsf{c}(n+1)$ of commutation classes of $w_0^{(n+1)}$. Some upper and lower bounds for $\mathsf{c}(n)$ were provided by Knuth [19, Section 9]. Felsner and Valtr [11, Theorem 2 and Proposition 1] found the following upper and lower bounds for $\mathsf{c}(n+1)$ improving Knuth's results: for a sufficiently large n,

$$2^{0.1887n^2} \leqslant \mathsf{c}(n) \leqslant 2^{0.6571n^2}.$$

The first few terms of $\mathsf{c}(n)$ are $1, 1, 2, 8, 62, 908, 24698, 1232944$, see A006245 in [24].

2.2 Wiring diagrams

There are several combinatorial models for the commutation classes of the longest element of $\mathfrak{S}_{n+1}$. For example, see Remark 11. We recall a well known combinatorial model, called a wiring diagram (cf. [15]).

Definition 4. Let $\mathbf{i} = (i_1, i_2, \ldots, i_\ell)$ be a word of $w \in \mathfrak{S}_{n+1}$.

1. The *wiring diagram* $G(\mathbf{i})$ of $\mathbf{i}$ is given by collections of line segments as follows.

   - For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n+1$, define G_i to be the collection of line segments $(A_j, B_{s_i(j)})$ connecting A_j and $B_{s_i(j)}$, where $A_j = (j,1)$ and $B_j = (j,0)$ are points in $\mathbb{R}^2$. The intersection of the segments (A_i, B_{i+1}) and (A_{i+1}, B_i) is called a *crossing* in column i. See Figure 1.
   - Define $G(\mathbf{i})$ to be the configuration obtained by arranging $G_{i_1}, G_{i_2}, \ldots, G_{i_\ell}$ vertically in this order. More precisely,
   $$G(\mathbf{i}) = \rho^{\ell-1}(G_{i_1}) \cup \rho^{\ell-2}(G_{i_2}) \cup \cdots \cup \rho^{0}(G_{i_\ell}),$$
   where ρ is the translation by $(0,1)$. See Figure 2.

2. The *jth row* of $G(\mathbf{i})$ is $\rho^{\ell-j}(G_{i_j})$. The points (j,ℓ) and $(j,0)$ are called the *jth starting point* and the *jth ending point* of $G(\mathbf{i})$, respectively. A *wire* of $G(\mathbf{i})$ is a path from a starting point to an ending point of $G(\mathbf{i})$ obtained by taking the union of ℓ segments one from each row. If a wire starts at the jth starting point, it is called the *jth wire* of $G(\mathbf{i})$.

3. We denote by $\mathcal{WD}(w)$ the set of wiring diagrams $G(\mathbf{i})$ for all reduced words $\mathbf{i}$ of w.

By the definition of $G(\mathbf{i})$, it is clear that the jth wire is from the jth starting point to the $w(j)$th ending point. One can reconstruct $\mathbf{i}$ from $G(\mathbf{i})$ because the unique crossing in row j of $G(\mathbf{i})$ is in column i_j. This gives a bijection between $\mathcal{R}(w)$ and $\mathcal{WD}(w)$. We define the equivalence relation '$\sim$' on $\mathcal{WD}(w)$ by $G(\mathbf{i}) \sim G(\mathbf{j})$ if and only if $\mathbf{i} \sim \mathbf{j}$. Equivalently, we have $G(\mathbf{i}) \sim G(\mathbf{j})$ if and only if $G(\mathbf{i})$ is obtained from $G(\mathbf{j})$ by a sequence of operations exchanging two adjacent rows in which the crossings are not in adjacent columns.

Let $[\mathcal{WD}(w)] = \mathcal{WD}(w)/\sim$. Then we have an obvious bijection between $[\mathcal{WD}(w)]$ and $[\mathcal{R}(w)]$ induced by the correspondence explained above.

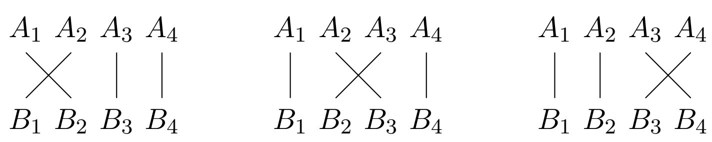


Figure 1: The configurations G_1, G_2, and G_3 (from left to right) for $n = 4$.

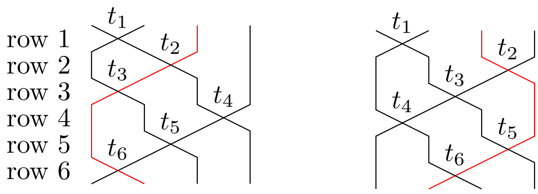


Figure 2: The wiring diagrams $G(\mathbf{i})$ (left) and $G(\mathbf{j})$ (right) for $\mathbf{i} = (1,2,1,3,2,1)$ and $\mathbf{j} = (1,3,2,1,3,2)$ in $\mathcal{R}(w_0^{(4)})$. The 3rd wire in each wiring diagram is colored red. The crossing in row j is labeled by t_j. The crossing t_4 in $G(\mathbf{i})$ (respectively, $G(\mathbf{j})$) is in column 3 (respectively, column 1).

2.3 Word posets

To study commutation classes, we associate a poset and a function on the poset to each reduced word $\mathbf{i} \in \mathcal{R}(w)$.

Definition 5. A *labeled poset* is a pair (P, f_P) of a poset P and a function $f_P : P \to \mathbb{Z}_{>0}$. Two labeled posets P and Q are *isomorphic*, denoted by $P \sim Q$, if there is a poset isomorphism $\phi : P \to Q$ such that $f_P(x) = f_Q(\phi(x))$ for all $x \in P$.

Following [25], we define word posets.

Definition 6. Let $\mathbf{i} = (i_1, \ldots, i_\ell) \in \mathcal{R}(w)$. The *word poset of* $\mathbf{i}$ is the labeled poset $(P_{\mathbf{i}}, f_{P_{\mathbf{i}}})$ such that

- $P_{\mathbf{i}}$ is the poset on $[\ell]$ which is the transitive closure of the binary relation containing (r, r) for $r \in [\ell]$ and (j, k) for $j, k \in [\ell]$ with $|i_j - i_k| = 1$ and $j < k$;
- $f_{P_{\mathbf{i}}} \colon P_{\mathbf{i}} \to \mathbb{Z}_{>0}$ satisfies $f_{P_{\mathbf{i}}}(j) = i_j$ for all $j \in P_{\mathbf{i}}$.

Denote by $\mathcal{P}(w)$ the set of labeled posets P such that $P \sim P_{\mathbf{i}}$ for some $\mathbf{i} \in \mathcal{R}(w)$. We also define $[\mathcal{P}(w)] = \mathcal{P}(w)/\sim$.

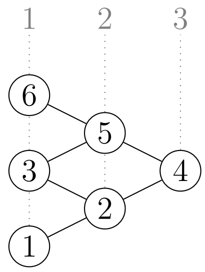
We will see later in this subsection that word posets are closely related to wiring diagrams. Throughout this paper, the following convention will be used when we draw the Hasse diagram of a word poset.

Convention. Let P be a word poset. For $j \in P$, if $f_P(j) = i$, we say that j *is in column* i. When we draw the Hasse diagram of P the element j will be placed in column i. For a subset A of P, we say that Q is the word poset obtained from P by *shifting* A to the left (respectively, right) by one column if P and Q are the same as posets and $f_Q(x) = f_P(x)$ if $x \notin A$, and $f_Q(x) = f_P(x) - 1$ (respectively, $f_Q(x) = f_P(x) + 1$) if $x \in A$.

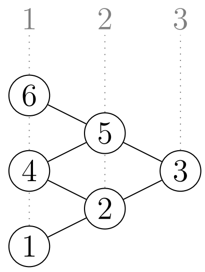
By the definition of word posets, every covering relation in a word poset $P \in \mathcal{P}(w)$ occurs between two adjacent columns. In other words, if $x \lessdot_P y$, then $|f_P(x) - f_P(y)| = 1$.

Example 7. We illustrate some examples of the word posets $P_{\mathbf{i}}$ associated with some reduced words $\mathbf{i} = (i_1, \ldots, i_\ell)$. We will arrange the elements of $P_{\mathbf{i}}$ so that j is in column $f_{P_{\mathbf{i}}}(j) = i_j$.

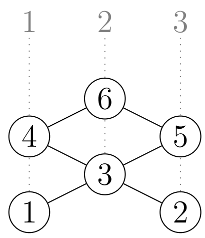
1. Let $\mathbf{i} = (1, 2, 1, 3, 2, 1)$. Then the Hasse diagram of the word poset $P_{\mathbf{i}}$ is shown below.



Here, the elements $1, 3, 6$ are in column 1, the elements $2, 5$ are in column 2, and the element 4 is in column 3. Note that if $\mathbf{i}' = (1, 2, 3, 1, 2, 1)$, then $\mathbf{i}' \sim \mathbf{i}$ and the word poset $P_{\mathbf{i}'}$ shown below is isomorphic to $P_{\mathbf{i}}$.



2. Let $\mathbf{j} = (1, 3, 2, 1, 3, 2)$. Then the Hasse diagram of the word poset $P_{\mathbf{j}}$ is given as follows.



A *linear extension* of a poset P is a permutation $p_1 p_2 \ldots p_n$ of the elements in P such that $j < k$ whenever $p_j <_P p_k$.

Proposition 8. *[25, Theorem 1.1] Let* $\mathbf{i}, \mathbf{j} \in \mathcal{R}(w)$.

1. *There is a bijection between the linear extensions of the poset* $P_{\mathbf{i}}$ *and the elements in the commutation class* $[\mathbf{i}]$ *given as follows. A linear extension* $p_1 p_2 \ldots p_n$ *of* $P_{\mathbf{i}}$ *corresponds to the word* $(i_{f_{P_{\mathbf{i}}}(p_1)}, i_{f_{P_{\mathbf{i}}}(p_2)}, \ldots, i_{f_{P_{\mathbf{i}}}(p_\ell)})$ *in* $[\mathbf{i}]$.

2. *We have* $\mathbf{i} \sim \mathbf{j}$ *if and only if* $P_{\mathbf{i}} \sim P_{\mathbf{j}}$.

We note that Proposition 8(1) has a similar result on partial commutation monoids, see [20, §5.1.2, Exercise 11], [27, Exercise 3.123], or [9, Proposition 4.11].

By Proposition 8(2), we can identify the commutation class $[\mathbf{i}]$ with $[P_{\mathbf{i}}]$.

Proposition 9. *The map* $[\mathbf{i}] \mapsto [P_{\mathbf{i}}]$ *is a bijection from* $[\mathcal{R}(w)]$ *to* $[\mathcal{P}(w)]$.

There is also a direct and natural correspondence between $[\mathcal{WD}(w)]$ and $[\mathcal{P}(w)]$, which we now explain. Let $[G] \in [\mathcal{WD}(w)]$. We define the corresponding word poset class $[P] \in [\mathcal{P}(w)]$ as follows.

- The elements of the underlying set P are the crossings in G.
- For two distinct elements $a, b \in P$, we have $a <_P b$ if there is a downward path from the crossing a to the crossing b in G. Here, a downward path means a path following wires (it is allowed to switch between the two wires at a crossing) in the direction that the y-coordinate decreases.
- For $a \in P$, define $f_P(a) = c$ if the crossing a is in column c in G.

