Cutting a cake for infinitely many guests

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Abstract

Fair division with unequal shares is an intensively studied resource allocation problem. For $i \in [n]$, let μ_i be an atomless probability measure on the measurable space $(C, \mathcal{S})$ and let t_i be positive numbers (entitlements) with $\sum_{i=1}^{n} t_i = 1$. A fair division is a partition of C into sets $S_i \in \mathcal{S}$ with $\mu_i(S_i) \geqslant t_i$ for every $i \in [n]$.

We introduce new algorithms to solve the fair division problem with irrational entitlements. They are based on the classical Last diminisher technique and we believe that they are simpler than the known methods. Then we show that a fair division always exists even for infinitely many players.

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1 Introduction

Cake cutting is a metaphor of the distribution of some inhomogeneous continuos goods and is intensively investigated by not just mathematicians but economists and political scientists as well. The preferences of the players P_i involved in the sharing are usually represented as atomless probability measures μ_i defined on a common σ-algebra $\mathcal{S} \subseteq \mathcal{P}(C)$ of the possible 'slices' of the 'cake' C. One option of how a division can be "good" is *proportionality*. This means that each of the n players gets at least one nth of the cake according to their own measurement, i.e. $C = \bigsqcup_{i=1}^{n} S_i$ with $\mu_i(S_i) \geqslant \frac{1}{n}$. The division is called *strongly proportional* if all these inequalities are strict. For $n = 2$ a proportional division can be found by the so called "Cut and choose" procedure. This was used by Abraham and Lot in the Bible to share Canaan. Abraham divided Canaan into two parts which have equal value for him and then Lot chose his favourite among these two parts leaving Abraham the other one. For a general n, Steinhaus challenged his students Banach and Knaster to find a solution that they successfully accomplished by developing the so called "Last Diminisher" procedure (see [12]). In this method P_1 picks a slice T_1 with $\mu_1(T_1) = \frac{1}{n}$. If $\mu_2(T_1) > \frac{1}{n}$, then P_2 diminishes T_1 in the sense that he takes an $T_2 \subseteq T_1$ with $\mu_2(T_2) = \frac{1}{n}$, otherwise he lets $T_2 := T_1$. They proceed similarly and slice T_n is allocated to the player who lastly diminished or to P_1 if nobody did so. Then the remaining cake worth at least $\frac{n-1}{n}$ for each of the remaining $n - 1$ players and they can continue using the same protocol.

A natural extension of the concept of proportional division is the so called "fair division with unequal shares". In this variant there are entitlements $t_i > 0$ associated to the players satisfying $\sum_i t_i = 1$. A division is called (strongly) *fair* if the slice S_i given to player P_i worths for him at least (more than) t_i, i.e. $\mu_i(S_i) \geqslant t_i$ ($\mu_i(S_i) > t_i$) holds for each i. If all of these entitlements are rational numbers, say $\frac{p_1}{q}, \ldots, \frac{p_n}{q}$, then a fair division according to them can be reduced to a proportional division problem for $\sum_{i=1}^{n} p_i$ players where measure μ_i is "cloned" to p_i copies. In the presence of irrational entitlements such a "player-cloning" argument is no more applicable.

Several finite procedures were developed to find a (strongly) fair division allowing irrational entitlements. In the special case of the problem where $C = [0, 1]$, $\mathcal{S}$ is the Borel σ-algebra and the measures μ_i are absolute continues, Shishido and Zeng developed and algorithm in [11]. In their protocol the players choose intervals that worth the same and exchange these intervals among each other exploiting the possible differences of their evaluations. A more recent algorithm in the same model but based on completely different ideas was given by Cseh and Fleiner in [5]. In a general step they reduce the problem to two sub-problems in one of which the number of players is smaller by one while in the other all the entitlements are rational and the number of players remains the same.

Our first contribution (Section 2) is two procedures solving the fair division problem with potentially irrational entitlements which we believe are simpler than the known methods. We keep working in the general settings we have already introduced which originated from Barbanel (see [2]). Our aim is to demonstrate that 'Last diminisher'-type of ideas are already powerful enough to design simple finite procedures. We provide two algorithms both of which solves the problem in finitely many steps. The first one reduces the problem to another one in which either the number of players is smaller by one or all the entitlements are rationals and the number of players is the same. This can be considered a direct improvement of the algorithm given in Section 7 of [5]. Taking rational numbers from non-degenerate intervals (which was used in the algorithms given in [5, 13]) is necessary in this procedure. In our second algorithm not even such a rational approximation is needed.

