Snarks with resistance n and flow resistance $2n$

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Abstract

We examine the relationship between two measures of uncolourability of cubic graphs – their resistance and flow resistance. The resistance of a cubic graph G, denoted by $r(G)$, is the minimum number of edges whose removal results in a 3-edge-colourable graph. The flow resistance of G, denoted by $r_f(G)$, is the minimum number of zeroes in a 4-flow on G. Fiol et al. [Electron. J. Combin. 25 (2018), #P4.54] made a conjecture that $r_f(G) \leqslant r(G)$ for every cubic graph G. We disprove this conjecture by presenting a family of cubic graphs G_n of order $34n$, where $n \geqslant 3$, with resistance n and flow resistance $2n$. For $n \geqslant 5$ these graphs are nontrivial snarks.

Mathematics Subject Classifications: 05C

1 Introduction

Snarks are 2-connected cubic graphs whose edges cannot be properly coloured with three colours. The significance of this class of graphs derives mainly from the fact that it may contain counterexamples to several important and long-standing conjectures in graph

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theory, such as the cycle double cover conjecture, the 5-flow conjecture, Fulkerson's conjecture, and others. While most of these conjectures are easy for 3-edge-colourable graphs, they are exceedingly difficult for snarks in general. On the other hand, a number of recent results confirm that some of these conjectures become tractable for snarks that are in a certain sense close to 3-edge-colourable graphs, see for example [6, 10, 15]. In this situation it is natural to focus on the study of invariants of cubic graphs that express – in various ways – to what extent a graph differs from a 3-edge-colourable graph. Such invariants are called *measures of uncolourability*. A deeper examination of relations between various uncolourability measures may provide new insights into the studied conjectures and lead to interesting partial results. An excellent survey on this topic, by Fiol et al. [4], is available and highly recommended.

One of the most prominent uncolourability measures is the *resistance* of a cubic graph. It is defined as the smallest number of edges (or, equivalently, vertices) whose removal from a graph results in a 3-edge-colourable graph [13, 14]; we denote the resistance of graph G by $r(G)$. Clearly, $r(G) = 0$ if and only if G is 3-edge-colourable. Moreover, $r(G) \geqslant 2$ whenever G is not 3-edge-colourable.

A similar measure of uncolourability is obtained by regarding snarks as cubic graphs that do not admit a nowhere-zero 4-flow. The *flow resistance* of a cubic graph G, denoted by $r_f(G)$, is the smallest number of zero-valued edges that any integer 4-flow on G can have. Like resistance, $r_f(G) = 0$ if and only if G is 3-edge-colourable. The similarity of these two measures is underscored by the fact that a cubic graph admits an integer nowhere-zero 4-flow if and only if it admits a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow, and the latter coincides with a proper 3-edge-colouring where colours are the non-zero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

In contrast to resistance, however, flow resistance can take value 1. In fact, cubic graphs with flow resistance 1 are very common, while those with flow resistance greater than 1, introduced by Jaeger [7, 8] as *strong snarks*, appear to be rare. The complete list of all cyclically 4-edge-connected snarks with girth at least 5 on up to 36 vertices generated by Brinkmann et al. [2] contains 64 326 024 items, only 32 of which are strong snarks, all with flow resistance 2. At the same time, all snarks in the list have resistance 2 (these facts easily follow from [2, Observations 4.10 and 4.14] and the definition of a strong snark).

The importance of flow resistance for the study of snarks is quite obvious because it offers a natural approach to Tutte's 5-flow-conjecture. A similar approach to Tutte's 3-flow conjecture has recently been taken by DeVos et al. [3] where the authors proved that every 3-edge-connected graph admits a 3-flow in which at most one sixth of the edges carries value zero.

In spite of these facts, flow resistance has so far attracted surprisingly little attention. The only explicit mention of flow resistance in the literature occurs in the cited survey [4], with Section 4.1 being completely devoted to this invariant. The authors of [4] note that flow resistance can be equivalently defined as the minimum number of edges that have to be contracted in order to obtain a graph that admits a nowhere-zero 4-flow [4, Theorem 33]. Moreover, they show in [4, Proposition 29] that $r_f(G)$ is bounded above by the minimum number of edges that any two perfect matchings of G can have, which is another useful uncolourability measure (denoted in [4] by $\gamma_2(G)$). They also propose the following interesting conjecture (Conjecture 51).

