Topology of augmented Bergman complexes

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Submitted: Sep 18, 2021; Accepted: Dec 30, 2021; Published: Feb 25, 2022

Abstract

The augmented Bergman complex of a matroid is a simplicial complex introduced recently in work of Braden, Huh, Matherne, Proudfoot and Wang. It may be viewed as a hybrid of two well-studied pure shellable simplicial complexes associated to matroids: the independent set complex and Bergman complex.

It is shown here that the augmented Bergman complex is also shellable, via two different families of shelling orders. Furthermore, comparing the description of its homotopy type induced from the two shellings re-interprets a known convolution formula counting bases of the matroid. The representation of the automorphism group of the matroid on the homology of the augmented Bergman complex turns out to have a surprisingly simple description. This last fact is generalized to closures beyond those coming from a matroid.

Mathematics Subject Classifications: 05C88, 05C89

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1 Introduction

Matroids are an abstraction of the combinatorial properties of linear dependence for a set of vectors in a vector space, introduced independently in the 1930s by H. Whitney and T. Nakasawa. Here we will work exclusively with a matroid M on a *finite* ground set E, which can be specified by either of these two subcollections of the subsets 2^E:

- the *independent sets* $\mathcal{I}(M)$, modeling the subsets of the vectors that are linearly independent, or
- the *flats* $\mathcal{F}(M)$, modeling the subsets of the vectors that are closed under taking linear span.

The independent sets and flats satisfy certain axioms I1, I2, I3 and F1, F2, F3, recalled in Section 2 below; see the book by Oxley [11] for further background on matroids. Both $\mathcal{I}(M)$ and $\mathcal{F}(M)$ give rise to well-studied abstract simplicial complexes associated to M, as we now explain.

The first two axioms I1, I2 for independent sets imply that $\mathcal{I}(M)$ forms an *abstract simplicial complex* on vertex set E. The third axiom I3 then implies that all inclusion-maximal independent sets (called *bases*) have the same cardinality $r(M)$, called the *rank* of the matroid. Thus the simplicial complex $\mathcal{I}(M)$ is *pure*, meaning that all of its inclusion-maximal faces, called *facets*, have the same dimension $r(M) - 1$.

On the other hand, one usually considers the collection of flats $\mathcal{F}(M)$ as a partial order via inclusion, and then one can construct its *order complex* $\Delta\mathcal{F}(M)$ as the simplicial complex whose vertices are the flats, and whose simplices are the linearly ordered collections of flats. There are two extreme flats $\overline{\varnothing}, E$ which are comparable to all the other flats, so that they form cone vertices in this simplicial complex $\Delta\mathcal{F}(M)$. One usually removes these two cone vertices to obtain the topologically more interesting *Bergman complex*[1] $\underline{\Delta}_M := \Delta(\mathcal{F}(M) \setminus \{\overline{\varnothing}, E\})$, which turns out to be pure of dimension $r(M) - 2$.

Both $\mathcal{I}(M)$ and $\underline{\Delta}_M$ were proven around 1980 (in work of Provan and Billera [13] and of Garsia [7]) to be not only pure, but also *shellable*, a property with strong topological and algebraic consequences. In particular, a pure d-dimensional shellable complex Δ is homotopy equivalent to a β-fold wedge of d-dimensional spheres, where β is its top (reduced homology) *Betti number*, or the absolute value $|\tilde{\chi}(\Delta)|$ of its (reduced) *Euler characteristic*. Shellability can be viewed as a connectivity property much stronger than *gallery-connectedness*. The hierarchy of purity, gallery-connectedness and shellability are reviewed in Subsection 2.3 below.

Our goal here is to study the topology of the following "hybrid" of $\mathcal{I}(M)$ and $\underline{\Delta}_M$, introduced in the monumental recent work of Braden, Huh, Matherne, Proudfoot and Wang [4, 5] that resolved several important conjectures in matroid theory.

Definition. Given a matroid M on ground set E, the *augmented Bergman complex* Δ_M is the abstract simplicial complex on vertex set

$$\{y_i : i \in E\} \sqcup \{x_F : \text{proper flats } F \in \mathcal{F}(M) \setminus \{E\}\}$$

[1] Also sometimes known as the order complex of the *proper part* $\mathcal{F}(M) \setminus \{\overline{\varnothing}, E\}$; see Subsection 2.2.

whose simplices are the subsets

$$\{y_i\}_{i\in I} \cup \{x_{F_1}, x_{F_2}, \ldots, x_{F_\ell}\} \tag{1}$$

for which $I \in \mathcal{I}(M)$ and the proper flats F_i satisfy $I \subseteq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell \ (\subsetneq E)$.

It is noted in [4, §2 Prop. 2.3] that Δ_M is pure of dimension $r(M)-1$, that it is gallery-connected, and that it contains as full-dimensional subcomplexes both the independent set complex $\mathcal{I}(M)$ (as the simplices in (1) with $\ell = 0$), and the cone over the Bergman complex $\underline{\Delta}_M$ with cone vertex $x_{\varnothing}$ (as the simplices in (1) with $\#I = 0$); we will denote the latter complex by $\mathrm{Cone}(\underline{\Delta}_M) = \Delta(\mathcal{F} \setminus \{E\})$.

This motivates our first main result, proven in Section 3.

Theorem 1. *The augmented Bergman complex Δ_M of a matroid is shellable, via two families of shellings:*

(i) one family that shells the facets of $\mathrm{Cone}(\underline{\Delta}_M)$ *first, and the facets of $\mathcal{I}(M)$ last,*

(ii) one family that shells the facets of $\mathcal{I}(M)$ first, and the facets of $\mathrm{Cone}(\underline{\Delta}_M)$ *last.*

Shellability immediately implies that the geometric realization of Δ_M is homotopy equivalent to a wedge of $(r(M)-1)$-dimensional spheres, and gives a combinatorial expression for the number of spheres. For example, the Provan-Billera shelling of $\mathcal{I}(M)$ shows that it is homotopy equivalent to a wedge of $(r(M)-1)$-spheres, and the number of spheres in the wedge is the evaluation $T_M(0,1)$ of the famous *Tutte polynomial* $T_M(x,y)$; see Björner [2, §7.3]. Similarly, Garsia's shelling of $\underline{\Delta}_M$ shows that it is homotopy equivalent to a $T_M(1,0)$-fold wedge of $(r(M)-2)$-dimensional spheres; see Björner [2, §7.4, 7.6].

Theorem 1 gives the following description of the homotopy type of Δ_M, proven in Section 4. It involves the set $\mathcal{B}(M)$ of *bases* of M, which we recall are the maximal independent sets, indexing the facets of $\mathcal{I}(M)$.

Corollary 2. *For a matroid M, the augmented Bergman complex Δ_M is homotopy equivalent to a β-fold wedge of $(r(M)-1)$-spheres, with two different expressions for β:*

$$\beta = \#\mathcal{B}(M) \quad (= T_M(1,1)) \tag{2}$$

$$\beta = \sum_{F\in\mathcal{F}(M)} T_{M|_F}(0,1)\cdot T_{M/F}(1,0). \tag{3}$$

Expressions (2), (3) *for β are predicted by the shellings in Theorem 1(i),(ii), respectively.*

Here $M|_F$ and M/F are the matroids obtained from M by *restriction* to the flat F and *contraction* on the flat F, respectively; see Subsection 2.1 below. We remark that the concordance between the two expressions for β in Corollary 2 comes from specializing a well-known Tutte polynomial convolution formula [6, 8]:

$$T_M(x,y) = \sum_{F\in\mathcal{F}(M)} T_{M|_F}(0,y)\cdot T_{M/F}(x,0).$$

Our last main result is a surprisingly simple description for the representation on the homology of the augemented Bergman complex given by the action of the matroid's *automorphism group*

$$\begin{aligned}\mathrm{Aut}(M) &:= \{\text{bijections } E \xrightarrow{\sigma} E : \sigma(I) \in \mathcal{I}(M) \text{ for all } I \in \mathcal{I}(M)\} \\ &= \{\text{bijections } E \xrightarrow{\sigma} E : \sigma(F) \in \mathcal{F}(M) \text{ for all } I \in \mathcal{F}(M)\}.\end{aligned}$$

Corollary 3. *The action of* $\mathrm{Aut}(M)$ *on the top (reduced) homology* $\tilde{H}_{r(M)-1}(\Delta_M, \mathbb{Z})$ *is the same as its action on the top oriented simplicial chain group* $\tilde{C}_{r(M)-1}(\mathcal{I}(M), \mathbb{Z})$ *for the complex* $\mathcal{I}(M)$.

