The Density of Fan-Planar Graphs

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Abstract

A topological drawing of a graph is fan-planar if for each edge e the edges crossing e form a star and no endpoint of e is enclosed by e and its crossing edges. A fan-planar graph is a graph admitting such a drawing. Equivalently, this can be formulated by three forbidden patterns, one of which is the configuration where e is crossed by two independent edges and the other two where e is crossed by two incident edges in a way that encloses some endpoint of e. A topological drawing is simple if any two edges have at most one point in common.

Fan-planar graphs are a new member in the ever-growing list of topological graphs defined by forbidden intersection patterns, such as planar graphs and their generalizations, Turán-graphs and Conway's thrackle conjecture. Hence fan-planar graphs fall into an important field in combinatorial geometry with applications in various areas of discrete mathematics. As every 1-planar graph is fan-planar and every fan-planar graph is 3-quasiplanar, they also fit perfectly in a recent series of works on nearly-planar graphs from the areas of graph drawing and combinatorial embeddings.

In this paper we show that every fan-planar graph on n vertices has at most $5n - 10$ edges, even though a fan-planar drawing may have a quadratic number of crossings. Our bound, which is tight for every $n \geqslant 20$, indicates how nicely fan-planar graphs fit in the row with planar graphs ($3n - 6$ edges) and 1-planar graphs ($4n - 8$ edges). With this, fan-planar graphs form an inclusion-wise largest non-trivial class of topological graphs defined by forbidden patterns, for which the maximum number of edges on n vertices is known exactly.

We demonstrate that maximum fan-planar graphs carry a rich structure, which makes this class attractive for many algorithms commonly used in graph drawing. Finally, we discuss possible extensions and generalizations of these new concepts.

Mathematics Subject Classifications: 05C10, 68R10

1 Introduction

Planarity of a graph is a well-studied concept in graph theory, computational geometry and graph drawing. The famous Euler formula characterizes for a certain embedding the relation between vertices, edges and faces, and many different algorithms e.g. [43] following different objectives have been developed to compute appropriate embeddings in the plane.

Because of the importance of planarity, a series of generalizations have been developed in the past. Topological graphs and topological drawings[1] respectively are being considered, i.e., the vertices are drawn as points in the plane and the edges drawn as Jordan curves between corresponding points without any other vertex as an interior point. In [30], the authors state "Finding the maximum number of edges in a topological graph with a forbidden crossing pattern is a fundamental problem in extremal topological graph theory" together with 9 citations from a large group of authors. In this paper we define fan-planar graphs as those admitting a topological drawing in which each edge is only allowed to be crossed by a fan of other edges. Equivalently, we forbid three crossing patterns as shown in Figure 1, one of which is an edge crossed by two independent edges. As our main result we show that every n-vertex planar graph has at most $5n - 10$ edges, and that this bound is tight for $n \geqslant 20$.

Most of the existent literature considers topological drawings that are *simple*, i.e., where any two edges have at most one point in common. In particular, two edges may not cross more than once and incident edges may not cross at all. We shall consider simple topological graphs only. Indeed, we shall argue in Section 4 that if we drop this assumptions and allow non-homeomorphic parallel edges, then even 3-vertex fan-planar graphs have arbitrarily many edges.

Related work. Most notably, there are k-planar graphs [37] and k-quasiplanar graphs [5]. A k-planar graph admits a topological drawing in which no edge is crossed more than k times by other edges, while a k-quasiplanar graph admits a drawing in which no k edges

[1]Formally, *topological graphs* and *topological drawings* are synonym and we use the term drawing only for emphasis.

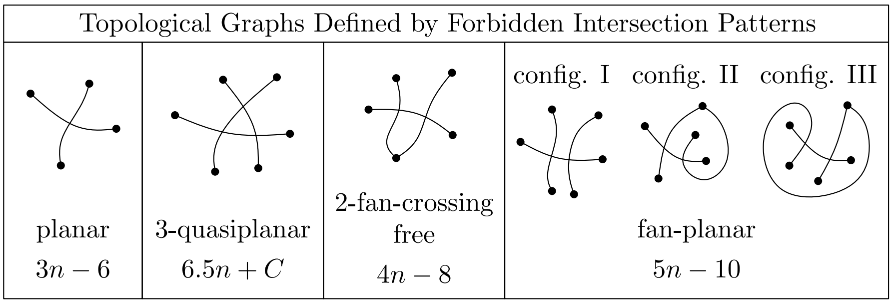


Figure 1: Topological graphs defined by forbidden patterns and the corresponding maximum number of edges in an n-vertex such graph.

pairwise cross each other.

The topic of k-quasiplanar graphs is almost classical [20]. A famous conjecture [20] states that for constant k the maximal number of edges in k-quasiplanar graphs is linear in the number of vertices. Note that 2-quasiplanar graphs correspond to planar graphs. A first linear bound for $k = 3$, i.e., 3-quasiplanar graphs, appeared in [5] and was subsequently improved in [37]. The current best bound for 4-quasiplanar graphs is $76(n-2)$ [1]. For the general case, the bounds have been gradually improved from $O(n(\log n)^{O(\log k)})$ [37], to $O(n \log n \cdot 2^{\alpha(n)^c})$ [39]. Better upper bounds are known if we restrict ourselves to simple topological drawings, i.e., where each pair of edges intersects at most once. In that case 3-quasiplanar graphs have at most $6.5n + O(1)$ edges [4], while k-quasiplanar graphs with any fixed $k \geqslant 2$ have at most $O(n \log n)$ edges [39]. Despite significant efforts, it remains open, whether the conjecture holds for general k, that is, whether k-quasiplanar graphs have at most linearly many edges.

A k-planar graph admits a topological drawing in which each edge has at most k crossings. The special case of 1-planar graphs were introduced by Ringel [38], who considered the chromatic number of these graphs. There is important work about the characterization on 1-planar graphs due to Suzuki [40], Thomassen [42] and Hong *et al.* [32]. Concerning computational questions, it is known that testing 1-planarity is NP-complete for the general case [31] while there are efficient algorithms for testing 1-planarity for a given rotation system [28]. Among the rich literature on k-planar graphs, let us also mention aspects like straight-line embeddings [6] and maximality [9, 19].

Closely related to 1-planar graphs are RAC-drawable graphs [8, 29], that is graphs that can be drawn in the plane with straight-line edges and right-angle crossings. For the maximum number of edges in such a graph with n vertices, a bound of $4n - 10$ has been proven [25], which is remarkably close to the $4n-8$ bound for the class of 1-planar graphs. A necessary condition for RAC-drawable graph is the absence of fan-crossings. An edge has a k-fan-crossing if it crosses k edges that have a common endpoint, cf. Figure 1. RAC-drawings do not allow 2-fan-crossings. Generalizing RAC-drawings, Cheong *et al.* [21] considered k-fan-crossing free graphs and gave bounds for their maximum number of edges. They obtain a tight bound of $4n - 8$ for n-vertex 2-fan-crossing free graphs, and a tight $4n - 9$ when edges are required to be straight-line segments. For $k > 2$, they prove an upper bound of $3(k-1)(n-2)$ edges, while all known examples of k-fan-crossing free graphs on n vertices have no more than kn edges.

Our results and more related work. As stated before we consider only simple topological drawings, i.e., any two edges have at most one point in common. We mostly consider only simple graphs, i.e., graphs without self-loops and parallel edges, and explicitly write *multigraph* otherwise. We consider here another variant of sparse non-planar graphs, somehow halfway between 1-planar graphs and quasiplanar graphs, where we allow more than one crossing on an edge e, but only if no endpoint of e is enclosed by e and its crossing edges. We call this a *fan-crossing* and the class of topological graphs obtained this way *fan-planar graphs*. Note that we do not differentiate on k-fan-crossings as it has been done by Cheong *et al.* [21].

The requirement that every edge in G is crossed only by a fan-crossing can be stated in terms of forbidden configurations. We define *configuration I* to be one edge that is crossed by two independent edges, and *configuration II*, respectively *configuration III*, to be an edge e that is crossed by adjacent edges, such that the triangle (highlighted in figure) formed by the three segments between the crossing points and the common endpoint encloses one of the endpoints of e, respectively both endpoints of e, see Figure 2. Note that for simple topological drawings, configurations II and III are well-defined. Now a simple topological graph is fan-planar if and only if neither configuration I nor II nor III occurs. Note that if we forbid only configurations I and III, then an edge may be crossed by the three edges of a triangle, which is actually not a star, nor a fan-crossing. However, if every edge is drawn as a straight-line segment, then neither configuration II nor III can occur and hence in this case it is enough to forbid configuration I.