Conjecture 1. (Fiol et al. [4]) If G is a bridgeless cubic graph, then $r_f(G) \leqslant r(G)$.

Intuitively, every individual zero-valued edge e of a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on a cubic graph G generates two faulty vertices of the corresponding 3-edge-colouring, the end-vertices of e. After removing one edge at each of the vertices a proper 3-edge-colouring is obtained. This observation seems to speak in favour of the conjecture. Nevertheless, the conjecture is false, as follows from the main result of the present paper.

Theorem 2. *For every integer $n \geqslant 3$ there exists a cubic graph G_n on $34n$ vertices with $r(G_n) = n$ and $r_f(G_n) = 2n$. Moreover, if $n \geqslant 5$, then G_n is cyclically 4-edge-connected and has girth 5.*

2 Preliminaries

A *semi-graph* G is a pair $G = (V, E)$ which consists of a set of vertices $V = V(G)$ and a set $E = E(G) \subseteq \mathcal{P}_2(V) \cup \mathcal{P}_1(V)$ consisting of edges and semi-edges; here $\mathcal{P}_i(A)$ denotes the set of all i-element subsets of a set A. In $E(G)$, the 2-element sets are called *edges* (as expected) while the 1-element sets are called *semi-edges*. Note that if E contains no elements from $\mathcal{P}_1(V)$, then G is simply a *graph*.

We denote the edge $\{u, v\}$ as uv and the semi-edge $\{u\}$ as (u). Furthermore, we define the *join* between two semi-edges (u) and (v) as the removal of semi-edges (u) and (v), and the addition of the edge uv. A semi-edge (u) and a vertex v may also join to form an edge uv, with semi-edge (u) being removed. The degree of a vertex v in a semi-graph G is defined as the combined total number of edges and semi-edges incident with v. Thus a cubic semi-graph is a semi-graph with each vertex having degree 3.

Essentially, semi-edges behave like edges except that they are associated with one vertex instead of two, with each vertex having at most one semi-edge. We say that a semi-graph G contains a semi-graph G' if $V(G') \subseteq V(G)$, $uv \in E(G')$ implies that $uv \in E(G)$, semi-edge $(u) \in E(G')$ implies semi-edge $(u) \in E(G')$ or there is an edge $uv \in E(G)$, and for every vertex $u \in V(G')$ the degree of u in G is greater than or equal to the degree of u in G'.

Let G be a semi-graph. An *edge colouring* of G is an assignment of colours to the elements of $E(G)$ such that adjacent elements receive distinct colours; such colourings are often termed *proper*. An edge colouring that uses k colours is a *k-edge-colouring*.

It is well known that every cubic graph can be properly edge-coloured with three or four colours. A *snark* is a 2-connected cubic graph that admits no proper 3-edge-colouring. Snarks with small edge cuts and short circuits are usually considered trivial. A snark is called *nontrivial* if it is cyclically 4-edge-connected and has no circuits of length smaller than 5. Recall that a connected graph is *cyclically k-edge-connected* if it contains no subset $S \subseteq E(G)$ of size $|S| < k$ such that $G - S$ is disconnected and has at least two components containing a circuit.

Let A be an abelian group. An *A-flow* on G is a pair (D, ϕ) where ϕ is an assignment of elements of A to the elements of $E(G)$, and D is an assignment of one of two directions to the elements of $E(G)$ such that, for every vertex v in G, the sum of values flowing into v equals the sum of values flowing out of v (*Kirchhoff's law*). A *nowhere-zero* A-flow is one which does not assign the zero element of A to any edge or semi-edge of G. Note that the choice of D is immaterial since one can reverse the orientation of any edge e and replace the value $\phi(e)$ with $-\phi(e)$ without violating the Kirchhoff law. Moreover, if each element $x \in A$ satisfies $x = -x$, then the assignment of an orientation can be omitted from the definition altogether. It is well known that the latter condition is satisfied if and only if $A \cong \mathbb{Z}_2^n$ for some $n \geqslant 1$.

