Modular and fractional L-intersecting families of vector spaces

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Abstract

This paper is divided into two logical parts. In the first part of this paper, we prove the following theorem which is the q-analogue of a generalized modular Ray-Chaudhuri-Wilson Theorem shown in [Alon, Babai, Suzuki, J. Combin. Theory Series A, 1991]. It is also a generalization of the main theorem in [Frankl and Graham, European J. Combin. 1985] under certain circumstances.

• Let V be a vector space of dimension n over a finite field of size q. Let $K = \{k_1, \ldots, k_r\}, L = \{\mu_1, \ldots, \mu_s\}$ be two disjoint subsets of $\{0, 1, \ldots, b-1\}$ with $k_1 < \cdots < k_r$. Let $\mathcal{F} = \{V_1, V_2, \ldots, V_m\}$ be a family of subspaces of V such that (a)

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for every $i \in [m]$, $\dim(V_i) \bmod b = k_t$, for some $k_t \in K$, and (b) for every distinct $i, j \in [m]$, $\dim(V_i \cap V_j) \bmod b = \mu_t$, for some $\mu_t \in L$. Moreover, it is given that neither of the following two conditions hold:

(i) $q + 1$ is a power of 2, and $b = 2$

(ii) $q = 2, b = 6$.

Then,

$$|\mathcal{F}| \leqslant \begin{cases} N(n,s,r,q), & \text{if } (s + k_r \leqslant n \text{ and } r(s-r+1) \leqslant b-1) \text{ or } (s < k_1 + r) \\ N(n,s,r,q) + \sum\limits_{t \in [r]} \begin{bmatrix} n \\ k_t \end{bmatrix}_q, & \text{otherwise,} \end{cases}$$

where $N(n,s,r,q) := \begin{bmatrix} n \\ s \end{bmatrix}_q + \begin{bmatrix} n \\ s-1 \end{bmatrix}_q + \cdots + \begin{bmatrix} n \\ s-r+1 \end{bmatrix}_q$.

In the second part of this paper, we prove q-analogues of results on a recent notion called *fractional L-intersecting family* of sets for families of subspaces of a given vector space over a finite field of size q. We use the above theorem to obtain a general upper bound to the cardinality of such families. We give an improvement to this general upper bound in certain special cases.

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1 Introduction

Let $[n]$ be the set of all natural numbers from 1 to n. A family $\mathcal{F}$ of subsets of $[n]$ is called *intersecting* if every set in $\mathcal{F}$ has a non-empty intersection with every other set in $\mathcal{F}$. One of the earliest studies on intersecting families dates back to the famous Erdős-Ko-Rado Theorem [Erdős et al., 1961] about maximal uniform intersecting families. Ray-Chaudhuri and Wilson [Ray-Chaudhuri and Wilson, 1975] introduced the notion of L-intersecting families. Let $L = \{l_1, \ldots, l_s\}$ be a set of non-negative integers. A family $\mathcal{F}$ of subsets of $[n]$ is said to be *L-intersecting* if for every distinct F_i, F_j in $\mathcal{F}$, $|F_i \cap F_j| \in L$. The Ray-Chaudhuri-Wilson Theorem states that if $\mathcal{F}$ is t-uniform (that is, every set in $\mathcal{F}$ is t-sized), then $|\mathcal{F}| \leqslant \binom{n}{s}$. This bound is tight as shown by the set of all s-sized subsets of $[n]$ with $L = \{0, \ldots, s-1\}$. Frankl-Wilson Theorem [Frankl and Wilson, 1981a] extends this to non-uniform families by showing that $|\mathcal{F}| \leqslant \sum\limits_{i=0}^{s} \binom{n}{i}$, where $\mathcal{F}$ is any family of subsets of $[n]$ that is L-intersecting. The collection of all the subsets of $[n]$ of size at most s with $L = \{0, \ldots s-1\}$ is a tight example to this bound. The first proofs of these theorems were based on the technique of higher incidence matrices. Alon, Babai, and Suzuki in [Alon et al., 1991] generalized the Frankl-Wilson Theorem using a proof that operated on spaces of multilinear polynomials. They showed that if the sizes of the sets in $\mathcal{F}$ belong to $K = \{k_1, \ldots, k_r\}$ with each $k_i > s - r$, then $|\mathcal{F}| \leqslant \binom{n}{s} + \cdots + \binom{n}{s-r+1}$. A modular version of the Ray-Chaudhuri-Wilson Theorem was shown in [Frankl and Wilson, 1981b]. This result was generalized in [Alon et al., 1991]. See [Liu and Yang, 2014] for a survey on L-intersecting families.

Researchers have also been working on similar intersection theorems for subspaces of a given vector space over a finite field. Hsieh [Hsieh, 1975], and Deza and Frankl [Deza and Frankl, 1983] showed Erdős-Ko-Rado type theorems for subspaces. Let V be a vector space of dimension n over a finite field of size q. The number of d-dimensional subspaces of V is given by the q-binomial coefficient (also known as Gaussian binomial coeffcient) $\begin{bmatrix} n \\ d \end{bmatrix}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-d+1}-1)}{(q^d-1)(q^{d-1}-1)\cdots(q-1)}$. The following theorem which is a q-analog of the Ray-Chaudhuri-Wilson Theorem by considering families of subspaces instead of subsets is due to [Frankl and Graham, 1985].

Theorem 1. *[Theorem 1.1 in [Frankl and Graham, 1985]] Let V be a vector space over of dimension n over a finite field of size q. Let $\mathcal{F} = \{V_1, V_2, \ldots, V_m\}$ be a family of subspaces of V such that dim(V_i)= k, for every $i \in [m]$. Let $0 \leqslant \mu_1 < \mu_2 < \cdots < \mu_s < b$ be integers such that $k \not\equiv \mu_t \pmod b$, for any t. For every $1 \leqslant i < j \leqslant m$, dim($V_i \cap V_j$) $\equiv \mu_t \pmod b$, for some t. Then,*

$$|\mathcal{F}| \leqslant \begin{bmatrix} n \\ s \end{bmatrix}_q$$

except possibly for $q = 2, b = 6, s \in \{3, 4\}$.

Example 2 (Remark 3.2 in [Frankl and Graham, 1985])**.** Let $n = k + s$. Let $\mathcal{F}$ be the family of all the k-dimensional subspaces of V, where V is an n-dimensional vector space over a finite field of size q. Observe that, for any two distinct $V_i, V_j \in \mathcal{F}$, $k - s \leqslant \dim(V_i \cap V_j) \leqslant k - 1$. This is a tight example for Theorem 1.

Alon et al. in [Alon et al., 1991] proved a generalization of the non-modular version of the above theorem. This result was subsequently strengthened in [Liu et al., 2018].