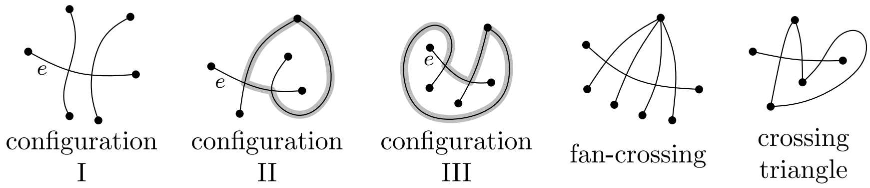


Figure 2: Crossing configurations

Obviously, 1-planar graphs are also fan-planar. Furthermore, fan-planar graphs are 3-quasiplanar since there are no three independent edges that mutually cross. So, the maximum number of edges in an n-vertex fan-planar graph is between $4n$ and $6.5n$. In the following, we will explore the exact bound.

Theorem 1. *Every simple fan-planar graph on $n \geqslant 3$ vertices has at most $5n - 10$ edges. This bound is tight for $n \geqslant 20$.*

We remark that fan-planar drawings graphs may have $\Omega(n^2)$ crossings, e.g., a straight-line drawing of $K_{2,n}$ with the bipartition classes placed on two parallel lines.

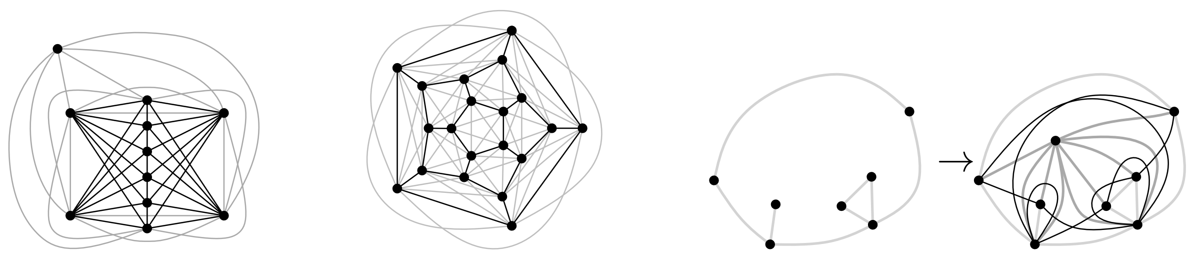
Very closely related to our approach is the research on forbidden grids in topological graphs, where a (k, l) grid denotes a k-subset of the edges pairwise intersected by an l-subset of the edges, see [36] and [41]. It is known that topological graphs without (k, l) grids have a linear number of edges if k and l are fixed. Note that configuration I, but also a 2-fan-crossing, is a $(2, 1)$ grid. Subsequently [2], "natural" (k, l) grids have been considered, which have the additional requirement that the k edges, as well as the l edges, forming the grid are pairwise disjoint. For natural grids, the achieved bounds are superlinear. Linear bounds on the number of edges have been found for the special case of forbidden natural $(k, 1)$ grids where the leading constant heavily depends on the parameter k. In particular, the authors give a bound of $65n$ for the case of forbidden natural $(2, 1)$ grids, which correspond to our forbidden configuration I. Additionally, the case of geometric graphs, i.e., graphs with straight-line edges, has been explored. For details and differences we refer to [2]. We remark that many arguments in this field of research are based on the probabilistic method, while we use a direct approach aiming on tight upper and lower bounds.

Remark. This paper initiates the study of fan-planar graphs. In fact, the first preliminary version of this paper dates back to 2014 [34]. Since then, fan-planarity has become a very popular subject of study among several researchers [3, 7, 10, 12–18, 22, 23, 33]. The interested reader may also have a look at the recent surveys on fan-planarity [11] or general beyond-planar graphs [26].

Note that in our preliminary version, we excluded configurations I and II only, believing that this would force every edge to be crossed only by a fan-crossing. Just recently, we have been contacted by the authors of [35] with a counterexample to our Lemma 3 in the original context. That example (see Figure 4(c) for a simpler such example) however contains what we now call configuration III, and is thus not fan-planar for its intended definition. Lemma 3 in fact holds true for every fan-planar graph — We now mention explicitly where the absence of configuration III is used in its proof.

2 Examples of Fan-Planar Graphs with Many Edges

The following examples of fan-planar graphs have n vertices and $5n - 10$ edges. The first one results from a $K_{4,n-5}$, where the $n - 5$ vertices form a path, see Figure 3(a). An easy calculation shows that this graph has $n - 1$ vertices and $4(n - 5) + (n - 6) = 5(n - 1) - 21$ edges. Indeed, one can add 10 edges to the graph, keeping fan-planarity, as well as additionally one vertex with 6 more incident edges and obtain a multigraph on n vertices and $5n - 10$ edges. We remark that this graph has parallel edges; however every pair of parallel edges is non-homeomorphic, that is, it surrounds at least one vertex of G. The second example is the (planar) dodecahedral graph where in each 5-face, we draw 5 additional edges as a pentagram, see Figure 3(b). This graph has $n = 20$ vertices and $5n - 10 = 90$ edges, and has already served as a tight example for 2-planar graphs [37].



(a) (b) (c)

Figure 3: (a) $K_{4,n-4}$ with $n - 4$ vertices on a path. (b) The dodecahedral graph with a pentagram in each face. (c) Adding 2-hops and spokes into a face.

Similarly to the idea with the dodecahedral graph, we can construct arbitrarily large n-vertex fan-planar graphs with $5n - 10$ by starting with an underlying planar graph and adding crossing edges and a vertex to each of its faces.

Proposition 2. *Every connected planar embedded multigraph H with each face of length at least 3 can be extended to a fan-planar multigraph G with $5|V(G)| - 10$ edges by adding* *an independent set of vertices and sufficiently many edges, such that the uncrossed edges of G are precisely the edges of H.*

Moreover, if H is 3-connected and each face has length at least 5, then G has no loops and parallel edges.

Proof. Let n and m be the number of vertices and edges of H, respectively, and F be the set of all faces of H. We construct the fan-planar graph G by adding one vertex and two sets of edges into each face $f \in F$. So let f be any face of H. Since H is connected, the boundary of f corresponds to a single closed walk $v_1, \ldots, v_s$, $s \geqslant 3$, in H around f, where vertices and edges may be repeated. We do the following, as illustrated in Figure 3(c).

(1) Add a new vertex v_f into f.

(2) For $i = 1, \ldots, s$ add a new edge $v_f v_i$ drawn in the interior of f.

(3) For $i = 1, \ldots, s$ add a new edge $v_{i-1}v_{i+1}$ (with indices modulo s) crossing the edge $v_f v_i$.

In (1) we added $|F|$ new vertices. In (2) we added $\deg(f)$ many "spoke edges" inside face f, in total $\sum_f \deg(f) = 2m$ new edges. And in (3) we added again $\deg(f)$ many "2-hop edges" inside face f, in total $\sum_f \deg(f) = 2m$ new edges. Thus we calculate $|V(G)| = n + |F|$ and $|E(G)| = m + 2m + 2m = 5m$, which together with Euler's formula $m = n + |F| - 2$ gives $|E(G)| = 5|V(G)| - 10$. It remains to see that no two edges in G are homeomorphic, and that G is fan-planar. Each "2-hop edge" $e = v_{i-1}v_{i+1}$ forms a shortcut for a path $v_{i-1} - v_i - v_{i+1}$ on the face f. Suppose some edge e' in G is parallel to e. In case e' is also a 2-hop edge in face f, then vertex v_f is surrounded by e and e'. Otherwise e' is an edge of H or a 2-hop edge of some other face, and either vertex v_f or vertex v_i is surrounded by e and e'.

For the fan-planarity of G, observe that each "spoke edge" $v_f v_i$ crosses only one 2-hop edge, and each 2-hop edge $v_{i-1}v_{i+1}$ crosses only three edges $v_{i-2}v_i$, $v_f v_i$ and $v_i v_{i+2}$, which form a fan-crossing.

Finally, note that if the planar graph H is 3-connected and each face has length at least 5, then the fan-planar graph G has no loops, nor parallel edges, nor crossing incident edges. Examples for such planar graphs are fullerene graphs [27]. Moreover, for every face f in H and the corresponding vertex v_f in G we have $\deg(f) = \deg(v_f)$. So, if every face in H has degree exactly 5 we can omit all vertices in step (1) and edges in (2) and obtain a fan-planar graph G' with $V(G') = V(H)$ and $5|V(H)| - 10$ edges. □

3 The $5n - 10$ Upper Bound For the Number of Edges

In this section Theorem 1 is proven. It suffices to consider simple topological graphs G that do not contain configuration I nor II nor III and further satisfy the following properties.