An A-flow (D, ϕ) where $A = \mathbb{Z}$ and $\phi(e) \in \{0, \pm 1, \ldots, \pm(k-1)\}$ for all $e \in E(G)$ is called an *integer k-flow*. A well-known useful result on flows is the following: A graph admits a nowhere-zero A-flow, for some finite abelian group A, if and only if it admits a nowhere-zero integer $|A|$-flow [11, 16]. In particular, a cubic semi-graph admits an integer 4-flow with m zeros if and only if it admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow with m zeros. This observation offers an additional advantage that in the study of flow resistance we do not need to bother with orientations of the graph in question.

It is easy to see that a nowhere zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on a cubic graph G corresponds exactly to a 3-edge-colouring. That is, if the assignment of elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ to the edges of G is interpreted as an edge colouring, and no edge is assigned the zero element, then the edge colouring is proper. This is easily seen to be true for semi-graphs as well. Furthermore, given any A-flow on a semi-graph, the sum of the flow values of its semi-edges must be zero. If G is cubic and $A = \mathbb{Z}_2 \times \mathbb{Z}_2$, the latter amounts to what is generally known as the *Parity Lemma*. These facts will be used implicitly throughout this paper.

3 Main result

Let X denote the semi-graph created from the Petersen graph by the removal of two adjacent vertices; it is shown in Figure 1. The semi-edges of X occur in two pairs a, b and c, d, each of them arising by the removal of the same vertex of the Petersen graph. The following properties of X, stated in Lemma 1, are well known. In fact, they easily follow from the Kirchhoff law and the fact that the Petersen graph is the smallest snark.

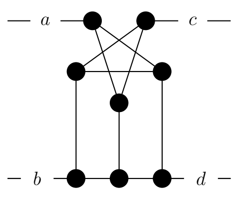

Figure 1: The semi-graph X

Lemma 3. *The following statements hold for the semi-graph X depicted in Figure 1.*

(i) $r(X) = 0$.

(ii) *If f is a proper 3-edge-colouring of X, then $f(a) = f(b)$ and $f(c) = f(d)$.*

Figure 2 depicts a semi-graph Y constructed from two instances X_1 and X_2 of X and from three additional vertices p, q, and r as follows. Denoting by x_i the semi-edge of X_i corresponding to the semi-edge x of X, with $x \in \{a, b, c, d\}$, we join c_1 and a_2 to p, next we join d_1 and b_2 to q, and finally we add the edges pr, qr, and the semi-edge $(r) = e$. We relabel the semi-edges a_1 and b_1 as a and b, respectively, and relabel the semi-edges c_2 and d_2 as c and d, respectively.

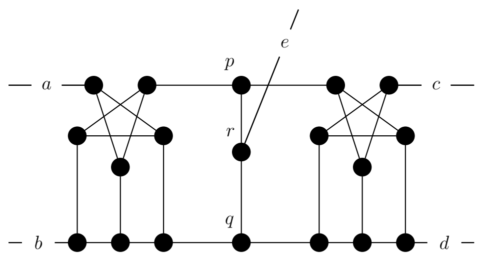

Figure 2: The semi-graph Y

Lemma 4. *The following statements hold for the semi-graph Y depicted in Figure 2.*

(i) $r(Y) = 1$.

(ii) $r_f(Y) = 1$.

Proof. (i): We first show that $r(Y) \geqslant 1$. It is clearly sufficient to prove that Y is not 3-edge-colourable. Suppose the contrary. Recall that Y contains two instances X_1 and X_2 of X, where X_1 is the one containing the semi-edges a and b and X_2 is the one containing the semi-edges c and d. The other two semi-edges of X_1 are c_1 and d_1 and those of X_2 are a_2 and b_2. By Lemma 3, every proper 3-edge-colouring of Y assigns the same colour to c_1 and d_1 and the same colour to a_2 and b_2. As a consequence, the end-vertex of the semi-edge e of Y has two edges with the same colour, a contradiction. Thus Y is not 3-edge-colourable. On the other hand, Y becomes 3-edge-colourable after removing either of the two edges adjacent to e, which implies that $r(Y) = 1$.

