Andrews-Beck type congruences for overpartitions

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Abstract

We prove Andrews-Beck type congruences for overpartitions concerning the \( D \)-rank and \( M_2 \)-rank. To prove congruences, we establish the generating function for weighted \( D \)-rank (respectively, \( M_2 \)-rank) moment of overpartitions and find a connection with the second \( D \)-rank (respectively, \( M_2 \)-rank) moment for overpartitions.

Mathematics Subject Classifications: 11P81, 05A17

1 Introduction

Ramanujan’s congruences for the partition function \( p(n) \) are one of remarkable results in the theory of partitions:

\[
\begin{align*}
p(5n + 4) & \equiv 0 \pmod{5}, \\
p(7n + 5) & \equiv 0 \pmod{7}, \\
p(11n + 6) & \equiv 0 \pmod{11},
\end{align*}
\]

Dyson [8] defined the rank of a partition, which is defined as the largest part minus the number of parts, conjectured combinatorial explanations for the Ramanujan congruences modulo 5 and 7, and conjectured the existence of a crank function for partitions that could provide a combinatorial proof of Ramanujan’s congruences modulo 11. Atkin and Swinnerton-Dyer [3] proved Dyson’s conjecture on the rank. Andrews and Garvan [2] found the crank function and proved that the crank explains all Ramanujan congruences modulo 5, 7 and 11.

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Let $NT(b, k, n)$ be the total number of parts in the partitions of $n$ with rank congruent to $b$ modulo $k$. Beck conjectured surprising congruences for the certain linear combinations among $NT(b, k, b)$. Andrews [1] has confirmed Beck’s conjectures: for all non-negative integers $n$,

$$NT(1, 5, 5n + i) + 2NT(2, 5, 5n + i)$$

$$-2NT(3, 5, 5n + i) - NT(4, 5, 5n + i) \equiv 0 \pmod{5}$$

for $i = 1$ or $4$, and

$$NT(1, 7, 7n + i) + 2NT(2, 7, 7n + i) + 3NT(3, 7, 7n + i)$$

$$-3NT(4, 7n + i) - 2NT(5, 7n + i) - NT(6, 7, 7n + i) \equiv 0 \pmod{7}$$

for $i = 1$ or $i = 5$. A crank version of Beck’s conjecture is confirmed by Chern [6]. For example, if $M_\omega(b, k, n)$ counts the total number of ones in the partitions of $n$ with crank congruent to $b$ modulo $k$, we have, for all integers $n \geq 0$,

$$M_\omega(1, 5, 5n + 4) + 2M_\omega(2, 5, 5n + 4) - 2M_\omega(3, 5, 5n + 4) - M_\omega(4, 5, 5n + 4) \equiv 0 \pmod{5}.$$

In the recent article [7], Chern also provided a list of over 70 Andrews-Beck type congruences involving $NT(b, k, b)$ and $M_\omega(b, k, n)$.

Now, we will consider overpartition analogue to Andrew-Beck type congruences. Recall that an overpartition is a partition in which the first occurrence of a number may be overlined. For example, the 14 overpartitions of 4 are

$$4, \overline{4}, 3 + 1, \overline{3} + 1, 3 + \overline{1}, 3, 2 + 2, 2 + 2,$$

$$2 + 1 + 1, \overline{2} + 1 + 1, 2 + \overline{1} + \overline{1}, \overline{2} + \overline{1} + 1, 1 + 1 + 1, \overline{1} + 1 + 1.$$

For an overpartition $\lambda$ of $n$, the $D$-rank of $\lambda$ [11] is defined as Dyson’s rank for ordinary partition,

$$D\text{-rank}(\lambda) = \ell(\lambda) - #(\lambda),$$

and the $M_2$-rank of $\lambda$ [12] is defined by

$$M_2\text{-rank}(\lambda) = \left\lceil \frac{\ell(\lambda)}{2} \right\rceil - #(\lambda) + #(\lambda_o) - \chi(\lambda),$$

where $\ell(\lambda)$ is the largest part of $\lambda$, $#(\lambda)$ is the number of parts in $\lambda$, $#(\lambda_o)$ is the number of odd non-overlined parts of $\lambda$, and $\chi(\lambda) = 1$ if the largest part of $\lambda$ is odd and non-overlined and $\chi(\lambda) = 0$ otherwise.

Let $\overline{NT}(b, k, n)$ denote the total number of parts in the overpartitions of $n$ with $D$-rank congruent to $b$ modulo $k$ and $\overline{NT2}(b, k, n)$ denote the total number of parts in the overpartitions of $n$ with $M_2$-rank congruent to $b$ modulo $k$. Then the following congruences are proved by Chan-Mao-Osburn [5]: for all $n \in \mathbb{N}$,

$$\overline{NT2}(1, 5, 5n+2) + 2\overline{NT2}(2, 5, 5n+2) - 2\overline{NT2}(3, 5, 5n+2) - \overline{NT2}(4, 5, 5n+2) \equiv 0 \pmod{5}$$

(1.1)
and
\[ NT(1, 3, 3n+i) - NT(2, 3, 3n+i) \equiv NT_2(1, 3, 3n+i) - NT_2(2, 3, 3n+i) \pmod{3} \] (1.2)
for \( i = 0 \) or 1.