It was shown in [6] based on Lyapunov's theorem that if not all the measures are identical, then a strongly proportional division exists. A constructive proof was obtained later in [13] which was then further developed for the case of unequal shares (i.e. strong fairness) in [2]. We point out in Section 3 that the strongly fair division problem (for potentially infinitely many players) can be actually reduced to the fair division problem in a completely elementary way, which reduction we need later.

In the last section (Section 4) we consider the (strongly) fair division problem for infinitely many players. Rational entitlements do not make this problem easier since representing them with a common denominator is impossible in general. Since the entitlements sum up to 1, they must converge to 0, thus extension of protocols in which one need to start with the smallest positive entitlement (like the one given in [5]) is problematic. By Last Diminisher-type of methods we are facing in addition the difficulty that diminishing infinitely often might be necessary in which case no "last diminisher" exists, moreover, we may end up with the empty set as a limit of the iterated trimmings. Eliminating one player and using induction for the rest is also not applicable for obvious reasons. Although the so called Fink protocol (see [7]) can be considered as such a player-eliminating recursive algorithm, it inspired our procedure that finds a fair division for infinitely many players:

Assume that $(C, \mathcal{S})$ is a measurable space and for $i \in \mathbb{N}$, μ_i is an atomless probability measure defined on $\mathcal{S}$ and t_i is a positive number such that $\sum_{i=0}^{\infty} t_i = 1$. Then there is a partition $C = \bigsqcup_{i=0}^{\infty} S_i$ such that $S_i \in \mathcal{S}$ with $\mu_i(S_i) \geqslant t_i$ for each $i \in \mathbb{N}$. Furthermore, if not all the μ_i are identical, then '$\mu_i(S_i) \geqslant t_i$' can be strengthened to '$\mu_i(S_i) > t_i$' for every $i \in \mathbb{N}$.

Let us mention that cake cutting problems have a huge literature and this particular model and notion of fairness that we consider is only a tiny fragment of it. About the so called exact, envy-free and equitable divisions (none of which are extendable to infinitely many players for obvious reasons) and the corresponding existence results a brief but informative survey can be found in [4]. For a more general picture about this field, including completely different mathematical models of the problem, we refer to [1], [3], [9] and [10].

2 'Last Diminisher'-type of procedures for fair division with irrational entitlements

Our aim is to find a fair division $S_1, \ldots, S_n$ for players $P_1, \ldots, P_n$ with respective atomless probability measures $\mu_1, \ldots, \mu_n$ and (potentially irrational) entitlements $0 < t_1 \leqslant t_2 \leqslant$

$\cdots \leqslant t_n < 1$ where $\sum_{i=1}^{n} t_i = 1$.

As it is standard in the cake cutting literature, algorithms use certain queries. We allow the following operations.

- The four basic arithmetical operations and comparison on $\mathbb{R}$.
- The set operations on $\mathcal{S}$.
- Computing $\mu_i(S)$ for some $i \in [n]$ where slice S is obtained in a previous step.
- Cutting a slice $S' \subseteq S$ with $\mu_i(S) = \alpha$ for an $i \in [n]$ and $\alpha \in [0, \mu_i(S)]$ where either $S = C$ or S is obtained in a previous step.[1]

2.1 Algorithm I

Player P_1 picks some T_1 with $\mu_1(T_1) = t_1$. If T_i is already defined for some $i < n$, we let $T_{i+1} := T_i$ if $\mu_{i+1}(T_i) \leqslant t_1$ and we define T_{i+1} to be a subset of T_i with $\mu_{i+1}(T_{i+1}) = t_1$ if $\mu_{i+1}(T_i) > t_1$. After the recursion is done, $\mu_i(T_n) \leqslant t_1$ holds for each i and there is equality for at least one index.

If $\mu_1(T_n) = t_1$, then we let $S_1 := T_n$ and remove player P_1 from the process. Since the rest of the cake worth at least $1 - t_1$ for all the players, dividing it fairly with respect to the entitlements $\frac{t_i}{1-t_1}$ for $1 < i \leqslant n$ leads to a fair division. Thus we invoke the algorithm for this sub-problem with less players.

