$\boldsymbol{H}$-Chromatic Symmetric Functions

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Abstract

We introduce H-chromatic symmetric functions, X_G^H, which use the H-coloring of a graph G to define a generalization of Stanley's chromatic symmetric functions. We say two graphs G_1 and G_2 are H-chromatically equivalent if $X_{G_1}^H = X_{G_2}^H$, and use this idea to study uniqueness results for H-chromatic symmetric functions, with a particular emphasis on the case H is a complete bipartite graph. We also show that several of the classical bases of the space of symmetric functions, i.e. the monomial symmetric functions, power sum symmetric functions, and elementary symmetric functions, can be realized as H-chromatic symmetric functions. Moreover, we show that if G and H are particular types of multipartite complete graphs we can derive a set of H-chromatic symmetric functions that are a basis for Λ^n. We end with some conjectures and open problems.

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1 Introduction

In this paper we introduce a new variation on chromatic symmetric functions: the H-chromatic symmetric functions, X_G^H. The original chromatic symmetric functions, X_G, were defined in 1995 by Stanley [15] using the colorings of a graph G to label the indeterminates in a symmetric function. The variation we define employs the H-colorings of a graph G, using H to restrict the colorings of G and thus produce a different symmetric function.

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Stanley's original paper produced two main conjectures, neither of which has been fully solved although significant progress has been made (e.g. [7], [8], [11], [13], [14]). The first concerned e-positivity of clawfree, incomparability graphs. The second concerned the uniqueness of chromatic symmetric functions for trees. We take inspiration from the second of these and look at uniqueness results for H-chromatic symmetric functions. We can approach this from two angles and the two overarching questions are:

1. Given a fixed graph H, do there exist two graphs, G_1 and G_2, such that $X^H_{G_1} = X^H_{G_2}$?

2. Given a fixed graph G, do there exist two graphs, H_1 and H_2, such that $X^{H_1}_G = X^{H_2}_G$?

With respect to the first question we introduce the term *H-chromatically equivalent* and say that two graphs G_1 and G_2 are H-chromatically equivalent if $X^H_{G_1} = X^H_{G_2}$. At the end of this section we detail the individual questions and results we explore.

The concept of a H-coloring has been well-studied in graph theory, particularly from a complexity angle (one of the classic papers in this area is [9]). Informally, given a graph H and a coloring of H, a H-coloring of a graph G colors the vertices of G using the colors on H while respecting the adjacencies in H. Formally, let ϕ be a labeling (i.e. an injection on the vertex set) of H. A *proper (H, ϕ)-coloring* of G is a map $\kappa : V(G) \to \phi(V(H))$ such that $v_1 \sim_G v_2$ implies $\phi^{-1}(\kappa(v_1)) \sim_H \phi^{-1}(\kappa(v_2))$. We say that G is *H-colorable* if there is a labeling of H, ϕ, such that there is a proper (H, ϕ)-coloring of G. We define our H-chromatic symmetric functions as follows:

Definition 1. Let G, H be two graphs and n be the number of vertices in H. Let $V(G) = \{v_1, v_2, \dots, v_k\}$. Then,

$$w^H_G := \sum_{\phi} \sum_{\kappa} x_{\kappa(v_1)} \dots x_{\kappa(v_k)},$$

where ϕ ranges over all labelings $\phi : V(H) \to [n]$ of H and κ ranges over all proper (H, ϕ)-colorings, is a symmetric polynomial in indeterminates $x_1, x_2, \dots, x_n$. Therefore, w^H_G extends naturally to a homogeneous symmetric function of degree k. We denote this symmetric function by X^H_G, and call it the *H-chromatic symmetric function* of G.

Note that it might have been tempting to define the H-chromatic symmetric function as $\sum_\phi \sum_\kappa \prod_{j=1}^n x_{\kappa(v_j)}$, where ϕ ranges over all possible labelings $\phi : V(H) \to \mathbb{N}$, κ ranges over all possible (H, ϕ)-colorings, and $V(G) = \{v_1, v_2, \dots, v_n\}$. However, unless we restrict the colours allowed for H, this definition will, in general, result in possibly infinite coefficients. A simple example that demonstrates this is $G = K_1$ and $H = K_2$, in which case the term x_1 appears infinitely many times.

When $n \geqslant k$, $X^{K_n}_G$ is simply a scalar multiple of X_G. In contrast to the ordinary chromatic symmetric functions, the H-chromatic symmetric functions may sometimes vanish—namely, when G is not H-colorable. It is natural to define $X^H_G = 0$ in that case. When H has a loop, however, then X^H_G never vanishes. While we could confine ourselves

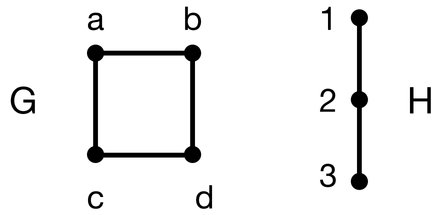

Figure 1: Example of a H-chromatic symmetric function: $8x_1x_2^2x_3 + 8x_1^2x_2x_3 + 8x_1x_2x_3^2 + 8x_1^2x_2^2 + 8x_2^2x_3^2 + 8x_1^2x_3^2 + \cdots$.

to simple H's, we will discuss a specific type of non-simple graph in Section 6.

Throughout the paper we will address the following specific questions:

- Given a pair of non-isomorphic graphs G_1, G_2, does there exist H such that $X_{G_1}^H \neq X_{G_2}^H$?(Yes, Corollary 23, Section 3.2)
- Given a set of (mutually non-isomorphic) graphs $\mathcal{G}$, what is the smallest size of a set of graphs $\mathcal{H}$ such that for any $G_1, G_2 \in \mathcal{G}$, there exists an $H \in \mathcal{H}$ with $X_{G_1}^H \neq X_{G_2}^H$? (Section 4)
- Given a fixed complete bipartite graph H, and two non-isomorphic graphs G_1, G_2, what is the easiest way to determine whether G_1 and G_2 are $K_{m,n}$-chromatically equivalent? (Section 5)
- Given a pair of non-isomorphic graphs, G_1, G_2, what are conditions that guarantee $X_{G_1}^{S_{n+1}^1} = X_{G_2}^{S_{n+1}^1}$, where S_{n+1}^1 is the star graph on $n+1$ vertices ($n \geqslant 2$) with an extra loop at its center? (Section 6)
- What is the description of a set of H-chromatic symmetric functions that is a basis for Λ^n, the space of symmetric functions of degree n? (Section 7)
- If G is the star graph S_n, are the H-chromatic symmetric functions unique? (Yes, up to degree sequences, Proposition 73, Section 8)
- If G is fixed, is it possible to find a set of H-chromatic functions that is a basis for Λ^n? (No, Proposition 74, Section 9)

As we said at the beginning of this introduction, there are two overarching questions to consider, one where we fix H and consider when $X_{G_1}^H = X_{G_2}^H$, and one where we fix G and consider when $X_G^{H_1} = X_G^{H_2}$. In fact, the majority of the problems we consider in this paper are in the first category. Among our results, those that involve determining when $X_G^{H_1} = X_G^{H_2}$, or related problems, are Results 12, 43–47, 51, 70–73, and 74. A possible avenue for future work would be to consider more widely when $X_G^{H_1} = X_G^{H_2}$.

The paper is organized as follows. Section 2 provides background definitions both from graph theory and symmetric function theory. Section 3 introduces the fundamental concepts specific to H-chromatic symmetric functions and establishes a selection of general results. Section 4 explores constructing a single H that distinguishes all graphs in a set $\mathcal{G}$. Section 5 considers questions about H-chromatic symmetric functions where H is a complete bipartite graph. Section 6 looks at uniqueness questions when H is an augmented star graph—a star graph with an extra loop at its center. Section 7 provides a basis for Λ_n using H-chromatic symmetric functions defined by complete multibipartite graphs. Section 8 builds on Section 7 and furthers our investigation into H-chromatic symmetric functions where H is complete bipartite, in particular, the star graph. We close in Section 9 with possible future directions of research.

2 Background

We will first recall some standard notation and concepts that will be useful later. See [2] and [12] for background details on graphs. A (undirected) *graph* consists of a *vertex set* $V(G)$, an *edge set* $E(G)$, and an *incidence relation* ψ_G that maps each edge to an unordered pair of vertices. These two vertices are called the *endpoints* of the edge. An edge is called a *loop* if its two endpoints are the same. A graph has *parallel edges*, also known as *multiple edges*, if there are two or more edges between the same two vertices in the graph. A graph is called *simple* if it has no loops or parallel edges. When G is a simple graph, we sometimes omit its incidence relation, and denote the edges (u, v) directly by their endpoints.

The number of vertices in a graph G is $|V| = |V(G)|$, and the number of edges is $|E| = |E(G)|$. Two vertices $v_1, v_2 \in V(G)$ are said to be *adjacent*, written $v_1 \sim_G v_2$, if they are the endpoints of an edge in $E(G)$. The *degree* of a vertex v, written $\deg(v)$, is the number of terminal points of edges incident with v. Therefore, a loop contributes 2 to the degree of a vertex. A graph $G' = (V(G'), E(G'), \psi_{G'})$ is a *subgraph* of a graph $G = (V(G), E(G), \psi_G)$ if $V(G') \subset V(G)$, $E(G') \subset E(G)$, and $\psi_{G'} = \psi_G|E(G')$. An *induced subgraph* G' of G is a subset of vertices in G paired with any edges from G whose endpoints are in this subset. More specifically, given a graph $G = (V, E)$, an induced subgraph, H, on the vertex set $V' \subset V$ has vertex set V' and edge set $E' = \{(i,j) | (i,j) \in \binom{V'}{2} \cap E\}$. A graph G is said to be *H-free* if H is not an induced subgraph of G.

Given two graphs G, H, we say G is *isomorphic* to H if there exists a pair of bijections $\phi : V(G) \to V(H), \theta : E(G) \to E(H)$, such that for every $e \in E(G)$, $\psi_G(e) = (u, v)$ if and only if $\psi_H(\theta(e)) = (\phi(u), \phi(v))$. We call such a pair of bijections an *isomorphism* from G to H. When G, H are both simple, an isomorphism can equivalently be represented by a single bijection $\phi : V(G) \to V(H)$ such that $\phi(u) \sim_H \phi(v)$ if and only if $u \sim_G v$ for every $u, v \in V(G)$. We shall also call such ϕ an isomorphism. An *automorphism* of G is an isomorphism from G to itself.

Let $G = (V, E, \psi_G)$ and $X \subset V$. The *coboundary* of X, denoted by ∂X, is the set $\{e \in E \mid \text{exactly one endpoint of } e \text{ is in } X\}$. A graph G is said to be *disconnected* if there exists a non-empty proper subset of V whose coboundary is empty. Otherwise, G is said to be *connected.* A *connected component*, or sometimes simply *component*, of G is a maximal (under set inclusion) connected subgraph of G. Two graphs are *disjoint* if their vertex sets (and edge sets) are disjoint. Let $G_1 = (V_1, E_1, \psi_{G_1}), G_2 = (V_2, E_2, \psi_{G_2})$ be two disjoint graphs, their disjoint union, denoted by $G_1 \uplus G_2$, is the graph $(V_1 \uplus V_2, E_1 \uplus E_2, \psi_{G_1 \uplus G_2})$, where $\psi_{G_1 \uplus G_2}$ is the map whose domain is $E_1 \uplus E_2$ and $\psi_{G_1 \uplus G_2}|E_1 = \psi_{G_1}$, $\psi_{G_1 \uplus G_2}|E_2 = \psi_{G_2}$.

Let $G = (V, E, \psi_G)$ and $X \subset V$. We say X is an *independent set* of G if the vertices in X are mutually non-adjacent and say X is a *clique* if the vertices in X are mutually adjacent. A *complete graph* on n vertices, denoted by K_n, is a simple graph on n vertices whose entire vertex set is a clique. An *edgeless graph* (or *null graph*) on n vertices, denoted by N_n, is the graph on n vertices whose edge set is empty. A *cycle* of size n, denoted by C_n, is a connected graph on n vertices such that every vertex has a degree of 2. (So C_1 is a single loop.) A *path* of size n, denoted by P_n, is C_n with an edge removed.

We say that a graph is *bipartite* if its vertices V can be partitioned into two independent sets, V_1 and V_2. A simple graph is *complete bipartite* if it is bipartite such that every vertex in V_1 is adjacent to every vertex in V_2. We denote by $K_{a,b}$ the complete bipartite graphs whose two parts contain a, b vertices, respectively. A *star graph* on $(n + 1)$ vertices, denoted by S_{n+1}, is the complete bipartite $K_{1,n}$. We call the vertex of degree n the *center* of the star, and the rest the *leaves.* A *complete n-partite* graph is a simple graph whose vertex set can be partitioned into n independent sets, such that there is an edge between any two vertices that are not in the same independent set. A graph is *cyclic* if it contains at least one subgraph that is a cycle. Otherwise, it is *acyclic.* A *forest* is an acyclic graph. A *tree* is a connected forest.

Let $G = (V, E, \psi_G)$ be a simple graph. The *complement* of G, denoted by $\overline{G}$, is $\overline{G} = (V, \binom{V}{2} \setminus E)$. That is, given $u \neq v$, $(u, v) \in E(\overline{G})$ if and only if $(u, v) \notin E(G)$.

Let $G = (V, E, \psi_G)$ be a graph. A *proper coloring*, or sometimes simply *coloring*, of a graph is a function $\kappa : V \to \mathbb{N}$ that satisfies $\kappa(u) \neq \kappa(v)$ if $u \sim v$. In this definition, we do not require $u \neq v$. Hence, a graph with a loop can never have a proper coloring. The *chromatic number* of G, denoted by $\chi(G)$, is $\min\{|\kappa(V)| \mid \kappa \text{ is a proper coloring of } G\}$.

Let $G = (V(G), E(G), \psi_G), H = (V(H), E(H), \psi_H)$ be two graphs. A *labeling* of H is an injective function $\phi : V(H) \to \mathbb{N}$. Let ϕ be a labeling of H. Recall from Section 1 that a *proper* (H, ϕ)*-coloring* of G is a map $\kappa : V(G) \to \phi(V(H))$ such that $v_1 \sim_G v_2$ implies $\phi^{-1}(\kappa(v_1)) \sim_H \phi^{-1}(\kappa(v_2))$. When we have a known fixed labeling ϕ, we sometimes omit it and write just H-coloring. We also use H-coloring as a name of the subject, even if no labelings may have been presented. This will be the main subject of this paper. We shall remark that we do not require G, H to be simple graphs. For example, when H contains a loop, then a proper H-coloring may assign the same color to two adjacent vertices in G. In particular, a proper coloring is not always a proper H-coloring.

We say that $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a *partition* of a positive integer n, written $\lambda \vdash n$, if λ is a list of weakly decreasing positive integers whose sum is n. We refer to each λ_i as a *part.* The function $r_i(\lambda)$ gives the number of parts of λ equal to i. The length of this partition is denoted by $l(\lambda) := k$. Let $\kappa : V(G) \to \mathbb{N}$ be a coloring. Let k_i be the number of vertices that are colored by i. If we arrange the non-zero terms in $\{k_i\}_{i \in \mathbb{N}}$ in weakly decreasing order, we will obtain a partition of n, where n is the number of vertices of G.

We call this partition the *type* of κ.