More explicitly, this means that the action is a signed permutation representation of $\mathrm{Aut}(M)$, in which $\mathbb{Z}$-basis elements $[B]$ indexed by bases B in $\mathcal{B}(M)$ are permuted up to (explicit) signs; see Remark 22 below.

Corollary 3 is generalized in Section 6 to a statement (Theorem 6.1) applying beyond matroids, to a finite set E equipped with an arbitrary *closure operator* $f : 2^E \longrightarrow 2^E$.

Example 4. Let M be the uniform matroid of rank 2 on ground set $E = \{1, 2, 3\}$, which has three bases $\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Figure 1 depicts the Hasse diagram of its lattice of flats $\mathcal{F}$ on the left, along with the simplicial complexes $\mathcal{I}(M), \mathrm{Cone}(\underline{\Delta}_M)$ and Δ_M from left to right. All three are all (pure) 1-dimensional complexes, that is, graphs. For graphs, shellability is equivalent to being connected.

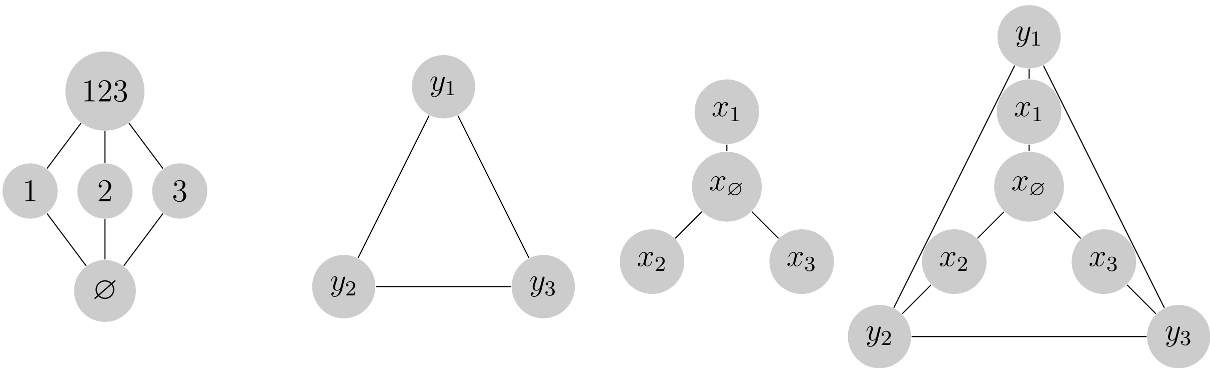

Figure 1: The poset $\mathcal{F}(M)$ and the complexes $\mathcal{I}(M), \mathrm{Cone}(\underline{\Delta}_M)$, and Δ_M from Example 4.

The augmented Bergman complex has homology $\tilde{H}_1(\Delta_M, \mathbb{Z}) \cong \mathbb{Z}^3$, canonically identified with the group of oriented 1-chains $\tilde{C}_1(\mathcal{I}(M), \mathbb{Z})$ via Corollary 3. This group of 1-chains has $\mathbb{Z}$-basis $\{[y_1, y_2], [y_1, y_3], [y_2, y_3]\}$, indexed by the three bases $\mathcal{B}(M)$; here one should interpret oriented edges as $[y_i, y_j] = -[y_j, y_i]$. The matroid M has automorphism group $G = \mathfrak{S}_3$ acting on the vertices via $g(y_i) = y_{g(i)}$. Corollary 3 then tells us, for example, that the transposition $g = (1, 3)$ in $\mathfrak{S}_3$ acts on $\tilde{H}_1(\Delta_M, \mathbb{Z})$ sending

$$\begin{aligned}g([y_1, y_3]) &= [y_3, y_1] = -[y_1, y_3], \\ g([y_1, y_2]) &= [y_3, y_2] = -[y_2, y_3].\end{aligned}$$

Example 5. The *Boolean matroid* M on $E = \{1, 2, \ldots, n\}$ of rank n has only one basis, E itself, that is, $\mathcal{B}(M) = \{E\}$. Here the augmented Bergman complex Δ_M triangulates an $(n-1)$-sphere that turns out to be isomorphic to the boundary complex of an n-dimensional convex polytope known as the *stellohedron*; see [4, Footnote 7] and [12, §10.4]. The examples with $n = 2, 3$ are depicted in Figure 2.

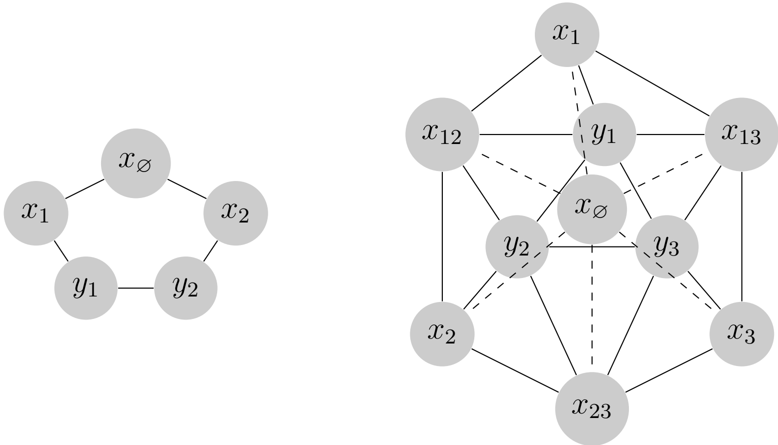

Figure 2: The complex Δ_M for a rank n Boolean matroid, where Δ_M is the boundary of the n-dimensional stellohedron. For $n = 2$, the stellohedron is a pentagon, shown at left. For $n = 3$ the stellohedron has 16 boundary triangles (not shaded here), shown at right.

Since there is only one basis for M, Corollary 3 implies that $\tilde{H}_{n-1}(\Delta_M, \mathbb{Z}) \cong \mathbb{Z}$, with $\mathbb{Z}$-basis element the oriented simplex $[E] = [y_1, \ldots, y_n]$; this corresponds to the homology orientation class of the boundary sphere of the stellohedron. Here $\mathrm{Aut}(M)$ is the symmetric group permuting $\mathfrak{S}_n$, and it acts via the sign representation: $g([E]) = \mathrm{sgn}(g) \cdot [E]$ for every permutation g in $\mathfrak{S}_n$.

2 Background

2.1 Matroids

We begin with two axiomatizations of matroids; see Oxley [11, Chap. 1] for other axioms.

Definition 6. (*Matroids defined by independent sets*) A *matroid* M on ground set E is a collection $\mathcal{I}(M) \subseteq 2^E$, called its *independent sets*, satisfying axioms:

I1. $\varnothing \in \mathcal{I}(M)$.

I2. $I \subseteq J$ and $J \in \mathcal{I}(M)$ implies $I \in \mathcal{I}(M)$.

I3. If $I, J \in \mathcal{I}(M)$ and $\#I < \#J$, then there exists $j \in J \setminus I$ with $I \cup \{j\} \in \mathcal{I}(M)$.

From Axioms I1, I2, one sees that the collection $\mathcal{I}(M)$ forms an abstract simplicial complex on vertex set E. Axiom I3 shows that it is pure of dimension $r-1$ where $r = r(M)$ is the cardinality of all maximal independent sets B, called the *bases* $\mathcal{B}(M)$.

Equivalently, one can define a matroid via flats.

Definition 7. (*Matroids defined by flats*) A *matroid* M on ground set E is a collection $\mathcal{F}(M) \subseteq 2^E$, called its *flats*, satisfying axioms:

F1. $E \in \mathcal{F}(M)$.

F2. $F, G \in \mathcal{F}(M)$ implies $F \cap G \in \mathcal{F}(M)$.

