Andrews-Beck type congruences for overpartitions

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Abstract

We prove Andrews-Beck type congruences for overpartitions concerning the D-rank and M_2-rank. To prove congruences, we establish the generating function for weighted D-rank (respectively, M_2-rank) moment of overpartitions and find a connection with the second D-rank (respectively, M_2-rank) moment for overpartitions.

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1 Introduction

Ramanujan's congruences for the partition function $p(n)$ are one of remarkable results in the theory of partitions:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

Dyson [8] defined the rank of a partition, which is defined as the largest part minus the number of parts, conjectured combinatorial explanations for the Ramanujan congruences modulo 5 and 7, and conjectured the existence of a crank function for partitions that could provide a combinatorial proof of Ramanujan's congruences modulo 11. Atkin and Swinnerton-Dyer [3] proved Dyson's conjecture on the rank. Andrews and Garvan [2] found the crank function and proved that the crank explains all Ramanujan congruences modulo 5, 7 and 11.

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Let $NT(b,k,n)$ be the total number of parts in the partitions of n with rank congruent to b modulo k. Beck conjectured surprising congruences for the certain linear combinations among $NT(b,k,b)$. Andrews [1] has confirmed Beck's conjectures: for all non-negative integers n,

$$\begin{aligned} &NT(1,5,5n+i)+2NT(2,5,5n+i) \\ &-2NT(3,5,5n+i)-NT(4,5,5n+i)\equiv 0 \pmod 5 \end{aligned}$$

for $i=1$ or 4, and

$$\begin{aligned} &NT(1,7,7n+i)+2NT(2,7,7n+i)+3NT(3,7,7n+i) \\ &-3NT(4,7n+i)-2NT(5,7,7n+i)-NT(6,7,7n+i)\equiv 0 \pmod 7 \end{aligned}$$

for $i=1$ or $i=5$. A crank version of Beck's conjecture is confirmed by Chern [6]. For example, if $M_\omega(b,k,n)$ counts the total number of ones in the partitions of n with crank congruent to b modulo k, we have, for all integers $n\geqslant 0$,

$$M_\omega(1,5,5n+4)+2M_\omega(2,5,5n+4)-2M_\omega(3,5,5n+4)-M_\omega(4,5,5n+4)\equiv 0 \pmod 5.$$

In the recent article [7], Chern also provided a list of over 70 Andrews-Beck type congruences involving0 $NT(b,k,b)$ and $M_\omega(b,k,n)$.

Now, we will consider overpartition analogue to Andrew-Beck type congruences. Recall that an overpartition is a partition in which the first occurrence of a number may be overlined. For example, the 14 overpartitions of 4 are

$$\begin{gathered} 4,\overline{4},3+1,\overline{3}+1,3+\overline{1},\overline{3}+\overline{1},2+2,\overline{2}+2, \\ 2+1+1,\overline{2}+1+1,2+\overline{1}+1,\overline{2}+\overline{1}+1,1+1+1+1,\overline{1}+1+1+1. \end{gathered}$$

For an overpartition λ of n, the D-rank of λ [11] is defined as Dyson's rank for ordinary partition,

$$D\text{-}rank(\lambda)=\ell(\lambda)-\#(\lambda),$$

and the M_2-rank of λ [12] is defined by

$$M_2\text{-}rank(\lambda)=\left\lceil\frac{\ell(\lambda)}{2}\right\rceil-\#(\lambda)+\#(\lambda_o)-\chi(\lambda),$$

where $\ell(\lambda)$ is the largest part of λ, $\#(\lambda)$ is the number of parts in λ, $\#(\lambda_o)$ is the number of odd non-overlined parts of λ, and $\chi(\lambda)=1$ if the largest part of λ is odd and non-overlined and $\chi(\lambda)=0$ otherwise.

Let $\overline{NT}(b,k,n)$ denote the total number of parts in the overpartitions of n with D-rank congruent to b modulo k and $\overline{NT2}(b,k,n)$ denote the total number of parts in the overpartitions of n with M_2-rank congruent to b modulo k. Then the following congruences are proved by Chan-Mao-Osburn [5]: for all $n\in\mathbb{N}$,

$$\overline{NT2}(1,5,5n+2)+2\overline{NT2}(2,5,5n+2)-2\overline{NT2}(3,5,5n+2)-\overline{NT2}(4,5,5n+2)\equiv 0 \pmod 5 \tag{1.1}$$

and

$$\overline{NT}(1,3,3n+i)-\overline{NT}(2,3,3n+i) \equiv \overline{NT2}(1,3,3n+i)-\overline{NT2}(2,3,3n+i) \pmod 3 \quad (1.2)$$

for $i = 0$ or 1.

In this paper, we prove Andrews-Beck type congruence on $\overline{NT}(b,k,n)$ and $\overline{NT2}(b,k,n)$ modulo 4 and 8 as follows.

Theorem 1. *For all integers $n \geqslant 0$,*

$$\overline{NT}(1,4,2n+1)+2\overline{NT}(2,4,2n+1)+3\overline{NT}(3,4,2n+1) \equiv 0 \pmod 4.$$

Theorem 2. *For all integers $n \geqslant 0$,*

$$\overline{NT2}(1,4,4n+1)+2\overline{NT2}(2,4,4n+1)+3\overline{NT2}(3,4,4n+1) \equiv 0 \pmod 4,$$
$$\overline{NT2}(1,4,4n+2)+2\overline{NT2}(2,4,4n+2)+3\overline{NT2}(3,4,4n+2) \equiv 0 \pmod 4,$$

and

$$\begin{aligned}&\overline{NT2}(1,8,4n+2)+2\overline{NT2}(2,8,4n+2)+3\overline{NT2}(3,8,4n+2)\\ &+4\overline{NT2}(4,8,4n+2)+5\overline{NT2}(5,8,4n+2)+6\overline{NT2}(6,8,4n+2)\\ &\qquad +7\overline{NT2}(7,8,4n+2) \equiv 0 \pmod 8.\end{aligned}$$

Lastly, we also prove a congruence between $\overline{NT}(b,k,n)$ and $\overline{NT2}(b,k,n)$.