Our paper is divided into two logical parts. In the first part (i.e., Section 2), we prove the following theorem which is a generalization of Theorem 1 due to Frankl and Graham under certain circumstances. It is also the q-analogue of a generalized modular Ray-Chaudhuri-Wilson Theorem shown in [Alon et al., 1991]. We assume that $\begin{bmatrix} a \\ b \end{bmatrix}_q = 0$, when $b < 0$ or $b > a$. Let

$$N(n, s, r, q) := \begin{bmatrix} n \\ s \end{bmatrix}_q + \begin{bmatrix} n \\ s-1 \end{bmatrix}_q + \cdots + \begin{bmatrix} n \\ s-r+1 \end{bmatrix}_q .$$

Theorem 3. *Let V be a vector space of dimension n over a finite field of size q. Let $K = \{k_1, \ldots, k_r\}, L = \{\mu_1, \ldots, \mu_s\}$ be two disjoint subsets of $\{0, 1, \ldots, b-1\}$ with $k_1 < \cdots < k_r$. Let $\mathcal{F} = \{V_1, V_2, \ldots, V_m\}$ be a family of subspaces of V such that (a) for every $i \in [m]$, dim(V_i) $\bmod b = k_t$, for some $k_t \in K$, and (b) for every distinct $i, j \in [m]$, dim($V_i \cap V_j$)$\bmod b = \mu_t$, for some $\mu_t \in L$. Moreover, it is given that neither of the following two conditions hold:*

(i) $q + 1$ is a power of 2, and $b = 2$

(ii) $q = 2, b = 6$

Then,

$$|\mathcal{F}| \leqslant \begin{cases} N(n,s,r,q), & \text{if } (s + k_r \leqslant n \text{and} r(s-r+1) \leqslant b-1) \text{ or} (s < k_1 + r) \\ N(n,s,r,q) + \sum\limits_{t\in[r]} \begin{bmatrix} n \\ k_t \end{bmatrix}_q, & \textit{otherwise.} \end{cases}$$

In the second part (i.e., Section 3), we study a notion of fractional L-intersecting families which was introduced in [Balachandran et al., 2019]. We say a family $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ of subsets of $[n]$ is a *fractional L-intersecting family*, where L is a set of irreducible fractions between 0 and 1, if for every distinct $i, j \in [m]$, $\frac{|F_i \cap F_j|}{|F_i|} \in L$ or $\frac{|F_i \cap F_j|}{|F_j|} \in L$. In this paper, we extend this notion from subsets to subspaces of a vector space over a finite field.

Definition 4. Let $L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\}$ be a set of positive irreducible fractions, where every $\frac{a_i}{b_i} < 1$. Let $\mathcal{F} = \{V_1, \ldots, V_m\}$ be a family of subspaces of a vector space V over a finite field. We say $\mathcal{F}$ is a *fractional L-intersecting family of subspaces* if for every two distinct $i, j \in [m]$, $\frac{\dim(V_i \cap V_j)}{\dim(V_i)} \in L$ or $\frac{\dim(V_i \cap V_j)}{\dim(V_j)} \in L$.

When every subspace in $\mathcal{F}$ is of dimension exactly k, it is an L'-intersecting family where $L' = \{\frac{a_1 k}{b_1}, \ldots, \frac{a_s k}{b_s}\}$. Applying Theorem 1, we get $|\mathcal{F}| \leqslant \begin{bmatrix} n \\ s \end{bmatrix}_q$. A tight example to this is the collection of all k-dimensional subspaces of V with $L = \{\frac{0}{k}, \ldots, \frac{k-1}{k}\}$. However, the problem of bounding the cardinality of a fractional L-intersecting family of subspaces becomes more interesting when $\mathcal{F}$ contains subspaces of various dimensions. In Section 3, we obtain upper bounds for the cardinality of a fractional L-intersecting family of subspaces that are q-analogs of the results in [Balachandran et al., 2019]. With the help of Theorem 3 that we prove in Section 2, we obtain the following result in Section 3.

Theorem 5. *Let* $L = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_s}{b_s}\}$ *be a collection of positive irreducible fractions, where every* $\frac{a_i}{b_i} < 1$. *Let* $\mathcal{F}$ *be a fractional L-intersecting family of subspaces of a vector space V of dimension n over a finite field of size q. Let* $t = \max_{i\in[s]} b_i$, $g(t,n) = \frac{2(2t+\ln n)}{\ln(2t+\ln n)}$, *and* $h(t,n) = \min(g(t,n), \frac{\ln n}{\ln t})$. *Then,*

$$|\mathcal{F}| \leqslant 2g(t,n)h(t,n)\ln(g(t,n)) \begin{bmatrix} n \\ s \end{bmatrix}_q + h(t,n) \sum_{i=1}^{s-1} \begin{bmatrix} n \\ i \end{bmatrix}_q.$$

Further, if $2g(t,n)\ln(g(t,n)) \leqslant n+2$, *then*

$$|\mathcal{F}| \leqslant 2g(t,n)h(t,n)\ln(g(t,n)) \begin{bmatrix} n \\ s \end{bmatrix}_q.$$

Example 6. Let s be a constant, $L = \{\frac{0}{s}, \frac{1}{s}, \ldots, \frac{s-1}{s}\}$, and $\mathcal{F}$ be the family of all the s-sized subspaces of V. Clearly, $\mathcal{F}$ is a fractional L-intersecting family showing that the bound in Theorem 5 is asymptotically tight up to a multiplicative factor of $\frac{\ln^2 n}{\ln \ln n}$.

We improve the bound obtained in Theorem 5 for the special case when $L = \{\frac{a}{b}\}$, where b is a prime.

Theorem 7. *Let $L = \{\frac{a}{b}\}$, where $\frac{a}{b}$ is a positive irreducible fraction less than 1 and b is a prime. Let $\mathcal{F}$ be a fractional L-intersecting family of subspaces of a vector space V of dimension n over a finite field of size q. Then, we have $|\mathcal{F}| \leqslant (b-1)(\begin{bmatrix} n \\ 1 \end{bmatrix}_q + 1)\lceil \frac{\ln n}{\ln b} \rceil + 2$.*

Example 8. Let $L = \{\frac{1}{2}\}$. Let V be a vector space of dimension n over a finite field of size q. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. Let $V' := \mathit{span}(\{v_2, \ldots, v_n\})$ be an $(n-1)$-dimensional subspace of V. Let $\mathcal{F}$ be the set of all $\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$ 2-dimensional subspaces of V each of which is obtained by a span of v_1 and each of the $\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$ 1-dimensional subspaces of V'. This example shows that when b and q are constants, the bound in Theorem 7 is asymptotically tight up to a multiplicative factor of $\ln n$.

2 Generalized modular RW Theorem for subspaces

As mentioned before, in this part we prove Theorem 3. The approach followed here is similar to the approach used in proving Theorem 1.5, a generalized modular Ray-Chaudhuri-Wilson Theorem for subsets, in [Alon et al., 1991]. We start by stating the Zsigmondy's Theorem which will be used in the proof of Theorem 3.