For example, the wiring diagrams $G(\mathbf{i})$ and $G(\mathbf{j})$ in Figure 2 correspond to the word posets $P_{\mathbf{i}}$ and $P_{\mathbf{j}}$ in Example 7.

Proposition 10. *Let* $w \in \mathfrak{S}_{n+1}$*. Then the map* $[G] \mapsto [P]$ *described above is a bijection from* $[\mathcal{WD}(w)]$ *to* $[\mathcal{P}(w)]$.

Proof. We first show that the map $[G] \mapsto [P]$ is well defined. Suppose that G' is obtained from G by exchanging two adjacent rows in which the crossings are not in adjacent columns. It is easy to see that a downward path from a crossing a to a crossing b in G remains a downward path from a to b in G'. This shows that the images of $[G]$ and $[G']$ under this map are identical. Thus the map is well defined.

Now we show that the map is a bijection. Since $[\mathcal{WD}(w)] = \{[G(\mathbf{i})] : \mathbf{i} \in \mathcal{R}(w)\}$ and $[\mathcal{P}(w)] = \{[P_{\mathbf{i}}] : \mathbf{i} \in \mathcal{R}(w)\}$ both in bijection with $[\mathcal{R}(w)]$, it suffices to show that $[G(\mathbf{i})] \mapsto [P_{\mathbf{i}}]$. This is straightforward to check by the construction of the map. We omit the details. $\square$

Remark 11. There are several different combinatorial models presenting commutation classes. For example, heaps for the longest element in $\mathfrak{S}_n$ (see [28, Section 2.2]); rhombic tilings of a regular $2n$-gon, where all side lengths of the rhombi and the $2n$-gon are the same (see [10]). We refer the reader to [8] and references therein for more information.

3 Contractions, extensions, and indices

In this section, we define A-, D-, and δ-indices of a word poset, and two operations on word posets, called contractions and extensions. We also provide some results on these objects which will be used in later sections.

From now on, we concentrate on the reduced words of the longest element $w_0^{(n+1)}$ in the symmetric group $\mathfrak{S}_{n+1}$ and the word posets in $\mathcal{P}(w_0^{(n+1)})$. In order to define indices, we prepare the following two lemmas.

Lemma 12 (cf. [14, §2.2]). *Let* $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$*. The wiring diagram* $G(\mathbf{i})$ *has the following properties.*

1. *The* i*th wire travels from the* i*th starting point to the* $(n+2-i)$*th ending point for* $1 \leqslant i \leqslant n+1$*.*
2. *Two different wires meet exactly once.*
3. *Every wire has exactly* n *crossings.*
4. *Let* $c_1, \ldots, c_n$ *be the crossings that lie on the* 1*st wire* (*respectively,* $(n+1)$*st wire*) *of* $G(\mathbf{i})$ *in this order, i.e., the row of* c_{i+1} *is lower than the row of* c_i*. Then each* c_i *is in column* i (*respectively,* $n+1-i$)*.*

Lemma 13. *Let* $\mathbf{i} = (i_1, \ldots, i_{\bar{n}}) \in \mathcal{R}(w_0^{(n+1)})$*. Then there are unique integers* $d_1, \ldots, d_n$ *such that*

$$1 \leqslant d_1 < \cdots < d_n \leqslant \bar{n}, \qquad (i_{d_1}, \ldots, i_{d_n}) = (n, n-1, \ldots, 2, 1), \tag{2}$$

and there are unique integers $a_1, \ldots, a_n$ *such that*

$$1 \leqslant a_1 < \cdots < a_n \leqslant \bar{n}, \qquad (i_{a_1}, \ldots, i_{a_n}) = (1, 2, \ldots, n-1, n). \tag{3}$$

Moreover, $|\{a_1, \ldots, a_n\} \cap \{d_1, \ldots, d_n\}| = 1$*.*

Proof. We will use the wiring diagram $G(\mathbf{i})$ of $\mathbf{i}$. Let $d_1 < \cdots < d_n$ be the row indices of the crossings in the $(n+1)$st wire of $G(\mathbf{i})$. Then, by Lemma 12, $d_1, \ldots, d_n$ satisfy (2). Similarly, let $a_1 < \cdots < a_n$ be the row indices of the crossings in the 1st wire of $G(\mathbf{i})$. Then, by Lemma 12, $a_1, \ldots, a_n$ satisfy (3). Since the 1st and the $(n+1)$st wires meet exactly once, we get $|\{a_1, \ldots, a_n\} \cap \{d_1, \ldots, d_n\}| = 1$. It remains to show that these integers are unique.

To show the uniqueness of $d_1, \ldots, d_n$ satisfying (2), suppose that $d'_1, \ldots, d'_n$ are integers such that $d_j \neq d'_j$ for some j and

$$1 \leqslant d'_1 < \cdots < d'_n \leqslant \bar{n}, \qquad (i_{d'_1}, \ldots, i_{d'_n}) = (n, n-1, \ldots, 2, 1).$$

Let k be the smallest integer such that $d_j = d'_j$ for $j < k$ and $d_k \neq d'_k$.

For simplicity, we write an (i,j)-crossing to mean a crossing in row i and column j. Since the $(n+1)$st wire passes through the $(d_{k-1}, n+2-k)$-crossing, the $(d_k, n+1-k)$-crossing, and the $(d_{k+1}, n-k)$-crossing, there is no $(j, n+1-k)$-crossing for all $d_{k-1} < j < d_{k+1}$ with $j \neq d_k$. Since $d_{k-1} = d'_{k-1} < d'_k$, this shows $d_{k+1} < d'_k$. See Figure 3.

By the same argument, we can deduce that $d_{j+1} < d'_j$ for $j = k, k+1, \ldots, n-1$. In particular, we have $d_n < d'_{n-1} < d'_n$. However, by the definition of d_n, the crossing in row d_n and column 1 is the lowest crossing in this column, which is a contradiction to $d_n < d'_n$. This shows that there are no such integers $d'_1, \ldots, d'_n$, so the uniqueness of $d_1, \ldots, d_n$ is proved.

Similarly, we can show the uniqueness of $a_1, \ldots, a_n$, and the proof is completed. $\square$

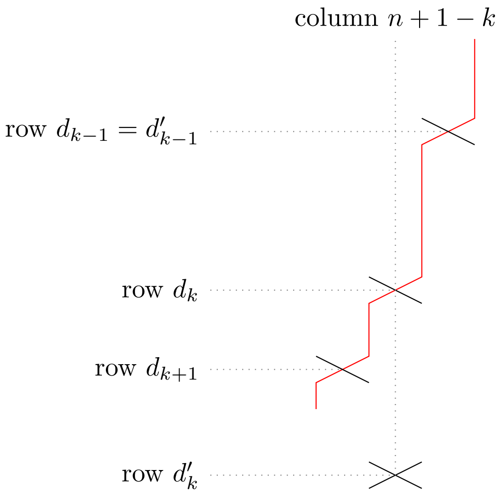


Figure 3: An illustration of a part of the $(n+1)$st wire (colored in red) in a wiring diagram and the crossings in rows d_{k-1}, d_k, d_{k+1}, and d'_k.

The previous lemma can be restated in terms of word posets as follows.

Proposition 14. *Let $P \in \mathcal{P}(w_0^{(n+1)})$. Then P contains a unique chain $\mathsf{D}(P)$ such that*

$$\mathsf{D}(P) = \{d_1 <_P d_2 <_P \cdots <_P d_n\}$$

and $f_P(d_i) = n+1-i$ for all $1 \leqslant i \leqslant n$. Similarly, P contains a unique chain $\mathsf{A}(P)$ such that

$$\mathsf{A}(P) = \{a_1 <_P a_2 <_P \cdots <_P a_n\}$$

and $f_P(a_i) = i$ for all $1 \leqslant i \leqslant n$. Moreover, we have $|\mathsf{D}(P) \cap \mathsf{A}(P)| = 1$.

Proof. Let $P \in \mathcal{P}(w_0^{(n+1)})$. By Proposition 9, there exists $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$ satisfying $P_{\mathbf{i}} \sim P$. Accordingly, it is enough to prove the statements for the word poset $P_{\mathbf{i}}$ for an arbitrary $\mathbf{i} = (i_1, \dots, i_{\bar{n}}) \in \mathcal{R}(w_0^{(n+1)})$. Then, using the map in Proposition 9, the statements for $P_{\mathbf{i}}$ that we need to prove can be reformulated as the statements for $\mathbf{i}$ in Lemma 13. Therefore the proof follows from this lemma. $\square$

We call $\mathsf{D}(P)$ the *descending chain* of P and $\mathsf{A}(P)$ the *ascending chain* of P. We now define an important notion in this paper called indices.

Definition 15. Let $P \in \mathcal{P}(w_0^{(n+1)})$. The D-*index* of P, denoted by $\mathrm{ind}_{\mathsf{D}}(P)$, is the number of elements in P above the descending chain $\mathsf{D}(P)$ in the Hasse diagram of P. Similarly, the A-*index* of P, denoted by $\mathrm{ind}_{\mathsf{A}}(P)$, is the number of elements in P above the ascending chain in the Hasse diagram of P. More precisely, if $\mathsf{D}(P) = \{d_1 <_P d_2 <_P \cdots <_P d_n\}$ and

$\mathsf{A}(P) = \{a_1 <_P a_2 <_P \cdots <_P a_n\}$, then

$$\mathrm{ind}_{\mathsf{D}}(P) = \sum_{i=1}^{n} \#\{k \in [\bar{n}] : k >_P d_i \text{ and } f_P(k) = f_P(d_i)\},$$
$$\mathrm{ind}_{\mathsf{A}}(P) = \sum_{i=1}^{n} \#\{k \in [\bar{n}] : k >_P a_i \text{ and } f_P(k) = f_P(a_i)\}.$$

For $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$, we also define

$$\mathrm{ind}_{\mathsf{D}}(\mathbf{i}) = \mathrm{ind}_{\mathsf{D}}(P_{\mathbf{i}}), \qquad \mathrm{ind}_{\mathsf{A}}(\mathbf{i}) = \mathrm{ind}_{\mathsf{A}}(P_{\mathbf{i}}).$$

Note that by the above definition,

$$|\{k \in [\bar{n}] : k >_P d_i \text{ and } f_P(k) = f_p(d_i)\}|$$

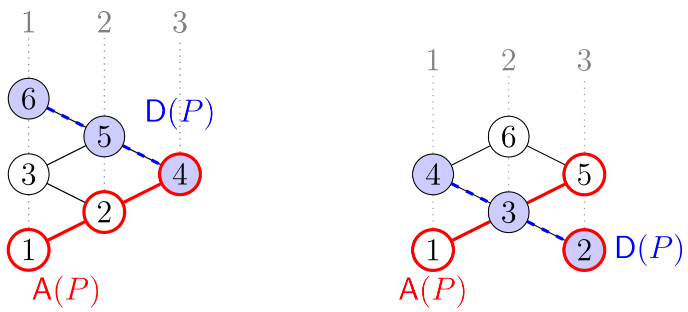
is the number of elements of P above the element d_i of $\mathsf{D}(P)$ in column $f_P(d_i) = n+1-i$ in the Hasse diagram of P.

In [7, Definition 3.4], the indices $\mathrm{ind}_{\mathsf{D}}(\mathbf{i})$ and $\mathrm{ind}_{\mathsf{A}}(\mathbf{i})$ of $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$ are defined without using word posets. It is not hard to see that the two definitions are equivalent.