If $\mu_1(T_n) < t_1$, then there must be a player who diminished the slice during the recursion. Let k be the largest index for which P_k is such a player. We allocate T_n to P_k but we do not remove P_k from the process unless $t_1 = t_k$. In order to satisfy P_k, he needs to get at least the $t'_k := \frac{t_k - t_1}{\mu_k(C \setminus T_n)}$ fraction of the rest of the cake $C \setminus T_n$ according to his measure μ_k, while for $i \neq k$ player P_i should get at least the fraction $t'_i := \frac{t_i}{\mu_i(C \setminus T_n)}$ of $C \setminus T_n$ w.r.t. μ_i. As we already noticed $\mu_i(T_n) \leqslant t_1$ and hence $\mu_i(C \setminus T_n) \geqslant 1 - t_1$ for every i, furthermore, the inequality is strict for $i = 1$ in this branch of the case distinction. Therefore

$$\sum_{i=1}^{n} t'_i < \frac{t_k - t_1}{1 - t_1} + \sum_{i \neq k} \frac{t_i}{1 - t_1} = \frac{\left(\sum_{i=1}^{n} t_i\right) - t_1}{1 - t_1} = 1.$$

Thus we can pick rational numbers $t''_i > t'_i$ with $\sum_{i \leqslant n} t''_i = 1$. Finally, we use a subroutine to divide $C \setminus T_n$ fairly among the players w.r.t. the rational entitlements t''_i to obtain a strongly fair division for the original problem.

2.2 Algorithm II

In this algorithm no 'rounding up to rationals' is necessary. We shall make several rounds and in each of them allocate a slice chosen in a 'Last diminisher' manner. The satisfied players are dropping out of the process. The algorithm itself is quite simple in this case as well but the proof of the correctness is somewhat more involved.

[1] It is well-defined because μ_i is atomless (see [8, Theorem 5]).

For $i \in [n]$, we denote by S_i^m the portion allocated to player P_i at the beginning of round m. We set $S_i^0 = \emptyset$ for every i. The rest of the cake is $C_m := C \setminus \bigcup_{i=1}^n S_i^m$. We also have improved entitlements t_i^m where $t_i^0 := t_i$. We say that player P_i is satisfied at the beginning of round m if $t_i \leqslant \mu_i(S_i^m)$. Let us define I_m as the set of indices of the players that are unsatisfied at the beginning of round m, i.e.

$$I_m := \{i \in [n] : \ t_i > \mu_i(S_i^m)\}.$$

If $I_m = \emptyset$, then the process terminates and the sets S_i^m for $i \in [n]$ form a fair division. If $I_m \neq \emptyset$, then the algorithm does the following. Let

$$c_m := \min_{i \in I_m} \frac{t_i^m - \mu_i(S_i^m)}{\mu_i(C_m)}$$

and let $i_m \in I_m$ the smallest index where this minimum is taken. Then player P_{i_m} picks a $T_1^m \subseteq C_m$ with $\mu_{i_m}(T_1^m) = t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)$. After this, players P_i for $i \in I_m \setminus \{i_m\}$ consider the (actual) slice one by one and diminish it or keep unchanged in the following way. If T_k^m is defined, $k < |I_m|$ and ℓ is the kth smallest element of $I_m \setminus \{i_m\}$, then let

$$T_{k+1}^m := \begin{cases} T_k^m & \text{if } \frac{\mu_\ell(T_k^m)}{\mu_\ell(C_m)} \leqslant c_m \\ S \text{ with } S \subseteq T_k^m \text{ and } \frac{\mu_\ell(S)}{\mu_\ell(C_m)} = c_m & \text{if } \frac{\mu_\ell(T_k^m)}{\mu_\ell(C_m)} > c_m. \end{cases}$$

Eventually they obtain $T_{|I_m|}^m =: R_m$ for which

$$\frac{\mu_i(R_m)}{\mu_i(C_m)} \leqslant \frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)} = c_m \tag{1}$$

for every $i \in I_m$ and there is equality for at least one index. Let $j_m := i_m$ if there is equality at (1) for i_m and let j_m be the smallest index in I_m for which we have equality if the inequality is strict for i_m. We allocate R_m to player P_{j_m}, formally $S_{j_m}^{m+1} := S_{j_m}^m \cup R_m$ and $S_i^{m+1} := S_i^m$ for $i \in [n] \setminus \{j_m\}$. For $i \in I_{m+1}$ let

$$t_i^{m+1} := \mu_i(S_i^{m+1}) + \frac{t_i^m - \mu_i(S_i^{m+1})}{\sum_{j \in I_{m+1}} \frac{t_j^m - \mu_j(S_j^{m+1})}{\mu_j(C_{m+1})}},$$

which completes the description of the general step of the algorithm.