Theorem 9 ([Zsigmondy, 1892])**.** *For any $q, b \in \mathbb{N}$, there exists a prime p such that $q^b \equiv 1 \pmod{p}, q^i \not\equiv 1 \pmod{p}\ \forall i,\ 0 < i < b$, except when (i) $q+1$ is a power of 2, $b = 2$, or (ii) $q = 2, b = 6$.*

2.1 Notations used in Section 2

Unless defined explicitly, in the rest of this section, the symbols $K = \{k_1, \ldots, k_r\}$, r, $L = \{\mu_1, \ldots, \mu_s\}$, s, q, V, $\mathcal{F}$, n, b, m, and $V_1, \ldots, V_m$ are defined as they are defined in Theorem 3. We shall use $U \subseteq V$ to denote that U is a subspace of V. Using Zsigmondy's Theorem, we find a prime p so that $q^i \not\equiv 1 \pmod{p}$ for $0 < i < b$ and $q^b \equiv 1 \pmod{p}$. This is possible except in the two cases specified in Theorem 9. We ignore these two cases from now on in the rest of Section 2.

2.2 Möbius inversion over the subspace poset

Consider the partial order defined on the set of subspaces of the vector space V over a finite field of size q under the 'containment' relation. Let α be a function from the set of subspaces of V to $\mathbb{F}_p$. A function β from the set of subspaces of V to $\mathbb{F}_p$ is the *zeta transform* of α if for every $W \subseteq V$, $\beta(W) = \sum_{U \subseteq W} \alpha(U)$. Then, applying the *Möbius inversion formula* we get for all $W \subseteq V$, $\alpha(W) = \sum_{U \subseteq W} \mu(U, W)\beta(U)$, where α is called the *Möbius transform* of β and $\mu(U,W)$ is the *Möbius function* for the subspace poset. In the proposition below, we show that the *Möbius function* for the subspace poset is defined as

$$\mu(X,Y) = \begin{cases} (-1)^d q^{\binom{d}{2}}, \text{if} X \subseteq Y \\ 0, \qquad \text{otherwise,} \end{cases}$$

$\forall X, Y \subseteq V$ with $d = \dim(Y) - \dim(X)$. The following proposition gives the Möbius inversion formula for the subspace lattice. See [Mathew et al., 2020] for a proof.

Proposition 10. *Let α and β be functions from the set of subspaces of V to $\mathbb{F}_p$. Then, $\forall\ W \subseteq V$,*

$$\beta(W) = \sum_{U \subseteq W} \alpha(U) \iff \alpha(W) = \sum_{\substack{U \subseteq W \\ d = \dim(W) - \dim(U)}} (-1)^d q^{\frac{d(d-1)}{2}} \beta(U).$$

Definition 11. Given two subspaces U and W of the vector space V, we define their union space $U \cup W$ as the span of union of sets of vectors in U and W.

The proposition below follows from the definitions of α and β. See [Mathew et al., 2020] for a proof.

Proposition 12. *Let α and β be functions as defined in Proposition* 10*. Then, $\forall\ W, Y$ such that $W \subseteq Y \subseteq V$,*

$$\sum_{\substack{T:\ W \subseteq T \subseteq Y \\ d = \dim(Y) - \dim(T)}} (-1)^d q^{\frac{d(d-1)}{2}} \beta(T) = \sum_{U:\ U \cup W = Y} \alpha(U).$$

Corollary 13. *For any non-negative integer g, the following are equivalent for functions α and β defined in Proposition* 10*:*

(i) $\alpha(U) = 0$, $\forall U \subseteq V$ *with* $\dim(U) \geqslant g$.

(ii) $\sum_{\substack{W \subseteq T \subseteq Y \\ d = \dim(Y) - \dim(T)}} (-1)^d q^{\frac{d(d-1)}{2}} \beta(T) = 0$, $\forall W, Y \subseteq V$ *with* $\dim(Y) - \dim(W) \geqslant g$.

Definition 14. Let $H = \{h_1, h_2, \ldots, h_t\}$ be a subset of $\{0, 1, \ldots, n\}$ where $h_1 < h_2 < \cdots < h_t$. We say H has a *gap* of size $\geqslant g$ if either $h_1 \geqslant g - 1$, $n - h_t \geqslant g - 1$, or $h_{i+1} - h_i \geqslant g$ for some $i \in [t-1]$.

Lemma 15. *Let α and β be functions as in Proposition* 10*. Let $H \subseteq \{0, 1, \ldots, n\}$ be a set of integers and g an integer, $0 \leqslant g \leqslant n$. Suppose we have the following conditions:*

(i) $\forall U \subseteq V$, *we have* $\alpha(U) = 0$ *whenever* $\dim(U) \geqslant g$.

(ii) $\forall T \subseteq V$, *we have* $\beta(T) = 0$ *whenever* $\dim(T) \notin H$.

(iii) H has a gap $\geqslant g+1$.

Then, $\alpha = \beta = 0$.

Proof. Let $H = \{h_1, h_2, \dots, h_{|H|}\}$. Suppose, for some $i \in [|H|]$, $h_i - h_{i-1} \geqslant g$ or $h_1 \geqslant g$, then we have $h_i \in H$ and $h_i - j \notin H$ for $1 \leqslant j \leqslant g$ and $h_i - g \geqslant 0$. Choose any two subspaces, say U and W, of V of dimensions h_i and $h_i - g$, respectively. Since $\dim(U) \geqslant g$, $\alpha(U) = 0$. We know from Corollary 13 that

$$\sum_{\substack{W \subseteq T \subseteq U \\ d = \dim(U) - \dim(T)}} (-1)^d q^{\frac{d(d-1)}{2}} \beta(T) = 0$$

But whenever $\dim(T) < h_i$, it lies between $h_i - g$ and $h_i - 1$, and hence $\beta(T) = 0$. Then,

$$\sum_{\substack{W \subseteq T \subseteq U \\ d = \dim(U) - \dim(T)}} (-1)^d q^{\frac{d(d-1)}{2}} \beta(T) = \beta(U) = 0$$

Since our choice of U was arbitrary, we may conclude that $\beta(U) = 0$, for all $U \subseteq V$ with $\dim(U) = h_i$. Thus, we can remove h_i from the set H, and then use the same procedure to further reduce the size of H till it is an empty set. If H is empty, $\beta(U) = 0$, for all $U \subseteq V$, giving $\alpha(U) = \beta(U) = 0$ as required.