Theorem 3. *For all integers $n \geqslant 0$, we have*

$$\sum_{j=1}^{7} j\overline{NT}(j,8,4n+1) \equiv \sum_{j=1}^{7} j\overline{NT2}(j,8,4n+1) \pmod 8.$$

The rest of the paper is organized as follows. In Section 2, we establish the generating function for weighted D-rank moment of overpartitions and find a relation with the second D-rank moment for overpartitions, from which we can prove Theorem 1. Also, we discover more congruences on $\overline{NT}(b,k,n)$. In Section 3, the generating function for weighted M_2-rank moment of overpartitions and a proof of Theorem 2 will be presented. Employing generalized Lambert series identities, we prove the congruence between $\overline{NT}(b,k,n)$ and $\overline{NT2}(b,k,n)$ in Section 4.

2 Weighted D-rank moments of overpartitions

Using standard combinatorial arguments in partition theory as [11, Proposition 1.1], we find that

$$\overline{R}(x,z,q) := \sum_{n\geqslant 0}\sum_{\lambda\in\overline{P}_n} x^{\#(\lambda)} z^{D\text{-}rank(\lambda)} q^n = \sum_{n\geqslant 0} \frac{(-1)_n x^n q^{n(n+1)/2}}{(zq, xq/z)_n}, \quad (2.1)$$

where $\overline{P}_n$ is the set of overpartitions of n.

Here and throughout the rest of the paper, we use the standard q-series notation,

$$(a)_n = (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}),$$
$$(a_1,\ldots,a_m)_n = (a_1,\ldots,a_m;q)_n := (a_1)_n\cdots(a_m)_n,$$

and

$$[a_1,\ldots,a_m]_n = [a_1,\ldots,a_m;q]_n = (a_1,q/a_1,\ldots,a_m,q/a_m)_n,$$

for $n\in\mathbb{N}_0\cup\{\infty\}$.

We will give two proofs for Theorem 1. For the first proof of Theorem 1, we will establish the generating function for the weighted D-rank moment of overpartitions and compare it with the ordinary and symmetrized D-rank moments. Here, the ordinary and symmetrized D-rank moments are defined by

$$\overline{N}_k(n) = \sum_{m\in\mathbb{Z}} m^k\overline{N}(m,n),$$
$$\overline{\eta}_k(n) = \sum_{m\in\mathbb{Z}} \binom{m-\lfloor\frac{k-1}{2}\rfloor}{k}\overline{N}(m,n),$$

where $\overline{N}(m,n)$ denotes the number of overpartitions of n with D-rank m.

Theorem 4. *We have*

$$\sum_{n\geqslant 0}\sum_{\lambda\in\overline{P}_n}\#(\lambda)D\text{-}rank(\lambda)q^n = -\sum_{n\geqslant 1}\frac{(-1)_n q^{n(n+1)/2}}{(q)_n^2}\sum_{m=1}^{n}\frac{q^m}{(1-q^m)^2},$$

which implies

$$\sum_{\lambda\in\overline{P}_n}\#(\lambda)D\text{-}rank(\lambda) = -\frac{1}{2}\overline{N}_2(n) = -\overline{\eta}_2(n).$$

It follows that

$$\sum_{n\geqslant 1}\frac{(-1)_n q^{n(n+1)/2}}{(q)_n^2}\sum_{m=1}^{n}\frac{q^m}{(1-q^m)^2} = 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geqslant 1}\frac{(-1)^{n+1}q^{n(n+1)}}{(1-q^n)^2}.$$

Proof. Applying $[\partial/\partial x]_{x=1}$ to the generating function $\overline{R}(x,z,q)$ (2.1), we have

$$\begin{aligned}\sum_{n\geqslant 0}\sum_{\lambda\in\overline{P}_n}\#(\lambda)z^{D\text{-}rank(\lambda)}q^n &= \frac{\partial}{\partial x}\left[\sum_{n\geqslant 0}\frac{(-1)_n x^n q^{n(n+1)/2}}{(zq,xq/z)_n}\right]_{x=1}\\ &= \sum_{n\geqslant 0}\left[\frac{(-1)_n x^n q^{n(n+1)/2}}{(zq,xq/z)_n}\frac{\partial}{\partial x}\log\left(\frac{x^n}{(xq/z)_n}\right)\right]_{x=1}\end{aligned}$$

$$= \sum_{n\geqslant 0} \frac{(-1)_n q^{n(n+1)/2}}{(zq, q/z)_n} \left[\frac{n}{x} + \sum_{m=1}^{n} \frac{q^m}{z - xq^m} \right]_{x=1}$$

$$= \sum_{n\geqslant 1} \frac{(-1)_n q^{n(n+1)/2}}{(zq, q/z)_n} \left(n + \sum_{m=1}^{n} \frac{q^m}{z - q^m} \right).$$

Then if we differentiate it by z and evaluate it at $z = 1$, we get

$$\sum_{n\geqslant 0} \sum_{\lambda \in \overline{P}_n} \#(\lambda) D\text{-}rank(\lambda) q^n = \frac{\partial}{\partial z} \left[\sum_{n\geqslant 1} \frac{(-1)_n q^{n(n+1)/2}}{(zq, q/z)_n} \left(n + \sum_{m=1}^{n} \frac{q^m}{z - q^m} \right) \right]_{z=1}$$

$$= -\sum_{n\geqslant 1} \frac{(-1)_n q^{n(n+1)/2}}{(q)_n^2} \sum_{m=1}^{n} \frac{q^m}{(1 - q^m)^2}, \tag{2.2}$$

which proves the first part.

