Modular and fractional \( L \)-intersecting families of vector spaces

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Abstract

This paper is divided into two logical parts. In the first part of this paper, we prove the following theorem which is the \( q \)-analogue of a generalized modular Ray-Chaudhuri-Wilson Theorem shown in [Alon, Babai, Suzuki, J. Combin. Theory Series A, 1991]. It is also a generalization of the main theorem in [Frankl and Graham, European J. Combin. 1985] under certain circumstances.

- Let \( V \) be a vector space of dimension \( n \) over a finite field of size \( q \). Let \( K = \{ k_1, \ldots, k_r \}, L = \{ \mu_1, \ldots, \mu_s \} \) be two disjoint subsets of \( \{0, 1, \ldots, b - 1 \} \) with \( k_1 < \cdots < k_r \). Let \( \mathcal{F} = \{ V_1, V_2, \ldots, V_m \} \) be a family of subspaces of \( V \) such that (a)
for every $i \in [m]$, $\dim(V_i) \mod b = k_i$, for some $k_i \in K$, and (b) for every distinct $i, j \in [m]$, $\dim(V_i \cap V_j) \mod b = \mu_j$, for some $\mu_j \in L$. Moreover, it is given that neither of the following two conditions hold:

(i) $q + 1$ is a power of 2, and $b = 2$

(ii) $q = 2$, $b = 6$.

Then,

$$|\mathcal{F}| \leq \begin{cases} N(n, s, r, q), & \text{if } (s + k_r \leq n \text{ and } r(s - r + 1) \leq b - 1) \text{ or } (s < k_1 + r) \\ N(n, s, q) + \sum_{k \in [r]} \binom{n}{k}, & \text{otherwise}, \end{cases}$$

where $N(n, s, r, q) := \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

In the second part of this paper, we prove $q$-analogues of results on a recent notion called fractional $L$-intersecting family of sets for families of subspaces of a given vector space over a finite field of size $q$. We use the above theorem to obtain a general upper bound to the cardinality of such families. We give an improvement to this general upper bound in certain special cases.

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1 Introduction

Let $[n]$ be the set of all natural numbers from 1 to $n$. A family $\mathcal{F}$ of subsets of $[n]$ is called intersecting if every set in $\mathcal{F}$ has a non-empty intersection with every other set in $\mathcal{F}$. One of the earliest studies on intersecting families dates back to the famous Erdős-Ko-Rado Theorem [Erdős et al., 1961] about maximal uniform intersecting families. Ray-Chaudhuri and Wilson [Ray-Chaudhuri and Wilson, 1975] introduced the notion of $L$-intersecting families. Let $L = \{l_1, \ldots, l_s\}$ be a set of non-negative integers. A family $\mathcal{F}$ of subsets of $[n]$ is said to be $L$-intersecting if for every distinct $F_i, F_j$ in $\mathcal{F}$, $|F_i \cap F_j| \in L$. The Ray-Chaudhuri-Wilson Theorem states that if $\mathcal{F}$ is $t$-uniform (that is, every set in $\mathcal{F}$ is $t$-sized), then $|\mathcal{F}| \leq \binom{n}{t}$. This bound is tight as shown by the set of all $s$-sized subsets of $[n]$ with $L = \{0, \ldots, s - 1\}$. Frankl-Wilson Theorem [Frankl and Wilson, 1981a] extends this to non-uniform families by showing that $|\mathcal{F}| \leq \sum_{i=0}^{s} \binom{n}{i}$, where $\mathcal{F}$ is any family of subsets of $[n]$ that is $L$-intersecting. The collection of all the subsets of $[n]$ of size at most $s$ with $L = \{0, \ldots, s - 1\}$ is a tight example to this bound. The first proofs of these theorems were based on the technique of higher incidence matrices. Alon, Babai, and Suzuki in [Alon et al., 1991] generalized the Frankl-Wilson Theorem using a proof that operated on spaces of multilinear polynomials. They showed that if the sizes of the sets in $\mathcal{F}$ belong to $K = \{k_1, \ldots, k_r\}$ with each $k_i > s - r$, then $|\mathcal{F}| \leq \binom{n}{s} + \cdots + \binom{n}{s-r+1}$. A modular version of the Ray-Chaudhuri-Wilson Theorem was shown in [Frankl and Wilson, 1981b]. This result was generalized in [Alon et al., 1991]. See [Liu and Yang, 2014] for a survey on $L$-intersecting families.
Researchers have also been working on similar intersection theorems for subspaces of a given vector space over a finite field. Hsieh [Hsieh, 1975], and Deza and Frankl [Deza and Frankl, 1983] showed Erdős-Ko-Rado type theorems for subspaces. Let $V$ be a vector space of dimension $n$ over a finite field of size $q$. The number of $d$-dimensional subspaces of $V$ is given by the $q$-binomial coefficient (also known as Gaussian binomial coefficient) $\left\lceil \binom{n}{d}_q \right\rceil = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-d+1}-1)}{(q^d-1)(q^{d-1}-1)\cdots(q-1)}$. The following theorem which is a $q$-analog of the Ray-Chaudhuri-Wilson Theorem by considering families of subspaces instead of subsets is due to Frankl and Graham, 1985.

**Theorem 1.** [Theorem 1.1 in [Frankl and Graham, 1985]] Let $V$ be a vector space over of dimension $n$ over a finite field of size $q$. Let $\mathcal{F} = \{V_1, V_2, \ldots, V_m\}$ be a family of subspaces of $V$ such that $\dim(V_i) = k$, for every $i \in [m]$. Let $0 \leq \mu_1 < \mu_2 < \cdots < \mu_s < b$ be integers such that $k \equiv \mu_t \pmod{b}$, for any $t$. For every $1 \leq i < j \leq m$, $\dim(V_i \cap V_j) \equiv \mu_t \pmod{b}$, for some $t$. Then,

$$|\mathcal{F}| \leq \left\lceil \frac{n}{s} \right\rceil_q$$

except possibly for $q = 2, b = 6, s \in \{3, 4\}$.

**Example 2** (Remark 3.2 in [Frankl and Graham, 1985]). Let $n = k + s$. Let $\mathcal{F}$ be the family of all the $k$-dimensional subspaces of $V$, where $V$ is an $n$-dimensional vector space over a finite field of size $q$. Observe that, for any two distinct $V_i, V_j \in \mathcal{F}$, $k - s \leq \dim(V_i \cap V_j) \leq k - 1$. This is a tight example for Theorem 1.

Alon et al. in [Alon et al., 1991] proved a generalization of the non-modular version of the above theorem. This result was subsequently strengthened in [Liu et al., 2018].

Our paper is divided into two logical parts. In the first part (i.e., Section 2), we prove the following theorem which is a generalization of Theorem 1 due to Frankl and Graham under certain circumstances. It is also the $q$-analog of a generalized modular Ray-Chaudhuri-Wilson Theorem shown in [Alon et al., 1991]. We assume that $\left\lceil \frac{a}{b} \right\rceil_q = 0$, when $b < 0$ or $b > a$. Let

$$N(n, s, r, q) := \left\lceil \frac{n}{s} \right\rceil_q + \left\lceil \frac{n}{s-1} \right\rceil_q + \cdots + \left\lceil \frac{n}{s-r+1} \right\rceil_q.$$  

**Theorem 3.** Let $V$ be a vector space of dimension $n$ over a finite field of size $q$. Let $K = \{k_1, \ldots, k_r\}, L = \{\mu_1, \ldots, \mu_s\}$ be two disjoint subsets of $\{0, 1, \ldots, b - 1\}$ with $k_1 < \cdots < k_r$. Let $\mathcal{F} = \{V_1, V_2, \ldots, V_m\}$ be a family of subspaces of $V$ such that (a) for every $i \in [m]$, $\dim(V_i) \pmod{b} = k_i$, for some $k_i \in K$, and (b) for every distinct $i, j \in [m]$, $\dim(V_i \cap V_j) \pmod{b} = \mu_t$, for some $\mu_t \in L$. Moreover, it is given that neither of the following two conditions hold:

(i) $q + 1$ is a power of $2$, and $b = 2$
Example 6. Let $q = 2, b = 6$

Then,

$$|\mathcal{F}| \leq \begin{cases} N(n, s, r, q), & \text{if } (s + k_r \leq n\text{and}(s - r + 1) \leq b - 1) \text{or } (s < k_1 + r) \\ N(n, s, r, q) + \sum_{i \in [r]} \left[ \frac{n}{h_i q} \right], & \text{otherwise.} \end{cases}$$

In the second part (i.e., Section 3), we study a notion of fractional $L$-intersecting families which was introduced in [Balachandran et al., 2019]. We say a family $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ of subsets of $[n]$ is a fractional $L$-intersecting family, where $L$ is a set of irreducible fractions between 0 and 1, if for every distinct $i, j \in [m], \frac{|F_i \cap F_j|}{|F_i|} \in L$ or $\frac{|F_i \cap F_j|}{|F_j|} \in L$. In this paper, we extend this notion from subsets to subspaces of a vector space over a finite field.