(i) The chosen embedding of G has the maximum number of uncrossed edges.

(ii) The addition of any edge to the given embedding violates the fan-planarity of G, that is, G is maximal fan-planar with respect to the given embedding.

So for the remainder of this paper let G be a maximal fan-planar graph with a fixed fan-planar embedding with the maximum number of uncrossed edges. Recall that the embedding of G is simple, i.e., any two edges have at most one point in common.

For such a fixed embedding of G we shall split the edges of G into three sets. The first set contains all uncrossed edges. We denote by H the subgraph of G with all vertices in V and all uncrossed edges of G. We may refer to H as the *planar subgraph of* G. Note that H might be disconnected even if G is connected. In the second set we consider every crossed edge whose endpoints lie in the same connected component of H. Note that two such edges may cross each other only if they correspond to the same connected component of H. And the third set contains all remaining edges, i.e., every crossed edge with endpoints in different components of H. We show how to count the edges in each of the sets and derive the upper bound.

3.1 Notation, definitions and preliminary results

We call a connected component of the plane after the removal of all vertices and edges of G a *cell of* G. Whenever we consider a subgraph of G we consider it together with its fan-planar embedding, which is inherited from the embedding of G. We will sometimes consider cells of a subgraph G' of G, even though those might contain vertices and edges of $G - G'$. The boundary of each cell c is composed of a number of edge segments and some (possibly none) vertices of G'. With slight abuse of notation we call the cyclic order of vertices and edge segments along c the *boundary of* c, denoted by ∂c. Note that vertices and edges may appear more than once in the boundary of a single cell. We define the *size of a cell* c, denoted by $||c||$, as the total number of vertices and edge segments in ∂c counted with multiplicity.

Note that by assumptions (i) and (ii) it follows that if two vertices are in the same cell c of G then they are connected by an uncrossed edge of G. However, this uncrossed edge does not necessarily bound cell c.

Lemma 3. *If two edges vw and ux cross in a point p, no edge at v crosses ux between p and u, and no edge at x crosses vw between p and w, then u and w are contained in the same cell of G.*

Proof. Let $e_0 = ux$ and $e_1 = vw$ be two edges that cross in point $p = p_1$ such that no edge at v crosses e_0 between p_1 and u, and no edge at x crosses e_1 between p_1 and w. If no edge of G crosses e_0 nor e_1 between p_1 and u, respectively w, then clearly u and w are bounding the same cell. So assume without loss of generality that some edge of G crosses e_1 between p_1 and w. By fan-planarity (specifically by the absence of configuration I) such edges are incident to u or x, where the latter is excluded by assumption. Let e_2 be the edge whose crossing with e_1 is closest to w, and let p_2 be the crossing point. As neither configuration II nor configuration III occurs, w is not surrounded by e_0, e_1, e_2. See

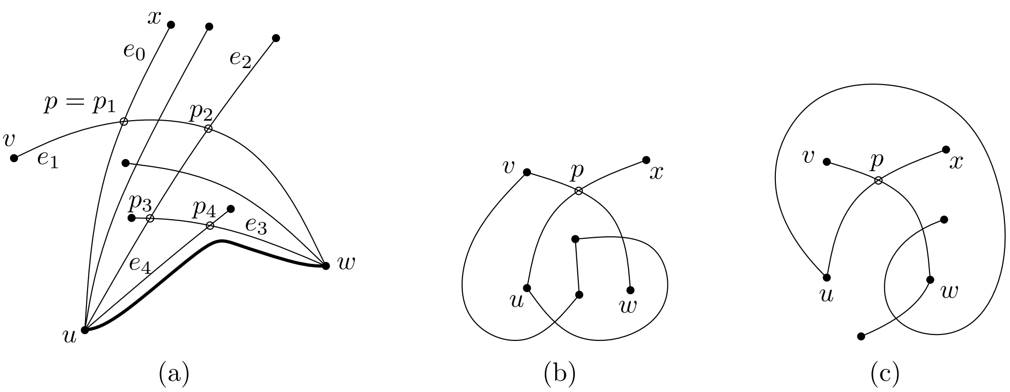


Figure 4: (a) Illustration of the proof of Lemma 3. (b) A counterexample to Lemma 3 when configuration II is allowed. (c) A counterexample to Lemma 3 when configuration III is allowed.

Figure 4(a) for an illustration of the actual situation, and Figures 4(b) and 4(c) for the potential problematic situations if configuration II or III was allowed.

No edge crosses e_1 between w and p_2. If e_2 is not crossed between u and p_2, then u and w are bounding the same cell and we are done. Otherwise let e_3 be the edge whose crossing with e_2 is closest to u, and let p_3 be the crossing point. By fan-planarity (the absence of configuration I) e_3 and e_1 have a common endpoint, and it is not v since e_3 does not cross e_0 between p_1 and u. (Here we use the absence of configuration II again.) So e_3 ends at w and by the absence of configuration III, u is not surrounded by e_1, e_2, e_3. By the choice of e_3, we have that e_2 is not crossed between u and p_3. Again, if u and w are not on the same cell then some edge crosses e_3 between p_3 and w. By fan-planarity (the absence of configuration I) any such edge e_4 has a common endpoint with e_2, and if it would not be u then either e_1 would be crossed by two independent edges (a configuration I occurs) or one v, w would be surrounded by e_2, e_3, e_4 (a configuration II or III occurs) – a contradiction to the fan-planarity of G. So all edges crossing e_3 between w and p_3 are incident to u. Let e_4 be such edge whose crossing with e_3 is closest to w, and let p_4 be the crossing point. Again, by absence of configuration III, v and w are not surrounded by e_2, e_3, e_4, cf. Figure 4(a) for an illustration.

Iterating this procedure until no edge crosses e_i nor e_{i-1} between p_i and u, w we see that u and w lie indeed on the same cell, which concludes the proof. □

Note that we use the absence of both configuration II and configuration III in the proof of Lemma 3. And indeed, as illustrated in Figure 4(b) and 4(c), the statement of the lemma is no longer true if configuration II or configuration III may occur. For better readability of the remainder, we shall from now on just argue "by fan-planarity" without each time explicitly referring to the specific forbidden configurations. Conclusively, in a fan-planar drawing each edge is either uncrossed or crossed by a fan.

Lemma 3 has a couple of nice consequences.

Corollary 4. *Any two crossing edges in G are connected by an uncrossed edge.*

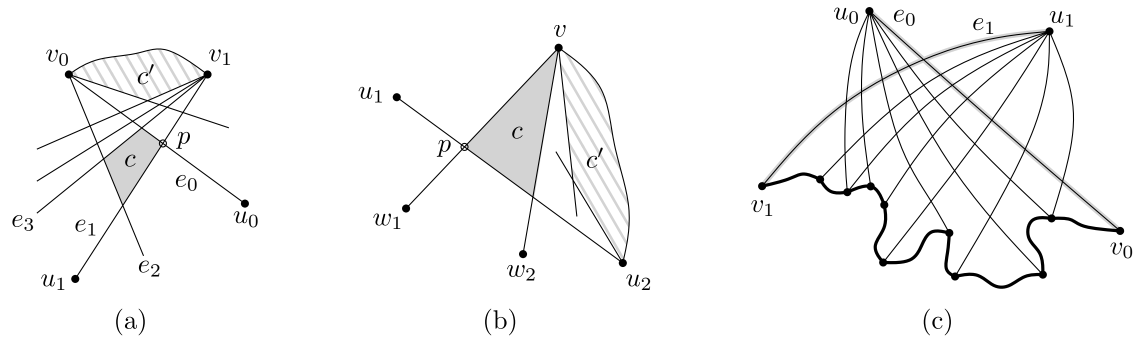


Figure 5: Illustration of the proofs of (a), (b) Corollary 5 and (c) Corollary 6.

Proof. Let ux and vw be the two crossing edges. By fan-planarity either no other edge at x or no other edge at u crosses the edge vw, say there is no such edge at x. Similarly, we may assume without loss of generality that no edge at v crosses the edge ux. However, this implies that ux and vw satisfy the requirements of Lemma 3 and we have that u and w are on the same cell. In particular, we can draw an uncrossed edge between u and w in this cell. Because G is maximally fan-planar, uw is indeed an edge of G. And since G is embedded with the maximum number of uncrossed edges, uw is also drawn uncrossed. □

Corollary 5. *If c is a cell of any subgraph of G, and $||c|| = 4$, then c contains no vertex of G in its interior.*

Proof. Let c be a cell of $G' \subseteq G$ with $||c|| = 4$. Then ∂c consists either of four edge segments or one vertex and three edge segments. Let us assume for the sake of contradiction that c contains a set $S \neq \emptyset$ of vertices in its interior.