In this paper, we prove Andrews-Beck type congruence on \( NT(b,k,n) \) and \( NT_2(b,k,n) \) modulo 4 and 8 as follows.

**Theorem 1.** For all integers \( n \geq 0 \),
\[ NT(1, 4, 2n + 1) + 2NT(2, 4, 2n + 1) + 3NT(3, 4, 2n + 1) \equiv 0 \pmod{4}. \]

**Theorem 2.** For all integers \( n \geq 0 \),
\[ NT_2(1, 4, 4n + 1) + 2NT_2(2, 4, 4n + 1) + 3NT_2(3, 4, 4n + 1) \equiv 0 \pmod{4}, \]
\[ NT_2(1, 4, 4n + 2) + 2NT_2(2, 4, 4n + 2) + 3NT_2(3, 4, 4n + 2) \equiv 0 \pmod{4}, \]

and
\[ NT_2(1, 8, 4n + 2) + 2NT_2(2, 8, 4n + 2) + 3NT_2(3, 8, 4n + 2) \]
\[ + 4NT_2(4, 8, 4n + 2) + 5NT_2(5, 8, 4n + 2) + 6NT_2(6, 8, 4n + 2) \]
\[ + 7NT_2(7, 8, 4n + 2) \equiv 0 \pmod{8}. \]

Lastly, we also prove a congruence between \( NT(b,k,n) \) and \( NT_2(b,k,n) \).

**Theorem 3.** For all integers \( n \geq 0 \), we have
\[ \sum_{j=1}^{7} jNT(j, 8, 4n + 1) \equiv \sum_{j=1}^{7} jNT_2(j, 8, 4n + 1) \pmod{8}. \]

The rest of the paper is organized as follows. In Section 2, we establish the generating function for weighted \( D \)-rank moment of overpartitions and find a relation with the second \( D \)-rank moment for overpartitions, from which we can prove Theorem 1. Also, we discover more congruences on \( NT(b,k,n) \). In Section 3, the generating function for weighted \( M_2 \)-rank moment of overpartitions and a proof of Theorem 2 will be presented. Employing generalized Lambert series identities, we prove the congruence between \( NT(b,k,n) \) and \( NT_2(b,k,n) \) in Section 4.

### 2 Weighted \( D \)-rank moments of overpartitions

Using standard combinatorial arguments in partition theory as [11, Proposition 1.1], we find that
\[ R(x, z, q) := \sum_{n \geq 0} \sum_{\lambda \in \mathcal{P}_n} x^{\#(\lambda)} z^{D\text{-rank}(\lambda)} q^n = \sum_{n \geq 0} \frac{(-1)^n x^n q^{n(n+1)/2}}{(zq, xq/z)_n}, \] (2.1)
where \( P_n \) is the set of overpartitions of \( n \).

Here and throughout the rest of the paper, we use the standard \( q \)-series notation,

\[
(a)_n = (a; q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}),
\]

\[
(a_1, \ldots, a_m)_n = (a_1, \ldots, a_m; q)_n := (a_1)_n \cdots (a_m)_n,
\]

and

\[
[a_1, \ldots, a_m]_n = [a_1, \ldots, a_m; q]_n = (a_1, q/a_1, \ldots, a_m, q/a_m)_n,
\]

for \( n \in \mathbb{N}_0 \cup \{\infty\} \).

We will give two proofs for Theorem 1. For the first proof of Theorem 1, we will establish the generating function for the weighted \( D \)-rank moment of overpartitions and compare it with the ordinary and symmetrized \( D \)-rank moments. Here, the ordinary and symmetrized \( D \)-rank moments are defined by

\[
\overline{N}_k(n) = \sum_{m \in \mathbb{Z}} m^k \overline{N}(m, n),
\]

\[
\overline{\eta}_k(n) = \sum_{m \in \mathbb{Z}} \left( m - \left\lfloor \frac{k-1}{2} \right\rfloor \right) \overline{N}(m, n),
\]

where \( \overline{N}(m, n) \) denotes the number of overpartitions of \( n \) with \( D \)-rank \( m \).

**Theorem 4.** We have

\[
\sum_{n \geq 0} \sum_{\lambda \in P_n} \#(\lambda) \text{D-rank}(\lambda) q^n = -\sum_{n \geq 1} \frac{(-1)_n q^{n(n+1)/2}}{(q^2)_n} \sum_{m \geq 1} \frac{q^m}{(1 - q^m)^2},
\]

which implies

\[
\sum_{\lambda \in P_n} \#(\lambda) \text{D-rank}(\lambda) = -\frac{1}{2} \overline{N}_2(n) = -\overline{\eta}_2(n).
\]

It follows that

\[
\sum_{n \geq 1} \frac{(-1)_n q^{n(n+1)/2}}{(q^2)_n} \sum_{m \geq 1} \frac{q^m}{(1 - q^m)^2} = 2\frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n+1)}}{(1 - q^n)^2}.
\]