Let us shade some more light on the running of the algorithm and on the ideas behind the formal definitions by a concrete example:

Example 1. Let the cake be the unit interval $[0, 1]$ and the slices are defined to be the Borel subsets. We have 3 players with respective entitlements $t_1 = \frac{1}{2}, t_2 = \frac{1}{3}$ and $t_3 = \frac{1}{6}$. The measure μ_1 is the uniform distribution on $[0, \frac{1}{2}]$, μ_2 is the same but on $[\frac{1}{2}, 1]$ and μ_3 is the uniform distribution on the whole cake $[0, 1]$.

Then the constant c_0 is simply the smallest entitlement $\frac{1}{6}$ and $i_0 = 3$. Player P_3 cuts off a slice which could be $T_1^0 = [0, \frac{1}{6}]$. Player P_1 diminishes this slice, he cuts off for example

$T_2^0 = [0, \frac{1}{12}]$ (this worths $\frac{1}{6}$ for him). Since $\mu_2(T_2^0) = 0$, player P_2 does not change this slice, i.e. $T_3^0 = T_2^0$. We have $j_0 = 1$ and allocate $R_0 := T_3^0$ to P_1, more precisely $S_1^1 := R_0$ and $S_2^1 := S_3^1 := \emptyset$.

Now P_1 still needs $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$. This is the $\frac{\frac{1}{3}}{\frac{5}{6}} = \frac{2}{5}$ fraction of the remaining cake according to his own measure. The removed part has no value for P_2, thus he still wants the $\frac{1}{3}$ fraction of the remaining part. Finally, P_3 wants the $\frac{\frac{1}{6}}{\frac{11}{12}} = \frac{2}{11}$ fraction of the rest according to his measure. Now the algorithm "scales" the ratios $\frac{2}{5}, \frac{1}{3}$ and $\frac{2}{11}$ in order to sum up to 1, i.e. considers

$$\frac{\frac{2}{5}}{\frac{2}{5} + \frac{1}{3} + \frac{2}{11}}, \ \frac{\frac{1}{3}}{\frac{2}{5} + \frac{1}{3} + \frac{2}{11}} \text{ and } \frac{\frac{2}{11}}{\frac{2}{5} + \frac{1}{3} + \frac{2}{11}},$$

norms the measures to be probability measures on the remaining cake and does the same that it initially did. The improved entitlements t_i^1 for $i = 1, 2, 3$ are defined in such a way that the scaled ratios are exactly the quantities $\frac{t_i^1 - \mu_i(S_i^1)}{\mu_i(C_1)}$.

We proceed by proving that the steps of the algorithm are well-defined and it always terminates:

Lemma 2. *The steps of Algorithm II can be done and it maintains the equation*

$$\sum_{i \in I_m} \frac{t_i^m - \mu_i(S_i^m)}{\mu_i(C_m)} = 1 \tag{2}$$

as well as the inequalities $t_i^m \geqslant t_i$ for every $i \in [n]$.

Proof. We use induction on m. For $m = 0$, (2) says $\sum_{i=1}^n t_i = 1$ which we assumed and $t_i^0 = t_i$ by definition. Suppose we are at the beginning of round m and (2) holds so far and $t_i^m \geqslant t_i$ for $i \in [n]$. By the definition of I_m and by $t_i^m \geqslant t_i$, the summands at (2) are all positive, thus we have $\frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)} \leqslant 1$. Therefore $t_{i_m}^m - \mu_{i_m}(S_{i_m}^m) \leqslant \mu_{i_m}(C_m)$ and hence there is indeed a $T_1^m \subseteq C_m$ with $\mu_{i_m}(T_1^m) = t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)$. If we know that the entitlements t_i^{m+1} for $i \in I_{m+1}$ are well-defined (i.e. no zeros in the denominators), then a direct calculation shows that they sum up to 1.

Observation 3. *If $j_m = i_m$, then player P_{i_m} will be satisfied after round m because in this case we have equality at (1) for i_m and $t_{i_m}^m \geqslant t_{i_m}$.*

If $I_m = \{i_m\}$, then we must have $j_m = i_m$. It follows by Observation 3 that $I_{m+1} = \emptyset$ and therefore the algorithm terminates, thus in this case there is nothing more to prove.