Case 1. ∂c consists of four edge segments. Let e_0, e_1, e_2, e_3 be the edges bounding c is this cyclic order. From the fan-planarity of G it follows that e_0 and e_2 have a common endpoint v_0. Similarly e_1 and e_3 have a common endpoint v_1. See Figure 5(a) for an illustration. If p denotes the crossing point of $e_0 = v_0u_0$ and $e_1 = v_1u_1$, then by fan-planarity no edge at u_i crosses e_{i+1} between p and v_{i+1}, where $i \in \{0, 1\}$ and indices are taken modulo 2. Hence by Lemma 3 there exists a cell c' of G that contains both v_0 and v_1.

Now consider the subgraph $G[S]$ of G on the vertices inside c. From the fan-planarity it follows that every edge between $G[S]$ and $G[V \setminus S]$ has as one endpoint v_0 or v_1. We now change the embedding of G by placing the subgraph $G[S]$ (keeping its inherited embedding) into the cell c' that contains v_0 and v_1. The resulting embedding of G is still fan-planar and moreover at least one edge between $G[S]$ and $\{v_0, v_1\}$ is now uncrossed – a contradiction to our assumption (i) that the embedding of G has the maximum number of uncrossed edges.

Case 2. ∂c consists of one vertex and three edge segments. Let v be the vertex and vw_1, vw_2, u_1u_2 be the edges bounding c. See Figure 5(b) for an illustration. If p denotes the crossing point of vw_1 and u_1u_2, then by fan-planarity either no edge at u_1 crosses vw_1

between p and v or no edge at u_2 crosses vw_1 between p and v. Moreover, for $i = 1, 2$ the edge vw_i is the only edge at w_i that crosses u_1u_2. Hence by Lemma 3 we have that either v and u_1 or v and u_2 are contained in the same cell of G – say cell c' contains v and u_2.

Now, similarly to the previous case, consider the subgraph $G[S]$ of G on the vertices inside c. From the fan-planarity, it follows that every edge between $G[S]$ and $G[V \setminus S]$ has as one endpoint v, u_1 or u_2. Moreover, every edge between a vertex in $G[S]$ and u_1 or u_2 is crossed only by edges incident to v, as otherwise u_1u_2 would be crossed by two independent edges. We now change the embedding of G by placing the subgraph $G[S]$ (keeping its inherited embedding) into the cell c' that contains v and u_2. The resulting embedding of G is still fan-planar and moreover at least one edge between $G[S]$ and u_2 is now uncrossed – a contradiction to (i). □

Corollary 6. *If $e_0 = u_0v_0$ and $e_1 = u_1v_1$ are two crossing edges of G such that every edge of G crossing e_i is crossed only by edges incident to u_{i+1}, where $i \in \{0, 1\}$ and indices are taken modulo 2, then v_0 and v_1 are in the same connected component of the planar subgraph H of G.*

Proof. Let p be the point in which e_0 and e_1 cross. For $i = 0, 1$ let S_i be the set of all edges crossing e_{i+1} between p and v_{i+1}. (All indices are taken modulo 2.) By assumption S_i is a star centered at u_i. Consider the embedding of the graph $S_0 \cup S_1$ inherited from G. By fan-planarity (specifically, the absence of both, configuration I and II) u_0 and u_1 are contained in the outer cell of $S_0 \cup S_1$. Moreover, every inner cell c of $S_0 \cup S_1$ has $||c|| = 4$ and thus by Corollary 5 all leaves of S_0 and S_1 are also contained in the outer cell c^* of $S_0 \cup S_1$.

We claim that no edge segment in the boundary ∂c^* of the outer cell is crossed by another edge in G. Indeed, if e' is an edge crossing some edge $e \in S_0 \cup S_1$ between the crossing of e and e_0 or e_1 and the endpoint of e different from u_0, u_1, then by assumption one endpoint of e' is u_0 or u_1 – say u_1. Moreover, since by Corollary 5 no cell c with $||c|| = 4$ contains any vertex, we have that e' crosses e_0 between p and v_0 and thus $e \in S_1$. See Figure 6(b).

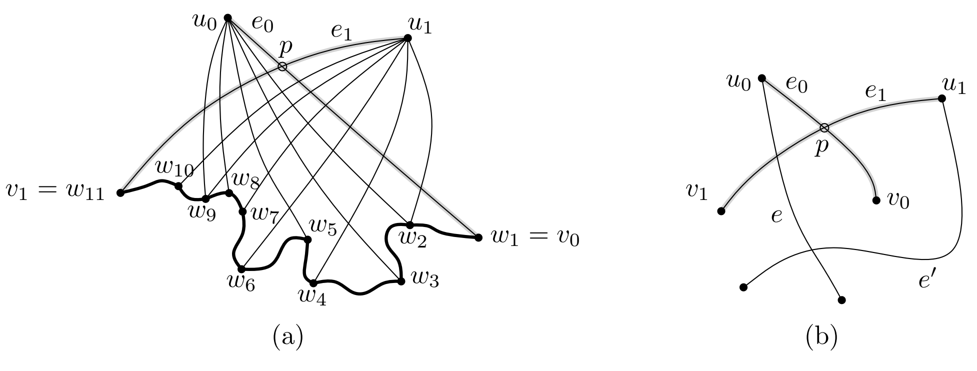


Figure 6: (a) The stars S_0 and S_1 in the proof of Corollary 6. (b) If an edge e' crosses $e \in S_0$ between the crossing of e and e_1 and the endpoint of e different from u_0, and $e' \notin S_1$, then v_0 is contained in a cell c bounded by e, e' and e_1 with $||c|| = 4$.

We conclude that if we label the vertices of $S_0 \cup S_1$ such that their cyclic order around c^* is $u_0, u_1, v_0 = w_1, w_2, \ldots, w_k = v_1$, then for each $j \in \{1, \ldots, k-1\}$ the vertices w_j and w_{j+1} are contained in the same cell of G and hence by maximality of G joint by an uncrossed edge. See Figure 6(a) for an illustration. □

Recall that H denotes the planar subgraph of G. For convenience we refer to the closure of cells of H as the *faces of* G. The boundary of a face f is a disjoint set of (not necessarily simple) closed walks in H, which we call *facial walks*. The *length of a facial walk* W, denoted by $|W|$, is the number of its edges counted with multiplicity. We remark that a facial walk may consist of only a single vertex, in which case its length is 0. See Figure 7(a) for an example.

For a face f and a facial walk W of f, we define $G(W)$ to be the subgraph of G consisting of the walk W and all edges that are drawn entirely inside f and have both endpoints on W. The set of cells of $G(W)$ that lie inside f is denoted by $C(W)$. Finally, the graph $G(W)$ is called a *sunflower* if $|W| \geqslant 5$ and $G(W)$ has exactly $|W|$ inner edges each of which connects two vertices at distance 2 on W. See Figure 7(b) for an example of a sunflower. We remark that for convenience we depict facial walks in our figures as simple cycles, even when there are repeated vertices or edges. Indeed, we can assume facial walks to be simple cycles as long as we bound the number of edges in terms of the length of facial walks and sizes of cells. Only in the final proof of Theorem 1 we bound the number of edges in terms of the number of vertices, and there those repetitions will be taken into account by Euler's formula.

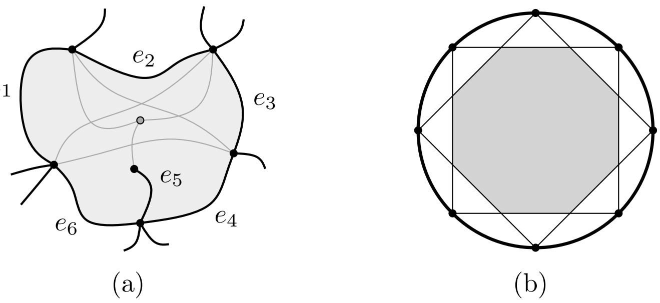


Figure 7: (a) A cell of H (drawn black) is shown in gray. The boundary of the cell is the union of the closed walk $e_1, e_2, e_3, e_4, e_5, e_5, e_6$ and the single vertex in the interior. (b) A sunflower on 8 vertices. The facial walk W is drawn thick. A cell bounded by 8 edge segments and no vertex is highlighted.

3.2 Counting the Number of Edges

We shall count the number of edges of G in three sets:

- Edges in H, that is all uncrossed edges.
- Edges in $E(G(W)) \setminus E(W)$ for every facial walk W.

- Edges between different facial walks of the same face f of G.

The edges in H will be counted in the final proof of Theorem 1 below. We start by counting the crossed edges, first within the same facial walk and afterwards between different facial walks. For convenience, for a facial walk W the edges in $E(G(W)) \setminus E(W)$ and their edge segments are called *inner edges* and *inner edge segments* of $G(W)$, respectively.