**Proof.** Applying \([\partial / \partial x]_{x=1}\) to the generating function \( \overline{R}(x, z, q) \) (2.1), we have

\[
\sum_{n \geq 0} \sum_{\lambda \in P_n} \#(\lambda) x^{D-\text{rank}(\lambda)} q^n = \frac{\partial}{\partial x} \left[ \sum_{n \geq 0} \frac{(-1)_n x^n q^{n(n+1)/2}}{(z, xq/z)_n} \right]_{x=1} = \sum_{n \geq 0} \left[ \frac{(-1)_n x^n q^{n(n+1)/2}}{(z, xq/z)_n} \frac{\partial}{\partial x} \log \left( \frac{x^n}{(xq/z)_n} \right) \right]_{x=1}
\]

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Theorem 4, we can give a proof of Theorem 1. The generating function of
and $S$ where the sequence $D$
from the following relation between the ordinary and symmetrized
by comparing with (2.2), which implies that

$$
\sum \left( \frac{(-1)_n q^{n(n+1)/2}}{(zq, q/z)_n} \right) \left[ \frac{n}{x} + \sum_{m=1}^{n} \frac{q^m}{z-xq^m} \right]_{x=1}
$$

Then if we differentiate it by $z$ and evaluate it at $z = 1$, we get

$$
\sum \sum_{n \geq 0 \lambda \in \mathcal{P}_n} \#(\lambda) D\text{-rank}(\lambda) q^n = \frac{\partial}{\partial z} \left[ \sum_{n \geq 0} (-1)_n q^{n(n+1)/2} (zq, q/z)_n \left( n + \sum_{m=1}^{n} \frac{q^m}{z-q^m} \right) \right]_{z=1}
$$

which proves the first part.

If we apply $[\frac{\partial}{\partial z}]_{z=1}$ to $R(1, z, q)$, then we have the generating function for the second $D$-rank moment as follows.

$$
\sum \sum_{n \geq 0 \lambda \in \mathcal{P}_n} \#(\lambda) D\text{-rank}(\lambda) q^n = \sum \sum_{n \geq 0 \lambda \in \mathcal{P}_n} D\text{-rank}(\lambda)^2 q^n
$$

by comparing with (2.2), which implies that $\sum_{\lambda \in \mathcal{P}_n} \#(\lambda) D\text{-rank}(\lambda) = -\frac{1}{2} \mathcal{N}_2(n)$. Also, from the following relation between the ordinary and symmetrized $D$-rank moments

$$
\mathcal{N}_{2k}(n) = \sum_{j=1}^{k} (2j) ! S^*(k, j) \tilde{\eta}_2(n),
$$

where the sequence $S^*(n, k)$ is defined recursively by $S^*(n+1, k) = S^*(n, k-1) + k^2 S^*(n, k)$ and $S^*(1, 1) = 1, S^*(n, k) = 0$ for $k \leq 0$ or $k > n$, we can see that $\frac{1}{2} \mathcal{N}_2(n) = \tilde{\eta}_2(n)$. Finally, the generating function of $\tilde{\eta}_2(n)$ [10, Theorem 2.1] gives

$$
\sum \frac{(-1)_n q^{n(n+1)/2}}{(q)_n^2} \sum_{m=1}^{n} \frac{q^m}{(1-q^m)^2} = 2 \frac{(q)_{\infty}}{(q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^n q^{n(n+1)}}{(1-q^n)^2}.
$$

Using the generating function for the weighted $D$-rank moment of overpartitions in Theorem 4, we can give a proof of Theorem 1.
Proof of Theorem 1. By Theorem 4, we have
\[
\sum_{n \geq 0} (NT(1, 4, n) + 2NT(2, 4, n) + 3NT(3, 4, n)) q^n \\
\equiv -2 \frac{(-q)_\infty}{(q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1}q^{n(n+1)}}{(1-q^n)^2} \pmod{4}.
\]
Using \((-q)_\infty/(q)_\infty \equiv 1 \pmod{2},
\[
2 \frac{(-q)_\infty}{(q)_\infty} \sum_{n \geq 1} \frac{(-1)^n q^{n(n+1)}}{(1-q^n)^2} \equiv 2 \sum_{n \geq 1} \frac{(-1)^n q^{n(n+1)}}{(1-q^n)^2} \pmod{4}
\[
= 2 \sum_{n \geq 1} \frac{(-1)^n q^{n(n+1)}}{(1-q^n)^2} \sum_{m \geq 1} mq^{n(m-1)}
\[
= 2 \sum_{n \geq 1} \sum_{m \geq 1} (2m-1)q^{4n^2-2n+(2m-2)(2n-1)} (q^{4n+2m-2} - 1)
\[
+ 2 \sum_{n \geq 1} \sum_{m \geq 1} (2m)q^{4n^2-2n+(2m-1)(2n-1)} (q^{4n+2m-1} - 1)
\]
\[
\equiv 2 \sum_{n \geq 1} \sum_{m \geq 1} q^{4n^2-2n+(2m-2)(2n-1)} (q^{4n+2m-2} - 1) \pmod{4},
\]
Since the last sum involves only terms where the power of \(q\) is even, the result follows. \(\square\)

From Theorem 1, we can have the following congruence for the second \(D\)-rank moment.

Corollary 5. For all integers \(n \geq 0\),
\[
\overline{N}_2(2n+1) \equiv 0 \pmod{2}.
\]

In fact, we prove more detailed results on congruences of \(NT(b, k, n)\), which also deduce the congruence in Theorem 1.