**Definition 4.** Let $L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\}$ be a set of positive irreducible fractions, where every $\frac{a_i}{b_i} < 1$. Let $\mathcal{F} = \{V_1, \ldots, V_m\}$ be a family of subspaces of a vector space $V$ over a finite field. We say $\mathcal{F}$ is a fractional $L$-intersecting family of subspaces if for every two distinct $i, j \in [m], \frac{\dim(V_i \cap V_j)}{\dim(V_i)} \in L$ or $\frac{\dim(V_i \cap V_j)}{\dim(V_j)} \in L$.

When every subspace in $\mathcal{F}$ is of dimension exactly $k$, it is an $L'$-intersecting family where $L' = \{\frac{a_1 k}{b_1}, \ldots, \frac{a_s k}{b_s}\}$. Applying Theorem 1, we get $|\mathcal{F}| \leq \left[ \frac{n}{s} \right]$. A tight example to this is the collection of all $k$-dimensional subspaces of $V$ with $L = \{0, \ldots, \frac{k-1}{k}\}$. However, the problem of bounding the cardinality of a fractional $L$-intersecting family of subspaces becomes more interesting when $\mathcal{F}$ contains subspaces of various dimensions. In Section 3, we obtain upper bounds for the cardinality of a fractional $L$-intersecting family of subspaces that are $q$-analogs of the results in [Balachandran et al., 2019]. With the help of Theorem 3 that we prove in Section 2, we obtain the following result in Section 3.

**Theorem 5.** Let $L = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_s}{b_s}\}$ be a collection of positive irreducible fractions, where every $\frac{a_i}{b_i} < 1$. Let $\mathcal{F}$ be a fractional $L$-intersecting family of subspaces of a vector space $V$ of dimension $n$ over a finite field of size $q$. Let $t = \max_{i \in [s]} b_i, g(t, n) = \frac{2(2t+\ln n)}{\ln(2t+\ln n)},$ and $h(t, n) = \min(g(t, n), \frac{\ln n}{mt})$. Then,

$$|\mathcal{F}| \leq 2g(t, n)h(t, n) \ln(g(t, n)) \left[ \frac{n}{s} \right] + h(t, n) \sum_{i=1}^{s-1} \left[ \frac{n}{i} \right].$$

Further, if $2g(t, n)\ln(g(t, n)) \leq n + 2$, then

$$|\mathcal{F}| \leq 2g(t, n)h(t, n) \ln(g(t, n)) \left[ \frac{n}{s} \right].$$

**Example 6.** Let $s$ be a constant, $L = \{\frac{0}{s}, \frac{1}{s}, \ldots, \frac{s-1}{s}\}$, and $\mathcal{F}$ be the family of all the $s$-sized subspaces of $V$. Clearly, $\mathcal{F}$ is a fractional $L$-intersecting family showing that the bound in Theorem 5 is asymptotically tight up to a multiplicative factor of $\frac{\ln^2 n}{\ln \ln n}$. 

THE ELECTRONIC JOURNAL OF COMBINATORICS 29(1) (2022), #P1.45
We improve the bound obtained in Theorem 5 for the special case when $L = \{ \frac{a}{b} \}$, where $b$ is a prime.

**Theorem 7.** Let $L = \{ \frac{a}{b} \}$, where $\frac{a}{b}$ is a positive irreducible fraction less than 1 and $b$ is a prime. Let $\mathcal{F}$ be a fractional $L$-intersecting family of subspaces of a vector space $V$ of dimension $n$ over a finite field of size $q$. Then, we have $|\mathcal{F}| \leq (b - 1)(\binom{n}{1}_q + 1)[\frac{\ln n}{\ln b}] + 2$.

**Example 8.** Let $L = \{ \frac{1}{2} \}$. Let $V$ be a vector space of dimension $n$ over a finite field of size $q$. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of $V$. Let $V' := \text{span}(\{v_2, \ldots, v_n\})$ be an $(n - 1)$-dimensional subspace of $V$. Let $\mathcal{F}$ be the set of all $\binom{n-1}{1}_q$ 2-dimensional subspaces of $V$ each of which is obtained by a span of $v_1$ and each of the $\binom{n-1}{1}_q$ 1-dimensional subspaces of $V'$. This example shows that when $b$ and $q$ are constants, the bound in Theorem 7 is asymptotically tight up to a multiplicative factor of $\ln n$.

## 2 Generalized modular RW Theorem for subspaces

As mentioned before, in this part we prove Theorem 3. The approach followed here is similar to the approach used in proving Theorem 1.5, a generalized modular Ray-Chaudhuri-Wilson Theorem for subsets, in [Alon et al., 1991]. We start by stating the Zsigmondy’s Theorem which will be used in the proof of Theorem 3.

**Theorem 9** ([Zsigmondy, 1892]). For any $q, b \in \mathbb{N}$, there exists a prime $p$ such that $q^b \equiv 1 \pmod{p}$, $q^i \not\equiv 1 \pmod{p}$ for all $0 < i < b$, except when (i) $q + 1$ is a power of 2, $b = 2$, or (ii) $q = 2, b = 6$.

### 2.1 Notations used in Section 2

Unless defined explicitly, in the rest of this section, the symbols $K = \{k_1, \ldots, k_r\}$, $r$, $L = \{\mu_1, \ldots, \mu_s\}$, $s$, $q$, $V$, $\mathcal{F}$, $n$, $b$, $m$, and $V_1, \ldots, V_m$ are defined as they are defined in Theorem 3. We shall use $U \subseteq V$ to denote that $U$ is a subspace of $V$. Using Zsigmondy’s Theorem, we find a prime $p$ so that $q^i \not\equiv 1 \pmod{p}$ for $0 < i < b$ and $q^b \equiv 1 \pmod{p}$. This is possible except in the two cases specified in Theorem 9. We ignore these two cases from now on in the rest of Section 2.

### 2.2 Möbius inversion over the subspace poset

Consider the partial order defined on the set of subspaces of the vector space $V$ over a finite field of size $q$ under the ‘containment’ relation. Let $\alpha$ be a function from the set of subspaces of $V$ to $\mathbb{F}_p$. A function $\beta$ from the set of subspaces of $V$ to $\mathbb{F}_p$ is the zeta transform of $\alpha$ if for every $W \subseteq V$, $\beta(W) = \sum_{U \subseteq W} \alpha(U)$. Then, applying the Möbius inversion formula we get for all $W \subseteq V$, $\alpha(W) = \sum_{U \subseteq W} \mu(U, W) \beta(U)$, where $\alpha$ is called...
the Möbius transform of $\beta$ and $\mu(U,W)$ is the Möbius function for the subspace poset. In the proposition below, we show that the Möbius function for the subspace poset is defined as

$$\mu(X,Y) = \begin{cases} (-1)^{q(\ell)}, & \text{if } X \subseteq Y \\ 0, & \text{otherwise}, \end{cases}$$

$\forall X, Y \subseteq V$ with $d = \dim(Y) - \dim(X)$. The following proposition gives the Möbius inversion formula for the subspace lattice. See [Mathew et al., 2020] for a proof.