Lemma 7. *Let W be any facial walk. If every inner edge segment of $G(W)$ bounds a cell of $G(W)$ of size 4 and no cell of $G(W)$ contains two vertices on its boundary not consecutive in W, then $G(W)$ is a sunflower.*

Proof. Let $v_0, \dots, v_k$ be the clockwise order of vertices around W. (In the following, indices are considered modulo $k+1$.) For any vertex v_i we consider the set of inner edges incident to v_i. Since no two non-consecutive vertices of W lie on the same cell, every v_i has at least one such edge. Moreover, note that for each edge $v_i v_{i+1}$ of W the unique cell c_i with $v_i v_{i+1}$ on its boundary has size at least 5. This implies that every v_i has indeed at least two incident inner edges. Finally, note that every inner edge is crossed, since otherwise there would be two non-consecutive vertices of W bounding the same cell of $G(W)$.

Now let us consider the clockwise first inner edge incident to v_i, denoted by e_i^1. Since an edge segment of e_i^1 bounds the cell c_i, there is a cell of size 4 on the other side of this segment. This means that e_i^1 and the clockwise next inner edge at v_i are crossed by some edge e. By fan-planarity e crosses only edges incident to v_i. Thus each endpoint of e bounds together with v_i some cell of $G(W)$. Since only consecutive vertices of W bound the same cell of $G(W)$, this implies that $e = v_{i-1}v_{i+1}$. Since this is true for every $i \in \{0, \dots, k\}$, we conclude that $G(W)$ is a sunflower. □

Recall that $C(W)$ denotes the set of all bounded cells of $G(W)$.

Lemma 8. *For every facial walk W with $|W| \geqslant 3$ we have*

$$|E(G(W)) \setminus E(W)| \leqslant 2|W| - 5 - \sum_{c \in C(W)} \max\{0, ||c|| - 5\}.$$

Proof. Without loss of generality we may assume that W is a simple cycle, since we bound the number of *inner* edges of W in terms of the length of W. We proceed by induction on the number of inner edges. As induction base we consider the case that $E(G(W)) \setminus E(W) = \emptyset$. Then $G(W) = W$ and $C(W)$ consists of a single cell c with $||c|| = 2|W|$. Thus

$$|E(G(W)) \setminus E(W)| = 0 = 2|W| - 5 - (||c|| - 5).$$

So assume that there is at least one inner edge e. First, consider any inner edge segment e^* of e and the two cells $c_1, c_2 \in C(W)$ containing e^* on their boundary. If c^* denotes the cell $c_1 \cup c_2$ of $G(W) \setminus e$, then

$$||c^*|| = ||c_1|| + ||c_2|| - 4$$

and thus

$$\max\{0, ||c^*|| - 5\} = \max\{0, ||c_1|| - 5\} + \max\{0, ||c_2|| - 5\} + x, \qquad (1)$$

where $x = 1$ if $||c_1|| \geqslant 5$ and $||c_2|| \geqslant 5$ and $x = 0$ otherwise.

Now, we shall distinguish three cases: $G(W)$ is a sunflower, some inner edge segment is not bounded by a cell of size 4, and some cell of $G(W)$ contains two vertices on its boundary that are not consecutive in W. By Lemma 7 this is a complete case distinction.

Case 1. $G(W)$ is a sunflower. Then by definition, $G(W)$ has exactly $|W|$ inner edges. Moreover, $C(W)$ contains exactly one cell c of size greater than 4 and for that cell we have $||c|| = |W|$. Thus

$$|E(G(W)) \setminus E(W)| = |W| = 2|W| - 5 - (|W| - 5).$$

Case 2. Some edge segment e^* of some inner edge e bounds two cells c_1, c_2 of size at least 5 *each.* Then applying induction to the graph $G' = G(W) \setminus e$ we get

$$\begin{aligned} |E(G(W)) \setminus E(W)| &= 1 + |E(G') \setminus E(W)| \\ &\leqslant 1 + 2|W| - 5 - \sum_{c \in C(G')} \max\{0, ||c|| - 5\} \\ &\overset{(1)}{\leqslant} 1 + 2|W| - 5 - \sum_{c \in C(W)} \max\{0, ||c|| - 5\} - 1. \end{aligned}$$

Case 3. Some cell of $G(W)$ contains two vertices u, w on its boundary that are not consecutive on W. Note that uw may or may not be an inner edge of $G(W)$. In the latter case we denote by c^* the unique cell that is bounded by u and w. In any case exactly two cells c_1, c_2 of $G(W) \cup uw$ are bounded by u and w and we have $||c^*|| = ||c_1|| + ||c_2|| - 4$, provided c^* exists.

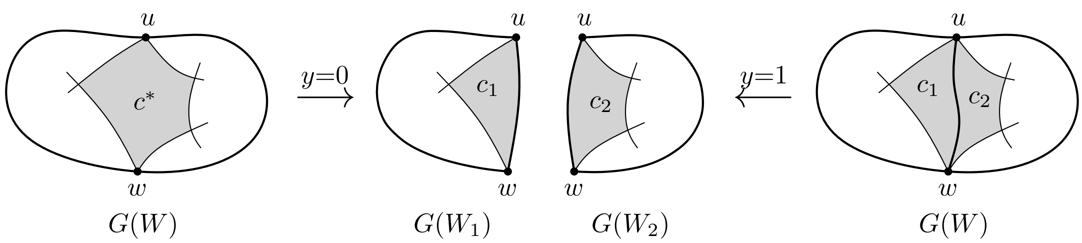


Figure 8: The graph $G(W)$ is split into two graphs $G(W_1)$ and $G(W_2)$ along two vertices u, w that are not consecutive on W but bound the same cell of $G(W)$.

We consider the two cycles W_1, W_2 in $W \cup uw$ that are different from W, such that W_1 surrounds c_1 and W_2 surrounds c_2. For $i = 1, 2$ consider $G(W_i)$, i.e., the subgraph of $G(W) \cup uw$ induced by W_i, see Figure 8. We have

$$\begin{aligned}
|W| &= |W_1| + |W_2| - 2, \\
|E(G(W)) \setminus E(W)| &= |(E(G(W_1)) \setminus E(W_1))| \\
&\quad + |(E(G(W_2)) \setminus E(W_2))| + y, \\
\sum_{c \in C(W)} \max\{0, ||c|| - 5\} &\overset{(1)}{=} \sum_{c \in C(W_1)} \max\{0, ||c|| - 5\} \\
&\quad + \sum_{c \in C(W_2)} \max\{0, ||c|| - 5\} + (1 - y),
\end{aligned}$$

where $y = 1$ if uw already was an inner edge of $G(W)$ and $y = 0$ otherwise. Now, applying induction to $G(W_1)$ and $G(W_2)$ gives the claimed bound. □

Recall that a face f of G is the closure of a cell of the planar subgraph H. The boundary of f consists of uncrossed edges of G forming a number of closed walks – the facial walks of f. Also recall that for such facial walk W we denote by $G(W)$ the subgraph of G on W and all edges with both endpoints on W that lie inside f, by $C(W)$ the set of all cells of $G(W)$. Note that some cell $c \in C(W)$ may contain further vertices of G, and if it does, then some other facial walk of f lies in c. This cell c might be the unbounded cell of $G(W)$. Conversely, if W_1, W_2 are two distinct facial walks for the same face f, then $G(W_1)$ is completely contained in a unique cell $c_1 \in C(W_2)$ and $G(W_2)$ is completely contained in a unique cell $c_2 \in C(W_1)$.

Now let us define by $C(f)$ the union of $C(W)$ for all facial walks W of f. Moreover, we partition $C(f)$ into $C_\emptyset(f)$ and $C_*(f)$, where a cell $c \in C(f)$ lies in $C_\emptyset(f)$ if and only if $(c \setminus \partial c) \cap V(G) = \emptyset$. I.e., cells in $C_\emptyset(f)$ do not have any vertex of G (equivalently, no other facial walk of f) in their open interior, whereas cells in $C_*(f)$ contain some vertex of G (equivalently, at least one other facial walk of f) in their interior. Without loss of generality we have that for each f, $C_*(f)$ is either empty or contains at least one bounded cell. This can be achieved by picking a cell of G that has the maximum number of surrounding Jordan curves of the form ∂c for $c \in \bigcup_f C_*(f)$, and defining it to be in the unbounded cell of G.