Suppose that $|I_m| > 1$. Then the right side of (1) is strictly smaller than 1 because it is one of the summands at (2) which are: all positive, there are at least two of them and they sum up to 1. By subtracting both sides of (1) from 1 and taking the reciprocates we obtain

$$\frac{\mu_i(C_m)}{\mu_i(C_{m+1})} \leqslant \frac{1}{1 - \frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)}} \tag{3}$$

for $i \in I_m$. In particular $\mu_j(C_{m+1}) > 0$ for $j \in I_{m+1}$. Since $t_j^m \geqslant t_j > \mu_j(S_j^{m+1})$, the entitlements t_i^{m+1} are indeed well-defined.

Claim 4. *We have*

$$\sum_{i \in I_{m+1}} \frac{t_i^m - \mu_i(S_i^{m+1})}{\mu_i(C_{m+1})} \leqslant 1$$

as well as $t_i^{m+1} \geqslant t_i^m$ *for every* $i \in I_{m+1}$*, moreover, all of these inequalities are strict if* $j_m \neq i_m$.

Proof.

$$\begin{aligned}
\sum_{i \in I_{m+1}} \frac{t_i^m - \mu_i(S_i^{m+1})}{\mu_i(C_{m+1})} &\overset{*}{\leqslant} \sum_{i \in I_m} \frac{t_i^m - \mu_i(S_i^{m+1})}{\mu_i(C_{m+1})} \\
&= \sum_{i \in I_m} \frac{t_i^m - \mu_i(S_i^{m+1})}{\mu_i(C_m)} \cdot \frac{\mu_i(C_m)}{\mu_i(C_{m+1})} \\
&\overset{(3)}{\leqslant} \sum_{i \in I_m} \frac{t_i^m - \mu_i(S_i^{m+1})}{\mu_i(C_m)} \cdot \frac{1}{1 - \frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)}} \\
&\overset{**}{\leqslant} \left[\left(\sum_{i \in I_m} \frac{t_i^m - \mu_i(S_i^m)}{\mu_i(C_m)} \right) - \frac{\mu_{j_m}(R_m)}{\mu_{j_m}(C_m)} \right] \cdot \frac{1}{1 - \frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)}} \\
&\overset{***}{=} \left[1 - \frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)} \right] \frac{1}{1 - \frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)}} = 1.
\end{aligned}$$

$*$ $I_{m+1} \subseteq I_m$ and the summands are non-negative,
$**$ $\mu_{j_m}(S_{j_m}^{m+1}) = \mu_{j_m}(S_{j_m}^m) + \mu_{j_m}(R_m)$ because $S_{j_m}^{m+1}$ is the disjoint union of $S_{j_m}^m$ and R_m, furthermore, $S_i^{m+1} = S_i^m$ for $i \in I_m \setminus \{j_m\}$,
$***$ (2) and there is equality at (1) for j_m.

The overestimation of $\frac{\mu_{i_m}(C_m)}{\mu_{i_m}(C_{m+1})}$ via (3) is strict if $i_m \neq j_m$. The part about the inequalities $t_i^{m+1} \geqslant t_i^m$ follows directly from the already proved part and the definition of t_i^{m+1}. □

□

Lemma 5. *Algorithm II terminates after finitely many steps.*

Proof. Suppose for a contradiction that the algorithm does not terminate for $\mu_1, \dots, \mu_n$ and $t_1, \dots, t_n$. Let k be the smallest number for which $I_k = I_m$ for every $m > k$. Then $j_k \neq i_k$ since otherwise we had $I_{k+1} = I_k \setminus \{i_k\} \subsetneq I_k$ (see Observation 3). By Claim 4 this implies $t_i^{k+1} > t_i^k \geqslant t_i$ for every $i \in I_k$. Let $(m_\ell)_{\ell \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers with $m_0 > k$ such that there are $i^*, j^* \in I_k$ with $i_{m_\ell} = i^*$ and $j_{m_\ell} = j^*$ for every ℓ. There cannot be a $\varepsilon > 0$ such that $\mu_{j^*}(R_{m_\ell}) \geqslant \varepsilon$ for infinitely many ℓ because then P_{j^*} would be eventually satisfied and removed from the process, contradicting the definition of k. Thus $\lim_{\ell\to\infty}\mu_{j^*}(R_{m_\ell})=0$. Since there is equality for j^* at (1) for each m_ℓ, we know that

$$\mu_{j^*}(R_{m_\ell})=[t_{i^*}^{m_\ell}-\mu_{i^*}(S_{i^*}^{m_\ell})]\,\frac{\mu_{j^*}(C_{m_\ell})}{\mu_{i^*}(C_{m_\ell})}.$$