**Proposition 10.** Let $\alpha$ and $\beta$ be functions from the set of subspaces of $V$ to $\mathbb{F}_p$. Then, $\forall W \subseteq V$,

$$\beta(W) = \sum_{U \subseteq W} \alpha(U) \iff \alpha(W) = \sum_{U \subseteq W} (-1)^d q^{\frac{d(d-1)}{2}} \beta(U).$$

**Definition 11.** Given two subspaces $U$ and $W$ of the vector space $V$, we define their union space $U \cup W$ as the span of union of sets of vectors in $U$ and $W$.

The proposition below follows from the definitions of $\alpha$ and $\beta$. See [Mathew et al., 2020] for a proof.

**Proposition 12.** Let $\alpha$ and $\beta$ be functions as defined in Proposition 10. Then, $\forall W, Y$ such that $W \subseteq Y \subseteq V$,

$$\sum_{T: W \subseteq T \subseteq Y} (-1)^d q^{\frac{d(d-1)}{2}} \beta(T) = \sum_{U: U \cup W = Y} \alpha(U).$$

**Corollary 13.** For any non-negative integer $g$, the following are equivalent for functions $\alpha$ and $\beta$ defined in Proposition 10:

(i) $\alpha(U) = 0$, $\forall U \subseteq V$ with $\dim(U) \geq g$.

(ii) $\sum_{W \subseteq T \subseteq Y} (-1)^d q^{\frac{d(d-1)}{2}} \beta(T) = 0$, $\forall W, Y \subseteq V$ with $\dim(Y) - \dim(W) \geq g$.

**Definition 14.** Let $H = \{h_1, h_2, \ldots, h_t\}$ be a subset of $\{0, 1, \ldots, n\}$ where $h_1 < h_2 < \cdots < h_t$. We say $H$ has a gap of size $\geq g$ if either $h_1 \geq g - 1$, $n - h_t \geq g - 1$, or $h_{i+1} - h_i \geq g$ for some $i \in [t - 1]$.

**Lemma 15.** Let $\alpha$ and $\beta$ be functions as in Proposition 10. Let $H \subseteq \{0, 1, \ldots, n\}$ be a set of integers and $g$ an integer, $0 \leq g \leq n$. Suppose we have the following conditions:

(i) $\forall U \subseteq V$, we have $\alpha(U) = 0$ whenever $\dim(U) \geq g$.

(ii) $\forall T \subseteq V$, we have $\beta(T) = 0$ whenever $\dim(T) \not\in H$. 
(iii) $H$ has a gap $\geq g + 1$.

Then, $\alpha = \beta = 0$.

**Proof.** Let $H = \{h_1, h_2, \ldots, h_{|H|}\}$. Suppose, for some $i \in [H]$, $h_i - h_{i-1} \geq g$ or $h_1 \geq g$, then we have $h_i \in H$ and $h_i - j \notin H$ for $1 \leq j \leq g$ and $h_i - g \geq 0$. Choose any two subspaces, say $U$ and $W$, of $V$ of dimensions $h_i$ and $h_i - g$, respectively. Since $\dim(U) \geq g$, $\alpha(U) = 0$. We know from Corollary 13 that

$$\sum_{\substack{W \subseteq T \subseteq U \\
d = \dim(U) - \dim(T)}} (-1)^d q^{\frac{d(d-1)}{2}} \beta(T) = 0$$

But whenever $\dim(T) < h_i$, it lies between $h_i - g$ and $h_i - 1$, and hence $\beta(T) = 0$. Then,

$$\sum_{\substack{W \subseteq T \subseteq U \\
d = \dim(U) - \dim(T)}} (-1)^d q^{\frac{d(d-1)}{2}} \beta(T) = \beta(U) = 0$$

Since our choice of $U$ was arbitrary, we may conclude that $\beta(U) = 0$, for all $U \subseteq V$ with $\dim(U) = h_i$. Thus, we can remove $h_i$ from the set $H$, and then use the same procedure to further reduce the size of $H$ till it is an empty set. If $H$ is empty, $\beta(U) = 0$, for all $U \subseteq V$, giving $\alpha(U) = \beta(U) = 0$ as required.

Now suppose $n - h_{|H|} \geq g$. In this case, we take $U$ of dimension $h_{|H|}$ and $W$ of dimension $h_{|H|} + g$ to show that $\beta(U) = 0$, and remove $h_{|H|}$ from $H$. Note that removing a number from the set $H$ can never reduce the gap. \hfill \square

### 2.3 Defining functions $f^{x,y}$ and $g^{x,y}$

Consider all the subspaces of the vector space $V$. We can impose an ordering on the subspaces of the same dimension, and use the natural ordering across dimensions, so that every subspace can be uniquely represented by a pair of integers $(d, e)$, indicating that it is the $e$th subspace of dimension $d$, $0 \leq d \leq n$, $1 \leq e \leq \left[\frac{n}{d}\right]_q$. Let us call that subspace $V_{d,e}$. Let $S$ be the number of subspaces of $V$ of dimension at most $s$, that is, $S = \sum_{t=0}^{s} \left[\frac{n}{t}\right]_q$. Let each subspace $V_{d,e}$ of dimension at most $s$ be represented as a 0-1 containment vector $v_{d,e}$ of $S$ entries, each entry of the vector denoting whether a particular subspace of dimension $\leq s$ is contained in $V_{d,e}$ or not.

$$v_{d,e}^{x,y} = \begin{cases} 
1, & \text{if } V_{x,y} \text{ is a subspace of } V_{d,e} \\
0, & \text{otherwise}
\end{cases}$$

The vector $v_{d,e}$ consists of $v_{d,e}^{x,y}$ values for $0 \leq x \leq s$, $1 \leq y \leq \left[\frac{n}{s}\right]_q$, making it a vector of size $S$. Thus, $v_{d,e}^{x,y}$ is simply the indicator function of whether $V_{x,y}$ is a subspace of $V_{d,e}$.
For $0 \leq x \leq s, 1 \leq y \leq \left[ \frac{n}{x} \right]_q$ we define functions $f^{x,y} : \mathbb{F}_2^s \to \mathbb{F}_p$ as

$$f^{x,y}(v) = f^{x,y}(v_1^0, v_1^1, \ldots, v_x^0, \ldots, v_x^1, \ldots, v_s^1, \ldots, v_s^0) := v^{x,y}.$$  

For $0 \leq x \leq s-r, 1 \leq y \leq \left[ \frac{n}{x} \right]_q$, we define functions $g^{x,y} : \mathbb{F}_2^s \to \mathbb{F}_p$ as

$$g^{x,y}(v) = f^{x,y}(v) \prod_{t \in [r]} \left( \sum_{j=1}^{[n/1]} v^{1,j} - \left[ k_t \right]_q \right).$$

Let $\Omega$ denote $\mathbb{F}_2^s$. The functions $f^{x,y}$ and $g^{x,y}$ reside in the space $\mathbb{F}_p^\Omega$. Note that the functions $g^{x,y}$ do not exist if $s < r$.

### 2.4 Swallowing trick: linear independence of functions $f^{x,y}$ and $g^{x,y}$

**Lemma 16.** Let $s + k_r \leq n$ and $r(s - r + 1) \leq b - 1$. The functions $g^{x,y}$, $0 \leq x \leq s - r, 1 \leq y \leq \left[ \frac{n}{x} \right]_q$, are linearly independent in the function space $\mathbb{F}_p^\Omega$ over $\mathbb{F}_p$.

**Proof.** If $s < r$, then the statement of the lemma is vacuously true. Assume $s \geq r$. We wish to show that the only solution to $\sum_{0 \leq x \leq s-r} \sum_{1 \leq y \leq \left[ \frac{n}{x} \right]_q} \alpha^{x,y} g^{x,y} = 0$ is the trivial solution $\alpha^{x,y} = 0, \forall x,y$. We define function $\alpha$ from the set of all subspaces of $V$ to $\mathbb{F}_p$ as:

$$\alpha(V_{d,e}) = \begin{cases} 
\alpha^{d,e}, & \text{if } 0 \leq d \leq s - r \\
0, & \text{if } d > s - r 
\end{cases}$$

We show that functions $\alpha$ and $\beta(U) := \sum_{T \subseteq U} \alpha(T)$ satisfy the conditions of Lemma 15, thereby implying $\alpha(U) = 0$, for all $U \subseteq V$, including $\alpha(V_{d,e}) = \alpha^{d,e} = 0$ for $0 \leq d \leq s - r$, which will in turn imply that the functions $g^{x,y}$ above are linearly independent.