Symmetric functions, $f(x)$, are formal power series in infinitely many variables, $x = (x_1, x_2, \ldots)$, that are invariant under permutations of the subscripts. See [10] or [16] for background on notation and terminology for symmetric functions. The terms of a symmetric function are referred to as *monomials*. A symmetric function is said to be a *homogeneous symmetric function of degree* k if the sum of the powers of each monomial is equal to k. The space of homogeneous symmetric functions of degree k is denoted Λ^k, and the graded vector space $\Lambda = \oplus_{k=0}^{\infty} \Lambda^k$ is the space of symmetric functions. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of k. The *monomial symmetric functions* are defined by

$$m_\lambda = \sum_{\sigma} x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \ldots x_{\sigma(l)}^{\lambda_l},$$

where σ varies over all l-tuples of positive integers. The set $\{m_\lambda \mid \lambda \vdash k\}$ is a basis for Λ^k.

Other classical symmetric functions are the *elementary* and *power sum symmetric functions*:

$$e_n = m_{(1^n)} = \sum_{i_1 < i_2 < \ldots < i_n} x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_n},$$

and

$$p_n = m_{(n)} = \sum_{i \geqslant 1} x_i^n$$

respectively. For $k < 0$, we define $p_k = e_k = 0$, and $p_0 = e_0 = 1$. Then we can define

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}$$

and

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}$$

for partition $\lambda \vdash n$. The *Schur functions* are defined as:

$$s_\lambda = \det(e_{\lambda_i' - i + j}),$$

where $i, j \in \{1, 2, \ldots, l\}$ and λ' is a conjugate partition of a given partition $\lambda = (\lambda_1, \ldots, \lambda_l)$.

In 1995, Stanley introduced a generalization of the chromatic symmetric polynomial of a graph—the *chromatic symmetric function*. We will recall its basic definition. For a more in-depth cover refer to his original paper, [15].

Let $G = (V, E, \psi)$ be a (not necessarily simple) graph. Let $V = \{v_1, v_2, \ldots, v_n\}$, we define *chromatic symmetric function* of G by

$$X_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \ldots x_{\kappa(v_n)}$$

where the sum ranges over all proper colorings of G. Throughout this paper, we will refer to Stanley's chromatic symmetric functions as "ordinary chromatic symmetric functions".

In this paper, we will mostly be concerned with simple graphs, although we will look at graphs with loops in some later sections. From now on, unless otherwise stated, whenever we say graphs, we *always* mean simple graphs.

Throughout this paper, the multinomial coefficients

$$\binom{n}{n_1, n_2, \ldots, n_k} := \frac{n!}{n_1! n_2! \ldots n_k!}$$

will be used for several times, where $n = \sum_{j=1}^{k} n_j$. For notational cleanness, however, we require only that n is greater than or equal to $\sum_{j=1}^{k} n_j$ and we interpret that

$$\binom{n}{n_1, n_2, \ldots, n_k} := \frac{n!}{n_1! n_2! \ldots n_k! (n - n_1 - n_2 - \ldots - n_k)!}.$$

3 H-chromatic symmetric functions

Before we explore the properties of H-chromatic symmetric functions we will introduce some additional concepts. As in Definition 1, computing the H-chromatic symmetric functions requires considering all labelings $\phi : V(H) \to [n]$. The total number of possible labelings is $n!$, which grows rapidly as $n \to \infty$. Nevertheless, the following Lemma allows us to compute X_G^H using only one labeling.

Lemma 2. *Let H be a graph on n vertices with an arbitrary fixed labeling ϕ. Suppose G is a graph on k vertices, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash k$. Let d_λ denote the number of (H, ϕ)-colorings of G of type λ. Then,*

$$X_G^H = \sum_{\lambda \vdash k, l(\lambda) \leqslant n} c_\lambda m_\lambda,$$

where

$$c_\lambda = \frac{d_\lambda n!}{\binom{n}{r_1(\lambda), r_2(\lambda), \ldots, r_k(\lambda)}}.$$

Note that $n \geqslant \sum_{i=1}^{k} r_i(\lambda)$ and we recall the convention of writing multinomial coefficients introduced in Section 2.

Proof. Since each labeling ϕ of H induces the same number of (H, ϕ)-colorings of type λ, there will be $d_\lambda n!$ terms of type λ in w_G^H, where w_G^H is defined in Definition 1. To compute the number of distinct monomials of type λ in indeterminates $x_1, x_2, \ldots, x_n$, we choose $r_1(\lambda)$ from the n indeterminates and assign them degree 1, and then $r_2(\lambda)$ from the rest of $(n - r_1(\lambda))$ indeterminates and assign them degree 2. This process is done sequentially for $j = 1, 2, \ldots, k$, where in each iteration, $r_j(\lambda)$ indeterminates are chosen from the remaining $(n - \sum_{i=1}^{j-1} r_i(\lambda))$ and assigned degree j. Therefore, there will be $\binom{n}{r_1(\lambda), r_2(\lambda), \ldots, r_k(\lambda)}$ distinct monomials of type λ. Hence, c_λ is the coefficient of any particular monomial of type λ in w_G^H. The conclusion follows from the definition of X_G^H. □

In light of Lemma 2, it is convenient to define the n-augmented monomials as follows. We include the n in the name to distinguish from the augmented monomials used by Stanley [15].

Definition 3. Given a number $n \in \mathbb{N}$ and a partition $\lambda \vdash k$ with $l(\lambda) \leqslant n$, we define the *n-augmented monomial* m_λ^n by

$$m_\lambda^n = \frac{n!}{\binom{n}{r_1(\lambda), r_2(\lambda), \ldots, r_k(\lambda)}} m_\lambda. \tag{1}$$

Suppose G, H are graphs on k, n vertices, respectively. Clearly, when k is less than or equal to n, the set $\{m_\lambda^n\}_{\lambda \vdash k}$ is a basis for Λ^k. When $k > n$, the only non-vanishing monomials in X_G^H must have type of length no larger than n. Therefore,

$$X_G^H \in \operatorname{span}\{m_\lambda\}_{\lambda \vdash k, l(\lambda) \leqslant n} = \operatorname{span}\{m_\lambda^n\}_{\lambda \vdash k, l(\lambda) \leqslant n}.$$

Therefore, we can always write X_G^H uniquely as $\sum_{\lambda \vdash k, l(\lambda) \leqslant n} d_\lambda m_\lambda^n$. In the rest of this paper, when we write $X_G^H = \sum_{\lambda \vdash k} d_\lambda m_\lambda^n$, we always assume $d_\lambda = 0$ for every $l(\lambda) > k$, although m_λ^n has not been formally defined. Lemma 2 says d_λ is the number of (H, ϕ)-colorings of G of type λ for any fixed ϕ.

It is worth mentioning the caveat that the transformation between m_λ and m_λ^n is not uniform over all λ. For example, if we write $X_G^H = \sum_\lambda c_\lambda m_\lambda = \sum_\lambda d_\lambda m_\lambda^n$, then it is *not* true in general that $c_{\lambda_1}/d_{\lambda_1} = c_{\lambda_2}/d_{\lambda_2}$ for any pair of partitions $\lambda_1, \lambda_2 \vdash k$. Furthermore, if we have $n_1, n_2 \geqslant l(\lambda)$, then

$$m_\lambda^{n_1} = \frac{(n_1 - \sum r_j(\lambda))!}{(n_2 - \sum r_j(\lambda))!} m_\lambda^{n_2} = \frac{(n_1 - l(\lambda))!}{(n_2 - l(\lambda))!} m_\lambda^{n_2}.$$

This relation, albeit elegant, is seldom used in the rest of this paper.

A natural question is whether we can actually obtain H-chromatic symmetric functions that are not ordinary chromatic symmetric functions? Indeed, we can. To see this, we observe that an ordinary chromatic symmetric function of degree k always contains the monomial $m_{(1^k)}$. By containing the monomial $m_{(1^k)}$ we mean that if we write the symmetric function in terms of the monomial basis, the coefficient of $m_{(1^k)}$ is non-zero. On the other hand, every monomial m_λ can be written as a multiple of a H-chromatic symmetric function, as the following proposition shows.

Proposition 4. *Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash k$, denote by K_λ the complete l-partite graph whose parts contain $\lambda_1, \lambda_2, \ldots, \lambda_l$ vertices, respectively. Then,*

$$X_{K_\lambda}^{K_l} = l! m_\lambda^l.$$

Note that in particular, if $\lambda = (n)$, then

$$X_{K_n}^{K_1} = m_{(n)}$$

Given Lemma 2, the proof of Proposition 4 is obvious, and thus omitted. We shall see a stronger version of the above proposition in Section 7.

In the setting of ordinary chromatic symmetric functions, we have the property that $X_{G_1 \uplus G_2} = X_{G_1} X_{G_2}$ [15]. This is, however, not true in general for H-chromatic symmetric functions when we fix a H. The easiest example to show this is $G_1 = G_2 = H = K_1$, in which case $X^H_{G_1} X^H_{G_2}$ contains the monomial $m_{(1,1)}$, whereas $X^H_{G_1 \uplus G_2}$ does not. Another nice property that holds for ordinary chromatic symmetric functions but not H-chromatic symmetric functions is the weighted deletion-contraction result. To formulate the exact statement the notion of weighted chromatic symmetric functions will be useful, which we will not introduce here. Interested readers should refer to Crew and Spirkl's paper for a detailed discussion [4]. The main difference in H-colorings is that a proper H-coloring of $G \setminus (u, v)$ that colors u, v differently is not necessarily a proper H-coloring of G. On the other hand, if we replace H-coloring by ordinary coloring, then the implication in the previous statement will hold, and is a key observation used in the proof of the weighted deletion-contraction result for ordinary chromatic symmetric functions.

In the previous paragraph, we see that there is very little we can say when G is written as a disjoint union of its components. When we consider the connected components of H, there is a simple but useful result. The restriction here is that G has to be connected.

Proposition 5. *Let $G, H_1, H_2, \ldots, H_l$ be connected graphs, and $H = \biguplus_{j=1}^{l} H_j$. Let $n_j = |V(H_j)|$ and $n = |V(H)|$. For each $1 \leqslant j \leqslant l$, we write $X_G^{H_j} = \sum_{\lambda \vdash k} c^j_\lambda m^{n_j}_\lambda$. Then,*

$$X^H_G = \sum_{\lambda \vdash k} c_\lambda m^n_\lambda,$$

where $c_\lambda = \sum_{j=1}^{l} c^j_\lambda$.

Proof. Fix a labeling ϕ on H. Since G is connected, we can only use one component of H to color G. Hence, the number of proper (H, ϕ)-colorings of G is the sum of the number of proper $(H_j, \phi_{|V(H_j)})$-colorings of G. The conclusion follows from Lemma 2. □

3.1 Some fundamental H-chromatic equivalence and uniqueness results

In this subsection, we deal with problems involving situations when H-chromatic symmetric functions of two graphs G_1 and G_2 for a fixed H are not equal. We explore results pertaining to when $X^H_{G_1} \neq X^H_{G_2}$.

Definition 6. We say that two graphs G_1 and G_2 are *H-chromatically equivalent* if and only if $X^H_{G_1} = X^H_{G_2}$. We note that this is an equivalence relation that partitions the set of graphs into equivalence classes. Elements in the same equivalence class are H-chromatically equivalent to each other.

Definition 7. We say that a graph G is *H-chromatically unique* in a set of graphs S if there is no graph $G_0 \in S$ such that $X^H_G = X^H_{G_0}$.

Proposition 8. *If two H-colorable graphs G_1 and G_2 differ in the minimum number of colors needed to H-color them or the maximum number of colors that can possibly H-color them, they cannot have the same H-chromatic symmetric function.*

Proof. Suppose two H-colorable graphs G_1 and G_2 differ in the minimum (resp. maximum) number of colors needed to H-color them. Suppose $X_{G_1}^H = \sum_{\lambda_a \vdash k} c_{\lambda_a} m_{\lambda_a}$. Let $h_{1,min}$ (resp. $h_{1,max}$) denote the minimum (resp. maximum) number of colors needed to H-color G_1. Then there exists some $m_{\lambda_{a_i}}$, $m_{\lambda_{a_j}}$ such that $\lambda_{a_i}, \lambda_{a_j}$ have length $h_{1,min}$ (resp. $h_{1,max}$).

Suppose $X_{G_2}^H = \sum_{\lambda_b \vdash k} c_{\lambda_b} m_{\lambda_b}$. Let $h_{2,min}$ (resp. $h_{2,max}$) denote the minimum (resp. maximum) number of colors needed to H-color G_2. Then there exists some $m_{\lambda_{b_i}}$, $m_{\lambda_{b_j}}$ such that $\lambda_{b_i}, \lambda_{b_j}$ have length $h_{2,min}$ (resp. $h_{2,max}$). Without loss of generality, suppose $h_{1,min} < h_{2,min}$ or $h_{1,max} < h_{2,max}$.

If $h_{1,min} < h_{2,min}$ or $h_{1,max} < h_{2,max}$, then $m_{\lambda_{b_1}}, m_{\lambda_{b_2}}, m_{\lambda_{b_3}}, \ldots$ will span monomial symmetric functions corresponding to partitions with length $h_{2,min}$ or $h_{2,max}$ but not those of length $h_{1,min}$ and $h_{1,max}$.

If the two graphs differ in the minimum (resp. maximum) number of colors needed to H-color them, then when we write $X_{G_1}^H$ and $X_{G_2}^H$ as linear combinations of monomial symmetric functions whose partitions correspond to viable H-colorings, we would see terms corresponding to partitions of different lengths so $X_{G_1}^H$ and $X_{G_2}^H$ cannot possibly be the same. □

Corollary 9. *Suppose G_1 and G_2 are both H-colorable, i.e. $X_{G_1}^H, X_{G_2}^H$ are non-zero. If G_1 is bipartite and G_2 is not bipartite then G_1 and G_2 cannot have the same H-chromatic symmetric functions.*

Proof. G_1 is 2-colorable, whereas G_2 is not. Applying Proposition 8 above gives us our result. □

Proposition 10. *Let H be a non-edgeless graph. Suppose B_1 is a connected bipartite graph such that $V(B_1)$ is partitioned into two disjoint sets V_1, V_2. Suppose B_2 is a connected bipartite graph such that $V(B_2)$ is partitioned into two disjoint sets W_1, W_2. Suppose $\{|V_1|, |V_2|\} \neq \{|W_1|, |W_2|\}$, i.e. the bipartitions of B_1 and B_2 are not the same. Then B_1 and B_2 are not H-chromatically equivalent, i.e. $X_{B_1}^H \neq X_{B_2}^H$.*

Proof. Suppose we color B_1 with two colors such that $|V_1|$ vertices are colored the same color and $|V_2|$ vertices are colored another color. We then have a partition $\lambda_1 = (|V_1|, |V_2|)$ that corresponds to 2-coloring B_1. Consider a similar partition $\lambda_2 = (|W_1|, |W_2|)$ that corresponds to 2-coloring B_2. Now the bipartition of a connected bipartite graph is unique, so its 2-coloring is unique up to switching colors. It follows that these are the only partitions corresponding to 2-coloring B_1 and B_2. The partitions of length 2 that correspond to 2-coloring B_1 and B_2 are not the same, so the corresponding m_λ's in $X_{B_1}^H$ and $X_{B_2}^H$ are not the same. Hence $X_{B_1}^H$ and $X_{B_2}^H$ cannot possibly be the same. □

Remark 11. We note that disconnected bipartite graphs can have non-unique bipartitions. So the analogue of Proposition 10 for non-connected graphs G_1, G_2 would involve comparing the set of possible bipartitions of graphs G_1, G_2.