If we apply $[\frac{\partial}{\partial z}(z\frac{\partial}{\partial z})]_{z=1}$ to $\overline{R}(1, z, q)$, then we have the generating function for the second D-rank moment as follows.

$$\sum_{n\geqslant 0} \overline{N}_2(n) q^n = \sum_{n\geqslant 0} \sum_{\lambda \in \overline{P}_n} D\text{-}rank(\lambda)^2 q^n$$

$$= \sum_{n\geqslant 0} \left[\frac{\partial}{\partial z} \left(z \frac{\partial}{\partial z} \frac{(-1)_n q^{n(n+1)/2}}{(zq, q/z)_n} \right) \right]_{z=1}$$

$$= \sum_{n\geqslant 0} \left[\frac{\partial}{\partial z} \frac{(-1)_n z q^{n(n+1)/2}}{(zq, q/z)_n} \sum_{m=1}^{n} \left(\frac{q^m}{1 - zq^m} + \frac{q^m}{zq^m - z^2} \right) \right]_{z=1}$$

$$= 2 \sum_{n\geqslant 1} \frac{(-1)_n q^{n(n+1)/2}}{(q)_n^2} \sum_{m=1}^{n} \frac{q^m}{(1 - q^m)^2},$$

by comparing with (2.2), which implies that $\sum_{\lambda \in \overline{P}_n} \#(\lambda) D\text{-}rank(\lambda) = -\frac{1}{2}\overline{N}_2(n)$. Also, from the following relation between the ordinary and symmetrized D-rank moments

$$\overline{N}_{2k}(n) = \sum_{j=1}^{k} (2j)! S^*(k, j) \overline{\eta}_{2j}(n),$$

where the sequence $S^*(n, k)$ is defined recursively by $S^*(n+1, k) = S^*(n, k-1) + k^2 S^*(n, k)$ and $S^*(1, 1) = 1$, $S^*(n, k) = 0$ for $k \leqslant 0$ or $k > n$, we can see that $\frac{1}{2}\overline{N}_2(n) = \overline{\eta}_2(n)$. Finally, the generating function of $\overline{\eta}_2(n)$ [10, Theorem 2.1] gives

$$\sum_{n\geqslant 1} \frac{(-1)_n q^{n(n+1)/2}}{(q)_n^2} \sum_{m=1}^{n} \frac{q^m}{(1 - q^m)^2} = 2 \frac{(-q)_\infty}{(q)_\infty} \sum_{n\geqslant 1} \frac{(-1)^{n+1} q^{n(n+1)}}{(1 - q^n)^2}. \qquad \square$$

Using the generating function for the weighted D-rank moment of overpartitions in Theorem 4, we can give a proof of Theorem 1.

Proof of Theorem 1. By Theorem 4, we have

$$\begin{aligned}&\sum_{n\geqslant 0}\left(\overline{NT}(1,4,n)+2\overline{NT}(2,4,n)+3\overline{NT}(3,4,n)\right)q^n\\&\equiv -2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geqslant 1}\frac{(-1)^{n+1}q^{n(n+1)}}{(1-q^n)^2}\pmod 4.\end{aligned}$$

Using $(-q)_\infty/(q)_\infty \equiv 1 \pmod 2$,

$$\begin{aligned}2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geqslant 1}\frac{(-1)^n q^{n(n+1)}}{(1-q^n)^2} &\equiv 2\sum_{n\geqslant 1}\frac{(-1)^n q^{n(n+1)}}{(1-q^n)^2} \pmod 4\\&= 2\sum_{n\geqslant 1}(-1)^n q^{n(n+1)}\sum_{m\geqslant 1} m q^{n(m-1)}\\&= 2\sum_{n\geqslant 1}\sum_{m\geqslant 1}(2m-1)q^{4n^2-2n+(2m-2)(2n-1)}\left(q^{4n+2m-2}-1\right)\\&\quad + 2\sum_{n\geqslant 1}\sum_{m\geqslant 1}(2m)q^{4n^2-2n+(2m-1)(2n-1)}\left(q^{4n+2m-1}-1\right)\\&\equiv 2\sum_{n\geqslant 1}\sum_{m\geqslant 1}q^{4n^2-2n+(2m-2)(2n-1)}\left(q^{4n+2m-2}-1\right) \pmod 4,\end{aligned}$$

Since the last sum involves only terms where the power of q is even, the result follows. □

From Theorem 1, we can have the following congruence for the second D-rank moment.

Corollary 5. *For all integers $n \geqslant 0$,*

$$\overline{N}_2(2n+1) \equiv 0 \pmod 2.$$

In fact, we prove more detailed results on congruences of $\overline{NT}(b,k,n)$, which also deduce the congruence in Theorem 1.

Theorem 6. *For all non-negative integers n,*

$$\begin{aligned}\overline{NT}(0,4,4n+i) &\equiv 0 \pmod 4 \quad \textit{for} \quad i=0,2,3,\\ \overline{NT}(2,4,n) &\equiv 0 \pmod 4,\\ \overline{NT}(2,4,4n+i) &\equiv 0 \pmod 8 \quad \textit{for} \quad i=1,2,3,\\ \overline{NT}(1,4,2n+1) &\equiv 0 \pmod 4,\\ \overline{NT}(3,4,2n+1) &\equiv 0 \pmod 4.\end{aligned}$$

Proof. Applying Proposition 2.2 in [5] (a generalization [1, Theorem 3]) with setting $d=1$ and $e\to 0$, we can rewrite (2.1) as follows.

$$\overline{R}(x,z,q) = \sum_{n\geqslant 0}\frac{(-1)_n x^n q^{n(n+1)/2}}{(zq, xq/z)_n}$$

$$= 1 - \frac{(-xq)_\infty}{(xq)_\infty} \sum_{n\geqslant 1} \frac{(xq,-1)_n}{(-xq)_n (q)_{n-1}} (-x)^n q^{n(n+1)} \left(\frac{1}{q^n(1-zq^n)} + \frac{x/z}{1-xq^n/z} \right).$$