Theorem 6. For all non-negative integers \(n\),
\[
\overline{NT}(0, 4, 4n + i) \equiv 0 \pmod{4} \quad \text{for } i = 0, 2, 3,
\]
\[
\overline{NT}(2, 4, n) \equiv 0 \pmod{4},
\]
\[
\overline{NT}(2, 4, 4n + i) \equiv 0 \pmod{8} \quad \text{for } i = 1, 2, 3,
\]
\[
\overline{NT}(1, 4, 2n + 1) \equiv 0 \pmod{4},
\]
\[
\overline{NT}(3, 4, 2n + 1) \equiv 0 \pmod{4}.
\]

Proof. Applying Proposition 2.2 in [5] (a generalization [1, Theorem 3]) with setting \(d = 1\) and \(e \to 0\), we can rewrite (2.1) as follows.
\[
\overline{R}(x, z, q) = \sum_{n \geq 0} \frac{(-1)_n x^n q^{n(n+1)/2}}{(zq, xq/z)_n}
\]
\[ n \text{ has only even powers of } i \pmod{4} \text{ for which } n \text{ has only terms with the powers of } z \]

For the case \( b \), which implies that

\[ \sum_{n \geq 1} \NT(b, k, n) q^n = -\frac{\partial}{\partial x} \left[ \sum_{n \geq 1} \frac{(xq_{n+1})\infty(-1)_n}{(xq_{n+1})\infty(q)_{n-1}} (-x)^n q^{n(n+1)} \frac{q^{(b-1)n} + x^{k-b}q^{(k-1-b)n}}{1 - q^{kn}} \right]_{x=1} \]

\[ = -\left[ \sum_{n \geq 1} \frac{(xq_{n+1})\infty(-1)_n}{(xq_{n+1})\infty(q)_{n-1}} (-x)^n q^{n(n+1)} \frac{q^{n(n+1)+(b-1)n}}{1 - q^{kn}} \frac{\partial}{\partial x} \log \frac{(-xq_{n+1})\infty x^n}{(xq_{n+1})\infty x^n} \right]_{x=1} \]

\[ - 2 \left( -\frac{q}{q_{\infty}} \right) \sum_{n \geq 1} (-1)^n q^{n(n+1)} \frac{1 - q^{(b-1)n} + q^{(k-1-b)n}}{1 + q^n} \frac{1}{1 - q^{kn}} \left( n + 2 \sum_{m \geq n} \frac{q^m}{1 - q^{2m}} \right) \]

Setting \( k = 4 \) and using \((1 - x)/(1 + x) \equiv 1 \pmod{2}\) yield

\[ \sum_{n \geq 0} \NT(b, 4, n) q^n \equiv 2 \sum_{n \geq 1} q^{2n(2n-1)} \frac{q^{(b-1)(2n-1)} + q^{(3-b)(2n-1)}}{1 - q^{4(2n-1)}} \]

\[ + 2b \sum_{n \geq 1} (-1)^n q^{n(n+1)} \frac{q^{(3-b)n}}{1 - q^{4n}} \pmod{4}. \]

For the case \( b = 0 \), we have

\[ \sum_{n \geq 0} \NT(0, 4, n) q^n \equiv 2 \sum_{n \geq 1} q^{(2n-1)^2} \pmod{4}, \]

which has only terms with the powers of \( q \) congruent 1 modulo 4. Hence, \( \NT(0, 4, 4n+i) \equiv 0 \pmod{4} \) for \( i = 0, 2, 3 \) and all integers \( n \geq 0 \). For the case \( b = 1 \) and 3, we find that

\[ \sum_{n \geq 0} \NT(b, 4, n) q^n \equiv 2 \sum_{n \geq 1} q^{2n(2n-1)} \frac{q^{(b-1)(2n-1)}}{1 - q^{4(2n-1)}} + 2 \sum_{n \geq 1} q^{2n(2n+1)} \frac{q^{(3-b)(2n)}}{1 - q^{8n}} \pmod{4} \]

has only even powers of \( q \), which implies that \( \NT(1, 4, 2n+1) \equiv \NT(3, 4, 2n+1) \equiv 0 \pmod{4} \) for all integers \( n \geq 0 \). Similarly, for the case \( b = 2 \), from (2.3),

\[ \sum_{n \geq 0} \NT(2, 4, n) q^n \equiv 4 \sum_{n \geq 1} q^{2n(2n-1)} \frac{q^{2n-1}}{1 - q^{4(2n-1)}} + 4 \sum_{n \geq 1} (-1)^n q^{n(n+1)} \frac{q^n}{1 - q^{4n}} \pmod{8} \]

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\[ 4 \sum_{n \geq 1} q^{2n(2n+1)} \frac{q^{2n}}{1 - q^{2n}} \quad (\text{mod } 8), \]

which includes terms only with the powers of \( q \) congruent 0 modulo 4. This proves that \( NT(2, 4, n) \equiv 0 \) (mod 4) and \( NT(2, 4, 4n + i) \equiv 0 \) (mod 8) for \( i = 1, 2, 3 \) for all integers \( n \) greater than 0. \qed