(ii): Suppose to the contrary that $r_f(Y) = 0$. Then Y admits a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. Since Y is cubic, the flow is also a proper 3-edge-colouring, and therefore $r(Y) = 0$, contradicting Statement (i). Hence, $r_f(Y) \geqslant 1$. It is easy to see that there exists a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on Y with e being the only zero edge. Therefore $r_f(Y) = 1$. □

Figure 3 displays a semi-graph Z constructed from an instance X_1 of X, an instance Y_2 of Y, and four additional vertices p, q, r, and s in a similar manner as Y was constructed from two instances of X. If we use the same convention for indices as before, then p, q, and r get the same incidences as previously. The semi-edges e_1 of X_1 and e_2 of Y_2 are then joined to the vertex s and a new semi-edge $(s) = e$ is added. As before, we relabel the semi-edges of Z inherited from X_1 and Y_2 as a, b, c, and d. The instance of X which contains the edge f, as seen in Figure 3, is called the *central instance* of X.

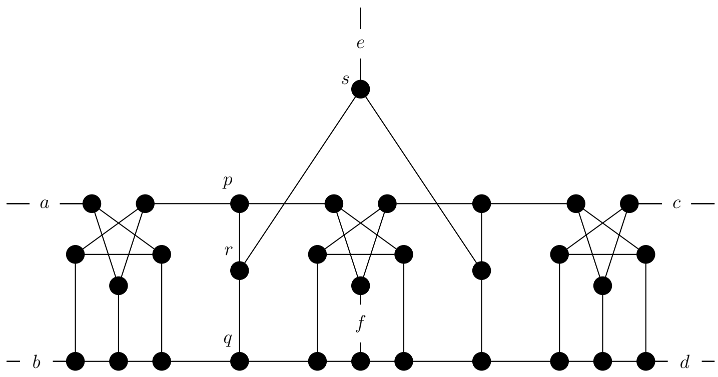

Figure 3: The semi-graph Z

Lemma 5. *The following statements hold for the semi-graph Z depicted in Figure 3.*

(i) $r(Z) = 1$.

(ii) $r_f(Z) = 2$.

Proof. (i): Since Z contains a copy of Y and $r(Y) = 1$, from Lemma 4 (i) we readily obtain that $r(Z) \geqslant 1$. Moreover, $Z - f$ is 3-edge-colourable for the edge f indicated in Figure 3, whence $r(Z) = 1$.

(ii): Lemma 4 (ii) implies that $r_f(Z) \geqslant 1$. Suppose that $r_f(Z) = 1$ and let ϕ be a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on Z with one zero. The zero-valued element of $E(Z)$ must be contained in the central instance of X, otherwise there would be an instance of Y which contains no zero edges, contradicting Lemma 4 (ii). By Lemma 3 and the fact that a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on a semi-graph corresponds to a proper 3-edge-colouring, we have that $\phi(a) = \phi(b)$ and $\phi(c) = \phi(d)$. Consequently, $\phi(a) + \phi(b) = 0$ and $\phi(c) + \phi(d) = 0$. Since the sum of the values assigned to the five semi-edges in Z must be zero, we conclude that $\phi(e) = 0$. This contradicts the assumption that $r_f(Z) = 1$. Furthermore, it is not difficult to find a $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow on Z with f and e being the only zero edges. Hence, $r_f(Z) = 2$. $\square$

Following on from this, it is straightforward to construct nontrivial snarks with flow resistance equal to twice their resistance. For each integer $n \geqslant 3$ we define a graph G_n on $34n$ vertices as follows: Let G_n contain n instances of Z, called $Z_0, Z_1, \ldots, Z_{n-1}$. In Z_i, let

a_i, b_i, c_i, d_i, and e_i denote the semi-edges corresponding, respectively, to the semi-edges a, b, c, d, and e of Z, and let f_i be the edge of Z_i corresponding to the edge f of Z shown in Figure 3. For each Z_i we also add the vertices u_i, v_i, and w_i to the graph G_n. The semi-edge a_i is then joined to the vertex u_i, the semi-edge b_i is joined to the vertex v_i, and the semi-edge e_i is joined to the vertex w_i. The edge u_iv_i is added. Finally, the semi-edge c_i is joined to u_{i+1}, the semi-edge d_i is joined to v_{i+1}, and the vertex w_i is joined to the vertex w_{i+1}, where the subscripts are reduced modulo n. The graph G_4 is illustrated in Figure 4.