Let $H = \{ x : 0 \leq x \leq n, x \equiv k_t \pmod{b}, t \in [r] \}$. We claim that $H$ has a gap of size at least $s - r + 2$. Suppose $n \geq b + k_1$. Then, $k_1 < k_2 < \cdots < k_r < b + k_1 \leq n$. Since it is given that $r(s - r + 1) \leq b - 1$, by pigeonhole principle, there is a gap of at least $s - r + 2$ between some $k_i$ and $k_{i+1}$, $i \in [r-1]$, or between $k_r$ and $b + k_1$. Suppose $s + k_r \leq n < b + k_1$. Then, there is a gap of at least $s + 1$ right above $k_r$. This proves the claim. We now need to show that for $T \subseteq V$, $\beta(T) = 0$ whenever $\dim(T) \notin H$, or whenever $\dim(T) \equiv k_t \pmod{b}$, for any $t \in [r]$. Suppose $v_T$ is the $S$-sized containment vector for $T$. When $\dim(T) \equiv k_t \pmod{b}$ for any $t \in [r]$, it follows from the property of
the prime $p$ given by Theorem 9 that $\sum_{1 \leq j \leq [q]} v_{i}^{1,j} - \left[ \frac{k_i}{1} \right] \neq 0$ in $\mathbb{F}_p$, for every $t \in [r]$.

\[
\beta(T) = \sum_{U \subseteq T} \alpha(U) = \sum_{\dim(U) \leq \dim(s-r)} \alpha(U) = \sum_{0 \leq d \leq s-r} \sum_{1 \leq e \leq [q]} \alpha(V^{d,e}) d_{v}(v_{i})
\]

Since $\sum_{1 \leq j \leq [q]} v_{i}^{1,j} - \left[ \frac{k_i}{1} \right] \neq 0$ in $\mathbb{F}_p$, for every $t \in [r]$, $d_{v}(v_{i}) = c(T) g^{d,e}(v_{i})$ where $c(T) \neq 0$.

Then,

\[
\beta(T) = c(T) \sum_{0 \leq d \leq s-r} \alpha(V^{d,e}) g^{d,e}(v_{i}) = c(T) \sum_{0 \leq d \leq s-r} \sum_{1 \leq e \leq [q]} \alpha(V^{d,e}) g^{d,e}(v_{i}) = c(T) \cdot 0 = 0.
\]

Since the set $H$ and the functions $\alpha$ and $\beta$ satisfy the conditions of Lemma 15, we have $\alpha = 0$. This proves the lemma.

Recall that we are given a family $\mathcal{F} = \{V_{1}, V_{2}, \ldots, V_{m}\}$ of subspaces of $V$ such that for every $i \in [m]$, $\dim(V_{i}) \mod b = k_{i}$, for some $k_{i} \in K$. Further, $\dim(V_{i} \cap V_{j}) \mod b = \mu_{t}$, for some $\mu_{t} \in L$ and $K$ and $L$ are disjoint subsets of $\{0, 1, \ldots, b-1\}$. Let $v_{i}$ be the containment vector of size $S$ corresponding to subspace $V_{i} \in \mathcal{F}$. We define the following functions from $\mathbb{F}_{q}^{m} \rightarrow \mathbb{F}_p$.

\[
g^{i}(v) = g^{i}(v^{0,1}, v^{1,1}, \ldots, v^{1,[q]}, \ldots, v^{s,1}, \ldots, v^{s,[q]})
\]

\[
:= \prod_{j=1}^{s} \left( \sum_{1 \leq y \leq [q]} \left( v^{1,j}_{i} v^{1,j}_{y} - \left[ \frac{\mu_{t}}{1} \right] \right) \right)
\]

Let $v = v_{j}$. Then, $\sum_{1 \leq y \leq [q]} (v^{1,j}_{i} v^{1,j}_{y})$ counts the number of 1-dimensional subspaces common to $V_{i}$ and $V_{j}$. That is, $\sum_{1 \leq y \leq [q]} v^{1,j}_{i} v^{1,j}_{y} = \left[ \dim(V_{i} \cap V_{j}) \right]_{q}$. In $\mathbb{F}_p$, $\left[ \dim(V_{i} \cap V_{j}) \right]_{q} \neq \left[ \frac{\mu_{t}}{1} \right]_{q}$ for any $1 \leq t \leq s$, if $i = j$, and $\left[ \dim(V_{i} \cap V_{j}) \right]_{q} = \left[ \frac{\mu_{s}}{1} \right]_{q}$ for some $1 \leq t \leq s$ if $i \neq j$.

Accordingly, $g^{i}(v_{j}) = \begin{cases} 0, & i \neq j \\ \neq 0, & i = j. \end{cases}$

**Lemma 17** (Swallowing trick 1). Let $s + k_{r} \leq n$ and $r(s-r+1) \leq b-1$. The collection of functions $g^{i}$, $1 \leq i \leq m$ together with the functions $g^{x,y}$, $0 \leq x \leq s-r$, $1 \leq y \leq \left[ \frac{n}{x} \right]_{q}$ are linearly independent in $\mathbb{F}_{q}^{m}$ over $\mathbb{F}_p$. 

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\[\text{THE ELECTRONIC JOURNAL OF COMBINATORICS 29(1) (2022), #P1.45} \]
Lemma 18. The collection of functions $g$, $0 \leq i \leq m$, $0 \leq y \leq \left[ \frac{n}{q} \right]_q$, also spans all the functions $g^x$, $0 \leq x \leq s - r$, $1 \leq y \leq \left[ \frac{n}{q} \right]_q$, as well as the functions $f^i$, $1 \leq i \leq m$.

Proof. Let $v \in \mathbb{F}_q^S$. The key observation here is that the product $f^x(v)f^{1,z}(v), 0 \leq x \leq s - 1, 1 \leq y \leq \left[ \frac{n}{x} \right]_q, 1 \leq z \leq \left[ \frac{n}{1} \right]_q$, may be replaced by the function $f^{x',w}(v)$, where $x \leq x' \leq x + 1, 1 \leq w \leq \left[ \frac{n}{x'} \right]_q$. If $V_{1,z} \subseteq V_{x,y}$, it is trivial that $f^x(v)f^{1,z}(v) = f^{x',w}(v)$, since $f^x(v) = 1$ only if $f^{1,z}(v) = 1$. If $V_{1,z} \not\subseteq V_{x,y}$, we let $V_{x',w}$ be the span of union of vectors of $V_{1,z}$ and $V_{x,y}$. Suppose, a vector space $U$ contains both $V_{1,z}$ and $V_{x,y}$. Then, it is clear that it must contain the span of their union as well. Similarly, a vector space $U$ that does not contain either $V_{1,z}$ or $V_{x,y}$ cannot contain $V_{x',w}$. Thus, $f^x(v)f^{1,z}(v) = f^{x',w}(v)$. To see why $x' = x + 1$ (in case $V_{1,z} \not\subseteq V_{x,y}$), the space $V_{x',w}$ may be obtained by taking any (non-zero) vector of $V_{1,z}$ and introducing it into the basis of $V_{x,y}$. The space spanned by this extended basis is exactly $V_{x',w}$ by definition, and the size of basis has increased by exactly 1.

By induction, it follows that,

$$f^{1,y_1}(v)f^{1,y_2}(v)\cdots f^{1,y_l}(v) = f^{x,y}(v)$$

for some $x, y$ where, $1 \leq x \leq l, 1 \leq y \leq \left[ \frac{n}{x} \right]_q$. That is, a product of $l$ functions of the form $f^{1,y}$ may be replaced by a single function $f^{x,y}$ where $x$ is at most $l$. 

2.5 Proof of Theorem 3: in the case when $s + k_r \leq \text{nand}(s - r + 1) \leq b - 1$

Lemma 18. The collection of functions $f^{x,y}$, $0 \leq x \leq s$, $1 \leq y \leq \left[ \frac{n}{q} \right]_q$, spans all the functions $g^x$, $0 \leq x \leq s - r$, $1 \leq y \leq \left[ \frac{n}{q} \right]_q$, as well as the functions $f^i$, $1 \leq i \leq m$.