By expanding the terms involving z in a geometric series as in the proof of [1, Corollary4], we have the generating function of $\overline{NT}(b,k,n)$, for $0 \leqslant b \leqslant k$,

$$\begin{aligned}
&\sum_{n\geqslant 0} \overline{NT}(b,k,n)q^n \\
&= -\frac{\partial}{\partial x} \left[\sum_{n\geqslant 1} \frac{(-xq^{n+1})_\infty (-1)_n}{(xq^{n+1})_\infty (q)_{n-1}} (-x)^n q^{n(n+1)} \left(\frac{q^{(b-1)n}}{1-q^{kn}} + \frac{x^{k-b} q^{(k-1-b)n}}{1-x^k q^{kn}} \right) \right]_{x=1} \\
&= -\left[\sum_{n\geqslant 1} \frac{(-xq^{n+1})_\infty (-1)_n}{(xq^{n+1})_\infty (q)_{n-1}} (-x)^n q^{n(n+1)} \frac{q^{n(n+1)+(b-1)n}}{1-q^{kn}} \frac{\partial}{\partial x} \log \left(\frac{(-xq^{n+1})_\infty}{(xq^{n+1})_\infty} x^n \right) \right]_{x=1} \\
&\quad - \left[\sum_{n\geqslant 1} \frac{(-xq^{n+1})_\infty (-1)_n}{(xq^{n+1})_\infty (q)_{n-1}} (-x)^n q^{n(n+1)} \frac{x^{k-b} q^{(k-1-b)n}}{1-x^k q^{kn}} \frac{\partial}{\partial x} \log \left(\frac{(-xq^{n+1})_\infty}{(xq^{n+1})_\infty} \frac{x^{n+k-b}}{1-x^k q^{kn}} \right) \right]_{x=1} \\
&= -2\frac{(-q)_\infty}{(q)_\infty} \sum_{n\geqslant 1} (-1)^n q^{n(n+1)} \frac{1-q^n}{1+q^n} \frac{q^{(b-1)n} + q^{(k-1-b)n}}{1-q^{kn}} \left(n + 2\sum_{m>n} \frac{q^m}{1-q^{2m}} \right) \\
&\quad - 2\frac{(-q)_\infty}{(q)_\infty} \sum_{n\geqslant 1} (-1)^n q^{n(n+1)} \frac{1-q^n}{1+q^n} \frac{q^{(k-1-b)n}}{1-q^{kn}} \left(\frac{k}{1-q^{kn}} - b \right). \qquad (2.3)
\end{aligned}$$

Setting $k = 4$ and using $(1-x)/(1+x) \equiv 1 \pmod 2$ yield

$$\begin{aligned}
\sum_{n\geqslant 0} \overline{NT}(b,4,n)q^n \equiv\ & 2\sum_{n\geqslant 1} q^{2n(2n-1)} \frac{q^{(b-1)(2n-1)} + q^{(3-b)(2n-1)}}{1-q^{4(2n-1)}} \\
&+ 2b \sum_{n\geqslant 1} (-1)^n q^{n(n+1)} \frac{q^{(3-b)n}}{1-q^{4n}} \pmod 4.
\end{aligned}$$

For the case $b = 0$, we have

$$\sum_{n\geqslant 0} \overline{NT}(0,4,n)q^n \equiv 2\sum_{n\geqslant 1} q^{(2n-1)^2} \pmod 4,$$

which has only terms with the powers of q congruent 1 modulo 4. Hence, $\overline{NT}(0,4,4n+i) \equiv 0 \pmod 4$ for $i = 0,2,3$ and all integers $n \geqslant 0$. For the case $b = 1$ and 3, we find that

$$\sum_{n\geqslant 0} \overline{NT}(b,4,n)q^n \equiv 2\sum_{n\geqslant 1} q^{2n(2n-1)} \frac{q^{(b-1)(2n-1)}}{1-q^{4(2n-1)}} + 2\sum_{n\geqslant 1} q^{2n(2n+1)} \frac{q^{(3-b)(2n)}}{1-q^{8n}} \pmod 4$$

has only even powers of q, which implies that $\overline{NT}(1,4,2n+1) \equiv \overline{NT}(3,4,2n+1) \equiv 0 \pmod 4$ for all integers $n \geqslant 0$. Similarly, for the case $b = 2$, from (2.3),

$$\sum_{n\geqslant 0} \overline{NT}(2,4,n)q^n \equiv 4\sum_{n\geqslant 1} q^{2n(2n-1)} \frac{q^{2n-1}}{1-q^{4(2n-1)}} + 4\sum_{n\geqslant 1} (-1)^n q^{n(n+1)} \frac{q^n}{1-q^{4n}} \pmod 8$$

$$\equiv 4\sum_{n\geqslant 1} q^{2n(2n+1)}\frac{q^{2n}}{1-q^{8n}} \pmod 8,$$

which includes terms only with the powers of q congruent 0 modulo 4. This proves that $\overline{NT}(2,4,n)\equiv 0 \pmod 4$ and $\overline{NT}(2,4,4n+i)\equiv 0 \pmod 8$ for $i=1,2,3$ for all integers n greater than 0. □

3 Weighted M_2-rank moments of overpartitions

As in Section 2, we find the generating function for the weighted M_2-rank moments of overpartitions and compare it with the ordinary and symmetrized M_2-rank moments. We have the ordinary and symmetrized M_2-rank moments defined by

$$\overline{N2}_k(n) = \sum_{m\in\mathbb{Z}} m^k \overline{N2}(m,n),$$

$$\overline{\eta 2}_k(n) = \sum_{m\in\mathbb{Z}} \binom{m-\lfloor\frac{k-1}{2}\rfloor}{k}\overline{N2}(m,n),$$

where $\overline{N2}(m,n)$ denotes the number of overpartitions of n with M_2-rank m. The generating function [12, Theorem 1.2] is

$$\overline{R2}(x,z,q) := \sum_{n\geqslant 0}\sum_{\lambda\in\overline{P}_n} x^{\#(\lambda)} z^{M_2\text{-}rank(\lambda)} q^n = \sum_{n\geqslant 0}\frac{(-1,-q;q^2)_n (xq)^n}{(zq^2, xq^2/z;q^2)_n}.$$