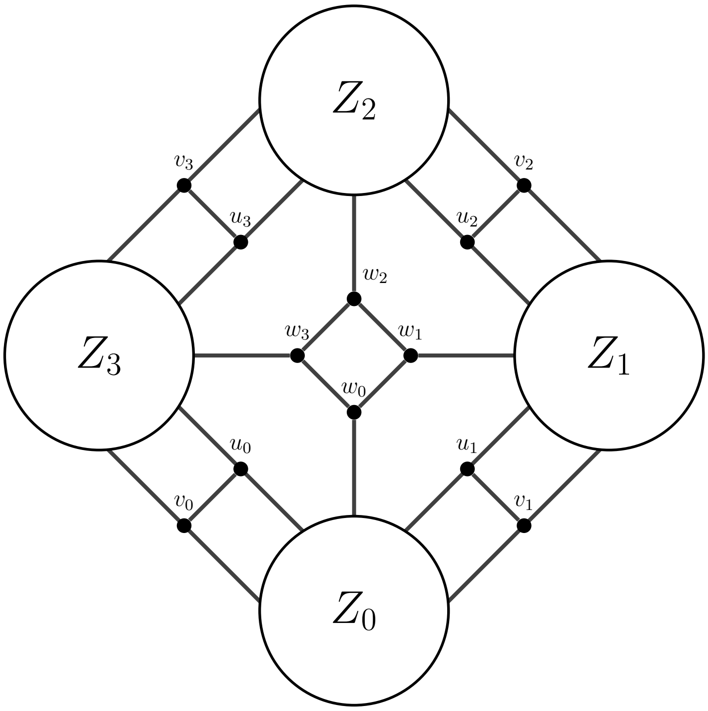

Figure 4: Graph G_4.

Theorem 6. *For every integer $n \geqslant 3$ there exists a cubic graph G_n on $34n$ vertices with $r(G_n) = n$ and $r_f(G_n) = 2n$. If $n \geqslant 5$, then G_n is a nontrivial snark.*

Proof. Clearly, G_n has girth 5 unless $n \leqslant 4$. It is straightforward to check that if $n \geqslant 4$, then G_n is cyclically 4-edge-connected. Indeed, from the construction of G_n it is clear that G_n has no bridges, 2-edge-cuts, and nontrivial 3-edge-cuts. Thus the smallest cycle-separating edge-cut is of size at least 4 (and one of size 4 can be easily identified). The details are left to the reader.

We now prove that each G_n has the stated values of resistance and flow resistance.

Since G_n contains n disjoint instances of Z, Lemma 5 (i) implies that $r(G_n) \geqslant n$. Moreover, since $Z_i - f_i$ is 3-edge-colourable for each i, it is easily seen that $G_n - \{f_1, \ldots, f_n\}$ is 3-edge-colourable, whence $r(G_n) = n$.

To prove that $r_f(G_n) \geqslant 2n$, observe that between any two instances of Z there exists at least one vertex. Therefore $r_f(G_n) \geqslant 2n$, otherwise there would be an instance of Z with fewer than two zero edges, contradicting Lemma 5. Furthermore, since each Z_i admits a 4-flow with the only zero edges being f_i and e_i for each i, it is easy to find a 4-flow on G_n with the only zero edges being from $\{f_1, \dots, f_n\}$ and from the set of n edges which join w_i to Z_i for each i. Therefore, $r_f(G_n) = 2n$. □

4 Remarks

4.1. Flow resistance is an uncolourability measure which certainly merits further study. One possible direction, motivated by an obvious approach to the 5-flow conjecture of Tutte [16], is bounding the number of zeros in a 4-flow. This line of research relates the 5-flow conjecture to other important conjectures in the area, in particular to the celebrated conjecture of Fulkerson [5]. Recall that Fulkerson's conjecture suggests that every bridgeless cubic graph admits a list of six perfect matchings that together cover every edge exactly twice. It is easy to see that once a bridgeless cubic graph fulfils the conjecture, it has a pair of perfect matchings whose intersection covers at most $1/15$ of the number of edges. Now we can use the inequality between r_f and γ_2 proved Proposition 29 of [4] to conclude that

$$r_f(G) \leqslant \gamma_2(G) \leqslant m/15 \tag{1}$$

where m is the number of edges. Moreover, the Petersen graph certifies that this bound is the best that one can hope for general bridgeless cubic graphs.