Example 16. Continuing Example 7, let $\mathbf{i} = (1, 2, 1, 3, 2, 1)$ and $\mathbf{j} = (1, 3, 2, 1, 3, 2)$. Then we have

$$\mathsf{A}(P_{\mathbf{i}}) = 1 <_{P_{\mathbf{i}}} 2 <_{P_{\mathbf{i}}} 4, \quad \mathsf{D}(P_{\mathbf{i}}) = 4 <_{P_{\mathbf{i}}} 5 <_{P_{\mathbf{i}}} 6,$$
$$\mathsf{A}(P_{\mathbf{j}}) = 1 <_{P_{\mathbf{j}}} 3 <_{P_{\mathbf{j}}} 5, \quad \mathsf{D}(P_{\mathbf{j}}) = 2 <_{P_{\mathbf{j}}} 3 <_{P_{\mathbf{j}}} 4.$$

The chains $\mathsf{D}(P)$ and $\mathsf{A}(P)$ for $P = P_{\mathbf{i}}$ (left) and $P = P_{\mathbf{j}}$ (right) are shown as follows.



Counting the number of elements above $\mathsf{A}(P)$ and $\mathsf{D}(P)$, we obtain

$$\mathrm{ind}_{\mathsf{D}}(\mathbf{i}) = 0, \quad \mathrm{ind}_{\mathsf{A}}(\mathbf{i}) = 3, \quad \mathrm{ind}_{\mathsf{D}}(\mathbf{j}) = 2, \quad \mathrm{ind}_{\mathsf{A}}(\mathbf{j}) = 2.$$

The following lemma shows that the elements of $P \in \mathcal{P}(w_0^{(n+1)})$ in each column form a chain.

Lemma 17. *Let $P \in \mathcal{P}(w_0^{(n+1)})$. Then $\{x \in P : f_P(x) = i\}$ is a chain in P for each $i \in [n]$.*

Proof. Since $P \in \mathcal{P}(w_0^{(n+1)})$, there is a reduced word $\mathbf{i} = (i_1, i_2, \ldots, i_{\bar{n}}) \in \mathcal{R}(w_0^{(n+1)})$ such that $P_{\mathbf{i}} \sim P$. Therefore it suffices to show that $\{x \in P_{\mathbf{i}} : f_{P_{\mathbf{i}}}(x) = i\}$ is a chain. Consider $x, y \in P_{\mathbf{i}}$ with $f_{P_{\mathbf{i}}}(x) = f_{P_{\mathbf{i}}}(y) = i$ and $x <_{\mathbb{Z}} y$. This means $i_x = i_y = i$. Since $\mathbf{i}$ is reduced there must be an integer z such that $x < z < y$ and $i_z \in \{i-1, i+1\}$. Then $x <_{P_{\mathbf{i}}} z <_{P_{\mathbf{i}}} y$ and therefore $x <_{P_{\mathbf{i}}} y$. Since any two elements $x, y \in P_{\mathbf{i}}$ with $f_{P_{\mathbf{i}}}(x) = f_{P_{\mathbf{i}}}(y)$ are comparable, $\{x \in P_{\mathbf{i}} : f_{P_{\mathbf{i}}}(x) = i\}$ is a chain and the lemma is proved. $\square$

Now we define two operations, called contractions, on word posets.

Definition 18. Let $P \in \mathcal{P}(w_0^{(n+1)})$ and let

$$\mathsf{D}(P) = \{d_1 <_P d_2 <_P \cdots <_P d_n\},$$
$$\mathsf{A}(P) = \{a_1 <_P a_2 <_P \cdots <_P a_n\}.$$

1. The D-*contraction* of P is the word poset $C_{\mathsf{D}}(P)$ obtained from P by removing the descending chain $\mathsf{D}(P)$ with the function $f_{C_{\mathsf{D}}(P)} \colon C_{\mathsf{D}}(P) \to \mathbb{Z}_{>0}$ given by

$$f_{C_{\mathsf{D}}(P)}(k) = \begin{cases} f_P(k) & \text{if } k \in I_{\mathsf{D}}(P), \\ f_P(k) - 1 & \text{if } k \in C_{\mathsf{D}}(P) \setminus I_{\mathsf{D}}(P), \end{cases}$$

   where $I_{\mathsf{D}}(P) = \{k \in P : k <_P d_i \text{ and } f_P(k) = f_P(d_i) \text{ for some } i \in [n]\}$. Here, the poset structure of $C_{\mathsf{D}}(P)$ is induced from that of P, i.e., for $x, y \in C_{\mathsf{D}}(P)$ we have $x <_{C_{\mathsf{D}}(P)} y$ if and only if $x <_P y$.

2. The A-*contraction* of P is the word poset $C_{\mathsf{A}}(P)$ obtained from P by removing the ascending chain $\mathsf{A}(P)$ with the function $f_{C_{\mathsf{A}}(P)} \colon C_{\mathsf{A}}(P) \to \mathbb{Z}_{>0}$ given by

$$f_{C_{\mathsf{A}}(P)}(k) = \begin{cases} f_P(k) - 1 & \text{if } k \in I_{\mathsf{A}}(P), \\ f_P(k) & \text{if } k \in C_{\mathsf{D}}(P) \setminus I_{\mathsf{A}}(P), \end{cases}$$

   where $I_{\mathsf{A}}(P) = \{k \in P : k <_P a_i \text{ and } f_P(k) = f_P(a_i) \text{ for some} i \in [n]\}$. Here, the poset structure of $C_{\mathsf{A}}(P)$ is induced from that of P.

Observe that $I_{\mathsf{D}}(P)$ (respectively, $I_{\mathsf{A}}(P)$) is the ideal consisting of the elements below the descending chain $\mathsf{D}(P)$ (respectively, ascending chain $\mathsf{A}(P)$) in the Hasse diagram of P. We call $I_{\mathsf{D}}(P)$ (respectively, $I_{\mathsf{A}}(P)$) the D-*contraction ideal* (respectively, A-*contraction ideal*) of P. Here, an *ideal* of a poset P means a subset I of P with the property that $x \in I$ and $y <_P x$ imply $y \in I$.

One may consider $C_{\mathsf{D}}(P)$ as the word poset whose Hasse diagram is obtained from that of P by removing the descending chain $\mathsf{D}(P)$ and shifting the part $(P \setminus \mathsf{D}(P)) \setminus I_{\mathsf{D}}(P)$ above $\mathsf{D}(P)$ to the left by one column. Similarly one may consider $C_{\mathsf{A}}(P)$ as the word poset whose Hasse diagram is obtained from that of P by removing the ascending chain $\mathsf{A}(P)$ and shifting the part $I_{\mathsf{A}}(P)$ below $\mathsf{A}(P)$ to the left by one column. See Figure 4.

Example 19. Let $\mathbf{i} = (4,3,4,2,3,4,1,2,5,4,3,2,1,4,5) \in \mathcal{R}(w_0^{(6)})$. The Hasse diagrams of $P_{\mathbf{i}}$, $C_{\mathsf{A}}(P_{\mathbf{i}})$, and $C_{\mathsf{D}}(P_{\mathbf{i}})$ are shown in Figure 4. Note that $\operatorname{ind}_{\mathsf{A}}(P_{\mathbf{i}}) = 2$ and $\operatorname{ind}_{\mathsf{D}}(P_{\mathbf{i}}) = 2$.

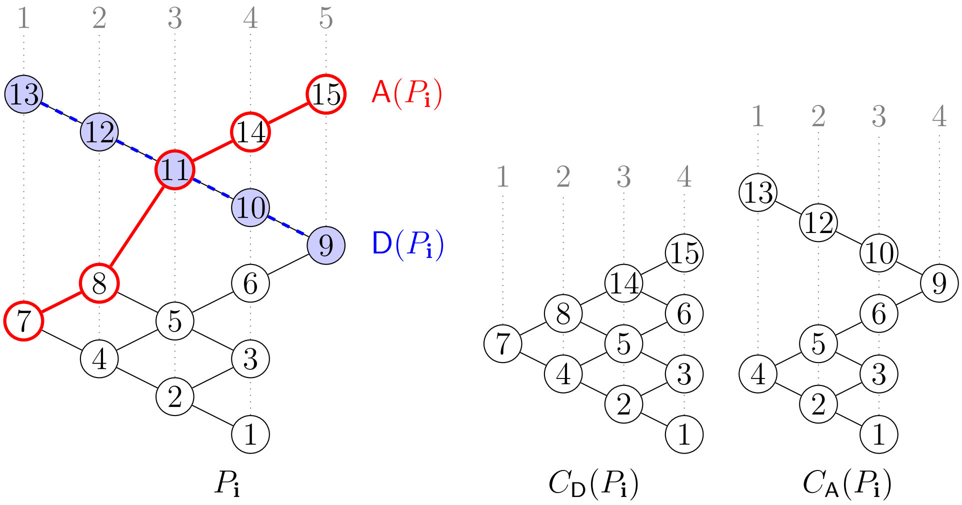


Figure 4: The Hasse diagrams of $P_{\mathbf{i}}$, $C_{\mathsf{D}}(P_{\mathbf{i}})$, and $C_{\mathsf{A}}(P_{\mathbf{i}})$ for $\mathbf{i} = (4,3,4,2,3,4,1,2,5,4,3,2,1,4,5)$. In each diagram, the element k is in column $f_{P_{\mathbf{i}}}(k)$, $f_{C_{\mathsf{D}}(P_{\mathbf{i}})}(k)$, or $f_{C_{\mathsf{A}}(P_{\mathbf{i}})}(k)$.

Now we define the reverse operations of the contractions.

Definition 20. Let $P \in \mathcal{P}(w_0^{(n)})$ and let I be an ideal of P.

1. The D-*extension* of P with respect to I is the word poset $E_{\mathsf{D}}(P,I)$ which is a poset on $P \sqcup \{d_1, d_2, \ldots, d_n\}$ with a function $f_{E_{\mathsf{D}}(P,I)} \colon E_{\mathsf{D}}(P,I) \to \mathbb{Z}_{>0}$ defined as follows.