Now suppose $n - h_{|H|} \geqslant g$. In this case, we take U of dimension $h_{|H|}$ and W of dimension $h_{|H|} + g$ to show that $\beta(U) = 0$, and remove $h_{|H|}$ from H. Note that removing a number from the set H can never reduce the gap. □

2.3 Defining functions $f^{x,y}$ and $g^{x,y}$

Consider all the subspaces of the vector space V. We can impose an ordering on the subspaces of same dimension, and use the natural ordering across dimensions, so that every subspace can be uniquely represented by a pair of integers $\langle d, e \rangle$, indicating that it is the e^{th} subspace of dimension d, $0 \leqslant d \leqslant n$, $1 \leqslant e \leqslant \begin{bmatrix} n \\ d \end{bmatrix}_q$. Let us call that subspace $V_{d,e}$. Let S be the number of subspaces of V of dimension at most s, that is, $S = \sum_{t=0}^{s} \begin{bmatrix} n \\ t \end{bmatrix}_q$. Let each subspace $V_{d,e}$ of dimension at most s be represented as a 0-1 containment vector $v_{d,e}$ of S entries, each entry of the vector denoting whether a particular subspace of dimension $\leqslant s$ is contained in $V_{d,e}$ or not.

$$v_{d,e}^{x,y} = \begin{cases} 1, \text{ if } V_{x,y} \text{ is a subspace of } V_{d,e} \\ 0, \text{ otherwise} \end{cases}$$

The vector $v_{d,e}$ consists of $v_{d,e}^{x,y}$ values for $0 \leqslant x \leqslant s$, $1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$, making it a vector of size S. Thus, $v_{d,e}^{x,y}$ is simply the indicator function of whether $V_{x,y}$ is a subspace of $V_{d,e}$.

For $0 \leqslant x \leqslant s, 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$ we define functions $f^{x,y} : \mathbb{F}_2^S \to \mathbb{F}_p$ as

$$f^{x,y}(v) = f^{x,y}(v^{0,1}, v^{1,1}, \ldots, v^{1, \begin{bmatrix} n \\ 1 \end{bmatrix}_q}, \ldots, v^{s,1}, \ldots, v^{s, \begin{bmatrix} n \\ s \end{bmatrix}_q}) := v^{x,y}.$$

For $0 \leqslant x \leqslant s - r, 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$, we define functions $g^{x,y} : \mathbb{F}_2^S \to \mathbb{F}_p$ as

$$g^{x,y}(v) = f^{x,y}(v) \prod_{t \in [r]} \left(\sum_{j=1}^{\begin{bmatrix} n \\ 1 \end{bmatrix}_q} v^{1,j} - \begin{bmatrix} k_t \\ 1 \end{bmatrix}_q \right)$$

Let Ω denote $\mathbb{F}_2^S$. The functions $f^{x,y}$ and $g^{x,y}$ reside in the space $\mathbb{F}_p^\Omega$. Note that the functions $g^{x,y}$ do not exist if $s < r$.

2.4 Swallowing trick: linear independence of functions $f^{x,y}$ and $g^{x,y}$

Lemma 16. *Let $s + k_r \leqslant n$ and $r(s - r + 1) \leqslant b - 1$. The functions $g^{x,y}$, $0 \leqslant x \leqslant s - r, 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$, are linearly independent in the function space $\mathbb{F}_p^\Omega$ over $\mathbb{F}_p$.*

Proof. If $s < r$, then the statement of the lemma is vacuously true. Assume $s \geqslant r$. We wish to show that the only solution to $\sum_{\substack{0 \leqslant x \leqslant s-r \\ 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q}} \alpha^{x,y} g^{x,y} = 0$ is the trivial solution $\alpha^{x,y} = 0,\ \forall x, y$. We define function α from the set of all subspaces of V to $\mathbb{F}_p$ as:

$$\alpha(V_{d,e}) = \begin{cases} \alpha^{d,e}, & \text{if } 0 \leqslant d \leqslant s - r \\ 0, & \text{if } d > s - r \end{cases}$$

We show that functions α and $\beta(U) := \sum_{T \subseteq U} \alpha(T)$ satisfy the conditions of Lemma 15, thereby implying $\alpha(U) = 0$, for all $U \subseteq V$, including $\alpha(V_{d,e}) = \alpha^{d,e} = 0$ for $0 \leqslant d \leqslant s - r$, which will in turn imply that the functions $g^{x,y}$ above are linearly independent.

Let $H = \{x : 0 \leqslant x \leqslant n, x \equiv k_t \pmod b, t \in [r]\}$. We claim that H has a gap of size at least $s - r + 2$. Suppose $n \geqslant b + k_1$. Then, $k_1 < k_2 < \cdots < k_r < b + k_1 \leqslant n$. Since it is given that $r(s - r + 1) \leqslant b - 1$, by pigeonhole principle, there is a gap of at least $s - r + 2$ between some k_i and k_{i+1}, $i \in [r - 1]$, or between k_r and $b + k_1$. Suppose $s + k_r \leqslant n < b + k_1$. Then, there is a gap of at least $s + 1$ right above k_r. This proves the claim. We now need to show that for $T \subseteq V$, $\beta(T) = 0$ whenever $\dim(T) \notin H$, or whenever $\dim(T) \not\equiv k_t \pmod b$, for any $t \in [r]$. Suppose v_T is the S-sized containment vector for T. When $\dim(T) \not\equiv k_t \pmod b$ for any $t \in [r]$, it follows from the property of the prime p given by Theorem 9 that $\sum_{1\leqslant j\leqslant \left[{n \atop 1}\right]_q} v_T^{1,j} - \begin{bmatrix} k_t \\ 1 \end{bmatrix}_q \neq 0$ in $\mathbb{F}_p$, for every $t \in [r]$.

$$\beta(T) = \sum_{U\subseteq T} \alpha(U) = \sum_{\substack{\dim(U)\leqslant s-r \\ U\subseteq T}} \alpha(U) = \sum_{\substack{0\leqslant d\leqslant s-r \\ 1\leqslant e\leqslant \left[{n \atop d}\right]_q}} \alpha(V^{d,e}) f^{d,e}(v_T)$$

Since $\sum_{1\leqslant j\leqslant \left[{n \atop 1}\right]_q} v_T^{1,j} - \begin{bmatrix} k_t \\ 1 \end{bmatrix}_q \neq 0$ in $\mathbb{F}_p$ for every $t \in [r]$, $f^{d,e}(v_T) = c(T)g^{d,e}(v_T)$ where $c(T) \neq 0$. Then,

$$\beta(T) = c(T) \sum_{\substack{0\leqslant d\leqslant s-r \\ 1\leqslant e\leqslant \left[{n \atop d}\right]_q}} \alpha(V^{d,e}) g^{d,e}(v_T) = c(T) \sum_{\substack{0\leqslant d\leqslant s-r \\ 1\leqslant e\leqslant \left[{n \atop d}\right]_q}} \alpha^{d,e} g^{d,e}(v_T) = c(T)\cdot 0 = 0.$$

Since the set H and the functions α and β satsify the conditions of Lemma 15, we have $\alpha = 0$. This proves the lemma. □

Recall that we are given a family $\mathcal{F} = \{V_1, V_2, \ldots, V_m\}$ of subspaces of V such that for every $i \in [m]$, $\dim(V_i) \bmod b = k_t$, for some $k_t \in K$. Further, $\dim(V_i \cap V_j) \bmod b = \mu_t$, for some $\mu_t \in L$ and K and L are disjoint subsets of $\{0, 1, \ldots, b-1\}$. Let v_i be the containment vector of size S corresponding to subspace $V_i \in \mathcal{F}$. We define the following functions from $\mathbb{F}_2^S \to \mathbb{F}_p$.