Proof. Let $v \in \mathbb{F}_q^S$. The key observation here is that the product $f^x(v)f^{1,z}(v), 0 \leq x \leq s - 1, 1 \leq y \leq \left[ \frac{n}{x} \right]_q, 1 \leq z \leq \left[ \frac{n}{1} \right]_q$, may be replaced by the function $f^{x',w}(v)$, where $x \leq x' \leq x + 1, 1 \leq w \leq \left[ \frac{n}{x'} \right]_q$. If $V_{1,z} \subseteq V_{x,y}$, it is trivial that $f^x(v)f^{1,z}(v) = f^{x',w}(v)$, since $f^x(v) = 1$ only if $f^{1,z}(v) = 1$. If $V_{1,z} \not\subseteq V_{x,y}$, we let $V_{x',w}$ be the span of union of vectors of $V_{1,z}$ and $V_{x,y}$. Suppose, a vector space $U$ contains both $V_{1,z}$ and $V_{x,y}$. Then, it is clear that it must contain the span of their union as well. Similarly, a vector space $U$ that does not contain either $V_{1,z}$ or $V_{x,y}$ cannot contain $V_{x',w}$. Thus, $f^x(v)f^{1,z}(v) = f^{x',w}(v)$. To see why $x' = x + 1$ (in case $V_{1,z} \not\subseteq V_{x,y}$), the space $V_{x',w}$ may be obtained by taking any (non-zero) vector of $V_{1,z}$ and introducing it into the basis of $V_{x,y}$. The space spanned by this extended basis is exactly $V_{x',w}$ by definition, and the size of basis has increased by exactly 1.

By induction, it follows that,

$$f^{1,y_1}(v)f^{1,y_2}(v)\cdots f^{1,y_l}(v) = f^{x,y}(v)$$

for some $x, y$ where, $1 \leq x \leq l, 1 \leq y \leq \left[ \frac{n}{x} \right]_q$. That is, a product of $l$ functions of the form $f^{1,y}$ may be replaced by a single function $f^{x,y}$ where $x$ is at most $l$. 

2.5 Proof of Theorem 3: in the case when $s + k_r \leq \text{nand}(s - r + 1) \leq b - 1$
Now consider functions
\[ g^i(v) = g^i(v^{0,1}, v^{1,1}, \ldots, v^{1,[\ell]}, \ldots, v^{s,1}, \ldots, v^{s,[\ell]}) \]
\[ = \prod_{j=1}^{s} \left( \sum_{1 \leq y \leq \left[\frac{n}{j}\right]} (v^{1,y}_j v^{1,y}_j) - \left[\frac{\mu_j}{1}\right]_q \right) \]
\[ = \prod_{j=1}^{s} \left( \sum_{1 \leq y \leq \left[\frac{n}{j}\right]} (v^{1,y}_j f^{1,y}(v)) - \left[\frac{\mu_j}{1}\right]_q \right) \]

Since the functions \( f^{x,y} \) only take 0/1 values, we can reduce any exponent of 2 or more on the function after expanding the product to 1. Moreover, the terms will all be products of the form \( f^{1,y}_j f^{1,y}_2 \ldots f^{1,y}_l(v), 1 \leq l \leq s \). These are replaced according to the observation above by single function of the form \( f^{x,y}(v) \), and thus the set of functions \( f^{x,y}, 0 \leq x \leq s, 1 \leq y \leq \left[\frac{n}{x}\right]_q \) span all functions \( g^i(v) \). Note that \( f^{0,1}(v) \) is the constant function 1.

Similarly, for \( 0 \leq x \leq s - r, 1 \leq y \leq \left[\frac{n}{x}\right]_q \),
\[ g^{x,y}(v) = f^{x,y}(v) \prod_{t \in [r]} \left( \sum_{j=1}^{\left[\frac{n}{j}\right]} v^{1,j}_t - \left[\frac{k_t}{1}\right]_q \right) \]
\[ = f^{x,y}(v) \prod_{t \in [r]} \left( \sum_{j=1}^{\left[\frac{n}{j}\right]} f^{1,j}(v) - \left[\frac{k_t}{1}\right]_q \right) \]
\[ = f^{x,y}(v) \left( \sum_{x'=0}^{r} \sum_{y'=1}^{\left[\frac{n}{x'}\right]} c_{x',y'} f^{x',y'}(v) \right) \quad (c_{x',y'} \text{ are constants}) \]
\[ = \sum_{x'=0}^{s} \sum_{y'=1}^{\left[\frac{n}{x'}\right]} c_{x',y'} f^{x',y'}(v) \quad (c_{x',y'} \text{ are constants}) \]

Thus, the set of function \( f^{x,y}, 0 \leq x \leq s, 1 \leq y \leq \left[\frac{n}{x}\right]_q \) span all functions \( g^{x,y}(v), 0 \leq x \leq s, 1 \leq y \leq \left[\frac{n}{x}\right]_q \).
This means that the above functions $g^{x,y}$ and $g^i$ belong to the span of functions $f^{x,y}$ which is a function space of dimension at most $S$. From Lemma 17, we know that $g^{x,y}$ and $g^i$ are together linearly independent. Thus,

$$\sum_{j=0}^{s-r} \left[ \begin{array}{c} n \\ j \\ q \end{array} \right] q + m \leq S = \sum_{j=0}^{s} \left[ \begin{array}{c} n \\ j \\ q \end{array} \right].$$

$$\Rightarrow |F| = m \leq \left[ \begin{array}{c} n \\ s \\ q \end{array} \right] + \left[ \begin{array}{c} n \\ s-1 \\ q \end{array} \right] + \cdots + \left[ \begin{array}{c} n \\ s-r+1 \\ q \end{array} \right].$$

### 2.6 Proof of Theorem 3

Let $X \subseteq \{0, \ldots, s-r\}$ be the set of those integers that are not congruent to any $k \in K$. The, in the following lemma, we show that the family $g^{x,y}$ with $x \in X$ is linearly independent.

**Lemma 19.** The collection of functions

$$\{g^{x,y} \mid 0 \leq x \leq s-r, 1 \leq y \leq \left[ \begin{array}{c} n \\ x \\ q \end{array} \right], \text{ andforall} \ t \in [r], x \not\equiv k_t \pmod{b}\}$$

are linearly independent in the function space $F^\Omega_p$ over $F_p$.

**Proof.** Recall that

$$g^{x,y}(v) = f^{x,y}(v) \prod_{t \in [r]} \left( \sum_{j=1}^{\left[ \begin{array}{c} n \\ 1 \\ q \end{array} \right]} v^{1,j} - \left[ \begin{array}{c} k_t \\ 1 \\ q \end{array} \right] \right).$$

The statement of the lemma is vacuously true, if $s < r$. Assume $s \geq r$. Assume, for the sake of contradiction, $\sum_{x \not\equiv k_t \pmod{p}, \forall t \in [r]} \alpha^{x,y} g^{x,y} = 0$ with at least one $\alpha^{x,y}$ as non-zero.

Let $\langle x_0, y_0 \rangle$ be the first subspace, based on the ordering of subspaces defined in Section 2.3, such that $\alpha^{x_0,y_0}$ is non-zero. Evaluating both sides on $v_{x_0,y_0}$, we see that all $f^{x,y}$ (and therefore $g^{x,y}$) with $\langle x, y \rangle$ higher in the ordering than $\langle x_0, y_0 \rangle$ will vanish (due to the virtue of our ordering), and so we get $\alpha^{x_0,y_0} = 0$ which is a contradiction. Here we have crucially used the fact that by ignoring $x \equiv k_t \pmod{p}$ cases, for any $t \in [r]$, we make sure that $v_{x_0,y_0}$ used above always has $x_0 \not\equiv k_t \pmod{b}$ and therefore

$$\left( \sum_{j=1}^{\left[ \begin{array}{c} n \\ 1 \\ q \end{array} \right]} v^{1,j}_{x_0,y_0} - \left[ \begin{array}{c} k_t \\ 1 \\ q \end{array} \right] \right) \not\equiv 0 \pmod{p}, \forall t \in [r].$$

THE ELECTRONIC JOURNAL OF COMBINATORICS 29(1) (2022), #P1.45
Lemma 20 (Swallowing trick 2). The collection of functions $g^t$, $1 \leq i \leq m$ together with the functions $g^{x,y}$, $0 \leq x \leq s - r$, $x \not\equiv k_t \pmod{b}$, for all $t \in [r]$, $1 \leq y \leq \left\lfloor \frac{n}{x} \right\rfloor$ are linearly independent in $\mathbb{F}_{p^2}^\Omega$ over $\mathbb{F}_p$.