Proposition 12. *Let H_1 and H_2 be two graphs that both have k vertices, but different number of edges. Suppose the number of edges in H_1 is given by $|E(H_1)|$ and the number of edges in H_2 is given by $|E(H_2)|$. Let G be a complete bipartite graph on at least two vertices. Suppose G is H_1-colorable and H_2-colorable, i.e. $X_G^{H_1}, X_G^{H_2}$ are non-zero. Then $X_G^{H_1} \neq X_G^{H_2}$.*

Proof. Let $G = K_{m,n}$ for some m, n. Consider the coefficient of the k-augmented monomial $m^k_{(m,n)}$. The number of ways of coloring G using two colors, which must by definition be adjacent in H_1 and H_2, is equivalent to counting the number of edges in each of H_1 and H_2. If $|E(H_1)| \neq |E(H_2)|$, then they will generate different coefficients of $m^k_{(m,n)}$. And since they are on the same number of vertices, then $X_G^{H_1}$ and $X_G^{H_2}$ will have different coefficients for $m^k_{(m,n)}$. □

Corollary 13. *Suppose H is a non-edgeless graph. The star graph is H-chromatically unique in the set of connected graphs.*

Proof. Suppose we partition the vertices of the star into two sets, V_1 and V_2. The star S_k is the only connected bipartite graph that corresponds to the case where $|V_1| = 1, |V_2| = k-1$. From Proposition 10, the star is not H-chromatically equivalent to other bipartite graphs. Similarly, Proposition 8 implies the star is not H-chromatically equivalent to non-bipartite graphs. □

Corollary 14. *Suppose a graph H can be used to H-color the complete graph K_n, i.e. $X_{K_n}^H \neq 0$. Then, K_n is H-chromatically unique in the set of connected graphs.*

Proof. K_n is the only n-partite graph on n vertices. All other graphs with n vertices are colorable with $n-1$ colors. So applying Proposition 8 gives us our result. □

In fact, this idea of H-chromatic uniqueness can be made even stronger.

Proposition 15. *Consider a finite set of graphs $\{G_j\}$ with corresponding H-chromatic symmetric functions $\{X_{G_j}^H\}$. Suppose each graph G_j has k_j vertices. Suppose we write each $X_{G_j}^H$ in the form $X_{G_j}^H = \sum_{\lambda \vdash k_j} c_{\lambda,j} m_\lambda$. If each $X_{G_\ell}^H$ written in this form contains a m_λ term (with fixed partition and nonzero coefficient $c_{\lambda,\ell}$) that the other H-chromatic symmetric functions in the set $\{X_{G_j}^H\} \setminus X_{G_\ell}^H$ do not have, then the functions in the set $\{X_{G_j}^H\}$ are linearly independent.*

Proof. Since the monomial symmetric functions $\{m_\lambda\}$ form a basis for Λ, the monomial symmetric function m_{λ_i} cannot be written as a linear combination of other monomial symmetric functions $\{m_{\lambda_k}\}$ where $\lambda_k \neq \lambda_i$. □

Corollary 16. *Suppose we write some non-zero X_G^H as follows:*

$$X_G^H = \sum_{\lambda \vdash k} c_\lambda m_\lambda$$

If G is H-chromatically unique in a set S due to having a component m_λ (with nonzero coefficient) that no other H-chromatic symmetric function of a graph in S has, then X_G^H cannot be written as a linear combination of other H-chromatic symmetric functions of graphs in S. Note that this is a stronger condition than H-chromatic uniqueness.

Proof. This follows from Proposition 15. □

Corollary 17. *Suppose H is a non-edgeless graph. The set of H-chromatic symmetric functions for connected bipartite graphs on the same number of vertices with different bipartitions is linearly independent. Let $B_{i,j}$ denote a bipartite graph with bipartition $\{i, j\}$. Suppose $i + j = n$. Then $\{X_{B_{i,j}}^H\}_{i=1,\ldots,\lfloor\frac{n}{2}\rfloor, j=n-i}$ is linearly independent.*

Proof. Apply Proposition 10 and Proposition 15. □

Corollary 18. *The H-chromatic symmetric functions for graphs with different number of vertices are linearly independent. Consider a finite set of graphs $\{G_i\}$ with G_i having i vertices and corresponding H-chromatic symmetric function $X_{G_i}^H$. Suppose each G_i is H-colorable, i.e. each $X_{G_i}^H$ is non-zero. Then $\{X_{G_i}^H\}_{i=1}^n$ is linearly independent.*

Proof. Apply Proposition 15. □

Corollary 19. *The set of H-chromatic symmetric functions for graphs with the same number of vertices but different chromatic numbers is linearly independent. Consider a finite set of graphs $\{G_i\}$ with G_i having chromatic number i and corresponding H-chromatic symmetric function $X_{G_i}^H$. Suppose each G_i is H-colorable, i.e. each $X_{G_i}^H$ is non-zero. Then $\{X_{G_i}^H\}_{i=1}^n$ is linearly independent.*

Proof. Apply Proposition 8 and Proposition 15. □

Corollary 20. *Suppose H is a graph such that an n-vertex star graph S_n and an n-vertex complete graph K_n are H-colorable, i.e. $X_{S_n}^H, X_{K_n}^H$ are non-zero. Then $X_{S_n}^H$ and $X_{K_n}^H$ cannot be written as a linear combination of H-chromatic symmetric functions of other connected graphs.*

Proof. Apply Proposition 13, and Corollaries 14, 16. The H-chromatic symmetric function of the star S_n is the only H-chromatic symmetric function of a n-vertex connected graph that contains the term $m_{(1,n-1)}$ when written as a linear combination of monomial symmetric functions m_λ. The H-chromatic symmetric function of a complete graph K_n is the only H-chromatic symmetric function of a n-vertex connected graph that has no monomials with partition length less than n. □

Note that even though these results on H-chromatic equivalence and H-chromatic uniqueness are true for cases where G, H are simple graphs, if we allow H to have a loop, then Propositions 8, 10, and Corollaries 9, 13, 14, 17, 19, 20 fail. If H were a graph with just one vertex and a loop, then we could H-color all the vertices in some graph G the same color. Thus G would have the same H-chromatic symmetric function as other graphs with the same number of vertices as G.

3.2 Automorphisms of G and X_G^H

We know that two non-isomorphic graphs can have the same chromatic symmetric functions [1, 15]. Given a H, can we always find two non-isomorphic graphs having the same H-chromatic symmetric function? The answer is affirmative: indeed, if H has n vertices, then any graph with a chromatic number larger than n will have a H-chromatic symmetric function of 0. A more interesting question is the following problem:

Problem 21. Given a pair of non-isomorphic graphs G_1, G_2, can we always find a H such that $X_{G_1}^H \neq X_{G_2}^H$?

To answer this question, it will be instructive to note that the coefficient of $m_{(1^k)}$ in X_G^H is closely related to the number of automorphisms of G.

Proposition 22. *Let G, H be two graphs and $X_G^H = \sum_{\lambda \vdash |V(G)|} c_\lambda m_\lambda^{|V(H)|}$. Then,*

$$c_{(1^{|V(G)|})} = S(G,H)\,|\mathrm{Aut}(G)|\,,$$

where $S(G,H)$ is the number of (not necessarily induced) subgraphs of H that are isomorphic to G and $\mathrm{Aut}(G)$ is the set of automorphisms of G.

Proof. We define an embedding of G into H to be an injective map $\psi : V(G) \to V(H)$ such that $u \sim_G v$ implies $\psi(u) \sim_H \psi(v)$. Given each embedding ψ, we can obtain a subgraph of H isomorphic to G by letting the vertex set be $\psi(V(G))$ and the edge set be $\{(\psi(u), \psi(v)) \mid (u,v) \in E(G)\}$. We denote this graph by $\Psi(\psi)$. In particular, $\Psi(\psi)$ is isomorphic to G. Conversely, given a subgraph H' of H that is isomorphic to G, and let $\psi : V(G) \to V(H')$ be an isomorphism. Then, we claim that $\Phi := \{\psi \circ \phi \mid \phi \in \mathrm{Aut}(G)\}$ is exactly the set of embeddings whose image under Ψ is H'. It is clear that every element in Φ is an embedding whose image under Ψ is H'. Conversely, let ψ' be an embedding of G into H and that $\Phi(\psi') = H'$. Then, $\psi^{-1} \circ \psi'$ is an automorphism of G, so $\psi' \in \Phi$. Hence, the total number of embeddings of G into H is $S(G,H)\,|\mathrm{Aut}(G)|$.

Now, we count the number of embeddings of G into H in another way. If we fix a labeling ϕ of H, then given an H-coloring of G of type $(1^{|V(G)|})$, it corresponds naturally to an injection from $V(G)$ to $V(H)$, and it is an embedding by the definition of an H-coloring. Given an embedding ψ, if we use the $\phi(\psi(u))$ to color u, then we will get a proper H-coloring. Hence, $c_{(1^{|V(G)|})}$ is the number of embeddings of G into H, and $c_{(1^{|V(G)|})} = S(G,H)\,|\mathrm{Aut}(G)|$. □

Corollary 23. *Let G_1, G_2 be two non-isomorphic graphs on k_1, k_2 vertices, respectively. Then,*

1. *If we write $X_{G_1}^{G_1} = \sum_{\lambda \vdash k_1} c_\lambda m_\lambda^{k_1}$, then, $c_{(1^{k_1})}$ is the number of automorphisms of G_1.*

2. *Let $X_{G_1}^{G_2} = \sum_{\lambda \vdash k_1} c_\lambda m_\lambda$. Then, $c_{(1^{k_1})} = 0$ if either $k_1 > k_2$ or $k_1 = k_2$ and $|E(G_1)| \geqslant |E(G_2)|$.*

3. *There exists a graph H such that $X_{G_1}^H \neq X_{G_2}^H$.*

Proof. The first claim is immediate from Proposition 22. The second claim is obvious when $k_1 > k_2$. When $k_1 = k_2$ and $|E(G_1)| \geqslant |E(G_2)|$, G_2 cannot possibly contain an subgraph isomorphic to G_1 and the conclusion follows from again from Proposition 22. To prove the third claim, without loss of generality, we may assume the assumptions in the second claim holds. (Otherwise, interchange G_1 and G_2.) Now let $H = G_2$ and use the first two statements. $\square$

Corollary 23 answers the question that we posed before Proposition 22. In fact, we can say even more. In the next section, we will show that given finitely many graphs $G_1, G_2, \ldots, G_l$, there exists a H such that $X_{G_i}^H \neq X_{G_j}^H$ for every $i \neq j$. The constructive proof will be heavily based on Proposition 5 and Corollary 23. We now formalize these discussions.

4 Graph distinguishers and uniform distinguishability of graphs

As shown in the previous section, given any two non-isomorphic graphs G_1, G_2, there exists an H such that $X_{G_1}^H \neq X_{G_2}^H$. In this section, we look at a generalization of this problem.

Problem 24. Given a set of (mutually non-isomorphic) graphs $\mathcal{G}$, what is the smallest size of a set of graphs $\mathcal{H}$ such that for any $G_1, G_2 \in \mathcal{G}$, there exists an $H \in \mathcal{H}$ with $X_{G_1}^H \neq X_{G_2}^H$?

Let us give such a $\mathcal{H}$ a name and introduce additional terminology.

Definition 25. Let $\mathcal{G}$ be a set of (mutually non-isomorphic) graphs. A set of graphs $\mathcal{H}$ is called a *$\mathcal{G}$-distinguisher* if for any $G_1, G_2 \in \mathcal{G}$, there exists an $H \in \mathcal{H}$ such that $X_{G_1}^H \neq X_{G_2}^H$. The graph $\mathcal{G}$ is said to be *finitely distinguishable* if there exists a finite $\mathcal{G}$-distinguisher. The graph $\mathcal{G}$ is said to be *uniformly distinguishable* if there exists a $\mathcal{G}$-distinguisher that contains exactly one graph.

Clearly, uniform distinguishability implies finite distinguishability, and the results in the last section shows that $\mathcal{G}$ is uniformly distinguishable if $|\mathcal{G}| = 2$ and, consequently, $\mathcal{G}$ is finitely distinguishable if $\mathcal{G}$ is finite. To extend these results, we will have to construct new H's. When both $\mathcal{G}$ and $\mathcal{H}$ are sets of connected graphs, Proposition 5 seems a useful tool. We introduce a lemma which shows we can assume a finite distinguisher consists only of connected graphs.

Lemma 26. *A set of connected graphs $\mathcal{G}$ is finitely distinguishable if and only if there exists a finite $\mathcal{G}$-distinguisher that consists of connected graphs only.*

Proof. Suppose $\mathcal{H}$ is a finite distinguisher of $\mathcal{G}$. Let $\mathcal{H}'$ be the set of all connected components of graphs in $\mathcal{H}$. Let $G_1, G_2 \in \mathcal{G}$ be given, and let $H \in \mathcal{H}$ be such that $X_{G_1}^H \neq X_{G_2}^H$. Let $H_1', H_2', \ldots, H_k'$ be its connected components. By Proposition 5, there must be an H_j' such that $X_{G_1}^{H_j'} \neq X_{G_2}^{H_j'}$. So $\mathcal{H}'$ is a finite $\mathcal{G}$-distinguisher. The other direction is obvious. $\square$

Before we write down the main result in this section we will introduce a final piece of terminology.

Definition 27. Let $\mathcal{G}$ and $\mathcal{H}$ be two sets of graphs. We say that $\mathcal{G}$ is *uniformly bounded with respect to* $\mathcal{H}$ if the set

$$\mathcal{C}_G^H := \{c_{G,\lambda}^H \mid G \in \mathcal{G}, H \in \mathcal{H}, X_G^H = \sum_{\lambda \vdash |V(G)|} c_{G,\lambda}^H m_\lambda^{|V(H)|}\}$$

is bounded.

In particular, when $\mathcal{G}, \mathcal{H}$ are finite, then $\mathcal{G}$ is uniformly bounded with respect to $\mathcal{H}$. We shall also remark that when $\mathcal{G}$ is uniformly bounded with respect to a finite set $\mathcal{H}$, then it is clear that $\mathcal{G}$ is also uniformly bounded with respect to $\mathcal{H}'$, which is defined in the proof of Lemma 26.

Proposition 28. *Let $\mathcal{G}$ be a set of connected graphs. Suppose there exists a finite $\mathcal{G}$-distinguisher $\mathcal{H}$, and $\mathcal{G}$ is uniformly bounded by $\mathcal{H}$. Then, $\mathcal{G}$ is uniformly distinguishable.*

Proof. By Lemma 26 and the remark before this proposition, it is safe to assume that $\mathcal{H} = \{H_1, H_2, \ldots, H_l\}$ is a finite $\mathcal{G}$-distinguisher that consists of connected graphs, and that $\mathcal{G}$ is uniformly bounded by $\mathcal{H}$. Let $N \in \mathbb{N}$ be an upperbound of $\mathcal{C}_G^H$ as in Definition 27. Now, set

$$H := \biguplus_{j=1}^{l} (2N)^j H_j,$$

where $nG = \uplus_{j=1}^n G$ is the graph obtained by taking the disjoint union of n copies of G. We claim that $\{H\}$ is a $\mathcal{G}$-distinguisher.