F3. For every $F \in \mathcal{F}(M)$ and $e \in E \setminus F$, there exists a unique $G \in \mathcal{F}(M)$ containing e that *covers* F in this sense: there does not exist $H \in \mathcal{F}(M)$ with $F \subsetneq H \subsetneq G$.

Assume from here on that E is **finite**. Then the flats $\mathcal{F}(M)$ let one define the *matroid closure* operator

$$\begin{array}{ccc} 2^E & \longrightarrow & 2^E \\ A & \longrightarrow & \overline{A} \end{array} \tag{4}$$

where $\overline{A}$ is the smallest flat containing A, namely

$$\overline{A} := \bigcap_{\substack{F \in \mathcal{F}(M): \\ F \supseteq A}} F.$$

We will also wish to view $\mathcal{F}(M)$ as a partially ordered set (*poset*) via inclusion, with unique bottom element $\overline{\varnothing}$ and top element E. In fact, $\mathcal{F}(M)$ is a *ranked lattice*, with rank function $r : \mathcal{F}(M) \to \{0, 1, 2, \ldots, r\}$ satisfying $r(\overline{\varnothing}) = 0$, $r(E) = r$, and $r(G) = r(F) + 1$ if there is no flat H satisfying $F \subsetneq H \subsetneq G$.

Passing between the independent sets and flats of a matroid M is not hard. First, given any subset $A \subseteq E$, one can define its *rank function* $r(A)$ either using $\mathcal{I}(M)$ or $\mathcal{F}(M)$ as follows:

$$\begin{aligned} r(A) &= \max\{\#I : I \in \mathcal{I}(M), I \subseteq A\} \\ &= r(\overline{A}). \end{aligned}$$

Then one can recover either $\mathcal{I}(M)$ or $\mathcal{F}(M)$ from the rank function r as follows:

$$\begin{aligned} \mathcal{I}(M) &:= \{I \subseteq E : r(I) = \#I\}, \\ \mathcal{F}(M) &:= \{F \subseteq E : r(F) < r(F \cup \{e\}) \text{ for all } e \in E \setminus F\}. \end{aligned}$$

The following two matroid constructions will turn out to be useful in the sequel.

Definition 8. Given a matroid M on ground set E, and a subset $A \subseteq E$, one can define two new matroids, the *restriction* $M|_A$, and the *contraction* M/A as follows:

$$\begin{aligned} \mathcal{I}(M|_A) &:= \{I \in \mathcal{I}(M) : I \subseteq A\} = \mathcal{I}(M) \cap 2^A, \\ \mathcal{F}(M/A) &:= \{F \setminus A : F \in \mathcal{F}(A), F \supseteq A\}. \end{aligned}$$

In particular, for a flat F, one has a poset isomorphism between $\mathcal{F}(M/F)$ and the poset interval

$$[F, E] := \{G \in \mathcal{F}(M) : F \subseteq G \subseteq E\}.$$

The isomorphism sends a flat G in the interval $[F, E]$ of $\mathcal{F}(M)$ to the flat $G \backslash F$ in $\mathcal{F}(M/F)$.

2.2 Order complexes

Several simplicial complexes that we will consider come from this construction.

Definition 9. Given any partially ordered set P, its *order complex* ΔP is the abstract simplicial complex on vertex set P, whose simplices are the totally ordered subsets of P.

An element of the poset P that is comparable to all others (such as a least element or greatest element) will give rise to a cone vertex in ΔP, so that ΔP is contractible. For this reason, such elements are often removed before forming the order complex. For example, since the poset of flats $\mathcal{F}(M)$ has least element $\overline{\varnothing}$ and greatest element E, they are removed before forming the *Bergman complex*

$$\underline{\Delta}_M := \Delta(\mathcal{F}(M) \setminus \{\overline{\varnothing}, E\}).$$

However, we sometimes put back in the bottom element $\overline{\varnothing}$ to consider the cone over $\underline{\Delta}_M$, which we denote

$$\mathrm{Cone}(\underline{\Delta}_M) := \Delta(\mathcal{F}(M) \setminus \{E\}).$$

We recall here from the Introduction that both $\underline{\Delta}_M$ and $\mathcal{I}(M)$ are subcomplexes of the following complex, introduced recently in the work of Braden, Huh, Matherne, Proudfoot and Wang [4, Def. 2.2], and central to their further work [5].

Definition 10. Given a matroid M on ground set E, the *augmented Bergman complex* Δ_M is the abstract simplicial complex on vertex set

$$\{y_i : i \in E\} \sqcup \{x_F : \text{proper flats } F \in \mathcal{F}(M) \setminus \{E\}\}$$

whose simplices are the subsets

$$\{y_i\}_{i \in I} \sqcup \{x_{F_1}, x_{F_2}, \ldots, x_{F_\ell}\} \tag{5}$$

for which $I \in \mathcal{I}(M)$ and the (possibly empty) proper flats F_i satisfy $I \subseteq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell \quad (\subsetneq E)$.

2.3 Purity, gallery-connectedness, shellability

We recall a hierarchy of simplicial complex properties.

Definition 11. A *facet* in a simplicial complex Δ is a face which is maximal under inclusion. One says that Δ is *pure* and d-dimensional if all of its facets have the same cardinality $d + 1$.

Definition 12. A pure d-dimensional simplicial complex Δ is *gallery-connected*[2], if for any two facets ϕ, ϕ' one has a sequence of facets $\phi = \phi_0, \phi_1, \ldots, \phi_t = \phi'$ with $\dim(\phi_i \cap \phi_{i-1}) = d-1$ for each $i = 1, 2, \ldots, t$.

Definition 13. A pure d-dimensional simplicial complex Δ is *shellable*[3] if one can order its facets $\phi_1, \phi_2, \phi_3, \ldots$ in a *shelling order*: for all $j \geqslant 2$, the intersection of the subcomplex generated by ϕ_j and the subcomplex generated by all previous facets $\{\phi_1, \ldots, \phi_{j-1}\}$ is a pure subcomplex of dimension $d-1$ inside ϕ_j. Here is a useful equivalent way to say that a total ordering $\prec$ on the facets of Δ is a shelling order:

$$\text{for all facets } \phi \prec \phi', \text{ there exists } \phi'' \prec \phi' \text{ such that } \phi \cap \phi' \subseteq \phi'' \cap \phi' \text{ and } \#\phi'' \cap \phi' = \#\phi' - 1.$$

Shellability of Δ determines the homotopy type of its *geometric realization* $\|\Delta\|$; see Kozlov [9, Chap. 12].

Definition 14. Let Δ be a shellable simplicial complex with shelling order $\phi_1, \phi_2, \phi_3, \ldots$ on its facets. The *restriction face* of facet ϕ_j is its subface $\mathscr{R}(\phi_j)$ containing these vertices:

$$\mathscr{R}(\phi_j) = \{x \in \phi_j : \text{ there exists } i \text{ with } 1 \leqslant i < j \text{ and } \phi_j \setminus \{x\} \subset \phi_i\}.$$

Call ϕ_j a *homology facet* in the shelling if $\mathscr{R}(\phi_j) = \phi_j$.

Lemma 15. *[9, Thm. 12.3] If a pure d-dimensional shellable complex Δ has a shelling with exactly β homology facets, then its geometric realization is homotopy equivalent to a β-fold wedge of d-spheres.*

Intuitively, each homology facet in the shelling "caps off" a d-sphere. Furthermore, the subcomplex obtained by removing all homology facets is contractible, as it is a shellable complex with no homology facets.

Example 16. As mentioned in the Introduction, $\mathcal{I}(M)$ and $\underline{\Delta}_M$ are shellable, and their shellings have $T_M(0,1), T_M(1,0)$ homology facets, respectively [2, §7.3, 7.4, 7.6], where $T_M(x,y)$ is the *Tutte polynomial*.

3 Proof of Theorem 1

We recall here the statement of the theorem.

Theorem 1. *The augmented Bergman complex Δ_M of a matroid is shellable, via two families of shellings:*

(i) one family that shells the facets of $\mathrm{Cone}(\underline{\Delta}_M)$ *first, and the facets of* $\mathcal{I}(M)$ *last,*

[2] Also known as *connected in codimension one* or *strongly connected* or *dually connected*.

[3] Here we restrict our shellable complexes to be pure; see Björner and Wachs [3] for the generalization to nonpure complexes.