Before we bound the number of edges between different facial walks of f we need one more lemma. Consider a face f of G with at least two facial walks and a cell $c \in C_*(f)$ that is inclusion-minimal. Let W_1 be the facial walk with $c \in C(W_1)$ and $W_2, \ldots, W_k$ be the facial walks that are contained in c. For $i = 1, \ldots, k$ let c_i be the cell of $G(W_i)$ that contains all walks W_j with $j \neq i$. In particular, we have $c_1 = c$. Moreover, we call an edge between two distinct facial walks W_i and W_j a *W_iW_j-edge*.

Lemma 9. *Exactly one of $c_1, \ldots, c_k$ has a vertex on its boundary.*

Proof. We proceed by proving a series of claims first.

Claim 1. If a W_iW_j-edge and a $W_{i'}W_{j'}$-edge cross, then $\{i, j\} = \{i', j'\}$.

Proof of Claim. Consider a W_iW_j-edge $e_1 = u_1v_1$ crossing a $W_{i'}W_{j'}$-edge $e_2 = u_1v_2$. By Corollary 4 one endpoint of e_1, say $u_1 \in W_i$, and one endpoint of e_2, say $u_2 \in W_{i'}$, are joint by an uncrossed edge. In particular, $W_i = W_{i'}$.

If, Case 1, e_1 is crossed by a second edge incident to v_2, then applying Lemma 3 gives an uncrossed edge u_1v_2, which is a contradiction to the fact that $W_{j'} \neq W_{i'}$, or an uncrossed edge v_1v_2, which implies $W_j = W_{j'}$ as desired. See Figure 9(a).

Otherwise, Case 2, e_1 is crossed only by edges at u_2, and by symmetry e_2 is crossed only by edges at u_1. Applying Corollary 6 we get that v_1 and v_2 are in the same connected component of H and hence $W_j = W_{j'}$, as desired. △

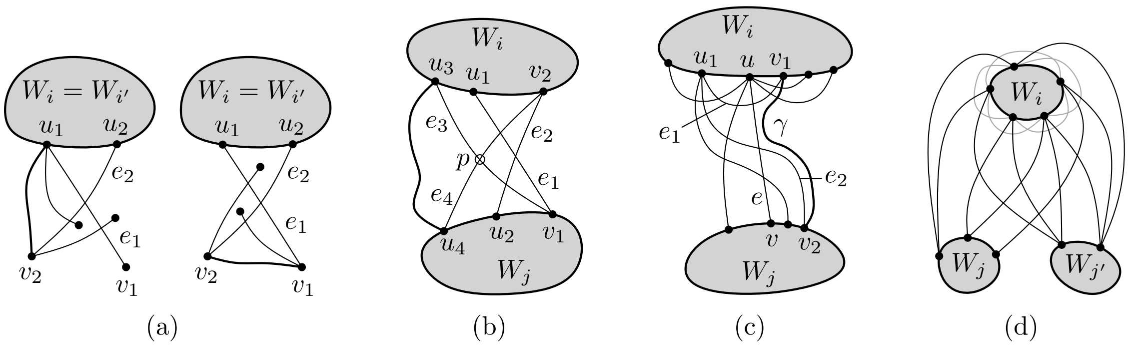


Figure 9: (a) Case 1 in the proof of Claim 1. Illustrations of the proofs of (b) Claim 2, (c) Claim 3 and (d) Claim 4.

Consider a vertex v on facial walk W_i with an incident W_iW_j-edge e for some j. As W_i and W_j are distinct facial walks, edge e is crossed; by some other $W_{i'}W_{j'}$-edges, by some edge of $G(W_i)$, or by some edge of $G(W_j)$. If no edge of $G(W_i)$ crosses e, then v must lie on the boundary of the cell c_i that contains W_j and all other facial walks. For a facial walk W_i and a vertex $v \in W_i$, let us call v *open* if v lies on ∂c_i. Moreover, a vertex $v \in W_i$ is called *closed* if v is not open but v is incident to some W_iW_j-edge for some $j \neq i$. So the endpoints of every W_iW_j-edge are open or closed, and by fan-planarity (the absence of configuration I) at least one endpoint is open.

Claim 2. If two W_iW_j-edges cross then both have exactly one open endpoint, which moreover are in the same facial walk.

Proof of Claim. Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be two crossing W_iW_j-edges. Assume for the sake of contradiction that e_1 has an open endpoint $u_1 \in W_i$ and e_2 has an open endpoint $u_2 \in W_j$. We consider the edges $e_3 = u_3v_1$ and $e_4 = u_4v_2$ that are incident to v_1 and v_2 respectively, cross each other and whose crossing point p is furthest away from v_1 and v_2. See Figure 9(b) for an illustration. Note that possibly $e_1 = e_3$ and/or $e_4 = e_2$.

Now u_3 is not in W_j because u_1 is an open endpoint and u_4 is not in W_i because u_2 is an open end. Hence by Claim 1 $u_3 \in W_i$ and $u_4 \in W_j$. Moreover, by Lemma 3 u_3u_4 is an uncrossed edge of G – a contradiction to the fact that W_i and W_j are distinct facial walks. △

Claim 2 implies that every edge between different facial walks has exactly one open endpoint and one closed endpoint. Indeed, by assumption (i) no cell of G has vertices from distinct facial walks on its boundary, meaning that every W_iW_j-edge e is crossed by some other edge e'. If e' runs between two facial walks, then e has a closed endpoint by Claim 2, and if e' runs within W_i or W_j, then e has a closed endpoint by definition. Let us remark that the second part of Claim 2 uses Lemma 3, which in turn uses the absence of configuration II, and indeed, the second part of Claim 2 is no longer true if configuration II may occur.

Claim 3. If a W_iW_j-edge has a closed endpoint $u \in W_i$ and w is the counterclockwise next open or closed vertex of W_i after u, then there exists a W_iW_j-edge incident to w with open endpoint in W_j.

Proof of Claim. Let $e = uv$ be a W_iW_j-edge that has a closed endpoint $u \in W_i$. By fan-planarity (absence of configuration I) v is an open vertex of W_j. As u is a closed endpoint, some edge of $G(W_i)$ crosses e. Let $e_1 = u_1v_1$ be the edge from $G(W_i)$ whose crossing with e is closest to v (meaning that this crossing lies on ∂c_i), where without loss of generality v_1 comes counterclockwise after u in W_i. Further assume without loss of generality that e is the W_iW_j-edge at u whose crossing with e_1 is closest to v_1.

Consider a Jordan curve γ between v_1 and the crossing of e_1 and e that runs along the left side of e_1 and is crossed only by those edges crossing e_1 on this stretch. Below we show how to extend γ without creating new crossings so that γ ends at some open vertex in W_j and argue that G together with γ seen as an edge at v_1 is fan-planar. Then the edge-maximality of G implies that an edge with the same endpoints as γ already exists in G, which will prove the claim. Note that by fan-planarity (absence of configuration I) every edge crossing γ is incident to v or u. If one such edge is incident to v, it is the desired W_iW_j-edge. Otherwise, all such edges are incident to u and by the choice of e the other endpoint lies also in W_i.

So let us first assume that e is not crossed between v and its crossing with e_1. In this case we can easily extend γ to end at v and we are done.

So e is crossed between its crossing with e_1 and v. Let e_2 be such a crossing edge whose crossing with e is closest to u. Then by fan-planarity (absence of configuration I) e_2 is incident to u_1 or v_1. Moreover, by Claim 1 and Claim 2 e_2 has a closed endpoint in W_i and an open endpoint in W_j. Thus if e_2 is incident to v_1, then e_2 is the desired W_iW_j-edge. So assume that $e_2 = u_1v_2$ for some $v_2 \in W_j$. We extend γ along the left side of e and e_2 all the way to v_2. We refer to Figure 9(c) for an illustration. If γ is not crossed on this stretch, we are done. Otherwise, by the choice of e_2, γ is crossed while running along e_2. Let e_3 be such a crossing edge. By fan-planarity, e_3 ends at u or v and by the choice of e, it does not end at u. Finally, by Claim 1 the endpoint of e_3 different from v lies in W_i, which makes e_3 our desired edge. △

Claim 3 together with Claim 2 implies that on each facial walk every closed vertex is followed by another closed vertex. In particular, the facial walks come in two kinds, one with open vertices only and one with closed vertices only. We remark that one can show that, if W_i has only closed vertices, then $G(W_i)$ is a sunflower.

Claim 4. Every facial walk with only closed vertices has edges to exactly one facial walk with only open vertices.