Let $G_1, G_2 \in \mathcal{G}$ be given. We may assume $|V(G_1)| = |V(G_2)|$, for otherwise the degree of $X_{G_1}^H$ and that of $X_{G_2}^H$ are different. Note also, in particular, that in order for $\mathcal{H}$ to be a $\mathcal{G}$-distinguisher, it is clearly not the case that $X_{G_1}^H = X_{G_2}^H = 0$. We write $X_{G_1}^{H_i} = \sum_\lambda a_{i,\lambda} m_\lambda^{|V(H_i)|}$ and $X_{G_2}^{H_i} = \sum_\lambda b_{i,\lambda} m_\lambda^{|V(H_i)|}$. By Proposition 5, $X_{G_1}^H = \sum_\lambda a_\lambda m_\lambda^{|V(H)|}$ and $X_{G_2}^H = \sum_\lambda b_\lambda m_\lambda^{|V(H)|}$, where $a_\lambda = \sum_{j=1}^l (2N)^j a_{j,\lambda}$ and $b_\lambda = \sum_{j=1}^l (2N)^j b_{j,\lambda}$.

We let m be the smallest number such that $\{H_m\}$ is a $\{G_1, G_2\}$-distinguisher, and let λ be such that $a_{m,\lambda} \neq b_{m,\lambda}$. Then,

$$\begin{aligned} 0 &< \left| \sum_{j=1}^{m} (2N)^j a_{j,\lambda} - \sum_{j=1}^{m} (2N)^j b_{j,\lambda} \right| \\ &= |(2N)^m a_{m,\lambda} - (2N)^m b_{m,\lambda}| \\ &= 2^m N^m \, |a_{m,\lambda} - b_{m,\lambda}| \\ &\leqslant 2^m N^{m+1}. \end{aligned}$$

But $\sum_{j=m+1}^l (2N)^j a_{j,\lambda} - \sum_{j=m+1}^l (2N)^j b_{j,\lambda}$ is a multiple of $2^{m+1} N^{m+1}$. So $a_\lambda - b_\lambda$ cannot possibly vanish. Hence, $X_{G_1}^H \neq X_{G_2}^H$. □

As remarked before, when $\mathcal{G}, \mathcal{H}$ are both finite, $\mathcal{G}$ is uniformly bounded with respect to $\mathcal{H}$. Hence, we have the following corollary.

Corollary 29. *Any finite set of connected graphs $\mathcal{G}$ is uniformly distinguishable. Equivalently, for any $k \in \mathbb{N}$, the set $\{G \mid |V(G)| \leqslant k,\ G \text{ is connected}\}$ is uniformly distinguishable.*

Proof. The second claim follows from the first one and the fact that there are finitely many (connected) graphs with no more than k-vertices up to isomorphism. To show the first claim, let $\mathcal{G}$ be a finite set of connected graphs. Then, there exists a finite distinguisher of $\mathcal{G}$ by Corollary 23. Hence, $\mathcal{G}$ is uniformly distinguishable by Proposition 28 and the remark before this corollary. □

In the previous discussions, we always assume that $\mathcal{G}$ is a set of connected graphs. Of course, a result that generalizes to arbitrary graphs will be more desirable. It turns out that the assumption that $\mathcal{G}$ contains only connected graphs in Corollary 29 is unnecessary. To remove this assumption, we will need to use a somewhat different strategy. In particular, we will be looking at the coefficient of $m_{(1^k)}$. Hence, instead of Proposition 5, the key tool here is Corollary 23.

Proposition 30. *Any finite set of (not necessarily connected) graphs $\mathcal{G}$ is uniformly distinguishable.*

Proof. Given a set of graphs $\mathcal{G}$, and let $\mathcal{H} = \{H_1, H_2, \ldots, H_l\}$ be the set of all connected components of graphs in $\mathcal{G}$ with isomorphic copies being identified. Without loss of generality, assume the graphs in $\mathcal{H}$ are indexed with respect to the following two rules:

1. If $i < j$, then $|V(H_i)| \geqslant |V(H_j)|$;
2. If $i < j$ and $|V(H_i)| = |V(H_j)|$, then $|E(H_i)| \geqslant |E(H_j)|$.

Therefore, by Corollary 23, a graph in $\mathcal{H}$ is not colorable by any other graph that comes after it in $\mathcal{H}$.

Now we construct an H we claim is a distinguisher of $\mathcal{G}$. Let C_1 be the maximum number of connected components of a graph in $\mathcal{G}$, $C_2 := \max\{|V(H_j)|\}_{j=1}^{l}$, and C_3 be the number of non-negative integer-valued vectors of length l whose sum of the entries are no larger than C_1. Let C_4 be the maximum number of proper H-colorings of G of type $(1^{|V(G)|})$ one can obtain if $G \in \mathcal{G}$ and H is a graph with at most C_1C_2 vertices. Clearly, C_j $(1 \leqslant j \leqslant 4)$ are all positive integers. Define $\{M_j\}_{j=1}^{l} \subset \mathbb{N}$ recursively as follows.

1. $M_1 = C_3C_4 + C_1 + 1$;
2. If $j \geqslant 2$, then $M_j = C_3C_4(\prod_{i=1}^{j-1} M_i)^{C_1} + C_1$.

Hence, we have $C_1 = M_1 \leqslant M_2 \leqslant \cdots \leqslant M_l$. Now, we set

$$H := \biguplus_{j=1}^{l} M_j H_j,$$

and we claim that H is a distinguisher for $\mathcal{G}$.

Suppose $G_1, G_2 \in \mathcal{G}$. Since H contains an isomorphic copy of each graph in $\mathcal{G}$, every graph in $\mathcal{G}$ is necessarily H-colorable. Thus, it is safe to assume that $k := |V(G_1)| = |V(G_2)|$. If we write $X_{G_1}^H = \sum_{\lambda \vdash k} a_\lambda m_\lambda^{|V(H)|}$ and $X_{G_2}^H = \sum_{\lambda \vdash k} b_\lambda m_\lambda^{|V(H)|}$, then it suffices to show that $a_{(1^k)} \neq b_{(1^k)}$. We note that $a := a_{(1^k)}$ is the number of proper H-colorings of G_1 of type (1^k). Suppose G_1 has m components $H_{j_1}, H_{j_2}, \ldots, H_{j_m}$, where $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_m$. Then, G_1 can be associated naturally with a vector of length l, $p(G_1) := (p_1, p_2, \ldots, p_l)$, where p_j is the number of isomorphic copies of H_j in G_1. Now, we can obtain a proper H-coloring of type (1^k) if we choose p_j copies of H_j from H to color *the* p_j copies of H_j in G_1. This is always possible since $M_j \geqslant C_1$ for every j, and it will give us $\prod_{j=1}^l (\alpha_j)^{p_j} P_{p_j}^{M_j}$ colorings of type (1^k), where α_j is the number of automorphisms of H_j and $P_{n_1}^{n_2}$ is the standard notation for the number of permutations that represents the number of ways to choose and permute n_1 items from n_2 distinct items:

$$P_{n_1}^{n_2} := \frac{n_2!}{(n_2 - n_1)!}.$$

We can similarly define $p(G_2) = (q_1, q_2, \ldots, q_l)$ and obtain $\prod_{j=1}^l (\alpha_j)^{q_j} P_{q_j}^{M_j}$ such colorings of type (1^k). We shall now study the two vectors $p(G_1)$ and $p(G_2)$.

Given two vectors $s = (s_1, s_2, \ldots, s_l), t = (t_1, t_2, \ldots, t_l)$, we say that s is reverse dominated by t, and write $s <_R t$, if and only if $s \neq t$ and if N is the largest number for which $s_N \neq t_N$, we have $s_N < t_N$. We shall also allow $>_R, \leqslant_R$, and $\geqslant_R$ to be defined naturally. Suppose $s <_R t$ and $\sum s_j, \sum t_j \leqslant C_1$. Let N be the largest number for which $s_N < t_N$. We have $s_N < t_N \leqslant C_1$. Therefore,

$$\begin{aligned}
\frac{P_{t_N}^{M_N}}{P_{s_N}^{M_N}} &= \frac{M_N!}{(M_N - t_N)!} \frac{(M_N - s_N)!}{M_N!} \\
&\geqslant M_N - s_N \\
&> M_N - C_1 \\
&\geqslant C_3 C_4 (\prod_{i=1}^{N-1} M_i)^{C_1}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\frac{\prod_{j=1}^l P_{t_j}^{M_j}}{\prod_{j=1}^l P_{s_j}^{M_j}} &= \frac{P_{t_N}^{M_N}}{P_{s_N}^{M_N}} \frac{\prod_{j=1}^{N-1} P_{t_j}^{M_j}}{\prod_{j=1}^{N-1} P_{s_j}^{M_j}} \\
&> C_3 C_4 (\prod_{i=1}^{N-1} M_i)^{C_1} \frac{\prod_{j=1}^{N-1} P_{t_j}^{M_j}}{\prod_{j=1}^{N-1} P_{s_j}^{M_j}} \\
&\geqslant \frac{C_3 C_4 (\prod_{i=1}^{N-1} M_i)^{C_1}}{\prod_{j=1}^{N-1} P_{s_j}^{M_j}}
\end{aligned}$$

$$
\begin{aligned}
&= C_3C_4 \prod_{j=1}^{N-1} \frac{M_j^{C_1}}{P_{s_j}^{M_j}} \\
&\geqslant C_3C_4.
\end{aligned}
$$

In particular, $\prod_{j=1}^{l} P_{t_j}^{M_j} > C_3C_4 \prod_{j=1}^{l} P_{s_j}^{M_j} \geqslant \prod_{j=1}^{l} P_{s_j}^{M_j}$.

Clearly, $p(G_1) \neq p(G_2)$, and without loss of generality, we assume $p(G_1) <_R p(G_2)$. We shall prove that $a_{(1^k)} < \prod_{j=1}^{l} P_{q_j}^{M_j} \leqslant \prod_{j=1}^{l} \alpha_j^{q_j} P_{q_j}^{M_j} \leqslant b_{(1^k)}$. The only non-trivial inequality is the first one. As a first step we will show the following:

$$
\begin{aligned}
a_{(1^k)} &= \sum_{v=(v_1,v_2,\ldots,v_l)} \beta_v \prod_{j=1}^{l} \binom{M_j}{v_j} \\
&\leqslant \sum_{v=(v_1,v_2,\ldots,v_l)} \beta_v \prod_{j=1}^{l} P_{v_j}^{M_j},
\end{aligned}
$$

where v ranges over all non-negative integer-valued vectors whose sum is no more than m and β_v is the total number of $\biguplus_{j=1}^{l} v_j H_j$-colorings of type (1^k) of G_1, and that all components in $H_v := \biguplus_{j=1}^{l} v_j H_j$ are used in the coloring. To see this, we write

$$
\sum_{v=(v_1,v_2,\ldots,v_l)} \beta_v \prod_{j=1}^{l} \binom{M_j}{v_j} = \sum_{s=1}^{m} \sum_{\substack{|v|=s \\ v=(v_1,v_2,\ldots,v_l)}} \beta_v \prod_{j=1}^{l} \binom{M_j}{v_j},
$$

where $|v| = \sum_{j=1}^{l} v_j$. On the right-hand side, the first sum controls the number of components in H that we use to color G, which shall never be larger than m. The second sum controls which (up to isomorphism) component and how many of it we choose. There are $\prod_{j=1}^{l} \binom{M_j}{v_j}$ ways we can obtain these components from H, and β_v gives the number of colorings of type (1^k) using these components.

As we have noted before, in order to have a proper H_j-coloring of type $(1^{|V(H_{j_i})|})$ of H_{j_i}, we must have that j is less than or equal to j_i. Hence, suppose $v = (v_1, v_2, \ldots, v_l) >_R p(G_1)$ and let N be the largest number for which $v_N > p_N$. The v_l components of H_l in H_v can only be used to color the p_l components of H_l in G_1. It then follows from induction that for any $j > N$, the v_j components of H_j in H_v can only be used to color the p_j components of H_j in G_1. Now, at least one component of H_N in H_v cannot be used to color any component in G_1. Consequently, $\beta_v = 0$. Hence,

$$
\begin{aligned}
a_{(1^k)} &\leqslant \sum_{v=(v_1,v_2,\ldots,v_l)} \beta_v \prod_{j=1}^{l} P_{v_j}^{M_j} \\
&\leqslant C_4 \sum_{\substack{v=(v_1,v_2,\ldots,v_l) \\ v \leqslant_R p(G_1)}} \prod_{j=1}^{l} P_{v_j}^{M_j}
\end{aligned}
$$

$$\leqslant C_4C_3\prod_{j=1}^{l}P_{p_j}^{M_j}$$

$$<\prod_{j=1}^{l}P_{q_j}^{M_j}.$$

Hence, $X_{G_1}^H \neq X_{G_2}^H$. □

Before we look into the next topic, we remark that, in general, one shall not expect that an infinite set of graphs is uniformly distinguishable, even if we assume that $\mathcal{G}$ only contains connected graphs. Indeed, if the chromatic numbers of graphs in $\mathcal{G}$ are not bounded from above, then for any H, there must be infinitely many graphs in $\mathcal{G}$ that are not H-colorable. This argument leads to the following result.

Proposition 31. *Let $\mathcal{G}$ be a set of graphs, and suppose $\{\chi(G) \mid G \in \mathcal{G}\}$ is not bounded. Then, $\mathcal{G}$ is not finitely distinguishable.*

5 K_{k_1,k_2}-chromatic symmetric functions

It is clear that the benefit of discussing H-chromatic symmetric functions over ordinary chromatic symmetric functions is our ability to vary H. On the other hand, we have seen in Section 3 that, for a general H, the H-chromatic symmetric functions do not inherit many nice properties from the ordinary chromatic symmetric functions. Nevertheless, many interesting results can be found by considering specific choices of H. By doing so, we can actually find explicit expressions for the coefficients in the monomial basis for the H-chromatic symmetric functions—something we cannot necessarily do for an arbitrary H. As our focus is on uniqueness results, we will choose graphs for which we have interesting uniqueness results. One such graph is the complete bipartite graph. We begin by first constructing an explicit formula for $K_{m,n}$-chromatic symmetric functions.

Proposition 32. *Let $H = K_{h_1,h_2}$ for some $h_1, h_2 \geqslant 1$. Then a connected graph G is H-colorable if and only if G is bipartite.*

Proof. Let G be connected and bipartite. In the case that G has only one vertex, the result is trivial. Suppose G has at least two vertices. Then G can be colored by any H as long as H is not edgeless. On the other hand, suppose G is not bipartite, then G has an odd cycle. Let H be a complete bipartite graph and let U and V be the two independent sets of H. If we can use H to color G, then at least two consecutive vertices in this odd cycle must be colored both by some $u \in U$ or both by some $v \in V$, which is a contradiction since U and V are independent sets in H. □

We need some machinery for the next result. Let G be a graph with connected components $G_1, \ldots G_l$, and label the maximally independent sets of each component V_j^i, where $1 \leqslant i \leqslant l$ and $j \in \{1, 2\}$. We adopt the following notation:

- $p = (p_1, \dots p_l)$, $p \in \{1,2\}^l$,
- $g^i_j = |V^i_j|$, $k_1 = \sum_i g^i_{p_i}$ and $k_2 = \sum_i g^i_{3-p_i} = |G| - k_1$,
- $\lambda = (\lambda_1, \dots, \lambda_j) \vdash k_1$ and $\mu = (\mu_1, \dots, \mu_d) \vdash k_2$,
- $r_i = r_i(\lambda)$ and $s_i = r_i(\mu)$.
- $\lambda + \mu$ is the partition of $|G|$ formed by appending μ to the end of λ,
- and $t_i = r_i(\lambda + \mu)$.

We will also need to recall that in each of the multinomial coefficients, say $\binom{k}{r_1(\lambda),\dots r_j(\lambda)}$, the term $k - r_1(\lambda) - \dots - r_j(\lambda)$ is implicitly included even if it is nonzero (so we mean $\binom{k}{r_1(\lambda),\dots r_j(\lambda), k-r_1(\lambda)-\dots-r_j(\lambda)}$).