3 Weighted \( M_2 \)-rank moments of overpartitions

As in Section 2, we find the generating function for the weighted \( M_2 \)-rank moments of overpartitions and compare it with the ordinary and symmetrized \( M_2 \)-rank moments. We have the ordinary and symmetrized \( M_2 \)-rank moments defined by

\[ N_2^k(n) = \sum_{m \in \mathbb{Z}} m^k \overline{N}_2(m, n), \]
\[ \eta_2^k(n) = \sum_{m \in \mathbb{Z}} \left( m - \frac{k-1}{2} \right) \overline{N}_2(m, n), \]

where \( \overline{N}_2(m, n) \) denotes the number of overpartitions of \( n \) with \( M_2 \)-rank \( m \). The generating function [12, Theorem 1.2] is

\[ \overline{R}_2(x, z, q) := \sum_{n \geq 0} \sum_{\lambda \in \mathcal{P}_n} x^{\#(\lambda)} z^{M_2\text{-rank}\(\lambda\)} q^n = \sum_{n \geq 0} \frac{(-1, -q; q^2)_n (xq)_n}{(zq^2, xq^2/z; q^2)_n}. \]

**Theorem 7.** We have

\[ \sum_{n \geq 0} \sum_{\lambda \in \mathcal{P}_n} \#(\lambda) \overline{N}_2\text{-rank}\(\lambda\) q^n = -\sum_{n \geq 1} \frac{(-1, -q; q^2)_n q^n}{(q^2; q^2)_n^2} \sum_{m=1}^{n} \frac{q^{2m}}{(1 - q^{2m})^2}, \]

which implies

\[ \sum_{\lambda \in \mathcal{P}_n} \#(\lambda) \overline{N}_2\text{-rank}\(\lambda\) = -\frac{1}{2} \overline{N}_2(n) = -\eta_2^2(n). \]

It follows that

\[ \sum_{n \geq 1} \frac{(-1, -q; q^2)_n q^n}{(q^2; q^2)_n^2} \sum_{m=1}^{n} \frac{q^{2m}}{(1 - q^{2m})^2} = 2 \frac{(-q)_\infty}{(q)_\infty} \sum_{n \geq 1} \frac{(-1)^n q^{n(n+2)}}{(1 - q^{2n})^2}. \]

**Proof of Theorem 7.** If we differentiate \( \overline{R}_2(x, z, q) \) by \( x \) and evaluate it at \( x = 1 \), we get

\[ \sum_{n \geq 0} \sum_{\lambda \in \mathcal{P}_n} \#(\lambda) z^{M_2\text{-rank}\(\lambda\)} q^n = \frac{\partial}{\partial x} \left[ \sum_{n \geq 0} \frac{(-1, -q; q^2)_n (xq)_n}{(zq^2, xq^2/z; q^2)_n} \right]_{x=1}. \]
\[
\begin{align*}
&= \sum_{n \geq 0} \left[ \frac{(-1, -q; q^2)_n (xq)^n}{(zq^2, xq^2/z; q^2)_n} \frac{\partial}{\partial x} \log \left( \frac{x^n}{(xq^2/z; q^2)_n} \right) \right]_{x=1} \\
&= \sum_{n \geq 1} \frac{(-1, -q; q^2)_n q^n}{(zq^2, q^2/z; q^2)_n} \left( n + \sum_{m=1}^{n} \frac{q^{2m}}{z - q^{2m}} \right).
\end{align*}
\]

Then applying \([\partial/\partial z]_{z=1}\) gives
\[
\sum_{n \geq 0} \sum_{\lambda \in \mathcal{P}_n} \#(\lambda) M_2\text{-rank}(\lambda) q^n = \frac{\partial}{\partial z} \left[ \sum_{n \geq 1} \frac{(-1, -q; q^2)_n q^n}{(zq^2, q^2/z; q^2)_n} \left( n + \sum_{m=1}^{n} \frac{q^{2m}}{z - q^{2m}} \right) \right]_{z=1}
= -\sum_{n \geq 1} \frac{(-1, -q; q^2)_n q^n}{(q^2; q^2)_n^2} \sum_{m=1}^{n} \frac{q^{2m}}{(1 - q^{2m})^2},
\]
which is the first part. Next, to compare with the \(M_2\)-rank moments, when we apply \([\partial/\partial z] (z \frac{\partial}{\partial z})\) \(z=1\) to \(\overline{R}_2(1, z, q)\), we find that
\[
\sum_{n \geq 0} \overline{N}_2(n) q^n = \sum_{n \geq 0} \sum_{\lambda \in \mathcal{P}_n} M_2\text{-rank}(\lambda)^2 q^n
= 2 \sum_{n \geq 1} \frac{(-1, -q; q^2)_n q^n}{(q^2; q^2)_n^2} \sum_{m=1}^{n} \frac{q^{2m}}{(1 - q^{2m})^2},
\]
(3.2)

Then the second part follows from comparing (3.1) with (3.2) and the following relation between the ordinary and symmetrized \(M_2\)-rank moments
\[
\overline{N}_{2k}(n) = \sum_{j=1}^{k} (2j)! S^*(k, j) \eta_{2j}(n).
\]
Lastly, we have the last identity by considering the generating function for \(\eta_{2j}(n)\) [10, Theorem 2.1].