If $\liminf_{\ell\to\infty} t_{i^*}^{m_\ell}-\mu_{i^*}(S_{i^*}^{m_\ell})=0$, then $\mu_{i^*}(S_{i^*}^{m_\ell})\geqslant t_{i^*}$ for some ℓ because $t_{i^*}^{m_0}>t_{i^*}$ and $t_{i^*}^{m_\ell}$ is increasing in ℓ, a contradiction. Therefore we must have $\lim_{\ell\to\infty}\frac{\mu_{j^*}(C_{m_\ell})}{\mu_{i^*}(C_{m_\ell})}=0$. Since $\mu_{i^*}(C_{m_\ell})\leqslant\mu_{i^*}(C_{m_0})$, this implies $\lim_{\ell\to\infty}\mu_{j^*}(C_{m_\ell})=0$. But then it follows from (2) that $\lim_{\ell\to\infty} t_{j^*}^{m_\ell}-\mu_{j^*}(S_{j^*}^{m_\ell})=0$. As earlier with i^*, this implies that player P_{j^*} will be eventually satisfied, which is a contradiction. □

3 From fairness to strong fairness, an elementary approach

Lemma 6. *Assume that $(C,\mathcal{S})$ is a measurable space, I is a countable index set, and for $i\in I$, μ_i is an atomless probability measure defined on $\mathcal{S}$ and t_i is a positive number such not all the μ_i are identical and $\sum_{i\in I}t_i=1$. Then there is a partition $C=C'\sqcup C''$ and $t_i',t_i''>0$ with $\sum_{i\in I}t_i'=\sum_{i\in I}t_i''=1$ such that $t_i'\cdot\mu_i(C')+t_i''\cdot\mu_i(C'')>t_i$ for each $i\in I$.*

Proof. Suppose that $j,k\in I$ and $C'\in\mathcal{S}$ such that $\mu_j(C')<\mu_k(C')$. It is enough to find $s_i',s_i''>0$ with $\sum_{i\in I}s_i',\sum_{i\in I}s_i''<1$ and $s_i'\cdot\mu_i(C')+s_i''\cdot\mu_i(C'')=t_i$ for every $i\in I$ because then

$$t_i':=\frac{s_i'}{\sum_{\ell\in I}s_\ell'}\text{ and }t_i'':=\frac{s_i''}{\sum_{\ell\in I}s_\ell''}$$

are as desired. We are looking for $\varepsilon,\delta>0$ for which the definitions

- $s_j':=t_j-\varepsilon$
- $s_j'':=t_j+\varepsilon\cdot\frac{\mu_j(C')}{\mu_j(C'')}$
- $s_k':=t_k+\delta\cdot\frac{\mu_k(C'')}{\mu_k(C')}$
- $s_k'':=t_k-\delta$
- $s_i'':=s_i':=t_i$ for $i\in\mathbb{N}\setminus\{j,k\}$

are suitable. Note that whatever ε and δ we choose, $s_i'\cdot\mu_i(C')+s_i''\cdot\mu_i(C'')=t_i$ will hold for each $i\in\mathbb{N}$. Thus the requirements $s_i',s_i''>0$ and $\sum_{i\in I}s_i',\sum_{i\in I}s_i''<1$ mean for ε and δ that they satisfy

$$\begin{aligned}&\varepsilon\in(0,t_j)\\&\delta\in(0,t_k)\\&\varepsilon>\delta\cdot\frac{\mu_k(C'')}{\mu_k(C')}\\&\delta>\varepsilon\cdot\frac{\mu_j(C')}{\mu_j(C'')}\end{aligned}$$

If $\mu_j(C') = 0$, then the last inequality is redundant and the existence of a solution is straightforward. Otherwise the last two inequalities demand

$$\frac{\mu_k(C'')}{\mu_k(C')} < \frac{\varepsilon}{\delta} < \frac{\mu_j(C'')}{\mu_j(C')}.$$

Since $\frac{\mu_k(C'')}{\mu_k(C')} < \frac{\mu_j(C'')}{\mu_j(C')}$ follows from $\mu_j(C') < \mu_k(C')$, the desired ε and δ exist in this case as well. □

Let μ_i' be the restriction of $\frac{\mu_i}{\mu_i(C')}$ to $\mathcal{S} \cap \mathcal{P}(C')$ if $\mu_i(C') \neq 0$ and an arbitrary atomless probability measure on $\mathcal{S} \cap \mathcal{P}(C')$ if $\mu_i(C') = 0$. We define μ_i'' analogously with respect to C''.