$$g^i(v) = g^i(v^{0,1}, v^{1,1}, \ldots, v^{1,\left[{n \atop 1}\right]_q}, \ldots, v^{s,1}, \ldots, v^{s,\left[{n \atop s}\right]_q})$$
$$:= \prod_{j=1}^{s} \left(\sum_{1\leqslant y\leqslant \left[{n \atop 1}\right]_q} \left(v_i^{1,y} v^{1,y}\right) - \begin{bmatrix} \mu_j \\ 1 \end{bmatrix}_q \right)$$

Let $v = v_j$. Then, $\sum_{1\leqslant y\leqslant \left[{n \atop 1}\right]_q} (v_i^{1,y} v^{1,y})$ counts the number of 1-dimensional subspaces common to V_i and V_j. That is, $\sum_{1\leqslant y\leqslant \left[{n \atop 1}\right]_q} v_i^{1,y} v^{1,y} = \begin{bmatrix} \dim(V_i \cap V_j) \\ 1 \end{bmatrix}_q$. In $\mathbb{F}_p$, $\begin{bmatrix} \dim(V_i \cap V_j) \\ 1 \end{bmatrix}_q \neq \begin{bmatrix} \mu_t \\ 1 \end{bmatrix}_q$ for any $1 \leqslant t \leqslant s$, if $i = j$, and $\begin{bmatrix} \dim(V_i \cap V_j) \\ 1 \end{bmatrix}_q = \begin{bmatrix} \mu_t \\ 1 \end{bmatrix}_q$ for some $1 \leqslant t \leqslant s$ if $i \neq j$. Accordingly, $g^i(v_j) = \begin{cases} 0, & i \neq j \\ \neq 0, & i = j. \end{cases}$

Lemma 17 (Swallowing trick 1)**.** *Let $s + k_r \leqslant n$ and $r(s-r+1) \leqslant b-1$. The collection of functions g^i, $1 \leqslant i \leqslant m$ together with the functions $g^{x,y}$, $0 \leqslant x \leqslant s-r$, $1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$ are linearly independent in $\mathbb{F}_p^{\Omega}$ over $\mathbb{F}_p$.*

Proof. Let

$$\sum_{1 \leqslant i \leqslant m} \alpha^i g^i + \sum_{\substack{0 \leqslant x \leqslant s-r \\ 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q}} \alpha^{x,y} g^{x,y} = 0 \tag{1}$$

We know that $g^i(v_j) = 0$ whenever $i \neq j$, and $g^{x,y}(v_i) = 0, 1 \leqslant i \leqslant m$. The latter holds because $\dim(V_i) \equiv k_t \pmod b$, say equal to $bl + k_t$, for some $t \in [r]$. Consequently, it follows that the number of 1-dimensional subspaces in V_i is $\begin{bmatrix} bl + k_t \\ 1 \end{bmatrix}_q$ which is equal to $\begin{bmatrix} k_t \\ 1 \end{bmatrix}_q$ in $\mathbb{F}_p$. Suppose we evaluate L.H.S. of Equation (1) on v_1, then all terms except the first one vanish. This gives us $\alpha^1 = 0$, and reduces the relation by one term from left. Next, we put $v = v_2$ to get $\alpha^2 = 0$, and so on. Finally, all α^i terms are zero, and we are left only with functions $g^{x,y}$. These $\alpha^{x,y}$ values are zero from Lemma 16. Therefore, we have shown that (1) implies that $\alpha^i = 0, 1 \leqslant i \leqslant m$ and $\alpha^{x,y} = 0, 0 \leqslant x \leqslant s - r, 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$, and hence the given functions are linearly independent. ☐

2.5 Proof of Theorem 3: in the case when $s + k_r \leqslant n \text{and} r(s - r + 1) \leqslant b - 1$

Lemma 18. *The collection of functions $f^{x,y}$, $0 \leqslant x \leqslant s$, $1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$, spans all the functions $g^{x,y}$, $0 \leqslant x \leqslant s - r$, $1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$ as well as the functions g^i, $1 \leqslant i \leqslant m$.*

Proof. Let $v \in \mathbb{F}_2^S$. The key observation here is that the product $f^{x,y}(v) f^{1,z}(v), 0 \leqslant x \leqslant s - 1, 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q, 1 \leqslant z \leqslant \begin{bmatrix} n \\ 1 \end{bmatrix}_q$ may be replaced by the function $f^{x',w}(v)$, where $x \leqslant x' \leqslant x + 1, 1 \leqslant w \leqslant \begin{bmatrix} n \\ x' \end{bmatrix}_q$. If $V_{1,z} \subseteq V_{x,y}$, it is trivial that $f^{x,y}(v) f^{1,z}(v) = f^{x,y}(v)$, since $f^{x,y}(v) = 1$ only if $f^{1,z}(v) = 1$. If $V_{1,z} \not\subseteq V_{x,y}$, we let $V_{x',w}$ be the span of union of vectors of $V_{1,z}$ and $V_{x,y}$. Suppose, a vector space U contains both $V_{1,z}$ and $V_{x,y}$. Then, it is clear that it must contain the span of their union as well. Similarly, a vector space U that does not contain either $V_{1,z}$ or $V_{x,y}$, cannot contain $V_{x',w}$. Thus, $f^{x,y}(v) f^{1,z}(v) = f^{x',w}(v)$. To see why $x' = x + 1$ (in case $V_{1,z} \not\subseteq V_{x,y}$), the space $V_{x',w}$ may be obtained by taking any (non-zero) vector of $V_{1,z}$ and introducing it into the basis of $V_{x,y}$. The space spanned by this extended basis is exactly $V_{x',w}$ by definition, and the size of basis has increased by exactly 1.

By induction, it follows that,

$$f^{1,y_1}(v) f^{1,y_2}(v) \cdots f^{1,y_l}(v) = f^{x,y}(v)$$

for some x, y where, $1 \leqslant x \leqslant l, 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$. That is, a product of l functions of the form $f^{1,y}$ may be replaced by a single function $f^{x,y}$ where x is at most l.