Proof. Proof is similar to the proof of Lemma 17.

Since $s < b$, for any $0 \leq x \leq s - r$ and for any $t \in [r]$, $x \not\equiv k_t \pmod{b}$ is equivalent to $x \not= k_t$. Combining Lemmas 19, 20 and 18, we have

$$\sum_{0 \leq j \leq s - r, j \not= k_t, t \in [r]} \left\lfloor \frac{n}{j} \right\rfloor + m \leq \sum_{j=0}^{s} \left\lfloor \frac{n}{j} \right\rfloor .$$

This implies,

$$|\mathcal{F}| = m \leq \begin{cases} N(n, s, r, q), & \text{if } s < k_1 + r \\ N(n, s, r, q) + \sum_{t \in [r]} \left\lfloor \frac{n}{k_t} \right\rfloor, & \text{otherwise.} \end{cases}$$

We thus have the following theorem which combined with the result in Section 2.5 yields Theorem 3.

Theorem 21. Let $V$ be a vector space of dimension $n$ over a finite field of size $q$. Let $K = \{k_1, \ldots, k_r\}$, $L = \{\mu_1, \ldots, \mu_s\}$ be two disjoint subsets of $\{0, 1, \ldots, b-1\}$ with $k_1 < \cdots < k_r$. Let $\mathcal{F} = \{V_1, V_2, \ldots, V_m\}$ be a family of subspaces of $V$ such that for all $i \in [m]$, $\dim(V_i) \equiv k_i \pmod{b}$, for some $k_i \in K$; for every distinct $i, j \in [m]$, $\dim(V_i \cap V_j) \equiv \mu_t \pmod{b}$, for some $\mu_t \in L$. Moreover, it is given that neither of the following two conditions hold:

(i) $q + 1$ is a power of 2, and $b = 2$
(ii) $q = 2, b = 6$

Then,

$$|\mathcal{F}| \leq \begin{cases} N(n, s, r, q), & \text{if } s < k_1 + r \\ N(n, s, r, q) + \sum_{t \in [r]} \left\lfloor \frac{n}{k_t} \right\rfloor, & \text{otherwise.} \end{cases}$$

3 Fractional $L$-intersecting families of subspaces

Let $L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\}$ be a collection of positive irreducible fractions, each strictly less than 1. Let $V$ be a vector space of dimension $n$ over a finite field of size $q$. Let $\mathcal{F}$ be a family of subspaces of $V$. Recall that, we call $\mathcal{F}$ a fractional $L$-intersecting family of subspaces if for all distinct $A, B \in \mathcal{F}$, $\dim(A \cap B) \in \{\frac{a_1}{b_1} \dim(A), \frac{a_2}{b_2} \dim(B)\}$, for some $\frac{a_1}{b_1}, \frac{a_2}{b_2} \in L$. In Section 3.1, we prove a general upper bound for the size of a fractional $L$-intersecting family using Theorem 3 proved in Section 2. In Section 3.2, we improve this upper bound for the special case when $L = \{\frac{a}{b}\}$ is a singleton set with $b$ being a prime number.
3.1 A general upper bound

The key idea we use here is to split the fractional $L$ intersecting family $\mathcal{F}$ into subfamilies and then use Theorem 3 to bound each of them.

Lemma 22. Let $L = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_s}{b_s}\}$, where every $\frac{a_i}{b_i}$ is an irreducible fraction in the open interval $(0, 1)$. Let $\mathcal{F} = \{V_1, \ldots, V_m\}$ be a fractional $L$-intersecting family of subspaces of a vector space $V$ of dimension $n$ over a finite field of size $q$. Let $k > 0$ and $p > \max(b_1, b_2, \ldots, b_s)$. Let $\mathcal{F}_k^p$ denote subspaces in $\mathcal{F}$ whose dimensions leave a remainder $k \pmod{p}$, where $p$ is a prime number. That is, $\mathcal{F}_k^p := \{W \in \mathcal{F} \mid \dim(W) \equiv k \pmod{p}\}$.

Then,

$$|\mathcal{F}_k^p| \leq \begin{cases} 
\left\lfloor \frac{n}{s} \right\rfloor, & \text{if } 2p \leq n + 2 \text{ or } (s < k + 1) \\
\left\lfloor \frac{n}{s} \right\rfloor + \left\lfloor \frac{n}{k} \right\rfloor, & \text{otherwise.}
\end{cases}$$

Proof. Apply Theorem 3 with family $\mathcal{F}$ replaced by $\mathcal{F}_k^p$, $K = \{k\}$, $r = 1$, $b$ replaced by $p$, and each $\mu_i$ replaced by $(\frac{a_i}{b_i}) \pmod{p} = (b_i^{-1}a_i,k) \pmod{p}$, where $b_i^{-1}$ is the multiplicative inverse of $b_i$ in $\mathbb{F}_p$. Let $s' \leq s$ be the number of distinct $\mu_i$'s. Notice that $k > 0$, and $p > b_i > a_i$ ensure that $k \not\equiv \frac{a_i}{b_i} \pmod{p}$ or $k \not\equiv \mu_i$. Thus $\mathcal{F}_k^p$ is a family of subspaces of $V$ such that (a) for every $W \in \mathcal{F}_k^p$, $\dim(W) \equiv k \pmod{p}$, and (b) for every distinct $U, W \in \mathcal{F}_k^p$, $\dim(U \cap W) \pmod{p} \in L$, where $L = \{\mu_1, \ldots, \mu_{s'}\}$ and $k \not\in L$. Moreover, since $s' \leq p - 1$ and $k \leq p - 1$, we have $s' + k \leq n$ if $2p \leq n + 2$. Since $p > b_i$ and every $b_i \geq 2$, we have $p > 2$. This avoids bad case (i) of Theorem 3. That $p$ is a prime avoids bad case (ii) of Theorem 3. Thus, we satisfy the premise of Theorem 3 and the conclusion follows.

Suppose $2p \leq n + 2$. The above lemma immediately gives us a bound of $|\mathcal{F}| \leq |\mathcal{F}_0^p| + (p - 1) \left\lfloor \frac{n}{s} \right\rfloor$. But it could be that most subspaces belong to $\mathcal{F}_0^p$. To overcome this problem, we instead choose a set of primes $P$ such that no subspace can belong to $\mathcal{F}_0^p$ for every $p \in P$. A natural choice is to take just enough primes in increasing order so that the product of these primes exceeds $n$, because then any subspace with dimension divisible by all primes in $P$ will have a dimension greater than $n$, which is not possible.