Proposition 33. *Let G be a graph with connected components $G_1, \dots G_l$. Let $H = K_{h_1,h_2}$ for some $h_1, h_2 \geqslant 1$. Then G is H-colorable if and only if each component G_i is bipartite. Furthermore, if G is H-colorable, then using the notation defined above, the H-chromatic symmetric function of G is given by*

$$X^H_G = \sum_p \sum_{\lambda,\mu} a^p_{\lambda,\mu} m_{\lambda+\mu},$$

where

$$a^p_{\lambda,\mu} = \frac{\binom{h_1}{r_1,\dots,r_{k_1}}\binom{k_1}{\lambda_1,\dots,\lambda_j}\binom{h_2}{s_1,\dots,s_{k_2}}\binom{k_2}{\mu_1,\dots,\mu_d}(h_1+h_2)!}{\binom{h_1+h_2}{t_1,\dots,t_{k_1+k_2}}},$$

Proof. Give H an arbitrary labeling ϕ. By Lemma 2, it suffices to count the number of (H,ϕ)-colorings of G. Let $p \in \{1,2\}^l$. Then we use p to make the following assignment of colors to independent sets in G: $V^i_{p_i}$ is assigned to V^H_1 and $V^i_{3-p_i}$ is assigned to V^H_2. From the h_1 colors that we have to color k_1 vertices partitioned by λ there are $\binom{h_1}{r_1,\dots,r_{k_1}}$ ways of choosing a color for λ_i vertices, and for each choice of colors, there are $\binom{k_1}{\lambda_1,\dots,\lambda_j}$ ways to color these vertices. To color the remaining k_2 vertices, we now have h_2 colors to pick from. Hence, there are $\binom{h_2}{s_1,\dots,s_{k_2}}$ ways of choosing a color or the μ_i vertices, and for each choice of colors there are $\binom{k_2}{\mu_1,\dots,\mu_d}$ ways to color the vertices. So the number of (H,ϕ)-colorings of G is given by

$$\alpha^p_{\lambda,\mu} = \binom{h_1}{r_1,\dots,r_{k_1}}\binom{k_1}{\lambda_1,\dots,\lambda_j}\binom{h_2}{s_1,\dots,s_{k_2}}\binom{k_2}{\mu_1,\dots,\mu_d}.$$

Now by Lemma 2, the coefficient of $m_{\lambda+\mu}$ is given by

$$a^p_{\lambda,\mu} = \frac{\alpha^p_{\lambda,\mu}(h_1+h_2)!}{\binom{h_1+h_2}{t_1,\dots,t_{k_1+k_2}}}.$$

Hence, summing over all possible λ, μ and p, the H-chromatic symmetric function of G is given by

$$X_G^H = \sum_p \sum_{\lambda,\mu} a_{\lambda,\mu}^p m_{\lambda+\mu}. \qquad \square$$

The following corollary is immediate by identifying $S_{n+1} = K_{1,n}$.

Corollary 34. *Let G be a bipartite graph and $G_1, G_2, \ldots, G_l$ be its connected components. Suppose $|V(G)| = k$. Let $H = S_{n+1}$ be a star with $n \geqslant 1$. Let V_1^i and V_2^i be the two independent sets of $V(G_i)$. (In the case where G_i contains only 1 vertex, we set V_1^i to be the set of that vertex and $V_2^j = \emptyset$.) For each partition $\lambda \vdash k$, and each $p = (p_1, p_2, \ldots, p_l) \in \{1,2\}^l$, define*

$$a_\lambda^p = \begin{cases} 0 & , \text{ if } \sum_j g_{p_j}^j \notin \lambda \text{ or } l(\lambda) > n+1 \\ \binom{n}{r'_1, r'_2, \ldots, r'_k} \binom{k - \sum_j g_{p_j}^j}{\lambda'_1, \lambda'_2, \ldots, \lambda'_{j-1}} (n+1)! / \binom{n+1}{r_1, r_2, \ldots, r_k} & , \text{ otherwise} \end{cases},$$

where λ' is the partition obtained from λ by removing one part that is equal to $\sum_j g_{p_j}^j$, $r_j = r_j(\lambda)$, and $r'_j = r_j(\lambda')$. Then, we have

$$X_G^H = \sum_\lambda \sum_p a_\lambda^p m_\lambda,$$

where λ ranges over all possible partitions of k, and p ranges over $\{1,2\}^l$.

It is a good idea to make sure that these H-chromatic symmetric functions are indeed new—that they are not just ordinary chromatic symmetric functions. The following corollary demonstrates this:

Corollary 35. *Let G be a bipartite graph (not necessarily connected) as defined in Proposition 33, and with no star components. Let $H = K_{m,n}$, with m fixed and $m < \min\{g_j^i \mid 1 \leqslant i \leqslant l,\ j = 1 \text{ or } 2\}$. Then for every $n \geqslant 1$, X_G^H cannot be written as a chromatic symmetric function of any graph G'.*

Proof. Suppose G and H are given as above. Then with the restriction that $m < \min\{g_j^i \mid 1 \leqslant i \leqslant l,\ j = 1 \text{ or } 2\}$, the coefficient of $m_{(1^k)}$ in X_G^H is 0 for any $n \geqslant 1$ by Proposition 33. However, the coefficient of $m_{(1^k)}$ in the chromatic symmetric function of a graph on k vertices is positive. $\square$

Although we are able to use a counting argument to derive Proposition 33, a simple observation will allow us to find a much neater expression, as well as some interesting uniqueness results.

Corollary 36. *Let G be a bipartite graph and $H = K_{h_1,h_2}$ be a complete bipartite graph. Then, using the same notations as in Proposition 33,*

$$X_G^H = \frac{1}{2} \sum_p X_{K_{k_1,k_2}}^H.$$

Proof. Define $\overline{p} = (3 - p_1, 3 - p_2, \ldots, 3 - p_l)$. Observe that $\sum\limits_{\lambda,\mu}(a^{p}_{\lambda,\mu} + a^{\overline{p}}_{\lambda,\mu})m_{\lambda+\mu}$ is simply just $X^{H}_{K_{k_1,k_2}}$. Summing over all possible p,

$$\sum_{p}\sum_{\lambda,\mu}(a^{p}_{\lambda,\mu} + a^{\overline{p}}_{\lambda,\mu})m_{\lambda+\mu} = \sum_{p} X^{H}_{K_{k_1,k_2}}$$

The left hand side is equivalent to $2X^{H}_{G}$, and so the claim follows. ☐

Next, we will discuss following general problem in the setting of $H = K_{m,n}$.

Problem 37. Given a fixed graph H and two non-isomorphic graphs G_1 and G_2, find conditions on G_1, G_2 that tell whether $X^{H}_{G_1} = X^{H}_{G_2}$.

In light of Definition 6, Problem 37 is simply asking whether G_1 and G_2 are H-chromatically equivalent. Of course, one way to provide the answer is to simply compute the two chromatic symmetric functions. The computation, however, is $\mathcal{NP}$-hard in general [9]. Therefore, any polynomial algorithms that answer this question are advantageous.

Corollary 36 hints towards a close relationship between G's based on their $K_{m,n}$-chromatic symmetric functions. Suppose we define an equivalence relation $\sim_{m,n}$ on all bipartite graphs as follows:

$$G_1 \sim_{m,n} G_2 \Leftrightarrow X^{K_{m,n}}_{G_1} = X^{K_{m,n}}_{G_2}.$$

Then, it is natural to ask whether $\sim_{m_1,n_1} = \sim_{m_2,n_2}$ for each $m_1, m_2, n_1, n_2 \in \mathbb{N}$. With these two questions in mind, we will first consider the more intuitive case of connected G, and then progress to include general G.

Proposition 38. *Let $H = K_{m,n}$ be a complete bipartite graph and G_1, G_2 be two connected bipartite graphs on k vertices. Then G_1 is H-chromatically equivalent to G_2 if and only if the sizes of the two parts of G_1 are the same as the sizes of the two parts of G_2. Moreover, the symmetric functions $\{X^{H}_{K_{j,k-j}} \mid 1 \leqslant j \leqslant \lfloor \frac{k}{2} \rfloor\}$ are linearly independent as elements in Λ^k.*

Proof. Observe that for any $G \in \mathcal{G}$, G is H-chromatically equivalent to $K_{j,k-j}$ if and only if X^{H}_{G} contains the term $m_{(j,k-j)}$. ☐

Our intuition would lead us to believe that in the case of general graphs, G_1 and G_2 are $K_{m,n}$-equivalent if they have the same number of vertices, same number of components, and same vertex partitions on said components. This seems obvious from the proof of Proposition 33. However, the latter condition is sufficient, but not necessary. For a more careful study, we will introduce the following definitions.

Definition 39. Let G be a bipartite graph. Following the notation of Proposition 33, we define

$$S(G) := \{\{\sum_{i} g^{i}_{p_i} \mid p \in \{1,2\}^{l}\}\}$$

and

$$\mathrm{diff}(G) := (\left|g^{1}_{1} - g^{1}_{2}\right|, \left|g^{2}_{1} - g^{2}_{2}\right|, \ldots, \left|g^{l}_{1} - g^{l}_{2}\right|).$$

Proposition 40. *Let G_1, G_2 be (not necessarily connected) bipartite graphs on k vertices and $H = K_{m,n}$ be a complete bipartite graph. Then G_1 is H-chromatically equivalent to G_2 if and only if $S(G_1) = S(G_2)$.*

Proof. Combine Corollary 36 and the linear independence result in Proposition 38. □

Proposition 40 answers our second question: $\sim_{m,n}$ is the same for all $m, n \in \mathbb{N}$. Let us pause and ask whether Proposition 40 provides an efficient method for determining $K_{m,n}$ equivalence. Not necessarily. Although the structure of $S(G)$ appears to be simpler than that of $X_G^{K_{m,n}}$, the size of $S(G)$ is still in $\mathcal{O}(2^k)$, where k is the number of vertices of G. The following algorithm does even better, giving a linear algorithm:

Proposition 41. *Let G_1, G_2 be two bipartite graphs on k vertices with the same number of connected components. Then, G_1 is $K_{m,n}$-chromatically equivalent to G_2 if and only if* $\operatorname{diff}(G_1)$ *is a permutation of* $\operatorname{diff}(G_2)$.

Proof. In this proof, we denote the g_i^j's associated with G_1 by a_i^j and those associated with G_2 by b_i^j. Without loss of generality, we also assume that $a_1^j \leqslant a_2^j$ and $b_1^j \leqslant b_2^j$ for every $1 \leqslant j \leqslant l$, so that the absolute value signs in the definitions of $\operatorname{diff}(G_1) = (d_1, d_2, \ldots, d_l)$ and $\operatorname{diff}(G_2) = (f_1, f_2, \ldots, f_l)$ can be omitted. Since permutation of the difference vectors are allowed, we also assume that the connected components are labeled so that $\operatorname{diff}(G_1)$ and $\operatorname{diff}(G_2)$ are in (weakly) increasing order.

Next, we observe that $s_1 := \min S(G_1) = \sum a_1^j = \frac{k-\sum d_j}{2}$ and $s_2 := \min S(G_2) = \sum b_1^j = \frac{k-\sum f_j}{2}$. Moreover, $S(G_1) = \{\{s_1 + \sum_{d\in D} d \mid D \in \mathcal{P}(\operatorname{diff}(G_1))\}\}$ and $S(G_2) = \{\{s_2 + \sum_{f\in F} f \mid F \in \mathcal{P}(\operatorname{diff}(G_2))\}\}$. Here, $\mathcal{P}$ is the power set if we identify $\operatorname{diff}(G)$ with a multiset naturally. If $d_j = f_j$ for every $1 \leqslant j \leqslant l$, then $s_1 = s_2$ and we clearly have $S(G_1) = S(G_2)$.

Conversely, assume $\operatorname{diff}(G_1) \neq \operatorname{diff}(G_2)$. If $\sum d_j \neq \sum f_j$, then $s_1 \neq s_2$ so $S(G_1) \neq S(G_2)$. So assume otherwise. Let j be the smallest number such that $d_j \neq f_j$. We have

$$\left\{\left\{s_1 + \sum_{d\in D} d \mid D \in \mathcal{P}((d_1, d_2, \ldots, d_{j-1}))\right\}\right\} = \left\{\left\{s_2 + \sum_{f\in F} f \mid F \in \mathcal{P}((f_1, f_2, \ldots, f_{j-1}))\right\}\right\},$$

where the quantity in the equality is denoted by S'. Now, it is clear that $\min(S(G_1)\setminus S') = s_1 + d_j \neq s_2 + f_j = \min(S(G_2) \setminus S')$. Hence, $S(G_1) \neq S(G_2)$ and the claim is proved. □

The beauty of Proposition 41 is that the size of $\operatorname{diff}(G)$ is in $\mathcal{O}(k)$, and is fairly easy to compute. In fact, the computation is not just in polynomial time, but also in linear time.

Corollary 42. *Let G_1, G_2 be two graphs on N vertices and M_1, M_2 edges, respectively. Then, for any complete bipartite graph $H = K_{m,n}$, it takes $\mathcal{O}(N+M_1+M_2)$ time to decide if G_1, G_2 are H-equivalent.*

Proof. Use breadth first search or depth first search to determine if G_1, G_2 are both bipartite graphs, and to compute their difference vectors. It suffices to note that the entries in the difference vectors are integers in the interval $[1, k-1]$, so we may bucket sort the two difference vectors. $\square$

So far our discussions have centered around a fixed H and varied G. There are no reasons to not ask if we can find any results for a fixed G and varied H. The problem of determining if $X_G^{H_1}$ and $X_G^{H_2}$ may not appear as interesting in the setting of $H = K_{m,n}$. Indeed, when G is a non-trivial bipartite and H_1, H_2 have distinct number of edges, then we always have $X_G^{H_1} \neq X_G^{H_2}$. A more reasonable question to ask here concerns X_G^H as vectors in the linear space Λ^k. We will first look at special case of $K_{m,n}$—namely, the star graphs S_{n+1}. Then, we shall briefly discuss the difficulties of dealing with general complete bipartite graphs. The following lemma is obvious from Corollary 34.

Lemma 43. *Let G be a connected bipartite graph and let k_1, k_2 be the sizes of its two maximal independent sets. Suppose $n > \max\{k_1, k_2\}$. Then,*

$$X_G^{S_{n+1}} = nX_G^{S_n}.$$

Moreover, when $n \leqslant \max\{k_1, k_2\}$, $X_G^{S_{n+1}} - nX_G^{S_n}$ contains monomials whose indexing partitions have length $n+1$ only.