(ii) *one family that shells the facets of $\mathcal{I}(M)$ first, and the facets of* $\mathrm{Cone}(\underline{\Delta}_M)$ *last.*

Before proving it, we identify and conveniently index facets of Δ_M. Recall from (1) that faces of Δ_M are

$$\phi = \{y_i\}_{i \in I} \cup \{x_{F_j}\}_{j=1}^{\ell}$$

where $I \in \mathcal{I}(M)$, each $F_j \in \mathcal{F}(M) \setminus \{E\}$, and $I \subseteq F_1 \subsetneq \cdots \subsetneq F_\ell$. This face ϕ is a facet if and only if both

- $\overline{I} = F_1$ (else one could add the vertex $x_{\overline{I}}$ to ϕ), and
- $\#I + \ell = r(M)$ (else $\overline{I} = F_1 \subsetneq \cdots \subsetneq F_\ell \subsetneq E$ is a non-maximal chain in interval $[F_1, E]$ of $\mathcal{F}(M)$).

In comparing facets ϕ, ϕ', with ϕ as above, and $\phi' = \{y_i\}_{i \in I'} \cup \{x_{F'_j}\}_{j=1}^{\ell'}$, we will use this abbreviated notation: letting $F_\bullet$ denote the chain of flats $F_1 \subsetneq \cdots \subsetneq F_\ell$, and similarly for $F'_\bullet$, write

$$\begin{aligned} \phi &\leftrightarrow (I, F_\bullet), \\ \phi' &\leftrightarrow (I', F'_\bullet). \end{aligned} \tag{6}$$

In our proofs that various linear orders $\prec$ on the facets of Δ_M are shellings, as in Definition 13, we will be given such a pair ϕ, ϕ' as in (6) with $\phi \prec \phi'$, and need to **construct**

$$\phi'' \leftrightarrow (I'', F''_\bullet) \text{ for which } \phi \cap \phi' \subseteq \phi'' \cap \phi' \text{ and } \#\phi'' \cap \phi' = \#\phi' - 1. \tag{7}$$

3.1 Flag-to-basis shellings

Definition 17. Call a total order $\prec$ on the facets ϕ of Δ_M a *flag-to-basis ordering* if two facets as in (6) have $\phi \prec \phi'$ whenever *either* of these conditions hold:

(a) $\#I < \#I'$, or

(b) $I = I'$, so $F_1 = \overline{I} = \overline{I}' = F'_1$, and $F_\bullet$ strictly precedes $F'_\bullet$ in some chosen shelling of $\underline{\Delta}_{M/F_1}$.

Note that in condition (b), we are identifying the flats $\mathcal{F}(M/F_1)$ with the poset interval $[F_1, E]$ in $\mathcal{F}(M)$.

Proof of Theorem 1(i). We check that any flag-to-basis ordering $\prec$ on the facets of Δ_M gives a shelling. Note Definition 17(a) ensures that $\prec$ orders facets of $\mathrm{Cone}(\underline{\Delta}_M)$ first and those of $\mathcal{I}(M)$ last. To check it is a shelling, given facets $\phi \prec \phi'$ as in (6), there are two cases to consider.

Case 1: $I = I'$. In this case, the shelling of Δ_{M/F_1} from Definition 17(b) provides the existence of a maximal chain $F''_\bullet$ shelled earlier than $F'_\bullet$ in $\mathcal{F}(M/F_1)$, and having $F_\bullet \cap F'_\bullet \subseteq F''_\bullet \cap F'_\bullet$ and $\#F''_\bullet \cap F'_\bullet = \#F'_\bullet - 1$. Thus taking $\phi'' \leftrightarrow (I, F''_\bullet)$ does the job for (7).

Case 2: $I \neq I'$. Consider the independent set $I \cap I' \subsetneq I'$, and use Axiom (I3) repeatedly to find $I'' \in \mathcal{I}(M)$ having $I \cap I' \subseteq I'' \subsetneq I'$ with $\#I'' = \#I' - 1$. Now let $F''_\bullet := \{\overline{I''}\} \cup F'_\bullet$. One can then check that $\phi'' \leftrightarrow (I'', F''_\bullet)$ has $\phi'' \prec \phi'$ (since $\#I'' < \#I'$), and does the job for (7). $\square$

3.2 Basis-to-flag shellings

Definition 18. Call a total order $\prec$ on the facets ϕ of Δ_M a *basis-to-flag ordering* if two facets as in (6) have $\phi \prec \phi'$ whenever any of these conditions (a),(b), or (c) hold:

(a) $\#I > \#I'$, or

(b) $F_1 = F'_1$ and $F_\bullet$ strictly precedes $F'_\bullet$ in some chosen shelling of $\underline{\Delta}_{M/F_1}$, or

(c) $F_\bullet = F'_\bullet$ (so that $\overline{I} = F_1 = F'_1 = \overline{I'}$), and I strictly precedes I' in some chosen shelling of $\mathcal{I}(M|_F)$.

As before, in condition (b), we identify $\mathcal{F}(M/F_1)$ with the interval $[F_1, E]$ in $\mathcal{F}(M)$, but now in condition (c), we also identify $\mathcal{I}(M|_F)$ with $\mathcal{I}(M) \cap 2^{F_1}$.

Proof of Theorem 1(ii). We check that any basis-to-flag ordering $\prec$ on the facets of Δ_M gives a shelling. Note Definition 18(a) ensures that $\prec$ orders facets of $\mathcal{I}(M)$ first and those of $\mathrm{Cone}(\underline{\Delta}_M)$ last. To check it is a shelling, given facets $\phi \prec \phi'$ as in (6), there are three cases to consider.

Case 1: $F_1 = F'_1$, but $F_\bullet \neq F'_\bullet$.

Then Definition 18(c) ensures that $F_\bullet$ strictly precedes $F'_\bullet$ in our chosen shelling of $\underline{\Delta}_{M/F_1}$. As before, this shelling of $\underline{\Delta}_{M/F_1}$ provides the existence of a maximal chain $F''_\bullet$ in $\mathcal{F}(M/F_1)$ having $F_\bullet \cap F'_\bullet \subseteq F''_\bullet \cap F'_\bullet$ and $\#F''_\bullet \cap F'_\bullet = \#F'_\bullet - 1$. Taking $\phi'' \leftrightarrow (I', F''_\bullet)$ does the job for (7).

Case 2: $F_\bullet = F'_\bullet$.

Here the shelling in Definition 18(c) provides the existence of an independent set I'' with $\overline{I''} = F_1$ having $I \cap I' \subseteq I'' \cap I'$ and $\#I'' \cap I' = \#I' - 1$. Thus taking $\phi'' \leftrightarrow (I'', F_\bullet)$ does the job for (7).

Case 3: $F_1 \neq F'_1$.

In this case, choose any element $i_0 \in F'_2 \setminus F'_1$; if $\#F'_\bullet = 1$ so that $F'_\bullet = \{F'_1\}$, choose any $i_0 \in E \setminus F'_1$. Then define $I'' := I' \cup \{i_0\}$ and $F''_\bullet := F'_\bullet \setminus \{F'_1\}$. One can then check that $\phi'' \leftrightarrow (I'', F''_\bullet)$ has $\phi'' \prec \phi'$ (since $\#I'' > \#I'$), and does the job for (7). $\square$

4 Proof of Corollary 2

We recall here the statement of the corollary.

Corollary 2. *For a matroid M, the augmented Bergman complex Δ_M is homotopy equivalent to a β-fold wedge of $(r(M)-1)$-spheres, with two different expressions for β:*

$$\beta = \#\mathcal{B}(M) \quad (= T_M(1,1))$$
$$\beta = \sum_{F \in \mathcal{F}(M)} T_{M|_F}(0,1) \cdot T_{M/F}(1,0).$$

The first and second expressions for β above (labeled (2), (3) in the Introduction) are predicted by the shellings in Theorem 1(i),(ii), respectively.

Proof. Recall β counts homology facets ϕ, that is, those with $\mathscr{R}(\phi) = \phi$, for the shellings in Theorem 1(i),(ii).