From the generating function for weighted \(M_2\)-rank moment of overpartitions, we can prove Theorem 2.

**Proof of Theorem 2.** By Theorem 7, we notice that
\[
\sum_{n \geq 0} \left( NT_2(1, 4, n) + 2 NT_2(2, 4, n) + 3 NT_2(3, 4, n) \right) q^n \equiv -2 \frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n+2)}}{(1 - q^{2n})^2} \pmod{4}.
\]

By applying \((-q)/q \equiv 1 \pmod{2}\),
\[
2 \frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^{n} q^{n(n+2)}}{(1 - q^{2n})^2} \equiv 2 \sum_{n \geq 1} \frac{(-1)^{n} q^{n(n+2)}}{(1 - q^{n})^2} \pmod{4}
\]
\[\equiv 2 \sum_{n \geq 1} \sum_{m \geq 1} \left( q^{4n(n+1)+8n(m-1)} - q^{4n^2-1+4(2n-1)(m-1)} \right) \quad (\text{mod 4}),\]

which contributes the term with powers congruent to 0 or 3 modulo 4. Then we have
\[
\begin{align*}
\mathcal{N}T^2(1, 4, 4n + 1) + 2\mathcal{N}T^2(2, 4, 4n + 1) + 3\mathcal{N}T^2(3, 4, 4n + 1) &\equiv 0 \quad (\text{mod 4}) \\
\mathcal{N}T^2(1, 4, 4n + 2) + 2\mathcal{N}T^2(2, 4, 4n + 2) + 3\mathcal{N}T^2(3, 4, 4n + 2) &\equiv 0 \quad (\text{mod 4}).
\end{align*}
\]

Also, by Theorem 7, we find that
\[\sum_{n \geq 0} \sum_{j=1}^{7} j\mathcal{N}T^2(j, 8, n)q^n \equiv -2\frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^{n+1}q^{n(n+2)}}{(1 - q^{2n})^2} \quad (\text{mod 8}).\]

Using
\[
\frac{(-q)_{\infty}}{(q)_{\infty}} = \frac{(q)_{\infty}}{(q)_{\infty}} \quad (\text{mod 4}) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2},
\]

we have
\[
2\frac{(-q)_{\infty}}{(q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^n q^{n(n+2)}}{(1 - q^{2n})^2} \equiv 2 \left( 1 + 2 \sum_{k \geq 1} (-1)^k q^{k^2} \right) \sum_{n \geq 1} \frac{(-1)^n q^{n(n+2)}}{(1 - q^{2n})^2} \quad (\text{mod 8})
\]
\[
= 2 \left( 1 + 2 \sum_{k \geq 1} (-1)^k q^{k^2} \right) \sum_{n \geq 1} (-1)^n q^{n(n+2)} \sum_{m \geq 1} mq^{2n(m-1)}
\]
\[
\equiv 2 \left( 1 + 2 \sum_{k \geq 1} (-1)^k q^{k^2} \right) \sum_{n \geq 1} \sum_{m \geq 1} (2m - 1) \left( q^{4n(n+1)+8n(m-1)} - q^{4n^2-1+4(2n-1)(m-1)} \right)
\]
\[
+ 2 \sum_{n \geq 1} \sum_{m \geq 1} (2m) \left( q^{4n(n+1)+4n(2m-1)} - q^{4n^2-1+2(2n-1)(2m-1)} \right) \quad (\text{mod 8}).
\]

Here,
\[
1 + 2 \sum_{k \geq 1} (-1)^k q^{k^2}
\]

only contributes to terms with powers of \(q\) congruent to 0 or 1 modulo 4, while the factor
\[
\sum_{n \geq 1} \sum_{m \geq 1} (2m - 1) \left( q^{4n(n+1)+8n(m-1)} - q^{4n^2-1+4(2n-1)(m-1)} \right)
\]

involves terms with powers of \(q\) congruent to 0 or 3 modulo 4. Therefore, the first term in (3.3) does not have terms with powers of \(q\) congruent to 2 modulo 4. The second term in (3.3) has only terms with powers of \(q\) congruent to 0 or 1 modulo 4. Hence, we obtain
\[
\mathcal{N}T^2(1, 8, 4n + 2) + 2\mathcal{N}T^2(2, 8, 4n + 2) + 3\mathcal{N}T^2(3, 8, 4n + 2)
\]
\[ +4N^2T_2(4, 8, 4n + 2) + 5N^2T_2(5, 8, 4n + 2) + 6N^2T_2(6, 8, 4n + 2) + 7N^2T_2(7, 8, 4n + 2) \equiv 0 \pmod{8}. \]

From Theorem 2 and (1.1), we can have congruences for the second \(M_2\)-rank moments as follows.