Proof. By Corollary 34, when $l(\lambda) \leqslant n$, the coefficients a_λ^p in $X_G^{S_n}$ and $X_G^{S_{n+1}}$ differ by a factor of n. When $n > \max\{k_1, k_2\}$, it is clear that $a_\lambda^p = 0$ in both $X_G^{S_n}$ and $X_G^{S_{n+1}}$ whenever $l(\lambda) > n$. The statement of the lemma follows. $\square$

Proposition 44. *Let G be a bipartite graph on k vertices. Then,* $\operatorname{span}\{X_G^{S_{n+1}} \mid n \geqslant 1\} = \operatorname{span}\{X_G^{S_{n+1}} \mid 1 \leqslant n \leqslant \max S(G)\}$ *is a subspace of Λ^k of dimension* $\max S(G)$. *In particular, the set $\{X_G^{S_{n+1}} \mid 1 \leqslant n \leqslant \max S(G)\}$ is linearly independent.*

Proof. By Lemma 43 and Corollary 36, $\operatorname{span}\{X_G^{S_{n+1}} \mid n \geqslant \max S(G)\} = \operatorname{span}\{X_G^{S_{S(G)+1}}\}$. To show $X_G^{S_2}, X_G^{S_3}, \ldots, X_G^{S_{\max S(G)+1}}$ are linearly independent, we note that whenever $n \leqslant \max S(G)$, $X_G^{S_{n+1}}$ contains a monomial whose corresponding partition has length $n+1$, whereas this is clearly impossible for a smaller star. $\square$

Corollary 45. *For a fixed $k \geqslant 2$, the subspace of Λ^k spanned by $\{X_G^{S_{n+1}} \mid |V(G)| = k, n \geqslant 1\}$ has dimension of at most $\frac{(\lceil k/2 \rceil + k - 1)\lfloor k/2 \rfloor}{2}$.*

Proof. By Corollary 36, $\operatorname{span}\{X_G^{S_{n+1}} \mid |V(G)| = k, n \geqslant 1\} = \operatorname{span}\{X_{K_{j,k-j}}^{S_{n+1}} \mid 1 \leqslant j \leqslant \lfloor \frac{k}{2} \rfloor, n \geqslant 1\}$. For a fixed j, by Proposition 44, $\dim(\operatorname{span}\{X_{K_{j,k-j}}^{S_{n+1}} \mid n \geqslant 1\}) = k - j$. It suffices to note that $(k-1) + (k-2) + \cdots + (k - \lfloor \frac{k}{2} \rfloor) = \frac{(\lceil k/2 \rceil + k - 1)\lfloor k/2 \rfloor}{2}$. $\square$

We know that the dimension of Λ^k is asymptotically equivalent to $\frac{1}{4k\sqrt{3}} e^{\pi\sqrt{2k/3}}$ as $k \to \infty$ [5]. Hence, the implication of the fact that $\dim \operatorname{span}\{X_G^{S_{n+1}} \mid |V(G)| = k, n \geqslant 1\}$

grows at a rate no faster than quadratic is that only very few symmetric functions can be written as a linear combination of star-chromatic symmetric functions.

Where does the argument from Lemma 43 to Corollary 45 start to collapse for general complete bipartite graphs? Unfortunately, it is Lemma 43 that fails in the general case. In fact, we have the following result which shows it is completely hopeless to obtain a similar result using the same logic.

Proposition 46. *Suppose G is a connected bipartite graph, and let g_1, g_2 be the sizes of its two maximal independent sets. Assume $\min\{g_1, g_2\} > 2$. Then, $X_G^{K_{m_1,n}}$ is not a (rational) scalar multiple of $X_G^{K_{m_2,n}}$ whenever $m_1 \neq m_2$, $2 < m_1, m_2$.*

Proof. We will compare ratios of coefficients of different monomials. Given an $m > 2$, we first compute the coefficient of $m_{(g_1,g_2)}$ in $X_G^{K_{m,n}}$. There are $2mn$ proper $K_{m,n}$-colorings of G that have type (g_1, g_2). So by Lemma 2, the coefficient of $m_{(g_1,g_2)}$ in $X_G^{K_{m,n}}$ is $c_m = (2mn)\alpha(m + n - 2)!$, where $\alpha = 1$ if $g_1 \neq g_2$ and $\alpha = 2$ otherwise. In particular,

$$\frac{c_{m+1}}{c_m} = \frac{(m+1)(m+n-1)}{m}.$$

Similarly, we can compute the coefficient of $m_{(g_1-1,g_2-1,1,1)}$ in $X_G^{K_{m,n}}$, which we denote by d_m. To count the number of proper colorings of this type, we need to first choose two colors from each independent set of $K_{m,n}$ to use, and then choose one vertex from each independent set of G to be colored differently. Hence, the total number of proper $K_{m,n}$-colorings of G that have type $(g_1 - 1, g_2 - 1, 1, 1)$ is $2m(m-1)n(n-1)g_1g_2$. By Lemma 2 again, we have $d_m = (2m(m-1)n(n-1)g_1g_2)\beta(m + n - 4)!$, where $\beta = 4!$, $3!$, or $2!$, depending on the values of g_1 and g_2. In particular,

$$\frac{d_{m+1}}{d_m} = \frac{(m+1)(m+n-3)}{m-1}.$$

Now, $(m+n-1)(m-1) = m^2 + mn - 2m - n + 1 > m^2 + mn - 3m = (m + n - 3)m$. So $c_{m+1}/c_m > d_{m+1}/d_m$, and the result follows by induction. $\square$

A similar argument to above leads to the following proposition.

Proposition 47. *Suppose G is a connected bipartite graph, and let g_1, g_2 be the sizes of its two maximal independent sets. Assume $|g_1 - g_2| \neq 1$, $\min\{g_1, g_2\} \geqslant 2$, and $n \geqslant 2$. Then, $X_G^{K_{m_1,n}}$ is not a (rational) scalar multiple of $X_G^{K_{m_2,n}}$ whenever $m_1 \neq m_2$, $m_1, m_2 \geqslant n$.*

6 S_k^1-chromatic symmetric functions

We have mentioned that the definition of H-chromatic symmetric functions does not require H to be simple. In this section, we work with a class of non-simple H. As in the previous question, the major question we are trying to answer is regarding the equivalence of different G's. We first define the principal subject of our discussion, which we will call *augmented stars*. We remark that the stars with loops have been studied in the literature of both mathematics and physics. One of the examples is the so-called Widom-Rowlinson model [17].

Definition 48. An augmented star on $(n+1)$ vertices, denoted by S^1_{n+1}, is the star S_{n+1} with an extra loop on its center.

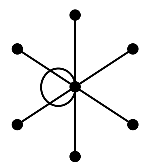

Figure 2: The augmented star S^1_7

We have seen that the only $K_{m,n}$-colorable graphs are the bipartite graphs. This is clearly not the case for augmented stars. Indeed, as long as H contains a loop, then there is always a proper H-coloring of G of the type (k), where k is the number of vertices in G. Nevertheless, some results in S^1_{n+1}-colorings can be derived from those in S_{n+1}-colorings. To do so, we will need to define a new class of symmetric functions that appear in Section 5, but whose formal definition has not been written down.

Definition 49. Given three integers $k_1, k_2, n \geqslant 1$. For each $\lambda \vdash k_1 + k_2$, we set

$$a_\lambda = \begin{cases} 0 & \text{, if } l(\lambda) > n+1 \text{ or } k_1 \notin \lambda \\ \binom{n}{r'_1, r'_2, \ldots, r'_k}\binom{k_2}{\lambda'_1, \lambda'_2, \ldots, \lambda'_{j-1}}(n+1)!/\binom{n+1}{r_1, r_2, \ldots, r_k} & \text{, otherwise} \end{cases}, \tag{2}$$

where $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{j-1}) \vdash k_2$ is obtained from λ by deleting one part of k_1, and $r_i = r_i(\lambda)$, $r'_i = r_i(\lambda')$. We define $X^{S_{n+1}}_{k_1,k_2} = \sum_\lambda a_\lambda m_\lambda$.

So, for example, by Corollary 34, we have $X^{S_{n+1}}_{K_{k_1,k_2}} = X^{S_{n+1}}_{k_1,k_2} + X^{S_{n+1}}_{k_2,k_1}$. We emphasize that k_1 and k_2 are not symmetric in the definition above. Informally speaking, $X^{S_{n+1}}_{k_1,k_2}$ is the symmetric function obtained by forcing k_1 vertices to be colored by the label on a fixed vertex in H and the remaining k_2 vertices to be colored arbitrarily by the labels on the other n vertices in H.

Before we use $X^{S_{n+1}}_{k_1,k_2}$ to compute the S^1_{n+1}-chromatic symmetric functions, we introduce a lemma on the linear independence of these symmetric functions. As we have seen in Section 5, such results can be very useful in the discussion of H-equivalence relations.

Lemma 50. *If $n \geqslant 2$, then $\{X^{S_{n+1}}_{1,k}, X^{S_{n+1}}_{2,k-1}, \ldots, X^{S_{n+1}}_{k,1}\}$ are linearly independent in Λ^{k+1}.*

Proof. When $n \geqslant k$, the statement is obvious since $m_{(j,1^{k-j})}$ exists in $X^{S_{n+1}}_{j,k-j}$ but vanishes in $X^{S_{n+1}}_{i,k-i}$ for every $i > j$. So assume $2 \leqslant n < k$. We set

$$c_1 X^{S_{n+1}}_{1,k} + c_2 X^{S_{n+1}}_{2,k-1} + \cdots + c_k X^{S_{n+1}}_{k,1} = 0.$$

We first observe that $m_{(j,k+1-j)}$ exists only in $X^{S_{n+1}}_{j,k+1-j}$ and $X^{S_{n+1}}_{k+1-j,j}$, and that its coefficients in the two symmetric functions are the same. Therefore, $c_j + c_{k+1-j} = 0$. When k is even, we also have $c_{\frac{k}{2}} = 0$. If we set $X^{S_{n+1}}_j := X^{S_{n+1}}_{j,k+1-j} - X^{S_{n+1}}_{k+1-j,j}$ for $1 \leqslant j \leqslant \lfloor \frac{k}{2} \rfloor$, then the equation can be rewritten as

$$c_1 X^{S_{n+1}}_1 + c_2 X^{S_{n+1}}_2 + \cdots + c_{\lfloor \frac{k}{2} \rfloor} X^{S_{n+1}}_{\lfloor \frac{k}{2} \rfloor} = 0.$$

Next, we make some observations of $X^{S_{n+1}}_j$. First, for each $1 \leqslant j \leqslant \lfloor \frac{k}{2} \rfloor$, the only monomial m_λ exists in $X^{S_{n+1}}_{j,k+1-j}$ where $k+1-j$ is a part of λ is $m_{(k+1-j,j)}$, whereas $X^{S_{n+1}}_{k+1-j,j}$ is a linear combination of monomials m_λ such that $k+1-j$ is a part of λ. Therefore, in $X^{S_{n+1}}_j$, all monomials m_λ with $k+1-j \in \lambda$ have non-positive coefficients, and all monomials m_λ with $k+1-j \notin \lambda$ have non-negative coefficients. Given this observation, we shall note that for $2 \leqslant j \leqslant \lfloor \frac{k}{2} \rfloor$, the monomial $m_{(k+1-j,j-1,1)}$ appears in $X^{S_{n+1}}_1$ with a positive coefficient, in $X^{S_{n+1}}_{k+2-j}$ with a positive coefficient, in $X^{S_{n+1}}_{k+1-j}$ with a negative coefficient, and in nowhere else among $X^{S_{n+1}}_1, X^{S_{n+1}}_2, \ldots, X^{S_{n+1}}_{\lfloor \frac{k}{2} \rfloor}$.

Now, we claim that $c_1 c_j \geqslant 0$ for every $2 \leqslant j \leqslant \lfloor \frac{k}{2} \rfloor$. We prove it by induction. When $j = 2$, the result follows from the fact that $m_{(k-1,1,1)}$ appears in $X^{S_{n+1}}_1$ with a positive coefficient and in $X^{S_{n+1}}_2$ with a negative coefficient, and nowhere else, as noted above. Assume the claim holds for j. The monomials $m_{(k-j,j,1)}$ appears in $X^{S_{n+1}}_1$ with a positive coefficient, in $X^{S_{n+1}}_j$ with a positive coefficient, in $X^{S_{n+1}}_{j+1}$ with a negative coefficient, and nowhere else. But $c_1 c_j \geqslant 0$, so c_1 and c_{j+1} cannot have different signs. The claim is proved.

Finally, consider the monomial $m_{(\lfloor \frac{k+1}{2} \rfloor, \lfloor \frac{k}{2} \rfloor, 1)}$. This monomial exists in $X^{S_{n+1}}_1$ since $n \geqslant 2$, and its coefficient is non-negative in all of $X^{S_{n+1}}_2, X^{S_{n+1}}_3, \ldots, X^{S_{n+1}}_{\lfloor \frac{k}{2} \rfloor}$, as noted at the beginning. Since $c_1 c_j \geqslant 0$, we must have $c_1 = 0$. Now, by an induction argument, suppose $c_i = 0$ for every $i \leqslant j$, then $X^{S_{n+1}}_{j+1}$ is the only possibly non-vanishing term among $X^{S_{n+1}}_1, X^{S_{n+1}}_2, \ldots, X^{S_{n+1}}_{\lfloor \frac{k}{2} \rfloor}$ that contains $m_{(k+1-j,j,1)}$, so $c_{j+1} = 0$. Hence, $c_j = 0$ for every $1 \leqslant j \leqslant k$, and we are done. □

Now, we are ready to compute the S^1_{n+1} chromatic symmetric functions and discuss its consequences. To do so, we will have to first introduce some notation. Given any graph G, denote by $\mathcal{I}(G)$ the set of all independent sets of G. Let $I(G)$ be the multiset $I(G) = \{\{|S| \mid S \in \mathcal{I}(G)\}\}$. We define $\tilde{\mathcal{I}}(G)$ and $\tilde{I}(G)$ similarly by requiring all independent sets to be maximal.

Proposition 51. *Given a graph G on k vertices that is not edgeless, and a natural number $n \geqslant 1$, we have*

$$X^{S^1_{n+1}}_G = n! m_{(k)} + \sum_{k_1 \in I(G)} X^{S_{n+1}}_{k-k_1,k_1}.$$

Proof. A proper S^1_{n+1}-coloring of G that is not of the type (k) is obtained by first picking an independent set S of G, coloring all vertices in S by the labels on the leaves, and coloring all vertices in $V(G) \setminus S$ by the label on the center of S^1_{n+1}. Once we find an independent set S, the monomials obtained by this manner sum to $X^{S_{n+1}}_{k-|S|,|S|}$. The values of $c(G)$ are obvious. $\square$

Given Lemma 50 and Proposition 51, the following result is immediate.

Corollary 52. *Suppose G_1, G_2 are two graphs on k vertices, and $n \geqslant 2$. Then, $X^{S^1_{n+1}}_{G_1} = X^{S^1_{n+1}}_{G_2}$ if and only if $I(G_1) = I(G_2)$, if and only if $\tilde{I}(G_1) = \tilde{I}(G_2)$.*

Proof. Combine Lemma 50 and Proposition 51 to get the first equivalence. For the second one, it suffices to note that $N_j = \sum_{i \geqslant j} \binom{i}{j} \tilde{N}_j$, where N_j is the number of independent sets of size j and $\tilde{N}_i$ is the number of maximal independent sets of size i. $\square$

We remark that Lemma 50 fails for $n = 1$. Indeed, $X^{S_2}_{j,k+1-j} = m_{(j,k+1-j)} = X^{S_2}_{k+1-j,j}$. Moreover, Corollary 52 does not hold for $n = 1$ either. The easiest counter-examples one can construct are G_1 and G_2 in Figure 3. Note that $I(G_1) = \{\{1,1,1,1,1,2,2,2,2,2\}\}$, whereas $I(G_2) = \{\{1,1,1,1,1,2,2,2,2,3\}\}$. However, we have $X^{S_2}_{G_1} = X^{S_2}_{G_2} = 5m_{(3,2)} + 5m_{(4,1)} + m_{(5)}$. Another way to see that $X^{S_2}_{G_1} = X^{S_2}_{G_2}$ without computing is by observing $X^{S_2}_{2,3} = X^{S_2}_{3,2}$.