   - The function $f_{E_{\mathsf{D}}(P,I)} \colon E_{\mathsf{D}}(P,I) \to \mathbb{Z}_{>0}$ is given by

   $$f_{E_{\mathsf{D}}(P,I)}(x) = \begin{cases} f_P(x) & \text{if } x \in I, \\ n+1-i & \text{if } x = d_i, \\ f_P(x)+1 & \text{if } x \in P \setminus I. \end{cases}$$

   - The covering relations of the poset $E_{\mathsf{D}}(P,I)$ on $P \sqcup \{d_1, d_2, \ldots, d_n\}$ are given by $x \lessdot_{E_{\mathsf{D}}(P,I)} y$ if and only if one of the following conditions holds:

     (a) $x, y \in I$ and $x \lessdot_P y$,

     (b) $x, y \in P \setminus I$ and $x \lessdot_P y$,

     (c) $x = d_i$ and $y = d_{i+1}$ for some $i \in [n-1]$,

     (d) x is a maximal element of I in P, $y \in \{d_1, \ldots, d_n\}$, and

     $$|f_{E_{\mathsf{D}}(P,I)}(x) - f_{E_{\mathsf{D}}(P,I)}(y)| = 1, \text{ or}$$

     (e) $x \in \{d_1, \ldots, d_n\}$, y is a minimal element of $P \setminus I$ in P, and

     $$|f_{E_{\mathsf{D}}(P,I)}(x) - f_{E_{\mathsf{D}}(P,I)}(y)| = 1.$$

2. The A-*extension* of P with respect to I is the word poset $E_{\mathsf{A}}(P,I)$ which is a poset on $P \sqcup \{a_1, a_2, \dots, a_n\}$ with a function $f_{E_{\mathsf{A}}(P,I)} \colon E_{\mathsf{A}}(P,I) \to \mathbb{Z}_{>0}$ defined as follows.
   - The function $f_{E_{\mathsf{A}}(P,I)} \colon E_{\mathsf{A}}(P,I) \to \mathbb{Z}_{>0}$ is given by
$$f_{E_{\mathsf{A}}(P,I)}(x) = \begin{cases} f_P(x)+1 & \text{if } x \in I, \\ i & \text{if } x = a_i, \\ f_P(x) & \text{if } x \in P \setminus I. \end{cases}$$
   - The covering relations of the poset $E_{\mathsf{A}}(P,I)$ on $P \sqcup \{a_1, a_2, \dots, a_n\}$ are given by $x \lessdot_{E_{\mathsf{A}}(P,I)} y$ if and only if one of the following conditions holds:
     (a) $x, y \in I$ and $x \lessdot_P y$,
     (b) $x, y \in P \setminus I$ and $x \lessdot_P y$,
     (c) $x = a_i$ and $y = a_{i+1}$ for some $i \in [n-1]$,
     (d) x is a maximal element of I in P, $y \in \{a_1, \dots, a_n\}$, and
$$|f_{E_{\mathsf{A}}(P,I)}(x) - f_{E_{\mathsf{A}}(P,I)}(y)| = 1, \text{ or}$$
     (e) $x \in \{a_1, \dots, a_n\}$, y is a minimal element of $P \setminus I$ in P, and
$$|f_{E_{\mathsf{A}}(P,I)}(x) - f_{E_{\mathsf{A}}(P,I)}(y)| = 1.$$

The extensions are the reverse operations of the contractions in the following sense.

Proposition 21. *Let $P \in \mathcal{P}(w_0^{(n+1)})$. Then*
$$P \sim E_{\mathsf{D}}(C_{\mathsf{D}}(P), I_{\mathsf{D}}(P)),$$
$$P \sim E_{\mathsf{A}}(C_{\mathsf{A}}(P), I_{\mathsf{A}}(P)).$$

Proof. This is straightforward to check using the definitions of contractions and extensions. We omit the details. □

We provide an example of A-extension in Figure 5. In [7, Definition 3.6], the D-contraction $C_{\mathsf{D}}(\mathbf{i})$ and the A-contraction $C_{\mathsf{A}}(\mathbf{i})$ of a reduced word $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$ are defined using wiring diagrams. For $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$, let $G(\mathbf{i})$ be the corresponding wiring diagram. Removing the $(n+1)$st wire from $G(\mathbf{i})$ and shifting the part below this wire to the left by one, we get a new wiring diagram such that the jth wire travels from the jth starting point to the $(n+1-j)$th ending point. Since the number of crossings decreases by n, the new wiring diagram represents a reduced word in $\mathcal{R}(w_0^{(n)})$. Similarly, removing the 1st wire from $G(\mathbf{i})$ also produces the wiring diagram of a reduced word in $\mathcal{R}(w_0^{(n)})$. One can check that $C_{\mathsf{D}}(P_{\mathbf{i}}) \sim P_{C_{\mathsf{D}}(\mathbf{i})}$ and $C_{\mathsf{A}}(P_{\mathbf{i}}) \sim P_{C_{\mathsf{A}}(\mathbf{i})}$. This leads us to the following result.

Proposition 22. *Let $P \in \mathcal{P}(w_0^{(n+1)})$. Then $C_{\mathsf{D}}(P)$ and $C_{\mathsf{A}}(P)$ are word posets in $\mathcal{P}(w_0^{(n)})$.*

Finally, we define the δ-index of a word poset.

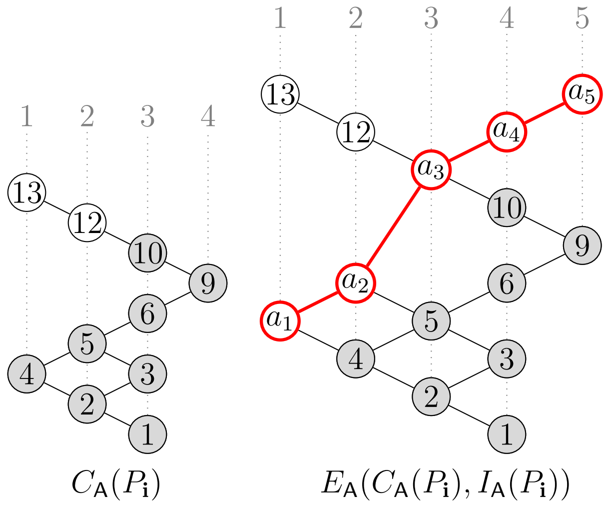


Figure 5: The Hasse diagram of $C_{\mathsf{A}}(P_{\mathbf{i}})$ with the ideal $I_{\mathsf{A}}(P_{\mathbf{i}})$ colored gray on the left; the A-extension of $C_{\mathsf{A}}(P_{\mathbf{i}})$ with respect to $I_{\mathsf{A}}(P_{\mathbf{i}})$ on the right for $\mathbf{i} = (4,3,4,2,3,4,1,2,5,4,3,2,1,4,5)$. The A-extension $E_{\mathsf{A}}(C_{\mathsf{A}}(P_{\mathbf{i}}), I_{\mathsf{A}}(P_{\mathbf{i}}))$ is isomorphic to $P_{\mathbf{i}}$ as a labeled poset (see Figure 4).

Definition 23. For a sequence $\delta = (\delta_1, \dots, \delta_{n-1}) \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$, the *$\delta$-index* of $P \in \mathcal{P}(w_0^{(n+1)})$ is the integer vector $\mathrm{ind}_\delta(P) = (I_1, \dots, I_{n-1})$ defined by

$$I_k \coloneqq \mathrm{ind}_{\delta_k}(C_{\delta_{k+1}} \circ C_{\delta_{k+2}} \circ \cdots \circ C_{\delta_{n-1}}(P)), \qquad 1 \leqslant k \leqslant n-1,$$

where $I_{n-1} = \mathrm{ind}_{\delta_{n-1}}(P)$. The *$\delta$-index* of $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$ is defined by $\mathrm{ind}_\delta(\mathbf{i}) = \mathrm{ind}_\delta(P_{\mathbf{i}})$.

Example 24. Let $\mathbf{i} = (4,3,4,2,3,4,1,2,5,4,3,2,1,4,5) \in \mathcal{R}(w_0^{(6)})$. Then, for a sequence $\delta = (\mathsf{A}, \mathsf{A}, \mathsf{A}, \mathsf{A}) \in \{\mathsf{A}, \mathsf{D}\}^4$, we obtain $\mathrm{ind}_\delta(\mathbf{i}) = (1,2,3,2)$ as shown in Figure 6.

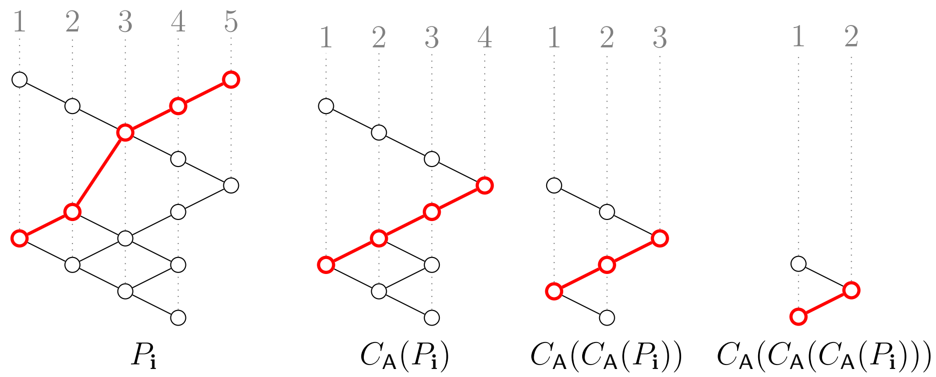


Figure 6: The Hasse diagrams of $P_{\mathbf{i}}$, $C_{\mathsf{A}}(P_{\mathbf{i}})$, $C_{\mathsf{A}}(C_{\mathsf{A}}(P_{\mathbf{i}}))$, and $C_{\mathsf{A}}(C_{\mathsf{A}}(C_{\mathsf{A}}(P_{\mathbf{i}})))$ for $\mathbf{i} = (4,3,4,2,3,4,1,2,5,4,3,2,1,4,5)$.

4 Gelfand–Cetlin type reduced words

In this section, we introduce Gelfand–Cetlin type reduced words. We give a recursive formula for the number of such reduced words using standard Young tableaux of shifted shapes in Theorem 37.

We first define Gelfand–Cetlin type word posets and Gelfand–Cetlin type reduced words.

Definition 25. A word poset $P \in \mathcal{P}(w_0^{(n+1)})$ is of *Gelfand–Cetlin type* if there exists $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ such that $\mathrm{ind}_\delta(P) = (0, 0, \dots, 0) \in \mathbb{Z}^{n-1}$. Denote by $\mathcal{P}_{\mathrm{GC}}(n)$ the set of Gelfand–Cetlin type word posets in $\mathcal{P}(w_0^{(n+1)})$ and let $[\mathcal{P}_{\mathrm{GC}}(n)] = \mathcal{P}_{\mathrm{GC}}(n)/\sim$.

Definition 26. A reduced word $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$ is of *Gelfand–Cetlin type* if $P_{\mathbf{i}}$ is of Gelfand–Cetlin type. A commutation class $[\mathbf{i}]$ is of *Gelfand–Cetlin type* if it contains a Gelfand–Cetlin type reduced word. Denote by $\mathcal{R}_{\mathrm{GC}}(n)$ the set of Gelfand–Cetlin type reduced words in $\mathcal{R}(w_0^{(n+1)})$ and let $[\mathcal{R}_{\mathrm{GC}}(n)] = \{[\mathbf{i}] : \mathbf{i} \in \mathcal{R}_{\mathrm{GC}}(n)\}$.

Remark 27. One can deduce from [7, Theorem A] that for $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$, the string polytope $\Delta_{\mathbf{i}}(\lambda)$ is combinatorially equivalent to a full dimensional Gelfand–Cetlin polytope of rank n if and only if $\mathbf{i}$ is of Gelfand–Cetlin type, see the proof of Corollary 38. This is why we say that such word posets and reduced words are *of Gelfand–Cetlin type*.

The following proposition easily follows from Proposition 9 and the definitions of $\mathcal{P}_{\mathrm{GC}}(n)$ and $\mathcal{R}_{\mathrm{GC}}(n)$.

Proposition 28. *The map $[\mathbf{i}] \mapsto [P_{\mathbf{i}}]$ is a bijection from $[\mathcal{R}_{\mathrm{GC}}(n)]$ to $[\mathcal{P}_{\mathrm{GC}}(n)]$. Accordingly,*

$$|[\mathcal{R}_{\mathrm{GC}}(n)]| = |[\mathcal{P}_{\mathrm{GC}}(n)]|.$$

We note that if $\mathrm{ind}_{\mathsf{A}}(P) = 0$, then $C_{\mathsf{A}}(P) = I_{\mathsf{A}}(P)$. Similarly, we have $C_{\mathsf{D}}(P) = I_{\mathsf{D}}(P)$ when $\mathrm{ind}_{\mathsf{D}}(P) = 0$. The succeeding lemma follows immediately from Proposition 21.