Proof of (2). Assume that $\prec$ is a flag-to-basis shelling order on the facets, as in Theorem 1(i). We will show that $\phi \leftrightarrow (I, F_\bullet)$ is a homology facet if and only if $F_\bullet = \varnothing$, that is, I is a basis.

For the "if" direction, assume $I = \{b_1, \ldots, b_r\}$ is a basis and $F_\bullet = \varnothing$, so $\phi = \{y_{b_1}, \ldots, y_{b_r}\}$. Then every vertex y_{b_i} lies in $\mathscr{R}(\phi)$, since $\phi \setminus \{y_{b_i}\} \subset \phi' \prec \phi$ where $\phi' \leftrightarrow (I', F'_\bullet)$ with $I' := I \setminus \{b_i\}$ and $F'_\bullet := \{\overline{I'}\}$. The fact that $\phi' \prec \phi$ uses Definition 17(a).

For the "only if" direction, assume that $F_\bullet = \{x_{F_1}, \ldots, x_{F_\ell}\} \neq \varnothing$, and we will show that $\mathscr{R}(\phi) \neq \phi$ because $x_{F_1} \notin \mathscr{R}(\phi)$. To see this, note that any facet $\phi' \leftrightarrow (I', F'_\bullet)$ containing $\phi \setminus \{x_{F_1}\}$ must either have $\#I' = \#I + 1$ (so $\phi' \succ \phi$), or have $I' = I$ and hence $F'_1 = \overline{I'} = \overline{I} = F_1$, which forces $\phi = \phi'$.

Proof of (3).

Assume that $\prec$ is a basis-to-flag shelling order on the facets, as in Theorem 1(ii). We will show that $\phi \leftrightarrow (I, F_\bullet)$ is a homology facet if and only if it satisfies the following conditions: considering the flat $F = \overline{I}$ (possibly $F = E$ when I is a basis; otherwise $F = F_1$) one has both that

(I) $F_\bullet$ is a homology facet in the chosen shelling for $\underline{\Delta}_{M/F}$, and

(II) I is a homology facet in the chosen shelling for $\mathcal{I}(M|_F)$.

This would prove (3), since then the homology facets for the order $\prec$ would be parametrized as follows: first choose the flat F in $\mathcal{F}(M)$ arbitrarily, then choose $F_\bullet$ from one of $T_{M/F}(1, 0)$ choices, and lastly choose I independently from one of $T_{M|F}(0, 1)$ choices; see Example 16

To check that conditions (I),(II) indeed describe the homology facets for the $\prec$ shelling order, first deal with the special case when $F := \overline{I} = E$, so that I is a basis and $F_\bullet = \varnothing$. It was already noted that Definition 18(a) implies $\prec$ shells the facets of the subcomplex $\mathcal{I}(M)$ first. Hence I will be a homology facet for the $\prec$ shelling if and only if it is a homology facet for the chosen shelling of $\mathcal{I}(M) = \mathcal{I}(M|_E)$, as in condition (II). Since $F_\bullet = \varnothing$, condition (I) above is vacuously satisfied in this case.

Now assume we are in the more generic case, when $F := \overline{I} = F_1 \neq E$. We need to understand whether or not a typical vertex x of ϕ lies in $\mathscr{R}(\phi)$, that is, whether $\phi \setminus \{x\}$ lies in some earlier facet $\phi' \prec \phi$. There are three cases to consider for x.

Case 1. $x = x_{F_1}$. In this case, one *always* has $x \in \mathscr{R}(\phi)$. To see this, pick any[4] $i_0 \in F_2 \setminus F_1$, and define the facet $\phi' \leftrightarrow (I', F'_\bullet)$ where $I' := I \cup \{i_0\}$ and $F'_\bullet := F_\bullet \setminus \{F_1\}$. Then $\phi' \supseteq \phi \setminus \{x\}$ and $\phi' \prec \phi$ since $\#I' > \#I$.

Case 2. $x = x_{F_j}$ for $j \geqslant 2$. In this case, Definition 18(b) shows that x lies in $\mathscr{R}(\phi)$ if and only if the vertex $x_{F_j \setminus F_1}$ lies in $\mathscr{R}(F_\bullet)$ in the chosen shelling for $\underline{\Delta}_{M/F_1}$.

[4]As in the proof of Theorem 1(ii), Case 3, if $\#F'_\bullet = 1$, so $F'_\bullet = \{F'_1\}$, then one chooses any $i_0 \in E \setminus F'_1$.

Case 3. $x = y_i$ for some $i \in I$. In this case, Definition 18(c) shows that x lies in $\mathscr{R}(\phi)$ if and only if y_i lies in $\mathscr{R}(I)$ in the chosen shelling for $\mathcal{I}(M|_{F_1})$.

Hence conditions (I),(II) above characterize homology facets for the $\prec$ shelling, completing the proof. $\square$

Remark 19. Tutte's original definition of the Tutte polynomial $T_M(x, y)$ involved choosing a linear order ω on the ground set E. From this he defined for each basis B in $\mathcal{B}(M)$ its *internal activity* $i_\omega(B)$ and *external activity* $e_\omega(B)$ with respect to ω, and then one has [2, §7.3,eqn (7.11)]

$$T_M(x, y) = \sum_{B \in \mathcal{B}(M)} x^{i_\omega(B)} y^{e_\omega(B)}.$$

In particular, $T_M(0, 1)$ and $T_M(1, 0)$ count the bases with internal and external activity zero, respectively. One can choose shelling orders for $\mathcal{I}(M)$ and $\underline{\Delta}_M$ having homology facets indexed by such bases; see Björner [2, §7.3, 7.6]. Consequently, one can choose the basis-to-flag shellings in Theorem 1(ii) so that their homology facets are indexed by triples (F, I, I') that combinatorially interpret the right side of (3):

- F is a flat,
- I a basis for $M|_F$ with internal activity zero, and
- I' a basis of M/F with external activity zero.

Bijections between the set $\mathcal{B}(M)$ and the set of triples $\{(F, I, I')\}$ above appear in [6, 8].

5 Proof of Corollary 3

We recall the statement of the corollary.

Corollary 3. *The action of* $\mathrm{Aut}(M)$ *on the top (reduced) homology* $\tilde{H}_{r(M)-1}(\Delta_M, \mathbb{Z})$ *is the same as its action on the top oriented simplicial chain group* $\tilde{C}_{r(M)-1}(\mathcal{I}(M), \mathbb{Z})$ *for the complex* $\mathcal{I}(M)$.

Recall that one can compute (reduced) simplicial homology $\tilde{H}_*(\Delta, \mathbb{Z})$ for a simplicial complex Δ using *oriented simplicial chains*; see, e.g., Munkres [10, §1.5]. The i^{th} chain group $\tilde{C}_d(\Delta, \mathbb{Z})$ has the following description. Fix for each i-dimensional simplex σ having vertex set $\{v_0, v_1, \ldots, v_i\}$ a *reference ordering* $(v_0, v_1, \ldots, v_i)$, and then $\tilde{C}_i(\Delta, \mathbb{Z})$ is a free abelian group having one $\mathbb{Z}$-basis element $[v_0, v_1, \ldots, v_i]$, called an *oriented simplex*, for each such σ, and for any permutation w in the symmetric group $\mathfrak{S}_{i+1}$, one sets

$$[v_{w(0)}, v_{w(1)}, \ldots, v_{w(i)}] := \mathrm{sgn}(w) \cdot [v_0, v_1, \ldots, v_i]$$

where $\mathrm{sgn}(w) \in \{+1, -1\}$ is the usual *sign* of the permutation w.

We claim that Corollary 3 will be another consequence of the flat-to-basis shellings of Δ_M from Theorem 1(i), similar to equation (2). The essential point is that matroid automorphisms permute the bases $\mathcal{B}(M)$, which index the homology facets for these shellings.

In fact, we will deduce Corollary 3 from a lemma that applies to a slightly more general notion of homology facets.

Definition 20. In a simplicial complex Δ, call a collection of its facets $\mathcal{B}$ a set of *homology facets* if the subcomplex $\Delta \setminus \mathcal{B}$ obtained by removing them is contractible.

This leads to the following generalization of Lemma 15.