Now consider functions

$$
\begin{aligned}
g^i(v) &= g^i(v^{0,1}, v^{1,1}, \cdots, v^{1, \left[{n \atop 1}\right]_q}, \cdots, v^{s,1}, \cdots, v^{s, \left[{n \atop s}\right]_q}) \\
&= \prod_{j=1}^{s} \left(\sum_{1 \leqslant y \leqslant \left[{n \atop 1}\right]_q} \left(v_i^{1,y} v^{1,y}\right) - \begin{bmatrix} \mu_j \\ 1 \end{bmatrix}_q \right) \\
&= \prod_{j=1}^{s} \left(\sum_{1 \leqslant y \leqslant \left[{n \atop 1}\right]_q} \left(v_i^{1,y} f^{1,y}(v)\right) - \begin{bmatrix} \mu_j \\ 1 \end{bmatrix}_q \right)
\end{aligned}
$$

Since the functions $f^{x,y}$ only take 0/1 values, we can reduce any exponent of 2 or more on the function after expanding the product to 1. Moreover, the terms will all be products of the form $f^{1,y_1} f^{1,y_2} \cdots f^{1,y_l}(v), 1 \leqslant l \leqslant s$. These are replaced according to the observation above by single function of the form $f^{x,y}(v)$, and thus the set of functions $f^{x,y}, 0 \leqslant x \leqslant s, 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$ span all functions $g^i(v)$. Note that $f^{0,1}(v)$ is the constant function 1.

Similarly, for $0 \leqslant x \leqslant s - r, 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$,

$$
\begin{aligned}
g^{x,y}(v) &= f^{x,y}(v) \prod_{t \in [r]} \left(\sum_{j=1}^{\left[{n \atop 1}\right]_q} v^{1,j} - \begin{bmatrix} k_t \\ 1 \end{bmatrix}_q \right) \\
&= f^{x,y}(v) \prod_{t \in [r]} \left(\sum_{j=1}^{\left[{n \atop 1}\right]_q} f^{1,j}(v) - \begin{bmatrix} k_t \\ 1 \end{bmatrix}_q \right) \\
&= f^{x,y}(v) \left(\sum_{x'=0}^{r} \sum_{y'=1}^{\left[{n \atop x'}\right]_q} c_{x',y'} f^{x',y'}(v) \right) && (c_{x',y'} \text{ are constants}) \\
&= \sum_{x'=0}^{s} \sum_{y'=1}^{\left[{n \atop x'}\right]_q} c_{x',y'} f^{x',y'}(v) && (c_{x',y'} \text{ are constants})
\end{aligned}
$$

Thus, the set of function $f^{x,y}, 0 \leqslant x \leqslant s, 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$ span all functions $g^{x,y}(v), 0 \leqslant x \leqslant$

$s - r, 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$. $\square$

This means that the above functions $g^{x,y}$ and g^i belong to the span of functions $f^{x,y}$ which is a function space of dimension at most S. From Lemma 17, we know that $g^{x,y}$ and g^i are together linearly independent. Thus,

$$\sum_{j=0}^{s-r} \begin{bmatrix} n \\ j \end{bmatrix}_q + m \leqslant S = \sum_{j=0}^{s} \begin{bmatrix} n \\ j \end{bmatrix}_q .$$

$$\Rightarrow |\mathcal{F}| = m \leqslant \begin{bmatrix} n \\ s \end{bmatrix}_q + \begin{bmatrix} n \\ s-1 \end{bmatrix}_q + \cdots + \begin{bmatrix} n \\ s-r+1 \end{bmatrix}_q .$$

2.6 Proof of Theorem 3

Let $X \subseteq \{0, \ldots, s-r\}$ be the set of those integers that are not congruent to any $k \in K$. The, in the following lemma, we show that the family $g^{x,y}$ with $x \in X$ is linearly independent.

Lemma 19. *The collection of functions*

$$\{g^{x,y} \mid 0 \leqslant x \leqslant s-r, 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q, \text{ andforall} t \in [r], x \not\equiv k_t \pmod{b}\}$$

are linearly independent in the function space $\mathbb{F}_p^{\Omega}$ over $\mathbb{F}_p$.

Proof. Recall that

$$g^{x,y}(v) = f^{x,y}(v) \prod_{t \in [r]} \left(\sum_{j=1}^{\begin{bmatrix} n \\ 1 \end{bmatrix}_q} v^{1,j} - \begin{bmatrix} k_t \\ 1 \end{bmatrix}_q \right) .$$

The statement of the lemma is vacuously true, if $s < r$. Assume $s \geqslant r$. Assume, for the sake of contradiction, $\sum_{\substack{0 \leqslant x \leqslant s-r \\ x \not\equiv k_t \pmod{p}, \forall t \in [r]}} \alpha^{x,y} g^{x,y} = 0$ with at least one $\alpha^{x,y}$ as non-zero. Let $\langle x_0, y_0 \rangle$ be the first subspace, based on the ordering of subspaces defined in Section 2.3, such that α^{x_0,y_0} is non-zero. Evaluating both sides on v_{x_0,y_0}, we see that all $f^{x,y}$ (and therefore $g^{x,y}$) with $\langle x, y \rangle$ higher in the ordering than $\langle x_0, y_0 \rangle$ will vanish (due to the virtue of our ordering), and so we get $\alpha^{x_0,y_0} = 0$ which is a contradiction. Here we have crucially used the fact that by ignoring $x \equiv k_t \pmod{p}$ cases, for any $t \in [r]$, we make sure that v_{x_0,y_0} used above always has $x_0 \not\equiv k_t \pmod{b}$ and therefore

$$\left(\sum_{j=1}^{\begin{bmatrix} n \\ 1 \end{bmatrix}_q} v_{x_0,y_0}^{1,j} - \begin{bmatrix} k_t \\ 1 \end{bmatrix}_q \right) \not\equiv 0 \pmod{p}, \ \forall t \in [r]. \quad \square$$

Lemma 20 (Swallowing trick 2). *The collection of functions* g^i, $1 \leqslant i \leqslant m$ *together with the functions* $g^{x,y}$, $0 \leqslant x \leqslant s-r, x \not\equiv k_t \pmod b$, *for all* $t \in [r], 1 \leqslant y \leqslant \begin{bmatrix} n \\ x \end{bmatrix}_q$ *are linearly independent in* $\mathbb{F}_p^{\Omega}$ *over* $\mathbb{F}_p$.

Proof. Proof is similar to the proof of Lemma 17. □

Since $s < b$, for any $0 \leqslant x \leqslant s-r$ and for any $t \in [r]$, $x \not\equiv k_t \pmod b$ is equivalent to $x \neq k_t$. Combining Lemmas 19, 20 and 18, we have

$$\sum_{\substack{0 \leqslant j \leqslant s-r, \\ j \neq k_t, t \in [r]}} \begin{bmatrix} n \\ j \end{bmatrix}_q + m \leqslant \sum_{j=0}^{s} \begin{bmatrix} n \\ j \end{bmatrix}_q .$$

This implies,

$$|\mathcal{F}| = m \leqslant \begin{cases} N(n,s,r,q), & \text{if } s < k_1 + r \\ N(n,s,r,q) + \sum\limits_{t \in [r]} \begin{bmatrix} n \\ k_t \end{bmatrix}_q, & \text{otherwise.} \end{cases}$$

We thus have the following theorem which combined with the result in Section 2.5 yields Theorem 3.