Theorem 7. *We have*

$$\sum_{n\geqslant 0}\sum_{\lambda\in\overline{P}_n} \#(\lambda) M_2\text{-}rank(\lambda) q^n = -\sum_{n\geqslant 1}\frac{(-1,-q;q^2)_n q^n}{(q^2;q^2)_n^2}\sum_{m=1}^{n}\frac{q^{2m}}{(1-q^{2m})^2},$$

which implies

$$\sum_{\lambda\in\overline{P}_n} \#(\lambda) M_2\text{-}rank(\lambda) = -\frac{1}{2}\overline{N2}_2(n) = -\overline{\eta 2}_2(n).$$

It follows that

$$\sum_{n\geqslant 1}\frac{(-1,-q;q^2)_n q^n}{(q^2;q^2)_n^2}\sum_{m=1}^{n}\frac{q^{2m}}{(1-q^{2m})^2} = 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geqslant 1}\frac{(-1)^{n+1}q^{n(n+2)}}{(1-q^{2n})^2}.$$

Proof of Theorem 7. If we differentiate $\overline{R2}(x,z,q)$ by x and evaluate it at $x=1$, we get

$$\sum_{n\geqslant 0}\sum_{\lambda\in\overline{P}_n} \#(\lambda) z^{M_2\text{-}rank(\lambda)} q^n = \frac{\partial}{\partial x}\left[\sum_{n\geqslant 0}\frac{(-1,-q;q^2)_n (xq)^n}{(zq^2, xq^2/z;q^2)_n}\right]_{x=1}$$

$$
\begin{aligned}
&= \sum_{n\geqslant 0}\left[\frac{(-1,-q;q^2)_n (xq)^n}{(zq^2, xq^2/z;q^2)_n}\frac{\partial}{\partial x}\log\left(\frac{x^n}{(xq^2/z;q^2)_n}\right)\right]_{x=1}\\
&= \sum_{n\geqslant 1}\frac{(-1,-q;q^2)_n q^n}{(zq^2, q^2/z;q^2)_n}\left(n+\sum_{m=1}^{n}\frac{q^{2m}}{z-q^{2m}}\right).
\end{aligned}
$$

Then applying $[\partial/\partial z]_{z=1}$ gives

$$
\begin{aligned}
\sum_{n\geqslant 0}\sum_{\lambda\in\overline{P}_n}\#(\lambda)M_2\text{-}rank(\lambda)q^n &= \frac{\partial}{\partial z}\left[\sum_{n\geqslant 1}\frac{(-1,-q;q^2)_n q^n}{(zq^2, q^2/z;q^2)_n}\left(n+\sum_{m=1}^{n}\frac{q^{2m}}{z-q^{2m}}\right)\right]_{z=1}\\
&= -\sum_{n\geqslant 1}\frac{(-1,-q;q^2)_n q^n}{(q^2;q^2)_n^2}\sum_{m=1}^{n}\frac{q^{2m}}{(1-q^{2m})^2}, \qquad (3.1)
\end{aligned}
$$

which is the first part. Next, to compare with the M_2-rank moments, when we apply $[\frac{\partial}{\partial z}(z\frac{\partial}{\partial z})]_{z=1}$ to $\overline{R2}(1,z,q)$, we find that

$$
\begin{aligned}
\sum_{n\geqslant 0}\overline{N2}_2(n)q^n &= \sum_{n\geqslant 0}\sum_{\lambda\in\overline{P}_n}M_2\text{-}rank(\lambda)^2 q^n\\
&= 2\sum_{n\geqslant 1}\frac{(-1,-q;q^2)_n q^n}{(q^2;q^2)_n^2}\sum_{m=1}^{n}\frac{q^{2m}}{(1-q^{2m})^2}. \qquad (3.2)
\end{aligned}
$$

Then the second part follows from comparing (3.1) with (3.2) and the following relation between the ordinary and symmetrized M_2-rank moments

$$
\overline{N2}_{2k}(n) = \sum_{j=1}^{k}(2j)!S^*(k,j)\overline{\eta 2}_{2j}(n).
$$

Lastly, we have the last identity by considering the generating function for $\overline{\eta 2}_2(n)$ [10, Theorem 2.1]. □

From the generating function for weighted M_2-rank moment of overpartitions, we can prove Theorem 2.

Proof of Theorem 2. By Theorem 7, we notice that

$$
\sum_{n\geqslant 0}\left(\overline{NT2}(1,4,n)+2\overline{NT2}(2,4,n)+3\overline{NT2}(3,4,n)\right)q^n \equiv -2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geqslant 1}\frac{(-1)^{n+1}q^{n(n+2)}}{(1-q^{2n})^2} \pmod 4.
$$

By applying $(-q)_\infty/(q)_\infty \equiv 1 \pmod 2$,

$$
2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geqslant 1}\frac{(-1)^n q^{n(n+2)}}{(1-q^{2n})^2} \equiv 2\sum_{n\geqslant 1}\frac{(-1)^n q^{n(n+2)}}{(1-q^n)^2} \pmod 4
$$

$$
\begin{aligned}
&= 2\sum_{n\geqslant 1}(-1)^n q^{n(n+2)}\sum_{m\geqslant 1} mq^{2n(m-1)} \\
&\equiv 2\sum_{n\geqslant 1}\sum_{m\geqslant 1}\left(q^{4n(n+1)+8n(m-1)} - q^{4n^2-1+4(2n-1)(m-1)}\right) \pmod 4,
\end{aligned}
$$

which contributes the term with powers congruent to 0 or 3 modulo 4. Then we have

$$
\begin{aligned}
&\overline{NT2}(1,4,4n+1) + 2\overline{NT2}(2,4,4n+1) + 3\overline{NT2}(3,4,4n+1) \equiv 0 \pmod 4 \\
&\overline{NT2}(1,4,4n+2) + 2\overline{NT2}(2,4,4n+2) + 3\overline{NT2}(3,4,4n+2) \equiv 0 \pmod 4.
\end{aligned}
$$