Lemma 21. *When Δ has a collection $\mathcal{B}$ of homology facets, it is homotopy equivalent to a wedge of spheres:*

$$\Delta \approx \bigvee_{\sigma \in \mathcal{B}} \mathbb{S}^{\dim(\sigma)}.$$

Furthermore, any group G of simplicial automorphisms of Δ preserving $\mathcal{B}$ setwise will act on $\tilde{H}_i(\Delta, \mathbb{Z})$ via its signed permutation representation on the $\mathbb{Z}$-submodule $\mathrm{span}_{\mathbb{Z}}\{[\sigma] : \sigma \in \mathcal{B}, \dim(\sigma) = i\}$. *within $\tilde{C}_i(\Delta, \mathbb{Z})$.*

Remark 22. More explicitly, if $g(\sigma) = \sigma'$, and σ, σ' with $[\sigma] = [v_0, v_1, \ldots, v_i]$ and $[\sigma'] = [v'_0, v'_1, \ldots, v'_i]$, then $g[\sigma] = \pm[\sigma']$ where the $\pm$ is $\mathrm{sgn}(w)$ for w defined by $(g(v_0), \ldots, g(v_i)) = (v'_{w(0)}, \ldots, v'_{w(i)})$. One can also view this signed permutation representation as a direct sum $\bigoplus_\sigma \mathrm{sgn}_\sigma \uparrow_{G_\sigma}^{G}$ of *induced representations*. Here σ runs through any choice of G-orbit representatives for $\mathcal{B}$, and G_σ is the subgroup of G setwise stabilizing the vertex set $\{v_0, v_1, \ldots, v_d\}$ of σ, with $\mathrm{sgn}_\sigma : G_\sigma \to \{+1, -1\}$ its sign character.

Proof of Lemma 21. (cf. proof of [2, Theorem 7.7.2]) Name $\Delta' := \Delta \setminus \mathcal{B}$. Then the homotopy type assertion is a consequence of what Björner calls the *Contractible Subcomplex Lemma* [1, Lemma 10.2]: for a contractible subcomplex $\Delta' \subset \Delta$, the projection $\|\Delta\| \twoheadrightarrow \|\Delta\|/\|\Delta'\|$ is a homotopy equivalence.

For the homology assertion, start with the long exact sequence in integral homology for the pair (Δ, Δ'),

$$\cdots \to \tilde{H}_i(\Delta') \to \tilde{H}_i(\Delta) \to \tilde{H}_i(\Delta, \Delta') \to \tilde{H}_{i-1}(\Delta') \to \cdots$$

Contractibility of Δ' implies $\tilde{H}_i(\Delta') = 0$ for all i, giving isomorphisms $\tilde{H}_i(\Delta) \cong \tilde{H}_i(\Delta, \Delta')$. On the other hand, since each simplex in $\mathcal{B} = \Delta \setminus \Delta'$ is a *facet* of Δ, lying in no higher-dimensional faces, the boundary maps in the complex $\tilde{C}_*(\Delta, \Delta')$ computing $\tilde{H}_*(\Delta, \Delta')$ are all zero. Hence $\tilde{H}_i(\Delta, \Delta') = \tilde{C}_i(\Delta, \Delta')$ for all i. Furthermore, our assumptions on G imply that all of these isomorphisms commute with the G-action. Lastly, note $\tilde{C}_i(\Delta, \Delta)$ has the same $\mathbb{Z}$-basis and G-action as $\mathrm{span}_{\mathbb{Z}}\{[\sigma] : \sigma \in \mathcal{B}, \dim(\sigma) = i\}$ within $\tilde{C}_i(\Delta)$. $\square$

Proof of Corollary 3. Apply Lemma 21 to the flat-to-basis shellings from Theorem 1(i) of Δ_M. The homology facets are indexed by the bases $\mathcal{B}(M)$, and preserved by the group $G = \mathrm{Aut}(M)$, and all have dimension $r(M) - 1$. Furthermore, note that within $\tilde{C}_{r(M)-1}(\Delta_M, \mathbb{Z})$, one has

$$\mathrm{span}_{\mathbb{Z}}\{[\sigma] : \sigma \in \mathcal{B}(M)\} = \tilde{C}_{r(M)-1}(\mathcal{I}(M), \mathbb{Z})$$

since the facets of Δ_M indexed by bases of M happen to be exactly the facets of $\mathcal{I}(M)$. $\square$

Remark 23. Corollary 3 is closely related to an identity of representations from work of Kook, Reiner and Stanton [8] on eigenspaces of *combinatorial Laplacians* for $\mathcal{I}(M)$. Specifically, taking $i = r(M) - 1$ in their [8, Thm. 19], asserts the following isomorphism of G-representations for $G = \mathrm{Aut}(M)$:

$$\tilde{C}_{r(M)-1}(\mathcal{I}(M)) \cong \bigoplus_{F \in \mathcal{F}(M)} \left[\tilde{H}_{r(F)-1}(\mathcal{I}(M|_F)) \ \otimes \ \tilde{H}^{r(M/F)-2}(\underline{\Delta}_{M/F})\right] \uparrow_{G_F}^{G} \tag{8}$$

where $G_F = \{g \in G : g(F) = F\}$, and $[-] \uparrow_{G_F}^{G}$ denotes induction of representations from G_F to G.

As we have seen, the flat-to-basis shelling in Theorem 1(i) led to Corollary 3, showing the G-action on the left side of (8) is the same as the one on $\tilde{H}_{r(M)-1}(\Delta_M)$. Similarly, with a bit more work (details omitted here), one can use the basis-to-flat shelling in Theorem 1(ii), and its resulting bases for $\tilde{H}_{r(M)-1}(\Delta_M)$ as in Björner [2, Thm. 7.7.2] to show that the G-action on $\tilde{H}_{r(M)-1}(\Delta_M)$ is isomorphic to the direct sum on the right side of (8).

Remark 24. The aforementioned work [8] showed a remarkable property for $\mathcal{I}(M)$ and its simplicial boundary maps $\{\partial_i\}_{i=1,2,\ldots}$: their associated *combinatorial Laplacian matrices* $\{\partial_i^T \partial_i\}$ have only *integer* eigenvalues. One might therefore ask whether Δ_M shares this property. Sadly, this fails already for the Boolean matroid M of rank 2, where Δ_M is the 5-cycle graph shown in Figure 2. One can check that its Laplacian matrix $\partial_1^T \partial_1$ has characteristic polynomial $x(x^2 - 5x + 5)^2$, whose eigenvalues are not all integers.

6 Augmented Bergman complexes for other closures

One can characterize a matroid M on ground set E in terms of its *matroid closure operator* $A \longmapsto \overline{A}$ defined in (4). This is an instance of the following more general notion.

Definition 25. Given a set E, a map $2^E \xrightarrow{f} 2^E$ is called a *closure operator* on E if it satisfies three axioms: for all subsets $A, B \subseteq E$,

- C1. $A \subseteq f(A)$
- C2. $A \subseteq B$ implies $f(A) \subseteq f(B)$
- C3. $f(f(A)) = f(A)$

Any closure operator on a finite set E has analogues of the complexes $\mathcal{I}(M), \underline{\Delta}_M, \Delta_M$, introduced next.

Definition 26. Given a closure operator f on a finite set E, define a subset $I \subseteq E$ to be *independent* if

$$f(I \setminus \{i\}) \subsetneq f(I) \text{ for all } i \in I.$$

Let $\mathcal{I}(f)$ denote the collection of all independent subsets $I \subseteq E$. It is not hard to check that $\mathcal{I}(f)$ always satisfies axioms I1, I2 from Definition 6, so that it defines a simplicial complex, also denoted $\mathcal{I}(f)$.

Definition 27. Given a closure operator f on a finite set E, define its *poset of closed sets*

$$\mathcal{F}(f) := \{F \subseteq E : f(F) = F\}$$

partially ordered via inclusion. It is not hard to check that $\mathcal{F}(f)$ always satisfies axioms $\mathsf{F1}, \mathsf{F2}$ from Definition 7, so that it becomes a lattice. Define the *Bergman complex*

$$\underline{\Delta}_f := \Delta(\ \mathcal{F}(f) \setminus \{f(\varnothing), E\}\ )$$

to be the order complex of the proper part of this lattice $\mathcal{F}(f)$.