Corollary 7. *Assume the settings of Lemma 6. If $\{S_i' : i \in I\}$ is a fair division with respect to μ_i', t_i' $(i \in I)$ and $\{S_i'' : i \in I\}$ is a fair divisions with respect to μ_i'', t_i'' $(i \in I)$, then for $S_i := S_i' \cup S_i''$, $\{S_i : i \in I\}$ is a strongly fair division with respect to μ_i, t_i $(i \in I)$.*

Proof. We have $\mu_i(S_i') \geqslant t_i' \cdot \mu_i(C')$ and $\mu_i(S_i'') \geqslant t_i'' \cdot \mu_i(C'')$ by fairness, thus by Lemma 6

$$\mu_i(S_i) = \mu_i(S_i' \sqcup S_i'') = \mu_i(S_i') + \mu_i(S_i'') \geqslant t_i' \cdot \mu_i(C') + t_i'' \cdot \mu_i(C'') > t_i.$$ □

4 Existence of a fair division for infinitely many players

We repeat the theorem here for convenience. Assume that $(C, \mathcal{S})$ is a measurable space and for $i \in \mathbb{N}$, μ_i is an atomless probability measure defined on $\mathcal{S}$ and t_i is a positive number such that $\sum_{i=0}^{\infty} t_i = 1$. Then there is a partition $C = \bigsqcup_{i=0}^{\infty} S_i$ such that $S_i \in \mathcal{S}$ with $\mu_i(S_i) \geqslant t_i$ for each $i \in \mathbb{N}$. Furthermore, if not all the μ_i are identical, then '$\mu_i(S_i) \geqslant t_i$' can be strengthened to '$\mu_i(S_i) > t_i$' for every $i \in \mathbb{N}$.

Proof. Without loss of generality we may look for a sub-partition instead of a partition, i.e. we can relax '$C = \bigsqcup_{i=0}^{\infty} S_i$' to '$C \supseteq \bigsqcup_{i=0}^{\infty} S_i$' since the remaining surplus part of the cake can be given to anybody. The last sentence of Theorem 1 follows from the rest of it via Corollary 7.

For $n \in \mathbb{N}$, we let $t_0^n, t_1^n, \dots, t_n^n$ to be the first $n+1$ entitlements scaled to sum up to 1, i.e.

$$t_i^n := \frac{t_i}{\sum_{j=0}^{n} t_j}.$$

Observation 8. $(1 - t_{n+1}^{n+1})t_i^n = t_i^{n+1}$ *and* $\lim_{n\to\infty} t_i^n = t_i$.

Proof.

$$\frac{t_i^{n+1}}{t_i^n} = \frac{\sum_{j=0}^{n} t_j}{\sum_{j=0}^{n+1} t_j} = \frac{\sum_{j=0}^{n+1} t_j - t_{n+1}}{\sum_{j=0}^{n+1} t_j} = 1 - t_{n+1}^{n+1},$$

$$\lim_{n\to\infty} t_i^n = \lim_{n\to\infty} \frac{t_i}{\sum_{j=0}^{n} t_j} = \frac{t_i}{\lim_{n\to\infty} \sum_{j=0}^{n} t_j} = t_i.$$ □

We shall define recursively $S_i^n \in \mathcal{S}$ for $i, n \in \mathbb{N}$ with $i \leqslant n$ in such a way that

(i) $C = \bigsqcup_{i \leqslant n} S_i^n$ for every n;

(ii) $\mu_i(S_i^n) \geqslant t_i^n$;

(iii) For every fixed $i \in \mathbb{N}$ the sequence $(S_i^n)_{n \geqslant i}$ is $\subseteq$-decreasing.

Observe that conditions (i) and (ii) say that for each fixed n the sets $S_0^n, S_1^n, \ldots, S_n^n$ form a fair division with respect to the measures μ_i and entitlements t_i^n. Although such a fair division can be found for every particular n, it cannot be guaranteed without condition (iii) that they have a meaningful "limit" which provides a fair division in the original settings.