Also, by Theorem 7, we find that

$$
\sum_{n\geqslant 0}\sum_{j=1}^{7} j\overline{NT2}(j,8,n)q^n \equiv -2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geqslant 1}\frac{(-1)^{n+1}q^{n(n+2)}}{(1-q^{2n})^2} \pmod 8.
$$

Using

$$
\frac{(-q)_\infty}{(q)_\infty} \equiv \frac{(q)_\infty}{(-q)_\infty} \pmod 4 = \sum_{k=-\infty}^{\infty}(-1)^k q^{k^2},
$$

we have

$$
\begin{aligned}
&2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geqslant 1}\frac{(-1)^n q^{n(n+2)}}{(1-q^{2n})^2} \\
&\equiv 2\left(1+2\sum_{k\geqslant 1}(-1)^k q^{k^2}\right)\sum_{n\geqslant 1}\frac{(-1)^n q^{n(n+2)}}{(1-q^{2n})^2} \pmod 8 \\
&= 2\left(1+2\sum_{k\geqslant 1}(-1)^k q^{k^2}\right)\sum_{n\geqslant 1}(-1)^n q^{n(n+2)}\sum_{m\geqslant 1} mq^{2n(m-1)} \\
&\equiv 2\left(1+2\sum_{k\geqslant 1}(-1)^k q^{k^2}\right)\sum_{n\geqslant 1}\sum_{m\geqslant 1}(2m-1)\left(q^{4n(n+1)+8n(m-1)} - q^{4n^2-1+4(2n-1)(m-1)}\right) \\
&\quad + 2\sum_{n\geqslant 1}\sum_{m\geqslant 1}(2m)\left(q^{4n(n+1)+4n(2m-1)} - q^{4n^2-1+2(2n-1)(2m-1)}\right) \pmod 8. \qquad (3.3)
\end{aligned}
$$

Here,

$$
1+2\sum_{k\geqslant 1}(-1)^k q^{k^2}
$$

only contributes to terms with powers of q congruent to 0 or 1 modulo 4, while the factor

$$
\sum_{n\geqslant 1}\sum_{m\geqslant 1}(2m-1)\left(q^{4n(n+1)+8n(m-1)} - q^{4n^2-1+4(2n-1)(m-1)}\right)
$$

involves terms with powers of q congruent to 0 or 3 modulo 4. Therefore, the first term in (3.3) dose not have terms with powers of q congruent to 2 modulo 4. The second term in (3.3) has only terms with powers of q congruent to 0 or 1 modulo 4. Hence, we obtain

$$
\overline{NT2}(1,8,4n+2) + 2\overline{NT2}(2,8,4n+2) + 3\overline{NT2}(3,8,4n+2)
$$

$$+4\overline{NT2}(4,8,4n+2)+5\overline{NT2}(5,8,4n+2)$$
$$+6\overline{NT2}(6,8,4n+2)+7\overline{NT2}(7,8,4n+2)\equiv 0 \pmod 8. \qquad \square$$

From Theorem 2 and (1.1), we can have congruences for the second M_2-rank moments as follows.

Corollary 8. *For all integers $n \geqslant 0$,*

$$\overline{N2}_2(4n+1)\equiv 0 \pmod 2,$$
$$\overline{N2}_2(4n+2)\equiv 0 \pmod 4,$$
$$\overline{N2}_2(5n+2)\equiv 0 \pmod 5.$$

4 Proof of Theorem 3

Let

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) := \sum_{n\geqslant 0} q^{n(n+1)/2}.$$

Then, by Jacobi's triple product identity and simple q-series manipulations, we have

$$\varphi(q) = \frac{(q^2;q^2)_\infty^5}{(q)_\infty^2(q^4;q^4)_\infty^2}, \quad \varphi(-q) = \frac{(q)_\infty^2}{(q^2;q^2)_\infty}, \quad \text{and} \quad \psi(q) = \frac{(q^2;q^2)_\infty^2}{(q)_\infty}$$

and

$$\varphi^2(q) = \varphi^2(q^2) + 4q\psi^2(q^4), \quad \varphi(q)\varphi(-q) = \varphi^2(-q^2), \quad \text{and} \quad \psi^2(q) = \varphi(q)\psi(q^2). \tag{4.1}$$

We will also use the following identities [4, Theorem 6.1, and 7.1].

$$\varphi^4(-q) = 1 + 8\sum_{n\geqslant 1} \frac{nq^{n(n+2)} + q^{n(n+1)} - nq^n}{(1+q^n)^2}, \tag{4.2}$$

$$\psi^4(q) = \sum_{n\geqslant 0} \frac{\left(2n+1+2q^{2n+1}-(2n+1)q^{4n+2}\right)q^{2n^2+2n}}{(1-q^{2n+1})^2}. \tag{4.3}$$

We prove Theorem 3 using generalized Lambert series identities with above theta function identities.