Lemma 29. *Let $P \in \mathcal{P}(w_0^{(n+1)})$. If $\mathrm{ind}_{\mathsf{A}}(P) = 0$, then*

$$P \sim E_{\mathsf{A}}(C_{\mathsf{A}}(P), C_{\mathsf{A}}(P)).$$

Similarly, if $\mathrm{ind}_{\mathsf{D}}(P) = 0$, then

$$P \sim E_{\mathsf{D}}(C_{\mathsf{D}}(P), C_{\mathsf{D}}(P)).$$

Lemma 29 implies that if $\mathrm{ind}_{\mathsf{A}}(P) = 0$ (respectively, $\mathrm{ind}_{\mathsf{D}}(P) = 0$), then P is completely determined by $C_{\mathsf{A}}(P)$ (respectively, $C_{\mathsf{D}}(P)$) up to isomorphism.

The following definition will be used frequently throughout this section.

Definition 30. For a word poset P, an element $x \in P$ is called a *top element* if x is the largest element in its column. In other words, $x \in P$ is a top element if $y \leqslant_P x$ for all $y \in P$ with $f_P(y) = f_P(x)$. For $i \in [n]$, denote by $m_i(P)$ the top element in column i.

The following lemma shows that if P is a Gelfand–Cetlin type word poset, the top elements of P must form a chain.

Lemma 31. *Let $P \in \mathcal{P}(w_0^{(n+1)})$. Then*

$$m_1(P) <_P m_2(P) <_P \cdots <_P m_n(P) \qquad \text{if } \mathrm{ind}_{\mathsf{A}}(P) = 0,$$

$$m_1(P) >_P m_2(P) >_P \cdots >_P m_n(P) \qquad \text{if } \mathrm{ind}_{\mathsf{D}}(P) = 0.$$

Moreover, $m_n(P)$ (respectively, $m_1(P)$) is the maximum element of P if $\mathrm{ind}_{\mathsf{A}}(P) = 0$ (respectively, $\mathrm{ind}_{\mathsf{D}}(P) = 0$).

Proof. We will only consider the case $\mathrm{ind}_{\mathsf{A}}(P) = 0$ because the other case $\mathrm{ind}_{\mathsf{D}}(P) = 0$ can be proved similarly.

Since $\mathrm{ind}_{\mathsf{A}}(P) = 0$, by Lemma 29, we have $P \sim E_{\mathsf{A}}(C_{\mathsf{A}}(P), C_{\mathsf{A}}(P))$. By definition, $Q \coloneqq E_{\mathsf{A}}(C_{\mathsf{A}}(P), C_{\mathsf{A}}(P))$ is the word poset obtained from $C_{\mathsf{A}}(P)$ by adding n elements $a_1, \dots, a_n$ with additional covering relations $a_1 \lessdot_Q \cdots \lessdot_Q a_n$ and $x \lessdot_Q a_i$ for each maximal element x in $C_{\mathsf{A}}(P)$ and $i \in [n]$ such that $|f_Q(x) - f_Q(a_i)| = 1$, where $f_Q(x) = f_{C_{\mathsf{A}}(P)}(x)$ for $x \in C_{\mathsf{A}}(P)$ and $f_Q(a_i) = i$ for $i \in [n]$. By the construction, we have $a_i = m_i(Q)$ for $i \in [n]$, and therefore $m_1(Q) <_Q \cdots <_Q m_n(Q)$. Since $P \sim Q$, this shows the first statement.

For the second statement, let x be an arbitrary element in Q. Suppose $f_Q(x) = i$. Since $Q \sim P \in \mathcal{P}(w_0^{(n+1)})$, by Lemma 17, $\{y \in Q \colon f_Q(y) = i\}$ is a chain in P for each $i \in [n]$. By definition $m_i(Q)$ is the maximum element in this chain. Then $x \leqslant_Q m_i(Q) \leqslant_Q m_n(Q)$. Thus $x \leqslant_Q m_n(Q)$, and therefore $m_n(Q)$ is the maximum element in Q. Since $P \sim Q$, this shows the second statement. □

By Lemma 31, if $P \in \mathcal{P}_{\mathrm{GC}}(n)$ and $n \geqslant 2$, then we have $\mathrm{ind}_{\mathsf{A}}(P) = 0$ or $\mathrm{ind}_{\mathsf{D}}(P) = 0$, but not both. This means that there is a unique $\delta = (\delta_1, \dots, \delta_{n-1}) \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ such that $\mathrm{ind}_\delta(P) = (0, \dots, 0)$. Therefore the map

$$\phi \colon [\mathcal{P}_{\mathrm{GC}}(n)] \to \{\mathsf{A}, \mathsf{D}\}^{n-1}$$

sending $[P]$ to such δ is well defined. This map is in fact a bijection.

Proposition 32. *For $n \geqslant 2$, the map $\phi \colon [\mathcal{P}_{\mathrm{GC}}(n)] \to \{\mathsf{A}, \mathsf{D}\}^{n-1}$ is a bijection.*

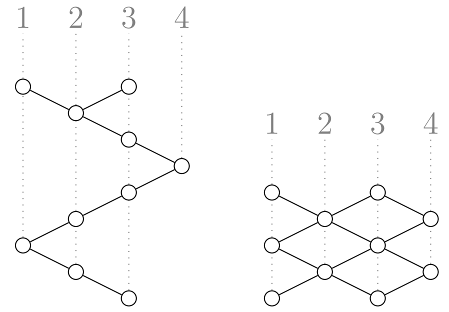
Proof. We first show that ϕ is injective. Suppose that $P \in \mathcal{P}_{\mathrm{GC}}(n)$ satisfies $\phi([P]) = \delta = (\delta_1, \dots, \delta_{n-1}) \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$. By definition, we have $\mathrm{ind}_\delta(P) = (0, \dots, 0)$. We will show that $[P]$ is determined by δ.

Define the word posets $P_k \in \mathcal{P}_{\mathrm{GC}}(k)$ for $k \in [n]$ recursively as follows. First, we set $P_n = P$. For $k \in [n-1]$, define

$$P_k = C_{\delta_k}(P_{k+1}).$$

Since $P_1 \in \mathcal{P}(w_0^{(2)})$, P_1 is a word poset with one element, say x, and $f_{P_1}(x) = 1$. By Lemma 29, for $k \in [n-1]$, we have

$$P_{k+1} \sim E_{\delta_k}(C_{\delta_k}(P_{k+1}), C_{\delta_k}(P_{k+1})) = E_{\delta_k}(P_k, P_k).$$



$P_{(3,2,1,2,3,4,3,2,3,1)}$. $P_{(1,3,2,1,4,3,4,2,3,1)}$.

Figure 7: The word posets $P_{(3,2,1,2,3,4,3,2,3,1)}$ and $P_{(1,3,2,1,4,3,4,2,3,1)}$.

Thus $P = P_n$ is determined uniquely by δ up to isomorphism. This shows that ϕ is injective.

To show that ϕ is surjective take an arbitrary sequence $\delta = (\delta_1, \dots, \delta_{n-1}) \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$. Define the word posets $P_k \in \mathcal{P}_{\mathrm{GC}}(k)$ for $k \in [n]$ by $P_1 \in \mathcal{P}(w_0^{(2)})$ and

$$P_{k+1} = E_{\delta_k}(P_k, P_k) \qquad \text{for } k \in [n-1].$$

Here we may choose any $P_1 \in \mathcal{P}(w_0^{(2)})$ because if $P_1, P_1' \in \mathcal{P}(w_0^{(2)})$ then $P_1 \sim P_1'$. It is easy to check that $\phi(P_n) = \delta$. Thus ϕ is surjective, which completes the proof. $\square$

The proof of the above proposition shows that if $P, Q \in \mathcal{P}(w_0^{(n+1)})$ satisfy $\mathrm{ind}_\delta(P) = \mathrm{ind}_\delta(Q) = (0, \dots, 0)$ for some $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$, then $P \sim Q$ since both P and Q are determined by δ. In general, the condition $\mathrm{ind}_\delta(P) = \mathrm{ind}_\delta(Q)$ for some $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ does not imply $P \sim Q$ as the following example shows.

Example 33. Consider the two words

$$\mathbf{i} = (3,2,1,2,3,4,3,2,3,1), \quad \mathbf{j} = (1,3,2,1,4,3,4,2,3,1) \in \mathcal{R}(w_0^{(5)}).$$

For any $(\delta_1, \delta_2) \in \{\mathsf{A}, \mathsf{D}\}^2$, we have $\mathrm{ind}_{(\delta_1,\delta_2,\mathsf{A})}(P_{\mathbf{i}}) = \mathrm{ind}_{(\delta_1,\delta_2,\mathsf{A})}(P_{\mathbf{j}})$, but $P_{\mathbf{i}} \not\sim P_{\mathbf{j}}$. See Figure 7 for these word posets. Accordingly, a single δ-index of P does not always determine the word poset class $[P]$.

Although a single δ-index of P is not enough to determine the word poset $[P]$, in the next section, we will show that the δ-indices $\mathrm{ind}_\delta(P)$ for all $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ determine $[P]$ (see Theorem 39). Note that in Example 33 we have $\mathrm{ind}_{\mathsf{D}}(P_{\mathbf{i}}) = 1 \neq 2 = \mathrm{ind}_{\mathsf{D}}(P_{\mathbf{j}})$, so $P_{\mathbf{i}}$ and $P_{\mathbf{j}}$ do not have the same δ-indices for all δ.

Proposition 32 immediately gives the cardinality of the Gelfand–Cetlin type commutation classes.

Corollary 34. *For $n \geqslant 2$, we have*

$$|[\mathcal{R}_{\mathrm{GC}}(n)]| = |[\mathcal{P}_{\mathrm{GC}}(n)]| = 2^{n-1}.$$

We note that Corollary 34 was proved in the recent paper [17, Proposition 20] using a different method.

Using the construction in the proof of Proposition 32 and standard Young tableaux, we find a recurrence relation for the number of Gelfand–Cetlin type reduced words. To this end, we need the following lemma, which allows us to draw the Hasse diagram of a Gelfand–Cetlin type word poset P corresponding to $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ using only δ.