**Corollary 8.** For all integers \(n \geq 0\),
\[
N^2_2(4n + 1) \equiv 0 \pmod{2}, \quad N^2_2(4n + 2) \equiv 0 \pmod{4}, \quad N^2_2(5n + 2) \equiv 0 \pmod{5}.
\]

### 4 Proof of Theorem 3

Let
\[
\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}.
\]

Then, by Jacobi’s triple product identity and simple \(q\)-series manipulations, we have
\[
\varphi(q) = \frac{(q^2; q^2)^5_\infty}{(q^2; q^2^4)^2_\infty}, \quad \varphi(-q) = \frac{(q^2)_\infty^2}{(q^2; q^2)_\infty^2}, \quad \text{and} \quad \psi(q) = \frac{(q^2; q^2)^2_\infty}{(q)_\infty^2}
\]
and
\[
\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4), \quad \varphi(q)\varphi(-q) = \varphi^2(-q^2), \quad \text{and} \quad \psi^2(q) = \varphi(q)\psi(q^2).
\] (4.1)

We will also use the following identities [4, Theorem 6.1, and 7.1].
\[
\varphi^4(-q) = 1 + 8 \sum_{n \geq 1} nq^{n(n+2)} + q^n(n^2) - nq^n \quad (1 + q^n)^2 \quad (4.2)
\]
\[
\psi^4(q) = \sum_{n \geq 0} \frac{(2n + 1 + 2q^{2n+1} - (2n + 1)q^{4n+2})q^{2n^2+2n}}{(1 - q^{2n+1})^2}. \quad (4.3)
\]

We prove Theorem 3 using generalized Lambert series identities with above theta function identities.

**Proof of Theorem 3.** Let
\[
a(n) = \sum_{j=1}^{7} jN^2T(j, 8, n) - \sum_{j=1}^{7} jNT_2(j, 8, n).
\]

By Theorem 4 and 7, we have
\[
\sum_{n \geq 0} a(n)q^n \equiv -2 \frac{(-q)_\infty}{(q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1}q^{n(n+1)}}{(1 - q^n)^2} + 2 \frac{(-q)_\infty}{(q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1}q^{n(n+2)}}{(1 - q^{2n})^2} \pmod{8}
\]
We consider the 4-dissection for \( \frac{(-q)_{\infty}}{(q)_{\infty}} \) [9],

\[
\frac{(-q)_{\infty}}{(q)_{\infty}} = \frac{(q^8; q^8)^{19}_{\infty}}{(q^4; q^4)^{14}_{\infty}(q^{16}; q^{16})^6_{\infty}} + 2q \frac{(q^8; q^8)^{13}_{\infty}}{(q^4; q^4)^{12}_{\infty}(q^{16}; q^{16})^2_{\infty}} + 4q^2 \frac{(q^8; q^8)^{7}_{\infty}(q^{16}; q^{16})^2_{\infty}}{(q^4; q^4)^{10}_{\infty}} + 8q^3 \frac{(q^8; q^8)^{3}_{\infty}(q^{16}; q^{16})^6_{\infty}}{(q^4; q^4)^{8}_{\infty}}.
\]

For the summation part in (4.4),

\[
\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)} - (-1)^n q^{n(n+2)}}{(1 - q^n)^2 - (1 - q^{2n})^2}
\]

\[
= 2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}(1 + q^n + 3q^{2n} + q^{3n})}{(1 - q^{4n})^2}
\]

\[
= 2 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}(1 + q^n + 3q^{2n} + q^{3n})}{(1 + q^{4n})^2} \pmod{8}
\]

(4.6)

using the fact

\[
\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)} q^{mn}}{(1 - q^{4n})^2} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)} q^{(6-m)n}}{(1 - q^{4n})^2}.
\]

By letting

\[
F_{a,b}(q) := \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+a)}}{(1 + q^{4n})^2} \quad \text{for } a = 1, 2, 3, \text{ and } 4,
\]

we can have 4-dissection of (4.6). Invoking (4.5) and the dissection of (4.6) into (4.4) and collecting only terms where the power of q is congruent to 1 modulo 4 yield

\[
\sum_{n\geq0} a(4n + 1)q^{4n+1}
\]