Lemma 23. Let $L = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_s}{b_s}\}$, where every $\frac{a_i}{b_i}$ is an irreducible fraction in the open interval $(0, 1)$. Let $\mathcal{F} = \{V_1, \ldots, V_m\}$ be a fractional $L$-intersecting family of subspaces of a vector space $V$ of dimension $n$ over a finite field of size $q$. Let $t := \max(b_1, b_2, \ldots, b_s)$ and $g(t, n) := \frac{2t}{2^{2t+1}n^2}$. Suppose $2g(t, n) \ln(g(t, n)) \leq n + 2$. Then,

$$|\mathcal{F}| \leq 2g^2(t, n) \ln(g(t, n)) \left\lfloor \frac{n}{s} \right\rfloor.$$
Proof. For some $\beta$ to be chosen later, choose $P$ to be the set $\{p_{\alpha+1}, p_{\alpha+2}, \ldots, p_{\beta}\}$ where $p_l$ denotes the $l^{th}$ prime number and $p_\alpha \leq t < p_{\alpha+1} < p_{\alpha+2} < \cdots < p_{\beta}$. Let $l#$ denote the product of all primes less than or equal to $l$. Thus, $p_l#$ which is known as the primorial function, is the product of the first $l$ primes. It is known that $p_l# = e^{(1+o(1))l\ln l}$ and $l# = e^{(1+o(1))l}$. We require the following condition for the set $P$:

$$\frac{p_\beta#}{l#} > n$$

Using the bounds for $p_l#$ and $l#$ discussed above, we find that it is sufficient to choose $\beta \geq 2(2^t + \ln n)$ where $t := g(t, n)$. From the Prime Number Theorem, it follows that $p_\beta$ (and so $p_{\alpha+1}, p_{\alpha+2}, \ldots, p_{\beta-1}$ as well) is at most $2g(t, n)\ln(g(t, n))$. We are given that $2p \leq 2p_\beta \leq n + 2$, for every $p \in P$. We apply Lemma 22 with $p = p_{\alpha+1}$ to get

$$|F| \leq |F_{p_{\alpha+1}}^0| + (p_{\alpha+1} - 1) \left[\begin{array}{c} n \\ s \end{array}\right]_q$$

Next, apply Lemma 22 on $F_{p_{\alpha+1}}^0$ with $p = p_{\alpha+2}$ and so on. As argued above, no subspace is left uncovered after we reach $p_\beta$. This means,

$$|F| \leq (p_{\alpha+1} + p_{\alpha+2} + \cdots + p_\beta - (\beta - \alpha)) \left[\begin{array}{c} n \\ s \end{array}\right]_q$$

$$< (\beta - \alpha)p_\beta \left[\begin{array}{c} n \\ s \end{array}\right]_q$$

$$< \beta p_\beta \left[\begin{array}{c} n \\ s \end{array}\right]_q$$

$$\leq 2g^2(t, n)\ln(g(t, n)) \left[\begin{array}{c} n \\ s \end{array}\right]_q$$

Lemma 24. Let $L = \left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_s}{b_s}\right\}$, where every $\frac{a_i}{b_i}$ is an irreducible fraction in the open interval $(0, 1)$. Let $F = \{V_1, \ldots, V_m\}$ be a fractional $L$-intersecting family of subspaces of a vector space $V$ of dimension $n$ over a finite field of size $q$. Let $t := \max(b_1, b_2, \ldots, b_s)$ and $g(t, n) := \frac{2(2^t + \ln n)}{\ln(2^t + \ln n)}$. Then,

$$|F| \leq 2g^2(t, n)\ln(g(t, n)) \left[\begin{array}{c} n \\ s \end{array}\right]_q + g(t, n) \sum_{i=1}^{s-1} \left[\begin{array}{c} n \\ i \end{array}\right]_q$$

Proof. Let $P = \{p_{\alpha+1}, p_{\alpha+2}, \ldots, p_{\beta}\}$, where $\beta = g(t, n)$ and $p_\beta \leq 2g(t, n)\ln(g(t, n))$. The proof is similar to the proof of Lemma 23. We apply Lemma 22 with $p = p_{\alpha+1}$ to show that

$$|F| \leq |F_{p_{\alpha+1}}^0| + (p_{\alpha+1} - 1) \left[\begin{array}{c} n \\ s \end{array}\right]_q + \sum_{i=1}^{s-1} \left[\begin{array}{c} n \\ i \end{array}\right]_q$$
Next, we apply Lemma 22 on $\mathcal{F}_0^{p_{\alpha+1}}$ with $p = p_{\alpha+2}$ and so on as shown in the proof of Lemma 23 to get the desired bound.

$$|\mathcal{F}| \leq (p_{\alpha+1} + p_{\alpha+2} + \cdots + p_{\beta} - (\beta - \alpha)) \left[n\right]_q + (\beta - \alpha) \sum_{i=1}^{s-1} \left[n\right]_q$$

$$< (\beta - \alpha) \left(p_{\beta} \left[n\right]_q + \sum_{i=1}^{s-1} \left[n\right]_q\right)$$

$$< \beta \left(p_{\beta} \left[n\right]_q + \sum_{i=1}^{s-1} \left[n\right]_q\right)$$

$$\leq 2g^2(t, n) \ln(g(t, n)) \left[n\right]_q + g(t, n) \sum_{i=1}^{s-1} \left[n\right]_q$$

Since $p_{\alpha+1} > t$, we have $p_{\alpha+1}p_{\alpha+2} \cdots p_{\beta} > t^{\beta - \alpha}$. This implies that, if $t^{\beta - \alpha} \geq n$, then the product of the primes in $P$ will be greater than $n$ as desired. Substituting $\beta - \alpha$ with $\ln t$ (and $p_{\beta}$ with $2g(t, n) \ln(g(t, n))$) in the second inequality above, we get another upper bound of $|\mathcal{F}| \leq 2g^2(t, n) \ln(n) \ln(g(t, n)) \left[n\right]_q + \frac{\ln t}{\ln n} \sum_{i=1}^{s-1} \left[n\right]_q$. We can do a similar substitution for $\beta - \alpha$ in the calculations done at the end of the proof of Lemma 23 to get a similar bound.

Combining all the results in this section, we get Theorem 5

### 3.2 An improved bound for singleton $L$

In this section, we improve the upper bound for the size of a fractional $L$-intersecting family obtained in Theorem 5 for the special case $L = \{\frac{a}{b}\}$, where $b$ is a constant prime. Before we give the proof, below we restate the the statement of Theorem 7.

**Statement of Theorem 7:** Let $L = \{\frac{a}{b}\}$, where $\frac{a}{b}$ is a positive irreducible fraction less than 1 and $b$ is a prime. Let $\mathcal{F}$ be a fractional $L$-intersecting family of subspaces of a vector space $V$ of dimension $n$ over a finite field of size $q$. Then, we have $|\mathcal{F}| \leq (b - 1)\left[\frac{n}{1}\right]_q + 1\left[\frac{\ln n}{\ln b}\right]_q + 2$.

**Proof.** We assume that all the subspaces in the family except possibly one subspace, say $W$, have a dimension divisible by $b$. Otherwise, $\mathcal{F}$ cannot satisfy the property of a fractional $\frac{a}{b}$-intersecting family. Let us ignore $W$ in the discussion to follow. For any subspace $V_i$ that is not the zero subspace, let $k$ be the largest power of $b$ that divides $\dim(V_i)$. Then, $\dim(V_i) = rb^{k+1} + jb^k$, for some $1 \leq j < b, r \geq 0$. Consider the subfamily, $\mathcal{F}^{j,k} = \{V_i : b^k \mid \dim(V_i), b^{k+1} \mid \dim(V_i), \dim(V_i) = rb^{k+1} + jb^k \text{ for some } r \geq 0, j \in [b - 1]\}$

The subfamily $\mathcal{F}^{j,k}, 1 \leq k \leq \left[\frac{\ln n}{\ln b}\right], 1 \leq j < b$, cover each and every subspace (except the zero subspace and the subspace $W$) of $\mathcal{F}$ exactly once. We will show that $|\mathcal{F}^{j,k}| \leq \left[\frac{n}{1}\right]_q + 1,$
which when multiplied with the number of values $j$ and $k$ can take will immediately imply the theorem.