Theorem 21. *Let* V *be a vector space of dimension* n *over a finite field of size* q. *Let* $K = \{k_1, \ldots, k_r\}, L = \{\mu_1, \ldots, \mu_s\}$ *be two disjoint subsets of* $\{0, 1, \ldots, b-1\}$ *with* $k_1 < \cdots < k_r$. *Let* $\mathcal{F} = \{V_1, V_2, \ldots, V_m\}$ *be a family of subspaces of* V *such that for all* $i \in [m]$, *dim(*V_i*)* $\equiv k_t \pmod b$, *for some* $k_t \in K$*; for every distinct* $i, j \in [m]$, *dim(*$V_i \cap V_j$*)* $\equiv \mu_t \pmod b$, *for some* $\mu_t \in L$. *Moreover, it is given that neither of the following two conditions hold:*

(i) $q+1$ *is a power of 2, and* $b = 2$

(ii) $q = 2, b = 6$

Then,

$$|\mathcal{F}| \leqslant \begin{cases} N(n,s,r,q), & \text{if } (s < k_1 + r) \\ N(n,s,r,q) + \sum\limits_{t \in [r]} \begin{bmatrix} n \\ k_t \end{bmatrix}_q, & \text{otherwise.} \end{cases}$$

3 Fractional *L*-intersecting families of subspaces

Let $L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\}$ be a collection of positive irreducible fractions, each strictly less than 1. Let V be a vector space of dimension n over a finite field of size q. Let $\mathcal{F}$ be a family of subspaces of V. Recall that, we call $\mathcal{F}$ a *fractional L-intersecting family of subspaces* if for all distinct $A, B \in \mathcal{F}$, $\dim(A \cap B) \in \{\frac{a_i}{b_i}\dim(A), \frac{a_i}{b_i}\dim(B)\}$, for some $\frac{a_i}{b_i} \in L$. In Section 3.1, we prove a general upper bound for the size of a fractional L-intersecting family using Theorem 3 proved in Section 2. In Section 3.2, we improve this upper bound for the special case when $L = \{\frac{a}{b}\}$ is a singleton set with b being a prime number.

3.1 A general upper bound

The key idea we use here is to split the fractional L intersecting family $\mathcal{F}$ into subfamilies and then use Theorem 3 to bound each of them.

Lemma 22. *Let $L = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_s}{b_s}\}$, where every $\frac{a_i}{b_i}$ is a irreducible fraction in the open interval $(0, 1)$. Let $\mathcal{F} = \{V_1, \dots, V_m\}$ be a fractional L-intersecting family of subspaces of a vector space V of dimension n over a finite field of size q. Let $k > 0$ and $p > \max(b_1, b_2, \dots, b_s)$. Let $\mathcal{F}^p_k$ denote subspaces in $\mathcal{F}$ whose dimensions leave a remainder $k \pmod p$, where p is a prime number. That is, $\mathcal{F}^p_k := \{W \in \mathcal{F} \mid \dim(W) \equiv k \pmod p\}$.*

Then,

$$|\mathcal{F}^p_k| \leqslant \begin{cases} \begin{bmatrix} n \\ s \end{bmatrix}_q, & \text{if } (2p \leqslant n+2) \text{ or } (s < k+1) \\ \begin{bmatrix} n \\ s \end{bmatrix}_q + \begin{bmatrix} n \\ k \end{bmatrix}_q, & \text{otherwise.} \end{cases}$$

Proof. Apply Theorem 3 with family $\mathcal{F}$ replaced by $\mathcal{F}^p_k$, $K = \{k\}$, $r = 1$, b replaced by p, and each μ_i replaced by $(\frac{a_i}{b_i}k) \bmod p = (b_i^{-1}a_i k) \bmod p$, where b_i^{-1} is the multiplicative inverse of b_i in $\mathbb{F}_p$. Let s' ($\leqslant s$) be the number of distinct μ_i's. Notice that $k > 0$, and $p > b_i > a_i$ ensure that $k \not\equiv \frac{a_i}{b_i}k \pmod p$ or $k \neq \mu_i$. Thus $\mathcal{F}^p_k$ is a family of subspaces of V such that (a) for every $W \in \mathcal{F}^p_k, \dim(W) \bmod p = k$, and (b) for every distinct $U, W \in \mathcal{F}^p_k, \dim(U \cap W) \bmod p \in L$, where $L = \{\mu_1, \dots, \mu_{s'}\}$ and $k \notin L$. Moreover, since $s' \leqslant p-1$ and $k \leqslant p-1$, we have $s' + k \leqslant n$ if $2p \leqslant n+2$. Since $p > b_i$ and every $b_i \geqslant 2$, we have $p > 2$. This avoids bad case (i) of Theorem 3. That p is a prime avoids bad case (ii) of Theorem 3. Thus, we satisfy the premise of Theorem 3 and the conclusion follows. □

Suppose $2p \leqslant n+2$. The above lemma immediately gives us a bound of $|\mathcal{F}| \leqslant |\mathcal{F}^p_0| + (p-1)\begin{bmatrix} n \\ s \end{bmatrix}_q$. But it could be that most subspaces belong to $\mathcal{F}^p_0$. To overcome this problem, we instead choose a set of primes P such that no subspace can belong to $\mathcal{F}^p_0$ for every $p \in P$. A natural choice is to take just enough primes in increasing order so that the product of these primes exceeds n, because then any subspace with dimension divisible by all primes in P will have a dimension greater than n, which is not possible.

Lemma 23. *Let $L = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_s}{b_s}\}$, where every $\frac{a_i}{b_i}$ is an irreducible fraction in the open interval $(0, 1)$. Let $\mathcal{F} = \{V_1, \dots, V_m\}$ be a fractional L-intersecting family of subspaces of a vector space V of dimension n over a finite field of size q. Let $t := \max(b_1, b_2, \dots, b_s)$ and $g(t, n) := \frac{2(2t+\ln n)}{\ln(2t+\ln n)}$. Suppose $2g(t, n)\ln(g(t, n)) \leqslant n+2$. Then,*

$$|\mathcal{F}| \leqslant 2g^2(t, n)\ln(g(t, n)) \begin{bmatrix} n \\ s \end{bmatrix}_q$$

Proof. For some β to be chosen later, choose P to be the set $\{p_{\alpha+1}, p_{\alpha+2}, \ldots, p_\beta\}$ where p_l denotes the l^{th} prime number and $p_\alpha \leqslant t < p_{\alpha+1} < p_{\alpha+2} < \cdots < p_\beta$. Let $l\#$ denote the product of all primes less than or equal to l. Thus, $p_l\#$ which is known as the *primorial function*, is the product of the first l primes. It is known that $p_l\# = e^{(1+o(1))l \ln l}$ and $l\# = e^{(1+o(1))l}$. We require the following condition for the set P:

$$\frac{p_\beta\#}{t\#} > n$$

Using the bounds for $p_l\#$ and $l\#$ discussed above, we find that it is sufficient to choose $\beta \geqslant \frac{2(2t+\ln n)}{\ln(2t+\ln n)} := g(t,n)$. Let $\beta = g(t,n)$. From the Prime Number Theorem, it follows that p_β (and so $p_{\alpha+1}, p_{\alpha+2}, \ldots, p_{\beta-1}$ as well) is at most $2g(t,n)\ln(g(t,n))$. We are given that $2p \leqslant 2p_\beta \leqslant n+2$, for every $p \in P$. We apply Lemma 22 with $p = p_{\alpha+1}$ to get

$$|\mathcal{F}| \leqslant |\mathcal{F}_0^{p_{\alpha+1}}| + (p_{\alpha+1}-1)\begin{bmatrix} n \\ s \end{bmatrix}_q$$