\[
\equiv 2 \frac{(q^8; q^8)^{13}_{\infty}}{(q^4; q^4)^{12}_{\infty}(q^{16}; q^{16})^2_{\infty}} \left[ \frac{(q^8; q^8)^{6}_{\infty}}{(q^4; q^4)^{2}_{\infty}(q^{16}; q^{16})^1_{\infty}}(F_{4,1}(q) + F_{4,3}(q)) 
+ 2q(F_{1,0}(q) + F_{1,3}(q) + F_{2,0}(q) + F_{2,2}(q))
+ 3F_{3,0}(q) + 3F_{3,1}(q) + F_{4,0}(q) + F_{4,2}(q) \right] \pmod{8}.
\]
Noting that
\[ F_{1,0}(q) + F_{3,0}(q) = \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{16n^2+4n} \frac{1}{1+q^{16n}} \] and
\[ F_{1,3}(q) + F_{3,1}(q) = -\sum_{n=-\infty \atop n \neq 0}^{\infty} q^{16n^2+28n+12} \frac{1}{1+q^{16n+12}}, \]
it turns out to be
\[
\sum_{n \geq 0} a(4n+1)q^n 
= 2 \left( \frac{q^2}{(q^2)^\infty} \right)^{13} \left[ - \frac{(q^2; q^2)^{13}}{(q^2)^\infty (q^4; q^4)^\infty} \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{n^2+3n+1} (1 + q^{2n+1})^2 + 2 \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{n^2+n} \frac{1}{1+q^{4n}} \right.
\left. - 2 \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{4n^2+7n+3} + 2 \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{n^2+n} \frac{1}{1+q^{2n} (1+q^{2n})^2} + 2 \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{n^2+2n} \frac{1}{(1+q^{2n})^2} \right] \pmod{8}.
\]
Replacing \( q \) by \( q^4 \) and setting \( b_1 = -1 \) and \( b_2 = -q^3 \) \((r = 0, s = 2)\) in [4, Theorem 2.1], we obtain that
\[
\frac{(q^4; q^4)^2}{[-1, -q^4; q^4]_\infty} = \frac{1}{[q^2; q^4]_\infty} \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{4n^2+n} + \frac{1}{[q^{-3}; q^4]_\infty} \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{4n^2+7n},
\]
which implies
\[
\sum_{n=-\infty \atop n \neq 0}^{\infty} q^{4n^2+n} - \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{4n^2+7n+3} = \frac{(q^2; q^4)^{5}}{(q^2)^\infty (q^4; q^4)^\infty} - \frac{1}{2} = \frac{1}{2} \varphi(-q) \varphi(q^2) - \frac{1}{2}. \quad (4.7)
\]
Similarly, replacing \( q \) by \( q^2 \) and setting \( a_1 = 1, b_1 = -1, \) and \( b_2 = -q \) \((r = 1, s = 2)\) in [4, Theorem 2.2], we find that
\[
- \frac{(q^2; q^4)^6}{(q^2)^\infty (q^4; q^4)^\infty} \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{n^2+3n+1} (1 + q^{2n+1})^2 + 2 \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{n^2+2n} \frac{1}{(1+q^{2n})^2} = \frac{1}{2} \varphi(q) \varphi^3(-q) - \frac{1}{2}. \quad (4.8)
\]
Lastly, we also have that
\[
2 \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{n^2+n} (1+q^{2n})^2 = \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{n^2+n} \frac{1}{1+q^{2n}}
= \sum_{n=-\infty \atop n \neq 0}^{\infty} (-1)^n q^{n^2+n} \frac{1}{1+q^{2n}} + 2 \sum_{n=-\infty \atop n \neq 0}^{\infty} q^{(2n+1)^2+(2n+1)} \frac{1}{1+q^{2(2n+1)}}
= \frac{1}{2} \varphi^2(-q^2) - \frac{1}{2} + 4 \sum_{n \geq 0} q^{(2n+1)^2+(2n+1)} \frac{1}{1+q^{2(2n+1)}}, \quad (4.9)
\]
where we have the last equality by using the following identity, which is the case \( r = 0 \) and \( s = 1 \) of [4, Theorem 2.1],

\[
\frac{(q)_{\infty}^2}{[a]_{\infty}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 - aq^n}.
\]

By (4.7), (4.8), and (4.9), we now need to prove that

\[
\varphi(q)\varphi^3(-q) + 2\varphi(-q)\varphi(q^2) + \varphi^2(-q^2) - 4 \equiv 0 \pmod{8}.
\]

The identities (4.1), (4.2), and (4.3) will give us

\[
\varphi(q)\varphi^3(-q) = \varphi^2(-q^2) \left( \varphi^2(q^2) - 4q^2 \varphi(q^4) \right)
= \varphi^4(-q^4) - 4q^4 \varphi^3(-q^2)
\equiv 1 - 4q \sum_{n \geq 0} \frac{(1 - q^{8n+1})q^{4n^2+4n}}{(1 - q^{4n+2})^2} \pmod{8}
\equiv 1 - 4 \sum_{n \geq 0} q^{(2n+1)^2} \pmod{8}.
\]

(4.10)

From the definition of \( \varphi(q) \) and (4.1), we obtain that

\[
2\varphi(-q)\varphi(q^2) + \varphi^2(-q^2)
= \varphi(-q) \left( 2\varphi(q^2) + \varphi(q) \right)
= \left( 1 + 2 \sum_{n \geq 1} (-1)^n q^{n^2} \right) \left( 3 + 4 \sum_{n \geq 1} q^{2n^2} + 2 \sum_{n \geq 1} q^{n^2} \right)
\equiv 3 + 4 \sum_{n \geq 1} q^{(2n-1)^2} + 4 \sum_{n \geq 1} q^{2n^2} + 4 \left( \sum_{n \geq 1} (-1)^n q^{n^2} \right) \left( \sum_{n \geq 1} q^{n^2} \right) \pmod{8}
= 3 + 4 \sum_{n \geq 1} q^{(2n-1)^2} + 4 \sum_{n \geq 1} (1 + (-1)^n) q^{2n^2} + 4 \sum_{n \geq 1} ((-1)^n + (-1)^m) q^{n^2+m^2}
\equiv 3 + 4 \sum_{n \geq 1} q^{(2n-1)^2} \pmod{8}.
\]

(4.11)

The last two congruences (4.10) and (4.11) imply the desired result.

Theorem 3 and (1.2).

Corollary 9. For all integers \( n \geq 0 \),

\[
\overline{N}_2(3n+i) \equiv \overline{N}_2(3n+i) \pmod{3}, \quad \text{for } i = 1, 2,
\overline{N}_2(4n+1) \equiv \overline{N}_2(4n+1) \pmod{4}.
\]
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References