Let $m^{j,k} = |\mathcal{F}^{j,k}|$. Let $M^{j,k}$ be an $m^{j,k} \times \left[\begin{array}{c} n \\ 1 \end{array}\right]_q$ 0-1 matrix whose rows correspond to the subspaces of $\mathcal{F}^{j,k}$ in any given order, whose columns correspond to the 1-dimensional subspaces of $V$ in any given order, and the $(i-1)^{th}$ entry is 1 if and only if the $i^{th}$ subspace of $\mathcal{F}^{j,k}$ contains the $i^{th}$ 1-dimensional subspace. Let $N^{j,k} = M^{j,k} \cdot (M^{j,k})^T$. Any diagonal entry $N^{j,k}_{i,i}$ is the number of 1-dimensional subspaces in the $i^{th}$ subspace in $\mathcal{F}^{j,k}$, and an off-diagonal entry $N^{j,k}_{i,j}$ is number of 1-dimensional subspaces common to the $i^{th}$ and $j^{th}$ subspaces of $\mathcal{F}^{j,k}$. In the rest of the proof, to reduce notational clutter, we shall use $G(x,y,z)$ to denote the Gaussian binomial coefficient $\left[\begin{array}{c} x \\ y \\ z \end{array}\right]$. We have

\[
N^{j,k}_{i,i} = G(r_1 b^{k+1} + jb^k, 1, q) = G(b^{k-1}, 1, q)G(r_1 b^2 + jb, 1, q^{b^{k-1}}),
\]

\[
N^{j,k}_{i,j} = G(r_2 ab^k + ja b^{k-1}, 1, q) = G(b^{k-1}, 1, q)G(r_2 ab + ja, 1, q^{b^{k-1}}),
\]

for some $r_1, r_2$ (may be different for different values of $i, j$). Let $P^{j,k}$ be the matrix over $\mathbb{R}$ obtained by dividing each entry of $N^{j,k}$ by $G(b^{k-1}, 1, q)$.

\[
\det(N^{j,k}) = G(b^{k-1}, 1, q)^{m^{j,k}} \det(P^{j,k})
\]

We will show that $\det(P^{j,k})$ is non-zero, thereby implying $\det(N^{j,k})$ is also non-zero. Consider $\det(P^{j,k}) \pmod{G(b, 1, q^{b^{k-1}})}$.

\[
P^{j,k}_{i,i} \equiv G(r_1 b^2 + jb, 1, q^{b^{k-1}}) \pmod{G(b, 1, q^{b^{k-1}})} \equiv 0 \pmod{G(b, 1, q^{b^{k-1}})},
\]

\[
P^{j,k}_{i,j} \equiv G(r_2 ab + ja, 1, q^{b^{k-1}}) \pmod{G(b, 1, q^{b^{k-1}})} \equiv G(r_3, 1, q^{b^{k-1}}) \pmod{G(b, 1, q^{b^{k-1}})},
\]

where $r_3 = ja \pmod{b}$ and $1 \leq r_3 \leq b - 1$ (since $j < b, a < b$, and $b$ is a prime, we have $1 \leq r_3 \leq b - 1$). We know that the determinant of an $r \times r$ matrix where diagonal entries are 0 and off-diagonal entries are all 1 is $(-1)^{r-1}(r-1)$.

\[
\det(P^{j,k}) \equiv (G(r_3, 1, q^{b^{k-1}}))^{m^{j,k}}(-1)^{(m^{j,k}-1)(m^{j,k} - 1)} \pmod{G(b, 1, q^{b^{k-1}})}
\]

Let $Q^{j,k}$ be the matrix formed by taking all but the last row and the last column of $P^{j,k}$.

\[
\det(Q^{j,k}) \equiv (G(r_3, 1, q^{b^{k-1}}))^{m^{j,k}(-1)^{(m^{j,k}-2)(m^{j,k} - 2)}} \pmod{G(b, 1, q^{b^{k-1}})}
\]

We will now show that one of $\det(P^{j,k})$ or $\det(Q^{j,k})$ is non-zero (mod $G(b, 1, q^{b^{k-1}})$) and therefore non-zero in $\mathbb{R}$. First, we show that $G(r_3, 1, q^{b^{k-1}})^{m^{j,k}}$ is not divisible by $G(b, 1, q^{b^{k-1}})$. Suppose $s_3 \equiv r_3^{-1} \pmod{b}$.

\[
G(r_3, 1, q^{b^{k-1}})^{m^{j,k}} G(s_3, 1, q^{s_3 b^{k-1}})^{m^{j,k}} = G(r_3 s_3, 1, q^{b^{k-1}})^{m^{j,k}}
\]

\[
G(r_3 s_3, 1, q^{b^{k-1}})^{m^{j,k}} \equiv G(1, 1, q^{b^{k-1}})^{m^{j,k}} \pmod{G(b, 1, q^{b^{k-1}})} \equiv 1 \pmod{G(b, 1, q^{b^{k-1}})}
\]
Therefore, $G(r_3, 1, q^{k-1})^{m^{j,k}}$ is invertible modulo $G(b, 1, q^{k-1})$, and hence the former is not divisible by the latter. Suppose $G(r_3, 1, q^{k-1})^{m^{j,k}}(-1)^{m^{j,k}-1}(m^{j,k}-1)$ is divisible by $G(b, 1, q^{k-1})$. We may ignore $(-1)^{m^{j,k}-1}$ for divisibility purpose. Then, there must be a product of prime powers that is equal to $(m^{j,k}-1)$ multiplied by $G(r_3, 1, q^{k-1})^{m^{j,k}}$ such that this product is divisible by $G(b, 1, q^{k-1})$. Observe that, $G(r_3, 1, q^{k-1})^{m^{j,k}-1}$ has only lesser powers of the same primes, and $m^{j,k}-1$ and $m^{j,k}-2$ cannot have any prime in common. So, the product $G(r_3, 1, q^{k-1})^{m^{j,k}-1}(m^{j,k}-2)$ cannot be divisible by $G(b, 1, q^{k-1})$, which is what we wanted to prove.

Therefore, either $P^{j,k}$ or $Q^{j,k}$ is a full rank matrix, or $rank(P^{j,k}) \geq m^{j,k} - 1$. Being a non-zero multiple of $P^{j,k}$, $rank(N^{j,k}) \geq m^{j,k} - 1$. But we know that $rank(AB) \leq \min(rank(A), rank(B))$, for any two matrices $A, B$.

\[
\begin{align*}
m^{j,k} - 1 & \leq rank(N^{j,k}) \leq \min(rank(M^{j,k}), rank((M^{j,k})^T)) \\
& = rank(M^{j,k}) \\
& \leq G(n, 1, q)
\end{align*}
\]

Or, $m^{j,k} \leq G(n, 1, q) + 1$, as required. It follows that,

\[
|\mathcal{F}| = m \leq 2 + \sum_{1 < k \leq \left\lfloor \frac{\ln n}{\ln b} \right\rfloor, 1 < j < b} m^{j,k} \leq (b-1)(G(n, 1, q) + 1) \left\lfloor \frac{\ln n}{\ln b} \right\rfloor + 2. \quad \square
\]

4 Concluding remarks

In Theorem 3, for $|\mathcal{F}|$ to be at most $N(n, s, r, q)$, one of the necessary conditions is $r(s - r + 1) \leq b - 1$. When $r = 1$, this condition is always true as $L \subseteq \{0, 1, \ldots, b-1\}$. However, when $r \geq 2$, it is not the case. Would it be possible to get the same upper bound for $|\mathcal{F}|$ without having to satisfy such a strong necessary condition? Another interesting question concerning Theorem 3 is regarding its tightness. From Example 2, we know that Theorem 3 is tight when $r = 1$. However, since Theorem 3 requires the sets $K$ and $L$ to be disjoint it is not possible to extend the construction in Example 2 to obtain a tight example for the case $r \geq 2$. Further, we know of no other tight example for this case. Therefore, we are not clear whether Theorem 3 is tight when $r \geq 2$.

We believe that the upper bounds given by Theorems 5 and 7 are not tight. Proving tight upper bounds in both the scenarios is a question that is obviously interesting. One possible approach to try would be to answer the following simpler question. Consider the case when $L = \{\frac{1}{2}\}$. We call such a family a bisection-closed family of subspaces. Let $\mathcal{F}$ be a bisection closed family of subspaces of a vector space $V$ of dimension $n$ over a finite field of size $q$. From Theorem 7, we know that $|\mathcal{F}| \leq \left( \begin{array}{c} n \\ 1 \end{array} \right)_q + 1 \log_2 n + 2$. We believe that

$|\mathcal{F}| \leq c \left( \begin{array}{c} n \\ 1 \end{array} \right)_q$, where $c$ is a constant. Example 8 gives a ‘trivial’ bisection-closed family of
size $\left[\begin{array}{c} n-1 \\ 1 \end{array}\right]_q$, where every subspace contains the vector $v_1$. It would be interesting to look for non-trivial examples of large bisection-closed families.

**References**