On the other hand, Kaiser et al. in [9] employ Edmonds' perfect matching polytope theorem [12, Theorem 25.1] to prove that every bridgeless cubic graph contains two perfect matchings M_1 and M_2 that together cover at least $3m/5$ edges. It follows that

$$r_f(G) \leqslant \gamma_2(G) \leqslant |M_1 \cap M_2| \leqslant 2m/3 - 3m/5 = m/15$$

without assuming Fulkerson's conjecture to be true. Thus the bound (1) holds for every bridgeless cubic graph, a fact in support of the validity of Fulkerson's conjecture.

4.2. Let G be a cubic graph. A 1-*reduction* of G is a graph obtained by the removal of adjacent vertices u and v from G, and the subsequent addition of edges to restore 3-regularity. In [1, Conjecture 1], it was conjectured that there exists a 1-reduction G' of every bridgeless cubic graph G with $r(G) > 2$, such that $r(G) > r(G')$. The graphs G_n for $n \geqslant 3$ as defined in this paper are counterexamples to this conjecture. Consider G_3, for example. Each of the three instances of Z contribute to the resistance by 1. Then, any 1-reduction of G_3 which could potentially reduce resistance must be contained in an instance of Z. Furthermore, the edge in the instance of Z which is being 1-reduced must be contained in the central instance of X, otherwise there remains an instance of Y, and resistance would not be affected. However, assuming that resistance is reduced after a 1-reduction in the central instance of X in an instance of Z, a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow, denoted by ϕ, would violate Kirchhoff's Law. Indeed, from Lemma 3 we would have

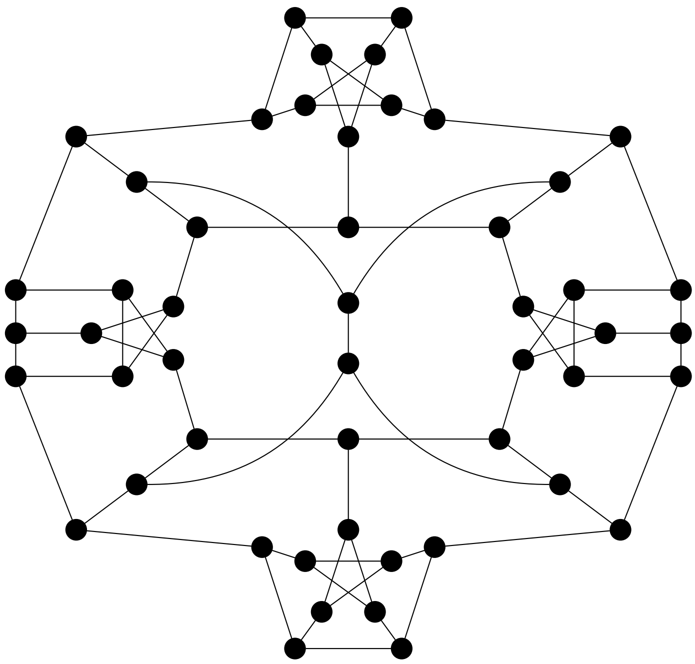

Figure 5: A counterexample to Conjecture 1 on 50 vertices, with cyclic connectivity 3, resistance 2, and flow resistance 3.

$\phi(a) = \phi(b)$ and $\phi(c) = \phi(d)$, implying that $\phi(e) = 0$, which contradicts the assumption that resistance has been reduced in that instance of Z. Therefore, G_n is a counterexample to the conjecture for $n \geqslant 3$.

4.3. The smallest nontrivial snark that provides a counterexample to Conjecture 1 and arises from the construction preceding Theorem 6 is the graph G_5, which has 170 vertices. A significantly smaller counterexample can be constructed in a manner similar to G_n for $n = 2$, with the only difference that the semi-edges e_1 of Z_1 and e_2 of Z_2 are not joined to the vertices w_1 and w_2, respectively, but instead are joined directly to each other. The resulting graph has 66 vertices, resistance 2, flow resistance 3, and is clearly a nontrivial snark. The number of vertices can be further decreased to 50 if we do not insist that the counterexample be a nontrivial snark, see Figure 5. Arguments are similar to those in the proof of Theorem 6.

Problem 7. Determine the smallest order of a snark G (trivial or nontrivial) with $r_f(G) > r(G)$.

Another related question to ask is the following.

Problem 8. Can the ratio of $r_f(G)$ to $r(G)$ be arbitrarily large?

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