Lemma 35. *Let* $P \in \mathcal{P}(w_0^{(n+1)})$ *and let* $\delta = (\delta_1, \dots, \delta_{n-1})$ *be the element in* $\{\mathsf{A}, \mathsf{D}\}^{n-1}$ *satisfying* $\mathrm{ind}_\delta(P) = (0, \dots, 0)$. *Let* $P_n = P$, *and for* $k \in [n-1]$, *define*

$$P_k = C_{\delta_k}(P_{k+1}).$$

Then P_1 *is isomorphic to the unique word poset (up to isomorphism) in* $\mathcal{P}(w_0^{(2)})$ *and, for* $k \in [n-1]$, *the word poset* P_{k+1} *is constructed as follows (see Figure 8).*

- *If* $\delta_{k-1} = \mathsf{A}$ *and* $\delta_k = \mathsf{A}$, *then the Hasse diagram of* P_{k+1} *is obtained from that of* P_k *by adding the chain* $m_1(P_{k+1}) \lessdot_{P_{k+1}} \cdots \lessdot_{P_{k+1}} m_{k+1}(P_{k+1})$ *with additional covering relations* $m_i(P_{k+1}) \gtrdot_{P_{k+1}} m_i(P_k)$ *for* $i \in [k]$.
- *If* $\delta_{k-1} = \mathsf{A}$ *and* $\delta_k = \mathsf{D}$, *then the Hasse diagram of* P_{k+1} *is obtained from that of* P_k *by adding the chain* $m_1(P_{k+1}) \gtrdot_{P_{k+1}} \cdots \gtrdot_{P_{k+1}} m_{k+1}(P_{k+1})$ *with an additional covering relation* $m_k(P_k) \lessdot_{P_{k+1}} m_{k+1}(P_{k+1})$.
- *If* $\delta_{k-1} = \mathsf{D}$ *and* $\delta_k = \mathsf{A}$, *then the Hasse diagram of* P_{k+1} *is obtained from that of* P_k *by adding the chain* $m_1(P_{k+1}) \lessdot_{P_{k+1}} \cdots \lessdot_{P_{k+1}} m_{k+1}(P_{k+1})$ *with an additional covering relation* $m_1(P_{k+1}) \gtrdot_{P_{k+1}} m_1(P_k)$.
- *If* $\delta_{k-1} = \mathsf{D}$ *and* $\delta_k = \mathsf{D}$, *then the Hasse diagram of* P_{k+1} *is obtained from that of* P_k *by adding the chain* $m_1(P_{k+1}) \gtrdot_{P_{k+1}} \cdots \gtrdot_{P_{k+1}} m_{k+1}(P_{k+1})$ *with additional covering relations* $m_i(P_k) \lessdot_{P_{k+1}} m_{i+1}(P_{k+1})$ *for* $i \in [k]$.

Proof. Consider the case that $\delta_{k-1} = \mathsf{A}$ and $\delta_k = \mathsf{A}$. Since $P_k = C_{\delta_k}(P_{k+1}) = C_{\mathsf{A}}(P_{k+1})$ and $\mathrm{ind}_{\mathsf{A}}(P_{k+1}) = 0$, by Lemma 29, we have

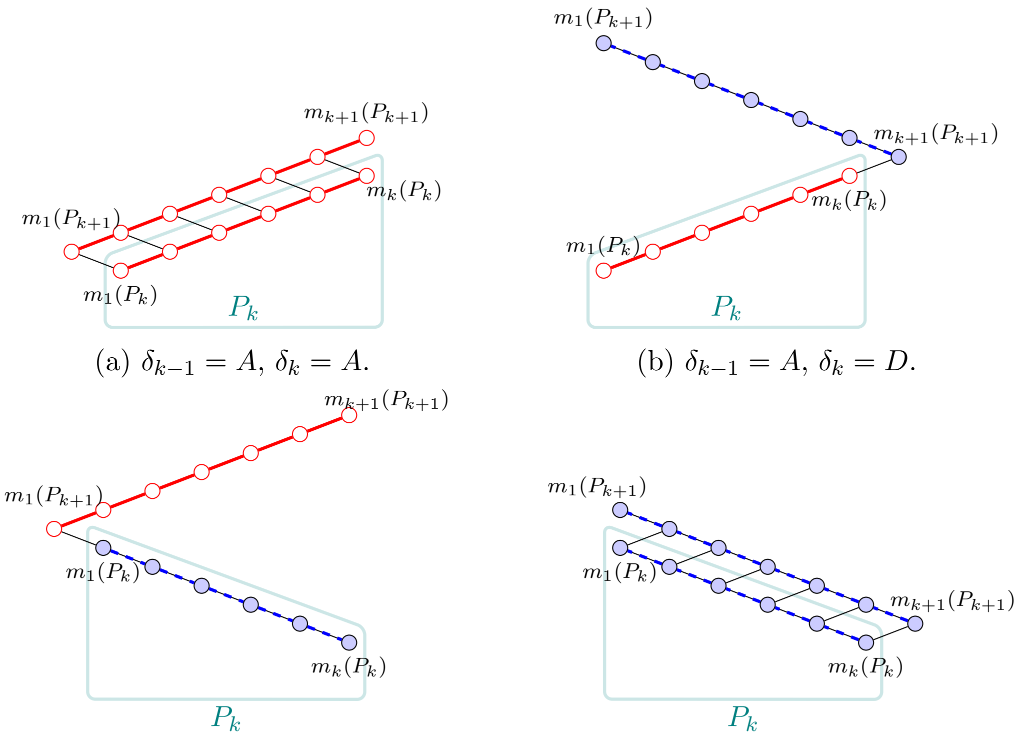
$$P_{k+1} \sim E_{\mathsf{A}}(P_k, P_k).$$

Then it is straightforward to check that we obtain the desired description for P_{k+1} by the definition of A-extension in Definition 20. The other three cases can be checked similarly. □

Now we define standard Young tableaux of shifted shape.

Definition 36. A *partition* of n is a weakly decreasing sequence $\mu = (\mu_1, \dots, \mu_t)$ of positive integers summing to n. A partition $\mu = (\mu_1, \dots, \mu_t)$ is *strict* if $\mu_1 > \cdots > \mu_t$. For a strict partition $\mu = (\mu_1, \dots, \mu_t)$, the *shifted diagram* of μ, denoted by μ^*, is the set

$$\mu^* := \{(i, j) \mid 1 \leqslant i \leqslant t, i \leqslant j \leqslant \mu_i + i - 1\}.$$



(a) $\delta_{k-1} = A$, $\delta_k = A$.

(b) $\delta_{k-1} = A$, $\delta_k = D$.

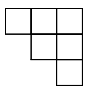
(c) $\delta_{k-1} = D$, $\delta_k = A$.

(d) $\delta_{k-1} = D$, $\delta_k = D$.

Figure 8: The Hasse diagrams of P_k and P_{k+1}.

We will identify μ^* as an array of squares where there is a square in row i and column j for each $(i, j) \in \mu^*$. For a strict partition $\mu = (\mu_1, \ldots, \mu_t)$ of n, a *standard Young tableau of shifted shape* μ is a bijection $T \colon \mu^* \to [n]$ such that $T(i, j) \leqslant T(i', j')$ if $i \leqslant i'$ and $j \leqslant j'$. We will represent a standard Young tableau T of shifted shape μ by filling $T(i, j)$ in the square in row i and column j of μ^*. Denote by g^μ the number of standard Young tableaux of shifted shape μ.

For example, the shifted diagram of shape $\mu = (3, 2, 1)$ is drawn as follows.

$\mu^* =$ 

There are two standard Young tableaux of shifted shape $\mu = (3, 2, 1)$:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 6 \\ \cline{3-3} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 3 & 5 \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 6 \\ \cline{3-3} \end{array}$$

Thrall [29] showed that the number g^μ of standard Young tableaux of shifted shape $\mu = (\mu_1, \ldots, \mu_t)$ is given as follows:

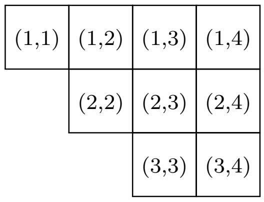
$$g^\mu = \frac{|\mu|!}{\mu_1! \mu_2! \cdots \mu_t!} \prod_{i<j} \frac{\mu_i - \mu_j}{\mu_i + \mu_j}. \tag{4}$$

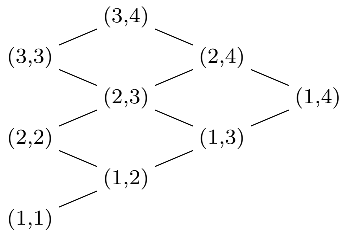
For instance, if $\mu = (3, 2, 1)$, then we have

$$g^{(3,2,1)} = \frac{6!}{3!2!1!} \cdot \frac{1 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3} = 2.$$

There is another formula for g^μ called the (shifted) hook length formula, see [23, p.267, eq.(2)].

For a strict partition $\mu = (\mu_1, \ldots, \mu_t)$, we define Q_μ to be the poset on μ^* with relations $(i, j) \leqslant_{Q_\mu} (i', j')$ if $i \leqslant i'$ and $j \leqslant j'$. For example, if $\mu = (4, 3, 2)$, then the poset Q_μ is given as follows.

$\mu^* =$ 

$Q_\mu =$ 

There is a natural bijection between the standard Young tableaux of shifted shape μ and the linear extensions of Q_μ. Therefore the number of linear extensions of Q_μ is equal to g^μ.

Now we are ready to state our first main theorem.

Theorem 37. *Let* $\mathsf{gc}(n)$ *be the number of Gelfand–Cetlin type reduced words in* $\mathcal{R}(w_0^{(n+1)})$, *i.e.,* $\mathsf{gc}(n) = |\mathcal{R}_{\mathrm{GC}}(n)|$. *Then* $\mathsf{gc}(0) = \mathsf{gc}(1) = 1$ *and for* $n \geqslant 2$, *we have*

$$\mathsf{gc}(n) = \sum_{i=1}^{n} g^{(n,n-1,\ldots,n-i+1)} \mathsf{gc}(n-i),$$

where g^μ *is the number of standard Young tableaux of shifted shape* μ *(see* (4)*).*

Proof. Clearly, we have $\mathsf{gc}(0) = \mathsf{gc}(1) = 1$. Suppose $n \geqslant 2$. Observe that

$$\mathsf{gc}(n) = |\mathcal{R}_{\mathrm{GC}}(n)| = \sum_{[\mathbf{i}] \in [\mathcal{R}_{\mathrm{GC}}(n)]} |[\mathbf{i}]|. \tag{5}$$

By Proposition 28, the map $[\mathbf{i}] \mapsto [P_\mathbf{i}]$ is a bijection from $[\mathcal{R}_{\mathrm{GC}}(n)]$ to $[\mathcal{P}_{\mathrm{GC}}(n)]$. Moreover, by Proposition 8, $|[\mathbf{i}]|$ is equal to the number of linear extensions of $P_\mathbf{i}$. This shows that we can rewrite (5) as

$$\mathsf{gc}(n) = \sum_{[P] \in [\mathcal{P}_{\mathrm{GC}}(n)]} e(P), \tag{6}$$

where $e(P)$ is the number of linear extensions of P.

Define $\mathsf{a}(n)$ and $\mathsf{d}(n)$ by

$$\mathsf{a}(n) = \sum_{\substack{[P] \in [\mathcal{P}_{\mathrm{GC}}(n)], \\ \mathrm{ind}_{\mathsf{A}}(P) = 0}} e(P), \qquad \mathsf{d}(n) = \sum_{\substack{[P] \in [\mathcal{P}_{\mathrm{GC}}(n)], \\ \mathrm{ind}_{\mathsf{D}}(P) = 0}} e(P).$$

We claim that

$$\mathsf{a}(n) = \sum_{i=1}^{n} g^{(n,n-1,\dots,n-i+1)} \mathsf{d}(n-i), \tag{7}$$

$$\mathsf{d}(n) = \sum_{i=1}^{n} g^{(n,n-1,\dots,n-i+1)} \mathsf{a}(n-i). \tag{8}$$

Since $\mathsf{gc}(n) = \mathsf{a}(n) + \mathsf{d}(n)$, the identity in this theorem is obtained by adding (7) and (8). Thus it suffices to show these two identities. We will only show (7) because (8) can be shown similarly.