Proof of Claim. Assume for the sake of contradiction that facial walk W_i with only closed vertices has edges to two different facials walks $W_j, W_{j'}$ with only open vertices. Claim 3 implies that if some closed vertex of W_i has an edge to W_j, then every closed vertex of W_i has an edge to W_j, and the same is true for $W_{j'}$. Hence, each of the at least three closed vertices in W_i has an edge to W_j and an edge to $W_{j'}$, which implies that some W_iW_j-edge and some $W_iW_{j'}$-edge must cross, see Figure 9(d). (Indeed, if any two such edges would not cross, then contracting W_j and $W_{j'}$ into a single point each and placing a new vertex in the middle of W_i with an edge to every closed vertex in W_i would give a planar drawing of $K_{3,3}$.) Thus by Claim 1 we have $W_j = W_{j'}$ – a contradiction to our assumption. △

We are now ready to prove that at most one facial walk has open vertices. Recall that by Claim 3 every facial walk is of one of two kinds: only open vertices or only closed vertices. Moreover, by fan-planarity (absence of configuration I) and Claim 2 no edge runs between two facial walks of the same kind. We consider a bipartite graph F whose black and white vertices correspond to facial walks of the first and second kind, respectively, and whose edges correspond to pairs W_i, W_j of facial walks for which there is at least one W_iW_j-edge in G. Since G is connected, F is connected, and by Claim 4 every white vertex is adjacent to exactly one black vertex. This means that F is a star and has exactly one black vertex, which concludes the proof. □

Now, we can bound the number of W_iW_j-edges. Recall that c is an inclusion-minimal cell in $C_*(f)$ for some face f of G, W_1 denotes the facial walk of f with $c \in C(W_1)$ and $W_2, \ldots, W_k$ denote the facial walks of f inside c. Further, for $i = 1, \ldots, k$ we denote by c_i the cell of $G(W_i)$ containing all W_j with $j \neq i$.

Lemma 10. *The number of edges between* $W_1, \ldots, W_k$ *is at most*

$$4(k-2) + \sum_{i=1}^{k} ||c_i||.$$

Proof. By Lemma 9 exactly one of $c_1, \ldots, c_k$ has vertices on its boundary, say W_1. Let U be the set of vertices on the boundary of c_1. For a vertex $u \in U$ and an index $i \in \{2, \ldots, k\}$ we call an edge between u and W_i a *uW_i-edge*. We define a bipartite graph J as follows. One bipartition class is formed by the vertices in U. In the second bipartition class there is one vertex w_i for each facial walk W_i, $i = 1, \ldots, k$. A vertex $u \in U$ is connected by an edge to w_i if and only if $i = 1$ or $i \geqslant 2$ and there is a uW_i-edge.

Claim 5. The graph J is planar.

Proof of Claim. We consider the following embedding of J. Afterwards we shall argue that this embedding is indeed a plane embedding. So take the position of every vertex $u \in U$ from the fan-planar embedding of G. For $i \geqslant 2$, we consider the drawing of W_i in the embedding of G, for each edge between a vertex $u \in U$ and the vertex w_i in J we take the drawing of one uW_i-edge in G, and then contract the drawing of W_i into a single point – the position for vertex w_i. Finally, we place the last vertex w_1 outside the cell c_1 and connect w_1 to each $u \in U$ in such a way that these edges do not cross any other edge in J. See Figure 10(a) for an illustrating example.

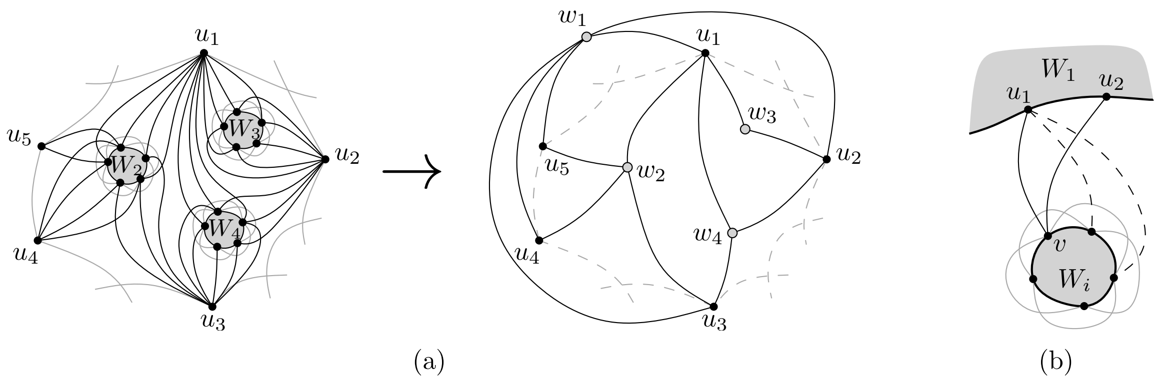


Figure 10: (a) Obtaining the graph J. (b) The contradiction in Claim 6.

Now the resulting drawing of J contains crossing edges only if a uW_i-edge crosses a $u'W_{i'}$-edge in G. However, by Lemma 9 the cells $c_2, \dots, c_k$ have no vertices on their boundary. Hence, for each $i = 2, \dots, k$ every uW_i-edge crosses an edge of $G(W_i)$. Now if a uW_i-edge e would cross a $u'W_{i'}$-edge with $u \neq u'$ and $i \neq i'$, then e would be crossed by two independent edges, which is configuration I and hence a contradiction to the fan-planarity of G. △

Since J is a planar bipartite graph with bipartition classes of size $|U|$ and k we have

$$|E(J)| = \sum_{i=1}^{k} \deg_J(w_i) \leqslant 2(|U| + k) - 4.$$

Claim 6. For each $i = 2, \dots, k$ the number of uW_i-edges is at most

$$||c_i|| + 2\deg_J(w_i).$$

Proof of Claim. Consider the vertices on W_i and the set $U' \subseteq U$ of vertices on W_1 that have a neighbor on W_i. For each $u \in U'$ consider the cyclic ordering of uW_i-edges around u. Since not every edge at u is a uW_i-edge (at least one edge ends in W_1) we obtain a linear order on the uW_i-edges going counterclockwise around u.

Now we claim that when we remove for each $u \in U'$ the last two uW_i-edges in the linear order for u, then every vertex v in W_i is the endpoint of at most one uW_i-edge. Indeed, if after the edges have been removed two vertices $u_1, u_2 \in U'$ have a common neighbor v on W_i, then at least two u_1W_i-edges, say e_1, e_2, cross the edge u_2v (or the other way around). As v is closed, u_2v is crossed by an edge e in $G(W_i)$. By fan-planarity (the absence of configurations I and II) e, e_1 and e_2 have a common endpoint on W_i, making e_1, e_2 a pair of parallel edges – a contradiction. So the number of uW_i-edges is at most $2|U'| + |W_i| = ||c_i|| + 2\deg_J(w_i)$. △

We can now bound the total number of uW_i-edges with $i \geqslant 2$ as follows.

$$\begin{aligned}\sum_{i=2}^{k} \#uW_i\text{-edges} &\leqslant \sum_{i=2}^{k}(||c_i|| + 2\deg_J(w_i)) \\ &= \sum_{i=2}^{k} ||c_i|| + 2|E(J)| - 2\deg_J(w_1) \\ &\leqslant \sum_{i=2}^{k} ||c_i|| + 4(|U| + k) - 8 - 2|U| \\ &= \sum_{i=2}^{k} ||c_i|| + 2|U| + 4(k-2) \leqslant \sum_{i=2}^{k} ||c_i|| + ||c_1|| + 4(k-2). \qquad \square\end{aligned}$$

We continue by bounding the total number of crossed edges of G that are drawn inside a fixed face f of G. To this end let k_f be the number of distinct facial walks of f and $|f|$ be the sum of lengths of facial walks of f, i.e., $|f| = \sum_{W \text{ facial walk of } f} |W|$.

Lemma 11. *The number of edges inside f is at most*

$$2|f| + 5(k_f - 2) - \sum_{c \in C_\emptyset(f)} \max\{0, ||c|| - 5\}.$$

Proof. We do induction on k_f. For $k_f = 1$, the face f is bounded by a unique facial walk W. Then by Lemma 8 there are at most

$$2|W| - 5 - \sum_{c \in C(W)} \max\{0, ||c|| - 5\}$$

edges inside f. With $|W| = |f|$ and $C_\emptyset(f) = C(W)$ this gives the claimed bound.