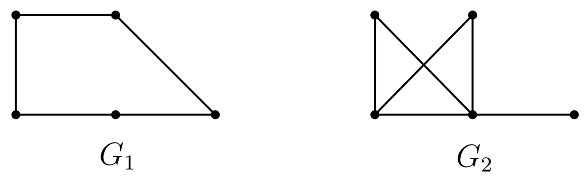

Figure 3: Example of G_1 and G_2 such that $I(G_1) = I(G_2)$.

When we have two graphs G_1 and G_2 and would like to know if they have the same chromatic symmetric functions with respect to an augmented star S^1_{n+1}, is Corollary 52 an easier characterization (i.e. it suggests a polynomial algorithm to the problem) of this equivalence relation? It is not quite clear yet. It seems hard to show if the problem is $\mathcal{NP}$-hard: even though computing $I(G)$ is $\mathcal{NP}$-hard, it is not clear yet whether comparing $I(G_1)$ to $I(G_2)$ still is. Nevertheless, we shall see that Corollary 52 does lead to some necessary *or* sufficient conditions of $X^{S^1_{n+1}}_{G_1} = X^{S^1_{n+1}}_{G_2}$ that will be hard to see directly from Proposition 51.

Corollary 53. *Suppose $n \geqslant 2$. Then,*

1. *If G_1, G_2 are two (simple) graphs on k vertices with $X^{S^1_{n+1}}_{G_1} = X^{S^1_{n+1}}_{G_2}$, then G_1, G_2 have the same number of edges.*

2. *All (simple) graphs in $\mathcal{G}_m := \{\overline{G} \mid |V(G)| = k, |E(G)| = m,\ G \text{ is } C_3\text{-free}\}$ have the same S^1_{n+1}-chromatic symmetric function, which is not equal to the S^1_{n+1}-chromatic symmetric function of any graph not in $\mathcal{G}_m$.*

3. *For each $1 \leqslant j \leqslant k$, all graphs in $\mathcal{T}_j := \{\overline{T} \mid T$ is a forest on k vertices with j connected components $\}$ have the same S^1_{n+1}-chromatic symmetric function. In particular, all tree-complements (on the same number of vertices) have the same S^1_{n+1}-chromatic symmetric function.*

Proof. Suppose $X^{S^1_{n+1}}_{G_1} = X^{S^1_{n+1}}_{G_2}$, then G_1 and G_2 have the same number of non-edges (i.e. independent sets of size 2), and therefore the same number of edges. If G_1, G_2 are C_3-free, then G_1, G_2 contain no cliques of size larger than 2, so $\mathcal{I}(\overline{G_1}) = \mathcal{I}(\overline{G_2})$ if and only if G_1, G_2 have the same number of edges. If $\overline{G} \notin \mathcal{G}_m$, then either $\overline{G}$ has different number of edges, or G contains a C_3 subgraph. Either way, G cannot have the same S^1_{n+1}-chromatic symmetric function as those of graphs in $\mathcal{G}_m$. 3 is a special case of 2. □

7 Classical bases of Λ as H-chromatic symmetric functions of complete multipartite graphs

In this section, we will show that classical bases of the algebra of symmetric functions, that is, the monomial symmetric functions, power sum symmetric functions, and elementary symmetric functions, can be realized as H-chromatic symmetric functions.

As promised in Section 3, we show a stronger result of Proposition 4. We denote by K_λ the complete $l(\lambda)$-partite graph whose parts contain $\lambda_1, \lambda_2, \dots, \lambda_l$ vertices, respectively. We denote by K_n^- the graph obtained by deleting an edge from the complete graph on n vertices.

Proposition 54. *Let $\lambda \vdash k$, $n = l(\lambda)$, and H be a graph that does not contain K^-_{n+1} as a subgraph. Let m be the number of subgraphs of H that are isomorphic to K_n. Then,*

$$X^H_{K_\lambda} = m \cdot n! m^{|V(H)|}_\lambda = m \cdot n! (\prod_{j=1}^{k} r_j(\lambda)!)(|V(H)| - \sum_{j=1}^{k} r_j(\lambda))! m_\lambda.$$

Proof. We denote the vertices in K_λ by $v^j_i, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant \lambda_i$, where $\{v^j_i\}^{\lambda_i}_{j=1}$ is an independent set. Fixing a labeling on H, we first show that λ is the only type of proper H-coloring possible. Let u_i be the vertex in H whose label is used to color v^1_i. Then, $\{u_i\}^n_{i=1}$ induces a clique K_n in H. Assume, by way of contradiction, that v^j_l is colored by the label on $u'_l \neq u_l$. Then, $u'_l \sim_H u_i$ for every $i \neq l$. But then $\{u_i\}^n_{i=1} \cup \{u'_l\}$ induces K^-_{n+1} or K_{n+1}, depending on whether $u_l \sim_H u'_l$, which is a contradiction either way. For each subgraph of H that is isomorphic to K_n, we can obtain $n!$ distinct proper H-colorings of G of type λ. Hence, under the fixed labeling of H, we can get $m \cdot n!$ such colorings in total. The first equality is proved. The second equality follows from the definition of n-augmented monomials. □

Example 55. We give an example illustrating the concept in the proof for Proposition 54. Consider a partition $\lambda = (4, 3, 2, 2) \vdash 11$. We'll show that if $G = K_{4,3,2,2}$ and $H = K_4$, then $X^H_G = c_{(4,3,2,2)} m_{(4,3,2,2)}$.

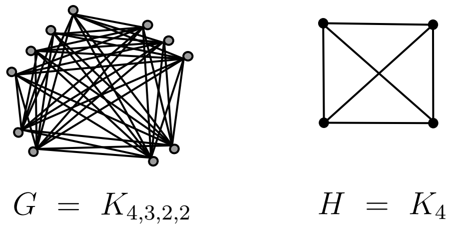

Figure 4: The G and H whose X_G^H corresponds to $m_{4,3,2,2}$

Let's denote the maximally independent sets of G by V_1, V_2, V_3, V_4. Suppose $|V_1| = 4, |V_2| = 3, |V_3| = 2, |V_4| = 2$. Denote the vertices in V_i by $\{v_{ij}\}_{j=1}^{|V_i|}$.

Now let's suppose we choose one vertex from each maximally independent set V_i. Suppose we choose $v_{11} \in V_1, v_{23} \in V_2, v_{31} \in V_3, v_{42} \in V_4$. If we look at the induced subgraph created by these vertices, we see that it is K_4 since $v_{ij} \in V_i$ must be connected to every $v_{ab} \in V_a$ (assuming $a \neq i$).

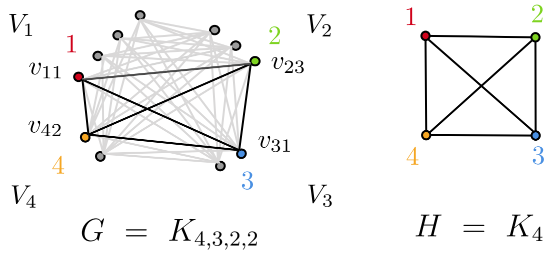

Figure 5: A Colored K_4 subgraph in $K_{4,3,2,2}$

Now suppose we keep $v_{11} \in V_1, v_{23} \in V_2, v_{42} \in V_4$ but replace $v_{31} \in V_3$ with $v_{32} \in V_3$. Now suppose without loss of generality that v_{11} is H-colored 1, v_{23} is H-colored 2, and v_{42} is H-colored 4, then the only option for H-coloring v_{32} is the color 3 because on the graph H, the only vertex adjacent to the vertices colored 1,2,4 is the vertex colored 3.

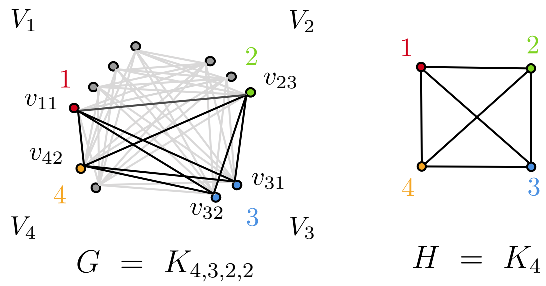

Figure 6: Two Colored K_4 subgraphs in $K_{4,3,2,2}$

If we repeat this line of reasoning until we have exhausted all vertices in all the maximally independent sets, then we can see the only way to H-color $K_{4,3,2,2}$ is by coloring

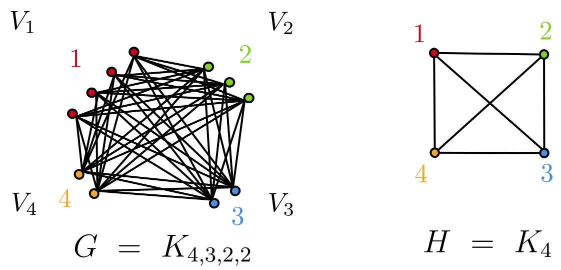

Figure 7: Coloring All the Vertices of $K_{4,3,2,2}$

all the vertices in V_1 the color 1, all the vertices in V_2 the color 2, all the vertices in V_3 the color 3, and all the vertices in V_4 the color 4.

So $m_{(4,3,2,2)}$ is the only m_λ term that will appear in X_G^H.

Remark 56. If $m_\lambda \neq m_{1^n}$, then it is not possible to write m_λ as a scalar multiple of a single chromatic symmetric function X_G. Recall that if G has n vertices, the coefficient of m_{1^n} in X_G is positive. From Proposition 54, we can write m_λ as a scalar multiple of a single X_G^H but not X_G. So Proposition 54 holds for H-chromatic symmetric functions but not chromatic symmetric functions.

Now we use Proposition 54 to prove that certain sets of H-chromatic symmetric functions form a basis for Λ^n. We also express the elementary symmetric functions and the power sum symmetric functions in terms of this basis.

Proposition 57. *The following set of H-chromatic symmetric functions is a basis for Λ^n:*

$$\beta = \{X_G^H : G = K_{\lambda_1,\lambda_2\ldots\lambda_k} \text{ where } \lambda = (\lambda_1, \lambda_2 \ldots \lambda_k) \vdash n, \lambda \neq (n), \text{ and } H = K_{l(\lambda)}\} \cup \{X_{K_n}^{K_1}\}$$

Proof. From Proposition 54, the set β corresponds to

$$\{c_\lambda m_\lambda : \lambda \vdash n, c_\lambda \neq 0\}$$

This is a basis for Λ^n since $\{m_\lambda : \lambda \vdash n\}$ is a basis for Λ^n. $\square$

We also have a basis for Λ in terms of X_G^H. In what follows, recall that $Par := \bigcup_{n\geqslant 0}\{\lambda | \lambda \vdash n\}$.

Corollary 58. *The set $\beta = \{X_G^H : G = K_{\lambda_1,\lambda_2\ldots\lambda_k}, \lambda = (\lambda_1, \lambda_2 \ldots \lambda_k) \in Par, \forall n, \lambda \neq (n)$ and $H = K_{l(\lambda)}\} \cup \left\{X_{K_n}^{K_1} : n \in \mathbb{N}\right\}$ is a basis for Λ.*

Corollary 59. *Let $\lambda = (\lambda_1, \lambda_2 \ldots \lambda_k)$. Then*

$$e_\lambda = \frac{1}{(\lambda_1)^2(\lambda_2)^2 \ldots (\lambda_k)^2} X_{K_{\lambda_1}}^{K_{\lambda_1}} X_{K_{\lambda_2}}^{K_{\lambda_2}} \ldots X_{K_{\lambda_k}}^{K_{\lambda_k}}$$

Proof. First note we can write the $\{m_\lambda\}$ basis in terms of X_G^H. Recall that $e_n = m_{1^n}$ and $e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_k}$. Also, $e_\lambda = m_{1^{\lambda_1}} m_{1^{\lambda_2}} \dots m_{1^{\lambda_k}}$.

We know that there are λ_i ways of (K_{λ_i}, ϕ)-coloring the graph $G = K_{\lambda_i}$. There are λ_i ways of fixing colorings ϕ on $H = K_{\lambda_i}$. So we know that $X_{K_{\lambda_i}}^{K_{\lambda_i}} = (\lambda_i)^2 m_{1^{\lambda_i}}$. Apply Proposition 54. $\square$

Corollary 60. *Let $\lambda = (\lambda_1, \lambda_2 \dots \lambda_k)$. Then,*

$$p_\lambda = X_{K_{\lambda_1}}^{K_1} X_{K_{\lambda_2}}^{K_1} \dots X_{K_{\lambda_k}}^{K_1}$$

Proof. First note that we can write the $\{m_\lambda\}$ basis in terms of X_G^H. Recall that $p_n = m_{(n)}$ and $p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_k}$. Also, $p_\lambda = m_{(\lambda_1)} m_{(\lambda_2)} \dots m_{(\lambda_k)}$.

Apply Proposition 54. Note that if H is only 1 vertex, then there is only one way to color the vertices of G, which is by coloring all the vertices of G the same color. $\square$

Corollaries 59 and 60 show it is possible to write e_λ and p_λ as the product of several X_G^H. An interesting question[1] would be whether, for some λ, e_λ or p_λ could be written as a scalar multiple of a single X_G^H.

Corollary 61. *Given an H-chromatic symmetric function of a graph G, X_G^H, it is possible to find sets of graphs $\{H_j | j \in \mathbb{N}\}$ and $\{G_i | i \in \mathbb{N}\}$ such that X_G^H can be expressed as a linear combination of $\{X_{G_i}^{H_j}\}$.*

Proof. Proposition 54 gives us a way of rewriting m_λ in terms of some X_G^H. Now since X_G^H is a symmetric function, it's in the span of $\{m_\lambda : \lambda \in Par\}$. So we can rewrite any X_G^H as a linear combination of other $X_{G_i}^{H_j}$. $\square$

8 *H*-chromatic symmetric functions of complete bipartite graphs

In the beginning of Section 7, we put restrictions on H so that each part in K_λ must be colored the same. Similarly, we can also put restrictions on H so that all but except for one part in K_λ must be colored the same. Consider K_{n+2}^{--}, which is defined to be the graph obtained by removing two non-adjacent edges from K_{n+2}. If H does not contain K_{n+2}^{--} as a subgraph, then it follows immediately that at most one part can be colored using different colors. In that case, we can think a proper H-coloring as if we first pick a subgraph of K_{n-1} from H to color $(n-1)$ parts of G. Then, we count the number of all other vertices that are adjacent to all these $(n-1)$ vertices. These are exactly the vertices that we can use to (freely) color the remaining part of G. For a general multi-partite G, this procedure is complicated to carry out and may not be fruitful. However, when G is bipartite, we are able to use this approach to simply characterize the H-colorings of G and produce some interesting results. We note that several of our remarks, lemmas, and corollaries in this section are results on H-coloring graphs, rather than specific results on chromatic symmetric functions. To the best of our knowledge these have not appeared in the graph theory literature before.

[1]We thank an anonymous referee for suggesting this question.

Remark 62. Let c be a (H, ϕ)-coloring of some graph G. Then c corresponds to a subgraph, C, of H, which is the subgraph induced on the vertices of H used in the coloring of G.

That is, every (H, ϕ)-coloring has an associated induced subgraph $C \subseteq H$. When we focus on complete bipartite G, we can determine exactly the type of subgraph C is.