To show (7), consider $[P] \in [\mathcal{P}_{\mathrm{GC}}(n)]$ with $\mathrm{ind}_{\mathsf{A}}(P) = 0$. By Proposition 32, there is a unique $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ such that $\mathrm{ind}_\delta(P) = (0, \dots, 0)$. Let $P_n = P$, and for $k \in [n-1]$, define

$$P_k = C_{\delta_k}(P_{k+1}).$$

Then, by Lemma 35, P_{k+1} is obtained from P_k by adding a descending or ascending chain of length $k+1$ depending on δ_{k-1} and δ_k. Since $\mathrm{ind}_{\mathsf{A}}(P) = 0$, there is a unique integer $i \in [n-1]$ such that $\delta_{n-1} = \delta_{n-2} = \cdots = \delta_{n-i} = \mathsf{A}$ and $\delta_{n-i-1} = \mathsf{D}$, where the second condition is ignored if $i = n-1$. By Lemma 35, one can easily see that $P = P_n$ is obtained from P_{n-i} by adding the poset $Q_{(n,n-1,\dots,n-i+1)}$ above it as shown in Figure 9. Since every element of P_{n-i} is smaller than every element of $Q_{(n,n-1,\dots,n-i+1)}$, we have

$$e(P) = e(P_{n-i}) e(Q_{(n,n-1,\dots,n-i+1)}) = e(P_{n-i}) g^{(n,n-1,\dots,n-i+1)}.$$

Note that $[P_{n-i}] \in [\mathcal{P}_{\mathrm{GC}}(n-i)]$ and $\mathrm{ind}_{\mathsf{D}}(P_{n-i}) = 0$. Conversely, for any such P_{n-i}, one can construct P in this way. This shows that

$$\mathsf{a}(n) = \sum_{i=1}^{n} g^{(n,n-1,\dots,n-i+1)} \sum_{\substack{[P_{n-i}] \in [\mathcal{P}_{\mathrm{GC}}(n-i)], \\ \mathrm{ind}_{\mathsf{D}}(P_{n-i}) = 0}} e(P_{n-i}),$$

which is the same as (7). Similarly, we obtain the formula (8) and the proof is completed. □

By (4), we have

$$\begin{aligned}
&g^{(2,1)} = 1, \quad g^{(2)} = 1, \\
&g^{(3,2,1)} = 2, \quad g^{(3,2)} = 2, \quad g^{(3)} = 1, \\
&g^{(4,3,2,1)} = 12, \quad g^{(4,3,2)} = 12, \quad g^{(4,3)} = 5, \quad g^{(4)} = 1.
\end{aligned}$$

Applying Theorem 37, we can compute $\mathsf{gc}(n)$ for $n = 2, 3, 4$ as follows.

$$\begin{aligned}
&\mathsf{gc}(2) = g^{(2,1)}\mathsf{gc}(0) + g^{(2)}\mathsf{gc}(1) = 1 + 1 = 2, \\
&\mathsf{gc}(3) = g^{(3,2,1)}\mathsf{gc}(0) + g^{(3,2)}\mathsf{gc}(1) + g^{(3)}\mathsf{gc}(2) = 2 + 2 + 2 = 6, \\
&\mathsf{gc}(4) = g^{(4,3,2,1)}\mathsf{gc}(0) + g^{(4,3,2)}\mathsf{gc}(1) + g^{(4,3)}\mathsf{gc}(2) + g^{(4)}\mathsf{gc}(3) = 12 + 12 + 10 + 6 = 40.
\end{aligned}$$

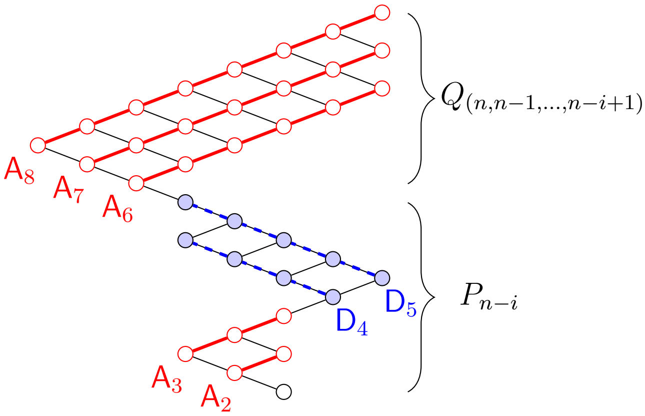


Figure 9: The Hasse diagram of a word poset $P \in \mathcal{P}_{\mathrm{GC}}(n)$ such that $\mathrm{ind}_\delta(P) = (0, \ldots, 0)$ for $\delta = (\mathsf{A}, \mathsf{A}, \mathsf{D}, \mathsf{D}, \mathsf{A}, \mathsf{A}, \mathsf{A})$. In this case, $\delta_{n-1} = \delta_{n-2} = \cdots = \delta_{n-i} = \mathsf{A}$ and $\delta_{n-i} = \mathsf{D}$, where $n = 8$ and $i = 3$. For $r = 2, 3, \ldots, n$, the set $P_r \setminus P_{r-1}$ forms is an ascending chain A_r or a descending chain D_r. The word poset $P = P_n$ is decomposed into two parts P_{n-i} and $P_n \setminus P_{n-i} \sim Q_{(n,n-1,\ldots,n-i+1)}$.

n	0	1	2	3	4	5	6	7	8
$\mathsf{gc}(n)$	1	1	2	6	40	916	102176	68464624	317175051664

Table 1: The first few terms of $\mathsf{gc}(n)$.

We present the first few terms of $\mathsf{gc}(n)$ in Table 1, which can also be found in A337699 in [24].

We close this section by presenting the following corollary of Theorem 37.

Corollary 38. *Let λ be a regular dominant weight of $\mathrm{SL}_{n+1}(\mathbb{C})$. The number of reduced words $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$ such that the string polytope $\Delta_{\mathbf{i}}(\lambda)$ is unimodularly equivalent to the Gelfand–Cetlin polytope $\mathrm{GC}(\lambda)$ is the same as $\mathsf{gc}(n)$.*

Proof. We first recall the known result from [7, Theorem A] that for $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$, the string polytope $\Delta_{\mathbf{i}}(\lambda)$ is unimodularly equivalent to the Gelfand–Cetlin polytope $\mathrm{GC}(\lambda)$ if and only if the string polytope $\Delta_{\mathbf{i}}(\lambda)$ has exactly $n(n+1)$ facets. Here, facets are codimension one faces. We note that the number of facets of any full dimensional Gelfand–Cetlin polytope of rank n is $n(n+1)$ (cf. [1]). Accordingly, if the string polytope $\Delta_{\mathbf{i}}(\lambda)$ is combinatorially equivalent to a full dimensional Gelfand–Cetlin polytope of rank n, then it is also unimodularly equivalent to $\mathrm{GC}(\lambda)$ because it has $n(n+1)$ facets. This proves the corollary. $\square$

5 Word posets are determined by δ-indices

In this section, we prove that the δ-indices of P for all $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ completely determine $P \in \mathcal{P}(w_0^{(n+1)})$ up to isomorphism as labeled posets in Definition 5. Equivalently, the δ-indices of $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$ for all $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ determine the commutation class $[\mathbf{i}] \in [\mathcal{R}(w_0^{(n+1)})]$.

Theorem 39. *Let $P, Q \in \mathcal{P}(w_0^{(n+1)})$. Then, $\operatorname{ind}_\delta(P) = \operatorname{ind}_\delta(Q)$ for all $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ if and only if $P \sim Q$.*

In order to prove Theorem 39, we need the following two lemmas.

Lemma 40. *Let $P \in \mathcal{P}(w_0^{(n+1)})$. Then*

$$\begin{aligned} \mathsf{A}(P) \cap C_{\mathsf{D}}(P) &= \mathsf{A}(C_{\mathsf{D}}(P)), \\ \mathsf{D}(P) \cap C_{\mathsf{A}}(P) &= \mathsf{D}(C_{\mathsf{A}}(P)). \end{aligned}$$

In other words, the ascending chain of P restricted to $C_{\mathsf{D}}(P)$ is the ascending chain of $C_{\mathsf{D}}(P)$, and similarly, the descending chain of P restricted $C_{\mathsf{A}}(P)$ is the descending chain of $C_{\mathsf{A}}(P)$.

Proof. By Proposition 14, we have

$$\begin{aligned} \mathsf{D}(P) &= \{d_1 <_P d_2 <_P \cdots <_P d_n\}, \\ \mathsf{A}(P) &= \{a_1 <_P a_2 <_P \cdots <_P a_n\}, \end{aligned}$$

where $f_P(a_i) = i$ and $f_P(d_i) = n+1-i$ for $1 \leqslant i \leqslant n$; $\mathsf{D}(P) \cap \mathsf{A}(P)$ has a unique element, say a_k. By the definition of contractions, $C_{\mathsf{D}}(P) = P \setminus \mathsf{D}$ and

$$\{a_1 <_{C_{\mathsf{D}}(P)} \cdots <_{C_{\mathsf{D}}(P)} a_{k-1} <_{C_{\mathsf{D}}(P)} a_{k+1} <_{C_{\mathsf{D}}(P)} \cdots <_{C_{\mathsf{D}}(P)} a_n\}. \tag{9}$$

Moreover, since $a_1, \ldots, a_{k-1}$ are below the descending chain $\mathsf{D}(P)$ and $a_{k+1}, \ldots, a_n$ are above $\mathsf{D}(P)$ in the Hasse diagram of P, we have

$$f_{C_{\mathsf{D}}(P)}(a_i) = \begin{cases} i & \text{if } 1 \leqslant i \leqslant k-1, \\ i-1 & \text{if } k+1 \leqslant i \leqslant n, \end{cases}$$

see Figure 10. Therefore, (9) is the ascending chain of $C_{\mathsf{D}}(P)$, which shows the first identity. The second identity can be proved similarly. $\square$

The following lemma shows that an ideal of a word poset $P \in \mathcal{P}(w_0^{(n+1)})$ is determined by the number of elements in each column.

Lemma 41. *Let $P \in \mathcal{P}(w_0^{(n+1)})$. Suppose that I and J are ideals of P such that*

$$|\{x \in I : f_P(x) = i\}| = |\{x \in J : f_P(x) = i\}|$$

for all $1 \leqslant i \leqslant n$. Then we have $I = J$.

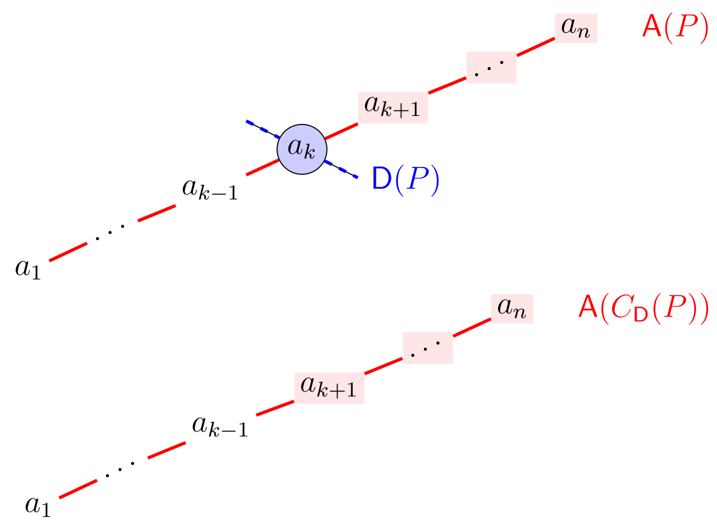


Figure 10: The ascending chain $\mathsf{A}(P)$ induces the ascending chain $\mathsf{A}(C_{\mathsf{D}}(P))$.