For $k_f \geqslant 2$, the face f has $k = k_f$ distinct facial walks $W_1, \ldots, W_k$. Let c be an inclusion-minimal cell in $C_*(f)$. Without loss of generality let W_1 be the facial walk with $c \in C(W_1)$ and $W_2, \ldots, W_j$ be the facial walks of f that lie inside c. In particular we have $2 \leqslant j \leqslant k$. Let G' be the graph that is obtained from G after removing all vertices that lie inside c. We consider G' with its fan-planar embedding inherited from G. Clearly, the face f' in G' corresponding to f in G has exactly $k - (j-1) < k$ facial walks and we have

$$|f| = |f'| + |W_2| + \cdots + |W_j|.$$

For $i = 1, \ldots, j$ we denote by c_i the cell of $G(W_i)$ containing W_1. (Hence $c_1 = c$.) Moreover, let $C = C(W_2) \cup \cdots \cup C(W_j)$. Then

$$C_\emptyset(f) = (C_\emptyset(f') \cup C) \setminus \{c_1, c_2, \ldots, c_j\}.$$

Further we partition the edges inside f into three disjoint sets E_1, E_2, E_3 as follows:

- The edges in E_1 are precisely the edges of G' inside f'.

- The edges in E_2 are precisely the edges of G between W_1 and $W_2 \cup \cdots \cup W_j$.
- $E_3 = (E(G(W_2)) \setminus E(W_2)) \cup \cdots \cup (E(G(W_j)) \setminus E(W_j))$.

Now by induction hypothesis,

$$|E_1| \leqslant 2|f'| + 5(k-j-1) - \sum_{c \in C_\emptyset(f')} \max\{0, ||c|| - 5\}.$$

By Lemma 10,

$$|E_2| \leqslant \sum_{i=1}^{j} ||c_i|| + 4(j-2) \leqslant \sum_{i=1}^{j} \max\{0, ||c_i|| - 5\} + 9j - 8,$$

and by Lemma 8,

$$|E_3| \leqslant 2(|W_2| + \cdots + |W_j|) - 5(j-1) - \sum_{c \in C} \max\{0, ||c|| - 5\}.$$

Plugging everything together we conclude that the number of edges of G inside f is at most

$$\begin{aligned} |E_1 \dot\cup E_2 \dot\cup E_3| &\leqslant 2|f'| + 5(k-j-1) - \sum_{c \in C_\emptyset(f')} \max\{0, ||c|| - 5\} \\ &\quad + \sum_{i=1}^{j} \max\{0, ||c_i|| - 5\} + 9j - 8 \\ &\quad + 2(|W_2| + \cdots + |W_j|) - 5(j-1) - \sum_{c \in C} \max\{0, ||c|| - 5\} \\ &= 2|f| + 5(k-2) - (j-2) - \sum_{c \in C_\emptyset(f)} \max\{0, ||c|| - 5\} \\ &\leqslant 2|f| + 5(k_f - 2) - \sum_{c \in C_\emptyset(f)} \max\{0, ||c|| - 5\}, \end{aligned}$$

which concludes the proof. □

Lemma 11 implies that inside a face f of H there are at most $2|f| + 5(k_f - 2)$ edges. Having this, we are now ready to prove our main theorem, namely that every simple fan-planar graph on $n \geqslant 3$ vertices has at most $5n - 10$ edges.

Proof of Theorem 1. Consider a fan-planar graph $G = (V, E)$ on n vertices with properties (i) and (ii). Let H be the spanning subgraph of G on all uncrossed edges. Note that $V(H) = V(G)$. We denote by $F(H)$ the set of all faces of H. Since every edge $e \in E(H)$ appears either exactly once in two distinct facial walks or exactly twice in the same facial walk, we have

$$\sum_{f \in F(H)} |f| = 2|E(H)|.$$

Further we denote by k_f the number of facial walks for a given face f, and by $CC(H)$ the number of connected components of H. Since a face with k facial walks gives rise to k connected components of H, we have

$$\sum_{f \in F(H)} (k_f - 1) = CC(H) - 1.$$

Hence we conclude

$$\begin{aligned} |E(G)| &\overset{\text{Lemma 11}}{\leqslant} |E(H)| + \sum_{f \in F(H)} (2|f| + 5(k_f - 2)) \\ &= |E(H)| + 2\sum_{f \in F(H)} |f| + 5\sum_{f \in F(H)} (k_f - 1) - 5|F(H)| \\ &= 5|E(H)| + 5CC(H) - 5|F(H)| - 5 = 5|V(H)| - 10, \end{aligned}$$

where the last equation is Euler's formula for the plane embedded graph H. With $|V(H)| = |V(G)| = n$ this concludes the proof. □

4 Discussion

We have shown that every simple n-vertex fan-planar graph has at most $5n-10$ edges. We have seen that maximum fan-planar graphs carry an underlying planar structure, possibly enhanced by relatively simple local (non-planar) substructures like stars or 2-hop edges along the boundary of each face, and combinations of both. Such properties make this class attractive for many algorithms commonly used in graph drawing that also use planar subgraphs as a base, and enhance the drawing by additional crossing edges.

The new concept of fan-planarity opens a variety of possible research directions. Of course, if we allow G to have parallel edges or self-loops, there could be arbitrarily many edges, even if the drawing of G is planar. However, if we only forbid configuration I, but allow configurations II and III, can an n-vertex topological graph have more than $5n - 10$ edges?

If we consider topological multigraphs with *non-homeomorphic parallel edges* and *non-trivial loops*, does our $5n - 10$ bound still hold? Here, two parallel edges are non-homeomorphic (a loop is non-trivial) if the bounded component of the plane described by the edges (the edge) contains at least one vertex. Note for instance that Euler's formula still holds for plane graphs with non-homeomorphic parallel edges and non-trivial loops, and in this case every face still has length at least 3. Hence such plane multigraphs still have at most $3n - 6$ edges. We strongly conjecture that our $5n - 10$ bound also holds for such fan-planar multigraphs.

If we allow non-simple topological graphs, i.e., allow edges to cross more than once and incident edges to cross, does every n-vertex fan-planar graph still have at most $5n - 10$ edges? We remark, that if we allow both, non-simple drawings and non-homeomorphic parallel edges, then there are 3-vertex topological graphs with arbitrarily many edges. Let us simply refer to Figure 11(a) for such an example. The idea is to start with an edge e_1 from u to v, and edge e_i starts clockwise next to e_{i-1} at u goes in parallel with e_{i-1} until e_{i-1} ends at v, where e_i goes a little further surrounding v once. No two parallel edges are homeomorphic.

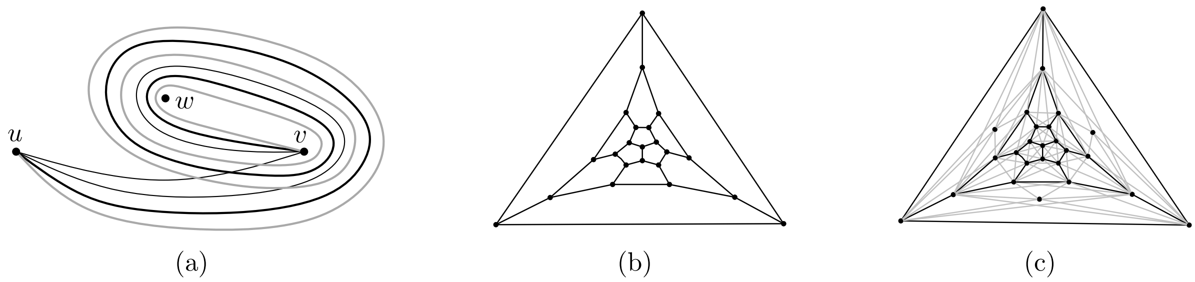


Figure 11: (a) A topological non-simple fan-planar graph with arbitrarily many edges. (b) The modified dodecahedral graph without extensions and (c) fully extended to $5n-11$ straight-line edges.

If we generalize fan-planarity to *k-fan-planarity*, where every edge may be crossed by at most k fan-crossings, $k > 1$, then a simple probabilistic argument from the analysis of natural grids shows that for fixed k every n-vertex k-fan-planar graph has at most $(3(k+1)^{k+1}/k^k)n$ edges, see Lemma 2.9 in [2]. However, exact bounds are not known.

Next, we suspect that every n-vertex *straight-line* fan-planar graph has at most $5n-11$ edges, similar to the $4n-9$ bound for straight-line 1-planar graphs [24]. The augmented dodecahedral graph from Figure 3(b) can be modified into a straight-line fan-planar graph with $5n-11$ edges: Replace one vertex of the dodecahedron by a triangle, which is used as the outer face. Draw the planar graph with convex faces, so that all pentagrams can be drawn straight-line, cf. Figure 11(b). The three pentagons that became hexagons are filled with 2-hops and spokes as explained in Proposition 2, i.e., by one additional vertex and 12 edges each.

Finally, how *few* edges can an edge-maximal fan-planar graph have? An n-cycle with 2-hop, respectively 3-hop, edges provides edge-maximal fan-planar graphs with no more than $3n$ edges if parallel edges are allowed and no more than $\frac{8}{3}n$ edges, otherwise. We suspect these examples to be best-possible.

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