We let $S_0^0 := C$ which obviously satisfies the conditions. Suppose that $S_0^n, S_1^n \ldots, S_n^n$ are already defined for some $n \in \mathbb{N}$. We need to find for each $i \leqslant n$ an $S_i^{n+1} \subseteq S_i^n$ with $\mu_i(S_i^{n+1}) \geqslant t_i^{n+1}$ in such a way that for

$$S_{n+1}^{n+1} := C \setminus \bigcup_{i \leqslant n} S_i^{n+1}$$

we have $\mu_{n+1}(S_{n+1}^{n+1}) \geqslant t_{n+1}^{n+1}$. For the last inequality it is enough to ensure that

$$\mu_{n+1}(S_i^n \setminus S_i^{n+1}) \geqslant \mu_{n+1}(S_i^n) \cdot t_{n+1}^{n+1} \text{ for } i \leqslant n. \tag{4}$$

Indeed, since

$$S_{n+1}^{n+1} = \bigsqcup_{i \leqslant n} S_i^n \setminus S_i^{n+1},$$

the inequalities (4) imply

$$\begin{aligned} \mu_{n+1}(S_{n+1}^{n+1}) &= \mu_{n+1}\left(\bigsqcup_{i \leqslant n} S_i^n \setminus S_i^{n+1}\right) = \sum_{i=0}^{n} \mu_{n+1}(S_i^n \setminus S_i^{n+1}) \geqslant \sum_{i=0}^{n} \mu_{n+1}(S_i^n) \cdot t_{n+1}^{n+1} \\ &= t_{n+1}^{n+1} \cdot \sum_{i=0}^{n} \mu_{n+1}(S_i^n) = t_{n+1}^{n+1} \cdot \mu_{n+1}(C) = t_{n+1}^{n+1} \cdot 1 = t_{n+1}^{n+1}, \end{aligned}$$

where we used (i) combined with the fact that μ_{n+1} is a probability measure. Therefore it is enough to find for every $i \leqslant n$ an $S_i^{n+1} \subseteq S_i^n$ such that

$$\mu_i(S_i^{n+1}) \geqslant t_i^{n+1} \tag{5}$$

$$\mu_{n+1}(S_i^n \setminus S_i^{n+1}) \geqslant \mu_{n+1}(S_i^n) \cdot t_{n+1}^{n+1}. \tag{6}$$

Let $i \leqslant n$ be fixed. If $\mu_{n+1}(S_i^n) = 0$, then we let $S_i^{n+1} := S_i^n$ which is clearly appropriate since $t_i^n \geqslant t_i^{n+1}$ (see Observation 8). Suppose that $\mu_{n+1}(S_i^n) > 0$ and note that $\mu_i(S_i^n) \geqslant t_i^n > 0$ by assumption. We claim that choosing S_i^{n+1} to be the slice corresponding to i in a fair division of S_i^n between P_i and P_{n+1} with respect to the restrictions of $\frac{\mu_i}{\mu_i(S_i^n)}$ and

$\frac{\mu_{n+1}}{\mu_{n+1}(S_i^n)}$ to $\mathcal{S} \cap \mathcal{P}(S_i^n)$ and respective entitlements $1 - t_{n+1}^{n+1}$ and t_{n+1}^{n+1} is suitable. Indeed, by the fairness of the obtained bipartition $\{S_i^{n+1},\ S_i^n \setminus S_i^{n+1}\}$ of S_i^n we have

$$\frac{\mu_i(S_i^{n+1})}{\mu_i(S_i^n)} \geqslant 1 - t_{n+1}^{n+1},$$
$$\frac{\mu_{n+1}(S_i^n \setminus S_i^{n+1})}{\mu_{n+1}(S_i^n)} \geqslant t_{n+1}^{n+1}.$$

Here the second inequality is equivalent with (6) and the first one implies (5) since

$$\mu_i(S_i^{n+1}) \geqslant (1 - t_{n+1}^{n+1})\mu_i(S_i^n) \geqslant (1 - t_{n+1}^{n+1})t_i^n = t_i^{n+1},$$

where we used $\mu_i(S_i^n) \geqslant t_i^n$ and Observation 8. The recursion is done.

We define $S_i := \bigcap_{n\geqslant i} S_i^n$ for $i \in \mathbb{N}$. Then for $i < j$ we have $S_i \cap S_j = \emptyset$ because $S_i \subseteq S_i^j,\ S_j \subseteq S_j^j$ and $S_i^j \cap S_j^j = \emptyset$ by (i). Furthermore,

$$\mu_i(S_i) = \mu_i\left(\bigcap_{n\geqslant i} S_i^n\right) = \lim_{n\to\infty} \mu_i(S_i^n) \geqslant \lim_{n\to\infty} t_i^n = t_i$$

by (iii), (ii) and Observation 8. This completes the proof of Theorem 1. □

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