Lemma 63. *Let G be a complete bipartite graph and H be a simple graph. Then for any H-coloring of G, any two colors used to color G must be adjacent to one another in H or adjacent to a common vertex in H.*

Proof. First note that because G is *complete* bipartite, this induced subgraph must be connected. Denote the maximally independent sets of G by V_1 and V_2. Trivially, since all vertices in V_1 are connected to all vertices in V_2, the colors in H used to color V_1 must be adjacent to the colors in H used to color V_2. So the only potential case where the lemma fails is within V_1 or V_2. Without loss of generality consider $v_1, v_2 \in V_1$, and suppose we color them by 1 and 2 in H respectively. Then since all of V_1 is adjacent to all of V_2, both v_1 and v_2 are adjacent to some vertex v_3 in V_2, which is colored, say, 3. Then it must be that both 1 and 2 are adjacent to 3 in H. □

This observation is not complicated, but has important implications.

Corollary 64. *Let H be a graph that does not contain C_4 as a subgraph and G be a complete bipartite graph. Then a H-coloring of G is an S-coloring of G and vice versa, where S is a star-subgraph of H.*

Proof. For any coloring ϕ of G, ϕ is associated with an induced subgraph on the vertices of H used in the coloring. Further, by Lemma 63, we know that these subgraphs must have diameter less than 3. With the restriction that H does not contain C_4 as a subgraph, the only possible subgraphs of H with diameter less than 3 are stars, or stars with additional edges between the leaf vertices. In the latter case, note that by carefully choosing a central vertex and deleting unnecessary edges, these too can be reduced to stars. Conversely, if ϕ is an S-coloring of G and S is a subgraph of H, then ϕ is also a H-coloring of G. □

In fact, if we restrict our attention from any complete bipartite G to the star, we can remove the condition that H does not contain C_4 as a subgraph.

Proposition 65. *Let G be a star graph. Then G cannot be colored using every single color in $H = C_4$.*

Proof. Label the vertices of C_4 by $1, 2, 3, 4$ cyclically. Suppose we are able to color G using all of the colors in C_4, and without loss of generality we color the central vertex of G using 1. Then there must exist a leaf vertex $v_1 \in G$ colored 3 by assumption. But 1 and 3 are not adjacent in C_4, so this is a contradiction. □

Corollary 66. *Let H be any simple graph and let G be a star graph. Then a H-coloring of G is an S-coloring of G and vice versa, where S is a star-subgraph of H.*

We now can define the key characteristic of a graph that will allow us to distinguish them: the degree sequence. The degree sequence and star sequence here is defined as in Martin and Wagner [11].

Definition 67. Let H be a graph on k vertices. Then we define the degree sequence of H as $D = (d_1, d_2, \ldots, d_k)$, where $d_i = |\{v \in V(T) \ : \ \deg(v) = i\}|$.

Definition 68. Let H be a graph on k vertices. Then we define the star sequence of H as $Z = (z_2, \ldots, z_k)$, where $z_i = z_i(T) =$ number of S_i-subgraphs of H.

Lemma 69. *Let H be a graph on k vertices with degree sequence $D = (d_1, \ldots, d_k)$. Then the star sequence of H, can be determined by the formula*

$$z_{i+1} = \sum_{i \leqslant j} \binom{j}{i} d_j$$

for $2 \leqslant i \leqslant k-1$.

Note that in the case of z_2, the formula above double counts the number of S_2-subgraphs of H, as the argument specifies the central vertex of the star. Of course, in the case of S_2, the star is symmetric regardless of choice of central vertex, hence the double counting. To amend this, either use the formula above and divide the result for z_2 by 2, or simply set z_2 to be the number of edges in H (a similar formula appears in [11]).

Lemma 70. *Let H be a graph on k vertices, and let $G = S_n$ for some n. Then the H-chromatic symmetric function of G is given by*

$$\sum_{\lambda} c_\lambda m_\lambda^n,$$

where we sum over all partitions $\lambda \vdash n$ with length no more than k, and where the coefficient c_λ is given by $c_\lambda := z_{l(\lambda)} \beta_\lambda$, with β_λ being the coefficient of m_λ^n in the corresponding $S_{l(\lambda)}$-chromatic symmetric function of G.

Proof. Every coloring of G by a graph H has an associated star subgraph in H, as noted in Corollary 66. Hence, for each n-augmented monomial m_λ^n, the coefficient counts the number of ways we can make the following choice. We need to choose an $S_{l(\lambda)}$ subgraph of H to color G, and then we need to choose a way of using this $S_{l(\lambda)}$ to color G. The number of ways of doing the first is $z_{l(\lambda)}$, and the number of ways of doing the second is the same as the coefficient of m_λ^n in the corresponding $S_{l(\lambda)}$-chromatic symmetric function of G. □

The next corollaries follow easily:

Corollary 71. *Let H_1 and H_2 both be graphs on k vertices. Let G be a star. If H_1 and H_2 share a degree sequence, then $X_G^{H_1} = X_G^{H_2}$.*

Corollary 72. *Let H_1 and H_2 both be graphs on k vertices. Suppose H_1 and H_2 have different degree sequences. Then let N be the first entry in the degree sequence where they differ. Then they also differ at the N-th entry of their star sequences (z_{N+1}). Let G be a complete bipartite graph on $n \geqslant N$ vertices. Then $X_G^{H_1} \neq X_G^{H_2}$.*

These two results can be combined to yield the following result:

Proposition 73. *Let $\mathcal{H}$ be a set of graphs on less than or equal to k vertices, and let $G = S_k$. Then the H-chromatic symmetric functions of G are distinct up to the degree sequences of $H \in \mathcal{H}$.*

Proof. For any pair of graphs H_1 and H_2 in $\mathcal{H}$ with different degree sequences, their star sequences will differ at some N less than or equal to k. Comparing $X_G^{H_1}$ and $X_G^{H_2}$ in Lemma 70, the coefficients of monomials with corresponding partition length N will be distinct. Since N less than or equal to k for every $H \in \mathcal{H}$, this choice of G works perfectly. □

Restricting G to be a star instead of any complete bipartite graph on k vertices allows us to classify more graphs, including those that contain C_4 subgraphs. However, with the restriction that we only consider H's that do not contain C_4 as a subgraph, all of the above can be translated naturally into analogues for general complete bipartite G.

9 Further avenues to explore and some conjectures

In this section, we shall present some open problems and conjectures in H-chromatic symmetric functions. Since the subject is still new, and experimentation with large graphs is, in general, impractical, we shall state most of them in the form of problems, as opposed to conjectures.

9.1 Possibility of constructing a basis for Λ using X_G^H where H is fixed

It is natural to ask whether Corollary 61 holds when G_i or H_j is fixed (and if not, for which graphs it's possible). In fact, it turns out it is impossible to generate a basis for Λ^n using the H-chromatic symmetric functions of a fixed graph G.

Proposition 74. *It is not possible to generate a basis for Λ^n using a set of H-chromatic symmetric functions, $\{X_G^{H_i}\}$, where G is a fixed graph and $\{H_i | i \in \mathbb{N}, 1 \leqslant i \leqslant n\}$ is finite set of graphs.*

Proof. Suppose $n \geqslant 4$. We will show it is not possible to generate a basis for Λ^n using a set of H-chromatic symmetric functions $\{X_G^{H_i}\}$ where G is a fixed graph.

If G has an edge connecting two distinct vertices, then it is not possible to color all the vertices of G the same color, i.e. we would not have the $m_{(n)}$ term in X_G^H, and $m_{(n)}$ would not be in the span of the H-chromatic symmetric functions $\{X_G^{H_i}\}$. Thus G must be the graph with no edges: $\overline{K_n}$.

Now we observe that if H_1 and H_2 have the same number of vertices, then $X^{H_1}_{\overline{K_n}} = X^{H_2}_{\overline{K_n}}$. From the definition of H-coloring, since $\overline{K_n}$ has no edges, we can color the vertices of $\overline{K_n}$ however we want and the edges of H do not affect the H-coloring of $\overline{K_n}$.

We can choose $X^{\overline{K_i}}_{\overline{K_n}}$ as a representative for the H-chromatic symmetric function of K_n where H is a graph on i vertices.

Here we compute $X^{\overline{K_i}}_{\overline{K_n}}$ for $i < n$.

$$X^{\overline{K_i}}_{\overline{K_n}} = \sum_{\lambda \vdash n, l(\lambda) \leqslant i} i! \binom{n}{\lambda_1, \lambda_2 \dots \lambda_k} m_\lambda$$

Here we compute $X^{\overline{K_i}}_{\overline{K_n}}$ for $i \geqslant n$.

$$X^{\overline{K_i}}_{\overline{K_n}} = \sum_{\lambda \vdash n} i! \binom{n}{\lambda_1, \lambda_2 \dots \lambda_k} m_\lambda$$

From the above formulas, we observe a scalar multiple property for $X^{\overline{K_i}}_{\overline{K_n}}$ where $i \geqslant n$. If $c > d \geqslant 0$, then:

$$X^{\overline{K_{i+c}}}_{\overline{K_n}} = \frac{(i+c)!}{(i+d)!} X^{\overline{K_{i+d}}}_{\overline{K_n}}$$

Let $p(n)$ denote the number of partitions of n. Recall that $\dim(\Lambda^n) = p(n)$.

We can show that if $n \geqslant 4$, $n < p(n)$. The following map on $\{1, 2, \dots n\}$ is defined on every $i \in \{1, 2, \dots n\}$:

$$f : i \mapsto \lambda \text{ such that } l(\lambda) = i$$

Note that if $n \geqslant 4, i = 2$ maps to both $(n-1, 1)$ and $(n-2, 2)$. Comparing the sizes of the domain and range gives us that $n < p(n)$.

If $B = \{X^{\overline{K_i}}_{\overline{K_n}}\}$ and $|B| = \dim(\Lambda^n) = p(n)$, then B cannot be linearly independent. Since $p(n) > n$, B must contain some $X^{\overline{K_{i+d}}}_{\overline{K_n}}$ and $X^{\overline{K_{i+c}}}_{\overline{K_n}}$ $(c > d \geqslant 0)$ such that

$$X^{\overline{K_{i+c}}}_{\overline{K_n}} = \frac{(i+c)!}{(i+d)!} X^{\overline{K_{i+d}}}_{\overline{K_n}}$$

We have shown that any set of H-chromatic symmetric functions of $\overline{K_n}$ that is of size $\dim(\Lambda^n) = p(n)$ contains at least 2 linearly dependent H-chromatic symmetric functions of $\overline{K_n}$.

It follows that we cannot create a basis for Λ^n using H-chromatic symmetric functions of a fixed G. □

But is it possible to generate a basis for Λ^n using a set of H-chromatic symmetric functions where the H is fixed rather than the G?

It turns out the answer is yes. As discussed in Section 3, when you choose $H = K_n$, then you can recover the chromatic symmetric functions up to a scalar multiple. Cho and van Willigenburg showed that $\{X_{G_\lambda} | \lambda = (\lambda_1, \lambda_2 \dots \lambda_k) \vdash n\}$ forms a basis of Λ [3]. Here, G_i is a connected graph on i vertices and $G_\lambda = G_{\lambda_1} \cup G_{\lambda_2} \dots \cup G_{\lambda_k}$.

But it is still unknown if we can create a basis for Λ^n using H-chromatic symmetric functions involving a fixed H and varying G, where H is not a complete graph.

So far, we have primarily considered cases where G, H are simple graphs. The proof of Proposition 74 depends on the fact that if G has an edge connecting two distinct vertices, it is not possible to H-color all the vertices of G the same color. However, if H could have a loop, this assumption is not true. An interesting question to explore is whether we can create a basis for Λ^n using H-chromatic symmetric functions of a fixed G where G, H do not necessarily have to be simple graphs.

9.2 Self-distinguishability of graphs

Given a graph G, the symmetric function X_G^G never vanishes. A natural question to ask is if this "self"-chromatic symmetric function can be used to distinguish all graphs up to isomorphism. It is known that two non-isomorphic graphs can have the same "self"-chromatic symmetric functions. For example, if $G_1 = K_1 \uplus K_2$ and $G_2 = P_3$, then $X_{G_1}^{G_1} = X_{G_2}^{G_2} = 12m_{(1,1,1)} + 4m_{(2,1)}$.

Nevertheless, if we restrict our attention to a smaller class of graphs, then there is more we can ask. For example, we have checked that up to $k = 9$, two trees on k vertices are self-distinguishable. Of course, when two trees have a different number of automorphisms, then they must have distinct "self"-chromatic symmetric functions by Proposition 22. However, this condition is by no means necessary. We shall propose the following conjecture.

Conjecture 75. Let T_1, T_2 be two non-isomorphic trees, then $X_{T_1}^{T_1} \neq X_{T_2}^{T_2}$.

9.3 Finite and Uniform distinguishability in a more general setting

In Section 4, we have seen that any finite set of graphs is uniformly distinguishable. We have also seen that not every set of graphs is uniformly, or even finitely, distinguishable. When our G's are connected, we also proved that finite distinguishability implies uniform distinguishability when a uniform bound on the coefficients of the H-chromatic symmetric function can be established.

However, we have not shown whether these results are sharp. Therefore, we may wish to ask the following questions.

Problem 76. Is finite distinguishability equivalent to uniform distinguishability? If not, can we pose restrictions on G's, such as connectedness, to make it true?

Problem 77. Is there a simple necessary and sufficient condition for a set of graph being finitely (uniformly) distinguishable? For example, we have shown that a set of graphs whose chromatic numbers are not bounded cannot be finitely distinguished. Is the converse true?

9.4 Monotonicity with respect to a basis

One of the big open questions in chromatic symmetric function theory concerns e-positivity and Schur-positivity of the chromatic symmetric functions (e.g. [6, 7, 14]) Clearly, we can ask similar questions here. Furthermore we say a symmetric function is *monotone* with respect to a basis b if and only if its coefficients in the b-expansion are all non-negative or all non-positive.

Problem 78. Find some necessary or sufficient (or both) conditions on G and H for X_G^H to be e-positive (monotone) or Schur-positive (monotone).

People are less interested about p-positivity of chromatic symmetric functions since it is known that $\omega(X_G)$ is always p-positive, where ω is the standard involution on Λ [15]. This fact, however, does not generalize to H-chromatic symmetric functions. For instance,

$$\omega(X_{P_4}^{C_7}) = 840p_{(4)} - 672p_{(3,1)} + 1260p_{(2,2)} - 168p_{(2,1,1)} + 84p_{(1,1,1,1)}.$$

Nevertheless, the result does hold when H is a star.

Conjecture 79. Let $n \geqslant 1$. Suppose $H = S_{n+1}$ is a star. Then, $\omega(X_G^H)$ is p-monotone for every graph G.

This is trivial when G is not a bipartite graph, in which case the symmetric function vanishes. We have also shown the result when G is a connected bipartite and n is sufficiently large. Indeed, we have the following formula that can be seen with little effort.

Proposition 80. *Suppose G is a connected bipartite graph and $H = S_{n+1}$. Assume $n \geqslant k_1, k_2$, where k_1, k_2 are the sizes of the two parts of G. If we set*

$$(X_1)_G^H := \sum_{k_1 \leqslant j \leqslant k_1+k_2} (-1)^{j-k_1} n! \binom{k_2}{j-k_1} p_{(j,1^{k_1+k_2-j})},$$

and

$$(X_2)_G^H := \sum_{k_2 \leqslant j \leqslant k_1+k_2} (-1)^{j-k_2} n! \binom{k_1}{j-k_2} p_{(j,1^{k_1+k_2-j})}.$$

Then, $X_G^H = (X_1)_G^H + (X_2)_G^H$.

From there, we can prove the result when G is a general bipartite graph and the sizes of the two parts of each connected component of G have the same parity, and that n is sufficiently large. However, a further generalization seems non-trivial.

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