Definition 28. Given a closure operator f on a finite set E, define its *augmented Bergman complex* Δ_f to be the abstract simplicial complex on vertex set

$$\{y_i : i \in E\} \sqcup \{x_F : F \in \mathcal{F}(f) \setminus \{E\}\}$$

whose simplices are the subsets

$$\{y_i\}_{i \in I} \cup \{x_{F_1}, x_{F_2}, \ldots, x_{F_\ell}\} \tag{9}$$

for which $I \in \mathcal{I}(f)$ and the F_i are all closed sets in $\mathcal{F}(f)$, satisfying $I \subseteq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell \quad (\subsetneq E)$.

As before with matroid closures, Δ_f always contains as subcomplexes both

- $\mathcal{I}(f)$ as the simplices in (9) with $\ell = 0$, and
- $\mathrm{Cone}(\underline{\Delta}_f) = \Delta(\ \mathcal{F}(F) \setminus \{E\}\ )$ having cone vertex $x_{f(\varnothing)}$, as the simplices in (9) with $\#I = 0$.

However, in contrast to matroid closures, the complexes $\mathcal{I}(f), \underline{\Delta}_f, \Delta_f$ need not be pure, nor shellable.

Example 29. Consider the closure operator $f : 2^E \to 2^E$ with $E = [5] = \{1, 2, 3, 4, 5\}$ whose poset of closed sets $\mathcal{F}(f)$ is depicted at top left in Figure 3. One can compute the closure $f(A)$ as the intersection of all $F \in \mathcal{F}(M)$ containing A; for example, $f(\{5\}) = \{4, 5\}$ and $f(\{1, 4\}) = \{1, 2, 3, 4, 5\}$. The complexes $\mathcal{I}(f)$ and $\underline{\Delta}_f$ are shown in the top row, in the middle and at right. The second row depicts the deletion $\Delta_f \setminus \{x_\varnothing\}$ of the vertex $x_\varnothing$ from the augmented Bergman complex Δ_f.

It is not hard to show that any finite lattice is isomorphic to $\mathcal{F}(f)$ for some closure f, and hence we cannot expect to say much about the homotopy type of the Bergman complex $\underline{\Delta}_f$ in general; we expect that the same holds for the independent set complex $\mathcal{I}(f)$.

Nevertheless, we claim that one still has the assertion of Corollary 3 on the topology of the augmented Bergman complex Δ_f, after appropriately defining *bases* and *automorphisms* for a closure.

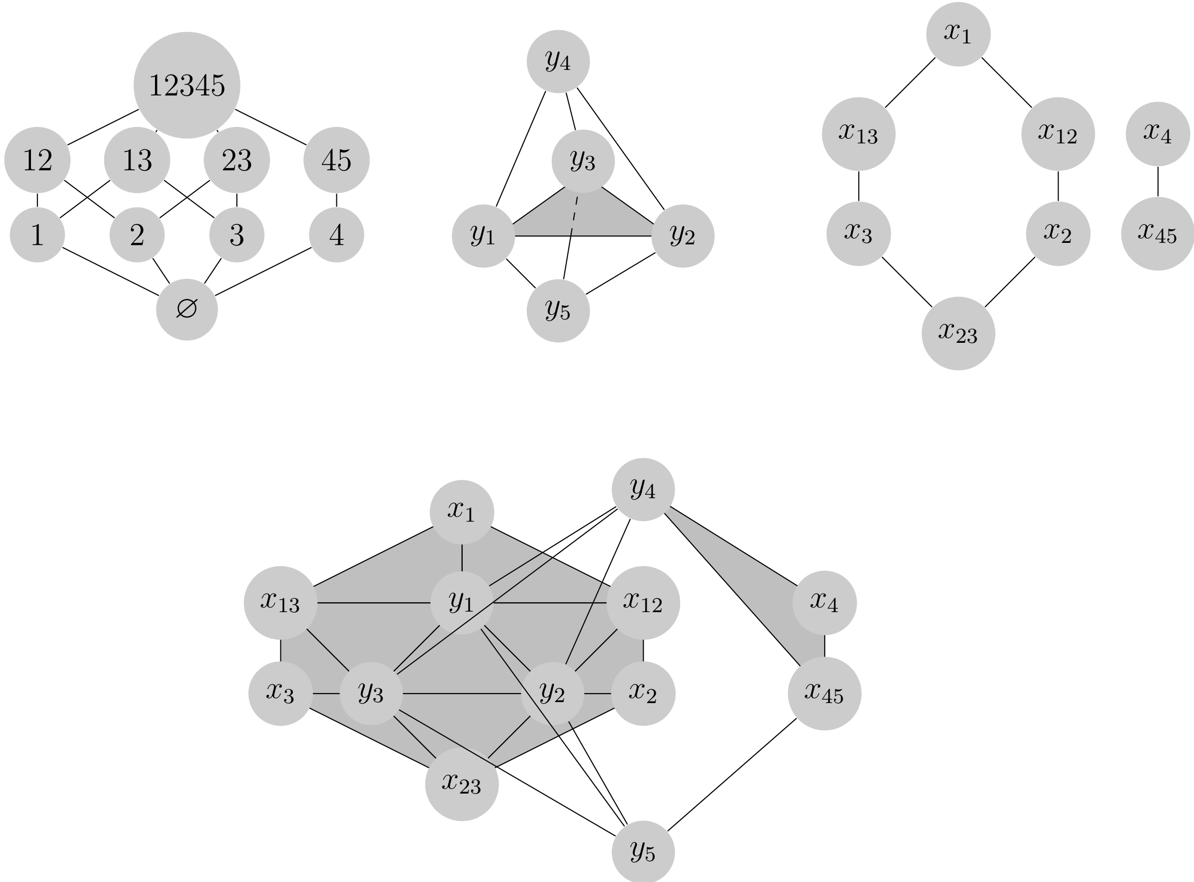

Figure 3: The lattice $\mathcal{F}(f)$, and the complexes $\mathcal{I}(f)$ and $\underline{\Delta}_f$, along with the complex $\Delta_f \setminus \{x_\varnothing\}$ for the closure in Example 29.

Definition 30. For a closure f on a finite set E, define the set $\mathcal{B}(f)$ of *bases*

$$\mathcal{B}(f) := \{B \in \mathcal{I}(f) : f(B) = E\}.$$

Definition 31. For a closure f on E, define its *automorphism group*

$$\mathrm{Aut}(f) := \{\text{bijections } E \xrightarrow{g} E : f(g(A)) = g(f(A)) \text{ for all } A \subseteq E\}$$

Note $\mathrm{Aut}(f)$ stabilizes $\mathcal{B}(f)$, and acts on $\mathcal{I}(f), \underline{\Delta}_f, \Delta_f$ via simplicial automorphisms.

Theorem 6.1. *For any closure operator on a finite set E, the bases $\mathcal{B}(f)$ index a collection of homology facets for Δ_f. Hence Δ is homotopy equivalent to a wedge of spheres $\bigvee_{B\in\mathcal{B}(f)} \mathbb{S}^{\#B-1}$. and the action of* $\mathrm{Aut}(f)$ *on* $\tilde{H}_i(\Delta, \mathbb{Z})$ *is the same as on* $\mathrm{span}_{\mathbb{Z}}\{[B] : B \in \mathcal{B}(f), \#B - 1 = i\}$. *within* $\tilde{C}_i(\Delta_f, \mathbb{Z})$.

Proof. In light of Lemma 21, it suffices to show that $\Delta = \Delta_f$ has its subcomplex $\Delta' := \Delta \setminus \mathcal{B}$ contractible. Our proof strategy introduces another simplicial complex Δ'', and shows it has these two properties:

(a) Δ'' is a simplicial subdivision of Δ', and hence homeomorphic to it.

(b) Δ'' is homotopy equivalent to the subcomplex $\mathrm{Cone}(\underline{\Delta}_f) = \Delta(\mathcal{F} \setminus \{E\})$ inside Δ_f.

Since cones are contractible, (b) would show Δ'' is contractible, and then (a) would show the same for Δ'. We define Δ'' to be the simplicial complex on vertex set

$$\{y_I : I \in \mathcal{I}(f) \setminus \mathcal{B}(f)\} \cup \{x_F : F \in \mathcal{F} \setminus \{E\}\}$$

whose simplices are subsets $\{y_{I_1}, \cdots, y_{I_k}\} \cup \{x_{F_1}, \ldots, x_{F_\ell}\}$ with $I_1 \subsetneq \cdots \subsetneq I_k \subseteq F_1 \subsetneq \cdots \subsetneq F_\ell (\subsetneq E)$.