Proof of Theorem 3. Let

$$a(n) = \sum_{j=1}^{7} j\overline{NT}(j,8,n) - \sum_{j=1}^{7} j\overline{NT2}(j,8,n).$$

By Theorem 4 and 7, we have

$$\sum_{n\geqslant 0} a(n)q^n \equiv -2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geqslant 1}\frac{(-1)^{n+1}q^{n(n+1)}}{(1-q^n)^2} + 2\frac{(-q)_\infty}{(q)_\infty}\sum_{n\geqslant 1}\frac{(-1)^{n+1}q^{n(n+2)}}{(1-q^{2n})^2} \pmod 8$$

$$= \frac{(-q)_\infty}{(q)_\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left[\frac{(-1)^n q^{n(n+1)}}{(1-q^n)^2} - \frac{(-1)^n q^{n(n+2)}}{(1-q^{2n})^2} \right]. \tag{4.4}$$

We consider the 4-dissection for $\frac{(-q)_\infty}{(q)_\infty}$ [9],

$$\begin{aligned} \frac{(-q)_\infty}{(q)_\infty} =& \frac{(q^8;q^8)_\infty^{19}}{(q^4;q^4)_\infty^{14}(q^{16};q^{16})_\infty^{6}} + 2q\frac{(q^8;q^8)_\infty^{13}}{(q^4;q^4)_\infty^{12}(q^{16};q^{16})_\infty^{2}} \\ &+ 4q^2\frac{(q^8;q^8)_\infty^{7}(q^{16};q^{16})_\infty^{2}}{(q^4;q^4)_\infty^{10}} + 8q^3\frac{(q^8;q^8)_\infty(q^{16};q^{16})_\infty^{6}}{(q^4;q^4)_\infty^{8}}. \end{aligned} \tag{4.5}$$

For the summation part in (4.4),

$$\begin{aligned} &\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left[\frac{(-1)^n q^{n(n+1)}}{(1-q^n)^2} - \frac{(-1)^n q^{n(n+2)}}{(1-q^{2n})^2} \right] \\ &= 2\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(n+1)}(1+q^n+3q^{2n}+q^{3n})}{(1-q^{4n})^2} \\ &\equiv 2\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(n+1)}(1+q^n+3q^{2n}+q^{3n})}{(1+q^{4n})^2} \pmod 8 \end{aligned} \tag{4.6}$$

using the fact

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(n+1)} q^{mn}}{(1-q^{4n})^2} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(n+1)} q^{(6-m)n}}{(1-q^{4n})^2}.$$

By letting

$$F_{a,b}(q) := \sum_{\substack{n=-\infty \\ n\neq 0 \\ n\equiv b \pmod 4}}^{\infty} \frac{(-1)^n q^{n(n+a)}}{(1+q^{4n})^2} \quad \text{for } a = 1,2,3, \text{ and } 4,$$

we can have 4-dissection of (4.6). Invoking (4.5) and the dissection of (4.6) into (4.4) and collecting only terms where the power of q is congruent to 1 modulo 4 yield

$$\begin{aligned} &\sum_{n\geqslant 0} a(4n+1)q^{4n+1} \\ &\equiv 2\frac{(q^8;q^8)_\infty^{13}}{(q^4;q^4)_\infty^{12}(q^{16};q^{16})_\infty^{2}} \Bigg[\frac{(q^8;q^8)_\infty^{6}}{(q^4;q^4)_\infty^{2}(q^{16};q^{16})_\infty^{4}} (F_{4,1}(q)+F_{4,3}(q)) \\ &\qquad + 2q\,(F_{1,0}(q)+F_{1,3}(q)+F_{2,0}(q)+F_{2,2}(q) \\ &\qquad\qquad +3F_{3,0}(q)+3F_{3,1}(q)+F_{4,0}(q)+F_{4,2}(q)) \Bigg] \pmod 8. \end{aligned}$$

Noting that

$$F_{1,0}(q)+F_{3,0}(q)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{q^{16n^2+4n}}{1+q^{16n}}\quad\text{and}\quad F_{1,3}(q)+F_{3,1}(q)=-\sum_{n=-\infty}^{\infty}\frac{q^{16n^2+28n+12}}{1+q^{16n+12}},$$

it turns out to be

$$\begin{aligned}&\sum_{n\geqslant 0}a(4n+1)q^n\\ &\equiv 2\frac{(q^2;q^2)_\infty^{13}}{(q)_\infty^{12}(q^4;q^4)_\infty^2}\left[-\frac{(q^2;q^2)_\infty^6}{(q)_\infty^2(q^4;q^4)_\infty^4}\sum_{n=-\infty}^{\infty}\frac{q^{n^2+3n+1}}{(1+q^{2n+1})^2}+2\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{q^{4n^2+n}}{1+q^{4n}}\right.\\ &\qquad\left.-2\sum_{n=-\infty}^{\infty}\frac{q^{4n^2+7n+3}}{1+q^{4n+3}}+2\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{q^{n^2+n}}{(1+q^{2n})^2}+2\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{q^{n^2+2n}}{(1+q^{2n})^2}\right]\pmod 8.\end{aligned}$$

Replacing q by q^4 and setting $b_1=-1$ and $b_2=-q^3$ ($r=0$, $s=2$) in [4, Theorem 2.1], we obtain that

$$\frac{(q^4;q^4)_\infty^2}{[-1,-q^3;q^4]_\infty}=\frac{1}{[q^3;q^4]_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{4n^2+n}}{1+q^{4n}}+\frac{1}{[q^{-3};q^4]_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{4n^2+7n}}{1+q^{4n+3}},$$

which implies

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{q^{4n^2+n}}{1+q^{4n}}-\sum_{n=-\infty}^{\infty}\frac{q^{4n^2+7n+3}}{1+q^{4n+3}}=\frac{(q)_\infty^2(q^4;q^4)_\infty^5}{2(q^2;q^2)_\infty^3(q^8;q^8)_\infty^2}-\frac{1}{2}=\frac{1}{2}\varphi(-q)\varphi(q^2)-\frac{1}{2}.\qquad(4.7)$$

Similarly, replacing q by q^2 and setting $a_1=1$, $b_1=-1$, and $b_2=-q$ ($r=1$, $s=2$) in [4, Theorem 2.2], we find that

$$-\frac{(q^2;q^2)_\infty^6}{(q)_\infty^2(q^4;q^4)_\infty^4}\sum_{n=-\infty}^{\infty}\frac{q^{n^2+3n+1}}{(1+q^{2n+1})^2}+2\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{q^{n^2+2n}}{(1+q^{2n})^2}=\frac{1}{2}\varphi(q)\varphi^3(-q)-\frac{1}{2}.\qquad(4.8)$$