Proof. Consider the ideals I and J in the statement of this lemma. Observe that I is the disjoint union of $\{x \in I : f_P(x) = i\}$ for $1 \leqslant i \leqslant n$. Thus it suffices to show that

$$\{x \in I : f_P(x) = i\} = \{x \in J : f_P(x) = i\} \qquad \text{for all } 1 \leqslant i \leqslant n. \tag{10}$$

We fix $1 \leqslant i \leqslant n$, and let

$$r = |\{x \in I : f_P(x) = i\}| = |\{x \in J : f_P(x) = i\}|.$$

By Lemma 17, $\{x \in P : f_P(x) = i\}$ is a chain and we denote it by

$$C \coloneqq \{x \in P : f_P(x) = i\} = \{c_1 <_P c_2 <_P \cdots <_P c_t\}.$$

Since I is an ideal, $I \cap C$ is also an ideal of C. Because C is a chain, we obtain

$$I \cap C = \{c_1 <_P c_2 <_P \cdots <_P c_r\}.$$

By the same argument we also have

$$J \cap C = \{c_1 <_P c_2 <_P \cdots <_P c_r\}.$$

Therefore $I \cap C = J \cap C$, which is (10). This completes the proof. □

We are now ready to prove Theorem 39.

Proof of Theorem 39. Let $P, Q \in \mathcal{P}(w_0^{(n+1)})$. By the definition of δ-indices, it is straightforward that if $P \sim Q$, then $\mathrm{ind}_\delta(P) = \mathrm{ind}_\delta(Q)$ for all $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$. Thus it is sufficient to prove the 'only if' part.

Let $P, Q \in \mathcal{P}(w_0^{(n+1)})$ such that $\mathrm{ind}_\delta(P) = \mathrm{ind}_\delta(Q)$ for all $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$. We will prove $P \sim Q$ by induction on n.

Since $P, Q \in \mathcal{P}(w_0^{(n+1)})$, there are reduced words $\mathbf{i}, \mathbf{j} \in \mathcal{R}(w_0^{(n+1)})$ with $P \sim P_\mathbf{i}$ and $Q \sim P_\mathbf{j}$. If $n = 1$, then $\mathcal{R}(w_0^{(n+1)})$ has only one element (1). Thus $\mathbf{i} = \mathbf{j}$ and $P \sim P_\mathbf{i} = P_\mathbf{j} \sim Q$. If $n = 2$, there are two reduced words $(1,2,1)$ and $(2,1,2)$ in $\mathcal{R}(w_0^{(n+1)})$. Since $\mathrm{ind}_\mathsf{A}(1,2,1) = 1$ and $\mathrm{ind}_\mathsf{A}(2,1,2) = 0$, if $\mathrm{ind}_\delta(P_\mathbf{i}) = \mathrm{ind}_\delta(P_\mathbf{j})$ for all $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$, we must have $\mathbf{i} = \mathbf{j}$. Therefore we also have $P \sim P_\mathbf{i} = P_\mathbf{j} \sim Q$.

Now let $n > 2$ and suppose that the statement holds for $n-1$. Since $\mathrm{ind}_\delta(P) = \mathrm{ind}_\delta(Q)$ for all $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$, by the definition of δ-indices, we have

$$\begin{aligned} \mathrm{ind}_\delta(C_\mathsf{D}(P)) &= \mathrm{ind}_\delta(C_\mathsf{D}(Q)) \quad \text{for all } \delta \in \{\mathsf{A}, \mathsf{D}\}^{n-2}, \\ \mathrm{ind}_\delta(C_\mathsf{A}(P)) &= \mathrm{ind}_\delta(C_\mathsf{A}(Q)) \quad \text{for all } \delta \in \{\mathsf{A}, \mathsf{D}\}^{n-2}. \end{aligned}$$

Thus, by the induction hypothesis, we have $C_\mathsf{D}(P) \sim C_\mathsf{D}(Q)$ and $C_\mathsf{A}(P) \sim C_\mathsf{A}(Q)$.

By Proposition 21,

$$\begin{aligned} P &\sim E_\mathsf{D}(C_\mathsf{D}(P), I_\mathsf{D}(P)), \\ Q &\sim E_\mathsf{D}(C_\mathsf{D}(Q), I_\mathsf{D}(Q)). \end{aligned}$$

Since $C_\mathsf{D}(P) \sim C_\mathsf{D}(Q)$, in order to show $P \sim Q$, it suffices to show that the word poset isomorphism $C_\mathsf{D}(P) \sim C_\mathsf{D}(Q)$ induces $I_\mathsf{D}(P) \sim I_\mathsf{D}(Q)$. By Lemma 41, in order to show $I_\mathsf{D}(P) \sim I_\mathsf{D}(Q)$, it suffices to show the following claim: for all $i \in [n]$,

$$|\{x \in I_\mathsf{D}(P) : f_{C_\mathsf{D}(P)}(x) = i\}| = |\{x \in I_\mathsf{D}(Q) : f_{C_\mathsf{D}(Q)}(x) = i\}|.$$

Since $f_R(x) = f_{C_\mathsf{D}(R)}(x)$ for all $x \in I_\mathsf{D}(R)$, where R is P or Q, the claim can be rewritten as

$$|\{x \in I_\mathsf{D}(P) : f_P(x) = i\}| = |\{x \in I_\mathsf{D}(Q) : f_Q(x) = i\}|. \tag{11}$$

Let R be either P or Q. By Proposition 14, $\mathsf{D}(R) \cap \mathsf{A}(R)$ has a unique element, say z. Suppose $f_R(z) = s$. For $i \in [n]$, define

$$\begin{aligned} a_i(R) &= |\{x \in I_\mathsf{D}(R) \setminus I_\mathsf{A}(R) : f_R(x) = i\}|, \\ b_i(R) &= |\{x \in I_\mathsf{D}(R) \cap I_\mathsf{A}(R) : f_R(x) = i\}|, \\ c_i(R) &= |\{x \in I_\mathsf{A}(R) \setminus I_\mathsf{D}(R) : f_R(x) = i\}|. \end{aligned}$$

See Figure 11.

By definition,

$$|\{x \in I_\mathsf{D}(R) : f_R(x) = i\}| = \begin{cases} a_i(R) + b_i(R) & \text{if } i < s, \\ b_i(R) & \text{if } i \geqslant s. \end{cases} \tag{12}$$

By Lemma 40, $\mathsf{D}(R) \setminus \{z\}$ is the descending chain of $C_\mathsf{A}(R)$, which is obtained from R by removing $\mathsf{A}(R)$ and shifting the elements below $\mathsf{A}(R)$ to the left by one column. This shows that

$$|\{x \in I_\mathsf{D}(C_\mathsf{A}(R)) : f_{C_\mathsf{A}(R)}(x) = i\}| = \begin{cases} a_i(R) - 1 + b_{i+1}(R) & \text{if } i < s, \\ b_{i+1}(R) & \text{if } i \geqslant s. \end{cases} \tag{13}$$

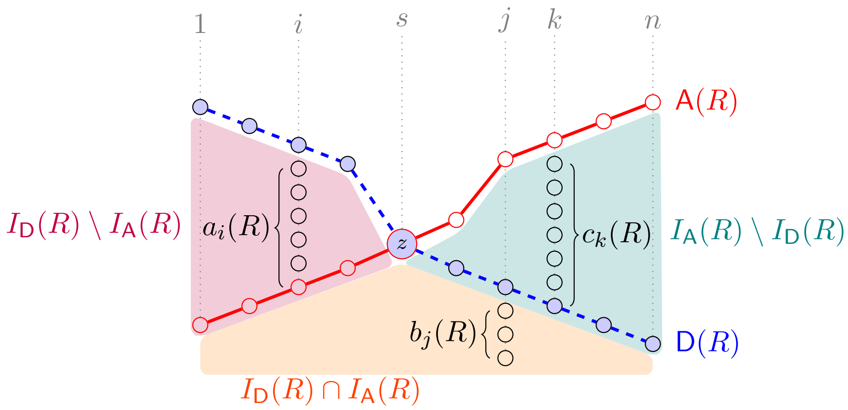


Figure 11: An illustration of $I_{\mathsf{D}}(R) \setminus I_{\mathsf{A}}(R)$, $I_{\mathsf{D}}(R) \cap I_{\mathsf{A}}(R)$, $I_{\mathsf{A}}(R) \setminus I_{\mathsf{D}}(R)$, $a_i(R)$, $b_k(R)$, and $c_j(R)$.

Similarly, we have

$$|\{x \in I_{\mathsf{A}}(C_{\mathsf{D}}(R)) : f_{C_{\mathsf{D}}(R)}(x) = i\}| = \begin{cases} b_i(R) & \text{if } i < s, \\ c_{i+1}(R) - 1 + b_{i+1}(R) & \text{if } i \geqslant s. \end{cases} \tag{14}$$

Since $C_{\mathsf{A}}(P) \sim C_{\mathsf{A}}(Q)$ (respectively, $C_{\mathsf{D}}(P) \sim C_{\mathsf{D}}(Q)$), the left hand side of (13) (respectively, (14)) is the same for both cases $R = P$ and $R = Q$. Comparing the right hand sides of (13) and (14) for the cases $R = P$ and $R = Q$, we obtain $b_i(P) = b_i(Q)$ for all $1 \leqslant i \leqslant n$, $a_i(P) = a_i(Q)$ for all $1 \leqslant i \leqslant s-1$, and $c_i(P) = c_i(Q)$ for all $s+1 \leqslant i \leqslant n$. By (12), this implies the claim (11) and the proof is completed. □

We note that Bédard [5] studied the combinatorics of commutation classes for Weyl groups of any Lie types by introducing a *level function* on a certain subset of positive roots of the corresponding root system. Indeed, for each reduced word a level function is defined, and this function distinguishes commutation classes, i.e., $\mathbf{i} \sim \mathbf{j}$ if and only if the corresponding level functions are the same.

Question 42. Recall that the indices have been defined for the reduced words of the longest element in $\mathfrak{S}_{n+1}$, which is the Weyl group of Lie type A. The level functions introduced by Bédard [5] and string polytopes are defined for any Lie type. In this regard, we may ask whether one can generalize the definitions of indices to other Lie types to provide more fruitful understanding of the combinatorics of string polytopes.

Remark 43. We have seen that the indices of reduced words are used to classify the string polytopes combinatorially equivalent to a Gelfand–Cetlin polytope. Recently, the combinatorics of string polytopes associated with reduced words of *small indices* has been studied in [6]. A reduced word $\mathbf{i} \in \mathcal{R}(w_0^{(n+1)})$ has small indices if $\operatorname{ind}_\delta(\mathbf{i}) = (0, \ldots, 0, k)$ for some $\delta \in \{\mathsf{A}, \mathsf{D}\}^{n-1}$ and $k \leqslant \kappa(\delta_{n-2}, \delta_{n-1})$. Here, $\kappa(\delta_{n-2}, \delta_{n-1}) = 2$ if $\delta_{n-2} = \delta_{n-1}$; and $\kappa(\delta_{n-2}, \delta_{n-1}) = n-1$ otherwise. In [6], Cho et al. found the number of codimension one faces and the description of the vertices for the string polytopes associated with reduced words having small indices. These examples show that the notion of indices may have a potential role to study the combinatorics of string polytopes.

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