Proof of assertion (a). One can view Δ'' as having been obtained from Δ' by performing *barycentric subdivision* [10, §2.15] $\sigma \mapsto \mathrm{Sd}(\sigma)$ to every simplex within the subcomplex $\mathcal{I}(f) \setminus \mathcal{B}(f)$ of Δ'; each simplex $\sigma = \{y_i\}_{i \in I}$ is replaced by the simplices $\{y_{I_1}, \ldots, y_{I_k}\}$ for which $I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_k \subseteq I$. More generally, the typical simplex of Δ' as in (9) is the *simplicial join* $\sigma * \sigma'$ [10, §8.62] of the simplex $\sigma = \{y_i\}_{i \in I}$ above and the simplex $\sigma' = \{x_{F_1}, \ldots, x_{F_\ell}\}$; one replaces this with the simplicial join $\mathrm{Sd}(\sigma) * \sigma'$ in Δ''.

Proof of assertion (b). Define a simplicial map $\Delta'' \xrightarrow{\pi} \Delta(\mathcal{F} \setminus \{E\})$ via this set map on vertices:

$$\begin{array}{rcll} x_F & \longmapsto & x_F & \text{for } F \in \mathcal{F}(f) \setminus \{E\} \\ y_I & \longmapsto & x_{f(I)} & \text{for } I \in \mathcal{I}(f) \setminus \mathcal{B}(f) \end{array}$$

It is not hard to check that π indeed carries simplices to simplices, that is, it is a well-defined simplicial map. One can also check that, for every element F in the poset $\mathcal{F}(f) \setminus \{E\}$, the inverse image under π of the order complex $\Delta\mathcal{F}(f)_{\leqslant F}$ of the principal order ideal $\mathcal{F}(f)_{\leqslant F}$ is the star of the vertex x_F within Δ'', and hence contractible. Thus by Quillen's Fiber Lemma [1, (10.5)(i)], the map π induces a homotopy equivalence. □

Remark 32. Note that $\mathrm{Cone}(\underline{\Delta}_M) = \Delta(\mathcal{F}(f) \setminus \{E\})$ can be identified with the subcomplex of Δ'' induced on the vertex subset $\{x_F : F \in \mathcal{F}(f) \setminus \{E\}\}$. Since these vertices x_F are all pointwise fixed by π, the same is true for this subcomplex $\mathrm{Cone}(\underline{\Delta}_M)$, so that the map π is actually a homotopy inverse to the inclusion map $\mathrm{Cone}(\underline{\Delta}_M) \hookrightarrow \Delta''$, showing that π is a *deformation retraction.*

In fact, if one removes the vertex $x_{f(\varnothing)}$ from both Δ', Δ'', one finds that $\Delta'' \setminus \{x_{f(\varnothing)}\}$ is a subdivision of $\Delta' \setminus \{x_{f(\varnothing)}\}$, and the simplicial map π also restricts to a deformation retraction

$$\Delta'' \setminus \{x_{f(\varnothing)}\} \xrightarrow{\pi} \underline{\Delta}_f. \tag{10}$$

This is depicted for the closure f from Example 29 in Figure 4. The top row shows $\Delta' \setminus \{x_{f(\varnothing)}\}$. The bottom row shows the subdivision $\Delta'' \setminus \{x_{f(\varnothing)}\}$ with the map π indicated by directed arrows along edges; to its right is the subcomplex $\underline{\Delta}_f$ onto which it retracts.

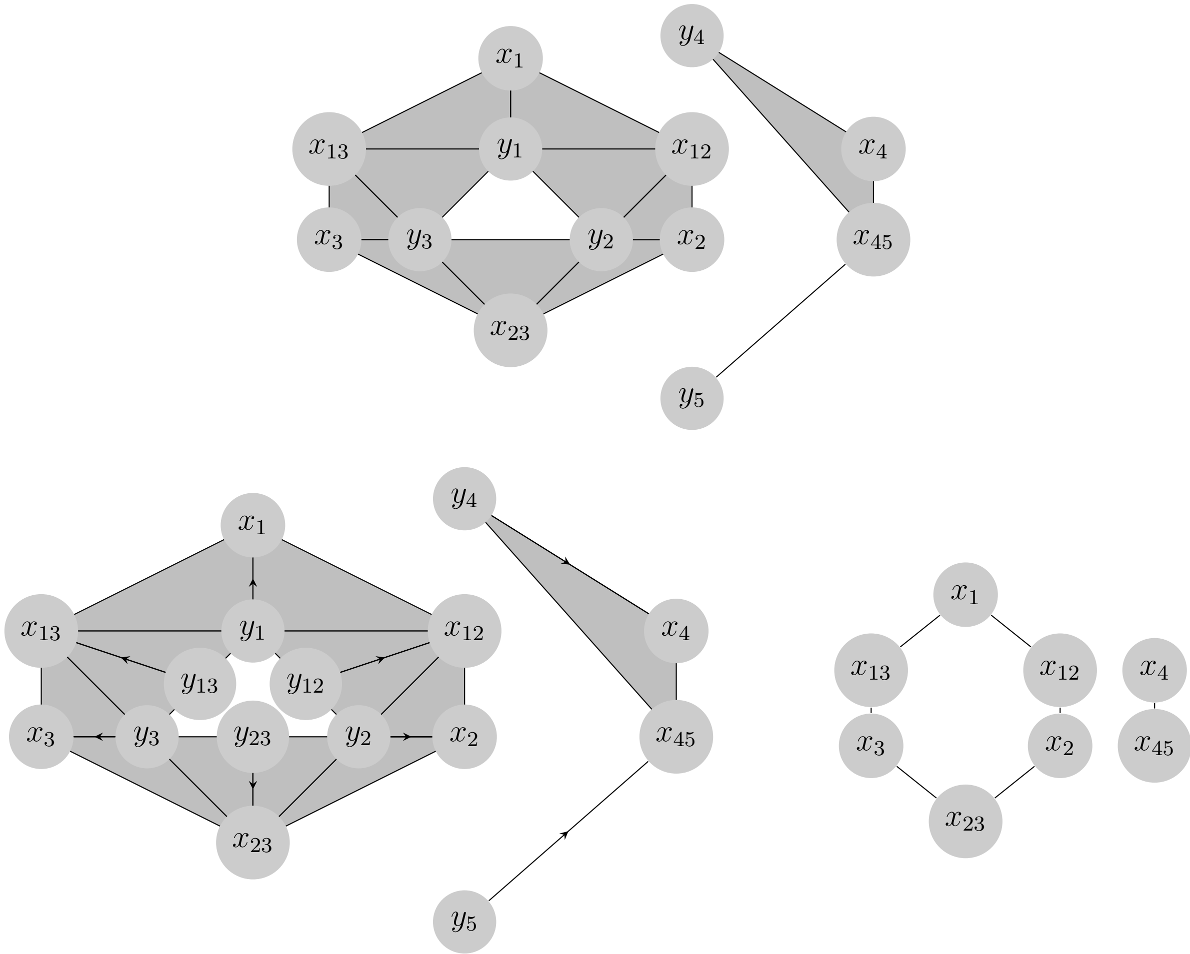

Figure 4: Continuing Example 29, the deleted subcomplex $\Delta' \setminus \{x_{f(\varnothing)}\}$ is shown at top. The second line shows its subdivision $\Delta'' \setminus \{x_{f(\varnothing)}\}$, along with directed arrows indicating the retraction $\pi : \Delta'' \setminus \{x_{f(\varnothing)}\} \to \underline{\Delta}_f$ from (10).

Acknowledgements

This work was performed during the University of Minnesota - Twin Cities School of Mathematics Summer 2021 REU program, supported by NSF RTG grant DMS-1148634. The authors thank Trevor Karn and Sasha Pevzner for presenting useful background and for valuable edits. They thank Amy Tao for initial discussions on this work. Lastly, they thank an anonymous referee for helpful edits and suggestions.

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