Lastly, we also have that

$$\begin{aligned}2\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{q^{n^2+n}}{(1+q^{2n})^2}&=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{q^{n^2+n}}{1+q^{2n}}\\ &=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^nq^{n^2+n}}{1+q^{2n}}+2\sum_{n=-\infty}^{\infty}\frac{q^{(2n+1)^2+(2n+1)}}{1+q^{2(2n+1)}}\\ &=\frac{1}{2}\varphi^2(-q^2)-\frac{1}{2}+4\sum_{n\geqslant 0}\frac{q^{(2n+1)^2+(2n+1)}}{1+q^{2(2n+1)}},\qquad(4.9)\end{aligned}$$

where we have the last equality by using the following identity, which is the case $r = 0$ and $s = 1$ of [4, Theorem 2.1],

$$\frac{(q)_\infty^2}{[a]_\infty} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 - aq^n}.$$

By (4.7), (4.8), and (4.9), we now need to prove that

$$\varphi(q)\varphi^3(-q) + 2\varphi(-q)\varphi(q^2) + \varphi^2(-q^2) - 4 \equiv 0 \pmod 8.$$

The identities (4.1), (4.2), and (4.3) will give us

$$\begin{aligned} \varphi(q)\varphi^3(-q) &= \varphi^2(-q^2)\left(\varphi^2(q^2) - 4q\psi^2(q^4)\right) \\ &= \left(\varphi(-q^2)\varphi(q^2)\right)^2 - 4q\left(\varphi(-q^2)\psi(q^4)\right)^2 \\ &= \varphi^4(-q^4) - 4q\psi^4(-q^2) \\ &\equiv 1 - 4q\sum_{n\geqslant 0} \frac{(1-q^{8n+4})q^{4n^2+4n}}{(1-q^{4n+2})^2} \pmod 8 \\ &\equiv 1 - 4\sum_{n\geqslant 0} q^{(2n+1)^2} \pmod 8. \end{aligned} \tag{4.10}$$

From the definition of $\varphi(q)$ and (4.1), we obtain that

$$\begin{aligned} &2\varphi(-q)\varphi(q^2) + \varphi^2(-q^2) \\ &= \varphi(-q)\left(2\varphi(q^2) + \varphi(q)\right) \\ &= \left(1 + 2\sum_{n\geqslant 1}(-1)^n q^{n^2}\right)\left(3 + 4\sum_{n\geqslant 1} q^{2n^2} + 2\sum_{n\geqslant 1} q^{n^2}\right) \\ &\equiv 3 + 4\sum_{n\geqslant 1} q^{(2n-1)^2} + 4\sum_{n\geqslant 1} q^{2n^2} + 4\left(\sum_{n\geqslant 1}(-1)^n q^{n^2}\right)\left(\sum_{n\geqslant 1} q^{n^2}\right) \pmod 8 \\ &= 3 + 4\sum_{n\geqslant 1} q^{(2n-1)^2} + 4\sum_{n\geqslant 1}\left(1 + (-1)^n\right) q^{2n^2} + 4\sum_{n>m\geqslant 1}\left((-1)^n + (-1)^m\right) q^{n^2+m^2} \\ &\equiv 3 + 4\sum_{n\geqslant 1} q^{(2n-1)^2} \pmod 8. \end{aligned} \tag{4.11}$$

The last two congruences (4.10) and (4.11) imply the desired result. □

Also, we have the following congruences between D-rank and M_2-rank moments by Theorem 3 and (1.2).

Corollary 9. *For all integers $n \geqslant 0$,*

$$\begin{aligned} \overline{N}_2(3n+i) &\equiv \overline{N2}_2(3n+i) \pmod 3, \quad \text{for } i = 1, 2, \\ \overline{N}_2(4n+1) &\equiv \overline{N2}_2(4n+1) \pmod 4. \end{aligned}$$

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References

[1] G. E. Andrews, The Ramanujan-Dyson Identities and George Beck's Congruence Conjectures, *Int. J. Number Theory* **17** (2021), 239–249.

[2] G. E. Andrews and F. G. Garvan, Dyson's crank of a partition, *Bull. Amer. Math. Soc. (N.S.)* **18** (1988), 167–171.

[3] A. O. L. Atkin and P. Swinnerton-Dyer, Some properties of partitions, *Proc. London Math. Soc. (3)* **4** (1954), 84–106.

[4] S. H. Chan, Generalized Lambert Series Identities, *Proc. London Math. Soc.* **91** (2005), 598–622.

[5] S. H. Chan, R. Mao, and R. Osburn, Variations of Andrews-Beck type congruences, *J. Math. Anal. Appl.* **495** (2021), Article 124771.

[6] S. Chern, Weighted partition rank and crank moments. I. Andrews-Beck type congruences, preprint.

[7] S. Chern, Weighted partition rank and crank moments. III. A list of Andrews-Beck type congruences modulo 5, 7, 11 and 13, *Int. J. Number Theory* **18** (2022), 141–163.

[8] F. J. Dyson, Some guesses in the theory of partitions, *Eureka (Cambridge)* **8** (1944), 10–15.

[9] M. D. Hirschhorn and J. Sellers, Arithmetic properties for overpartitions, *J. Comb. Math. Comb. Comput.* **53** (2005), 65–73.

[10] C. Jennings-Shaffer, Higher order SPT functions for overpartitions, overpartitions with smallest part even, and partitions with smallest part even and without repeated odd parts, *J. Number Theory* **149** (2015), 285–312.

[11] J. Lovejoy, Rank and conjugation for the Frobenius representation of an overpartition, *Ann. Comb.* **9** (2005), 321–334.

[12] J. Lovejoy, Rank and conjugation for a second Frobenius representation of an overpartition, *Ann. Comb.* **12** (2008), 101–113.