Next, apply Lemma 22 on $\mathcal{F}_0^{p_{\alpha+1}}$ with $p = p_{\alpha+2}$ and so on. As argued above, no subspace is left uncovered after we reach p_β. This means,

$$\begin{aligned} |\mathcal{F}| &\leqslant (p_{\alpha+1} + p_{\alpha+2} + \cdots + p_\beta - (\beta-\alpha))\begin{bmatrix} n \\ s \end{bmatrix}_q \\ &< (\beta-\alpha)p_\beta \begin{bmatrix} n \\ s \end{bmatrix}_q \\ &< \beta p_\beta \begin{bmatrix} n \\ s \end{bmatrix}_q \\ &\leqslant 2g^2(t,n)\ln(g(t,n))\begin{bmatrix} n \\ s \end{bmatrix}_q \end{aligned}$$

□

Lemma 24. *Let $L = \{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_s}{b_s}\}$, where every $\frac{a_i}{b_i}$ is an irreducible fraction in the open interval $(0,1)$. Let $\mathcal{F} = \{V_1, \ldots, V_m\}$ be a fractional L-intersecting family of subspaces of a vector space V of dimension n over a finite field of size q. Let $t := \max(b_1, b_2, \ldots, b_s)$ and $g(t,n) := \frac{2(2t+\ln n)}{\ln(2t+\ln n)}$. Then,*

$$|\mathcal{F}| \leqslant 2g^2(t,n)\ln(g(t,n))\begin{bmatrix} n \\ s \end{bmatrix}_q + g(t,n)\sum_{i=1}^{s-1}\begin{bmatrix} n \\ i \end{bmatrix}_q$$

Proof. Let $P = \{p_{\alpha+1}, p_{\alpha+2}, \ldots, p_\beta\}$, where $\beta = g(t,n)$ and $p_\beta \leqslant 2g(t,n)\ln(g(t,n))$. The proof is similar to the proof of Lemma 23. We apply Lemma 22 with $p = p_{\alpha+1}$ to show that

$$|\mathcal{F}| \leqslant |\mathcal{F}_0^{p_{\alpha+1}}| + (p_{\alpha+1}-1)\begin{bmatrix} n \\ s \end{bmatrix}_q + \sum_{i=1}^{s-1}\begin{bmatrix} n \\ i \end{bmatrix}_q$$

Next, we apply Lemma 22 on $\mathcal{F}_0^{p_{\alpha+1}}$ with $p = p_{\alpha+2}$ and so on as shown in the proof of Lemma 23 to get the desired bound.

$$\begin{aligned} |\mathcal{F}| &\leqslant (p_{\alpha+1} + p_{\alpha+2} + \cdots + p_\beta - (\beta-\alpha)) \begin{bmatrix} n \\ s \end{bmatrix}_q + (\beta - \alpha) \sum_{i=1}^{s-1} \begin{bmatrix} n \\ i \end{bmatrix}_q \\ &< (\beta-\alpha)\Big(p_\beta \begin{bmatrix} n \\ s \end{bmatrix}_q + \sum_{i=1}^{s-1} \begin{bmatrix} n \\ i \end{bmatrix}_q \Big) \\ &< \beta\Big(p_\beta \begin{bmatrix} n \\ s \end{bmatrix}_q + \sum_{i=1}^{s-1} \begin{bmatrix} n \\ i \end{bmatrix}_q \Big) \\ &\leqslant 2g^2(t,n)\ln(g(t,n)) \begin{bmatrix} n \\ s \end{bmatrix}_q + g(t,n) \sum_{i=1}^{s-1} \begin{bmatrix} n \\ s \end{bmatrix}_q \end{aligned}$$ □

Since $p_{\alpha+1} > t$, we have $p_{\alpha+1}p_{\alpha+2}\cdots p_\beta > t^{\beta-\alpha}$. This implies that, if $t^{\beta-\alpha} \geqslant n$, then the product of the primes in P will be greater than n as desired. Substituting $\beta - \alpha$ with $\frac{\ln n}{\ln t}$ (and p_β with $2g(t,n)\ln(g(t,n))$) in the second inequality above, we get another upper bound of $|\mathcal{F}| \leqslant 2g(t,n)\frac{\ln(n)\ln(g(t,n))}{\ln t} \begin{bmatrix} n \\ s \end{bmatrix}_q + \frac{\ln n}{\ln t} \sum_{i=1}^{s-1} \begin{bmatrix} n \\ i \end{bmatrix}_q$. We can do a similar substitution for $\beta - \alpha$ in the calculations done at the end of the proof of Lemma 23 to get a similar bound.

Combining all the results in this section, we get Theorem 5

3.2 An improved bound for singleton L

In this section, we improve the upper bound for the size of a fractional L-intersecting family obtained in Theorem 5 for the special case $L = \{\frac{a}{b}\}$, where b is a constant prime. Before we give the proof, below we restate the the statement of Theorem 7.

Statement of Theorem 7: Let $L = \{\frac{a}{b}\}$, where $\frac{a}{b}$ is a positive irreducible fraction less than 1 and b is a prime. Let $\mathcal{F}$ be a fractional L-intersecting family of subspaces of a vector space V of dimension n over a finite field of size q. Then, we have $|\mathcal{F}| \leqslant (b-1)(\begin{bmatrix} n \\ 1 \end{bmatrix}_q + 1)\lceil \frac{\ln n}{\ln b} \rceil + 2$.

Proof. We assume that all the subspaces in the family except possibly one subspace, say W, have a dimension divisible by b. Otherwise, $\mathcal{F}$ cannot satisfy the property of a fractional $\frac{a}{b}$-intersecting family. Let us ignore W in the discussion to follow. For any subspace V_i that is not the zero subspace, let k be the largest power of b that divides $\dim(V_i)$. Then, $\dim(V_i) = rb^{k+1} + jb^k$, for some $1 \leqslant j < b, r \geqslant 0$. Consider the subfamily,

$$\mathcal{F}^{j,k} = \{V_i \ : \ b^k | \dim(V_i),\ b^{k+1} \not| \dim(V_i),\ \dim(V_i) = rb^{k+1} + jb^k \text{ for some } r \geqslant 0, j \in [b-1]\}$$

The subfamily $\mathcal{F}^{j,k}, 1 \leqslant k \leqslant \lceil \frac{\ln n}{\ln b} \rceil, 1 \leqslant j < b$, cover each and every subspace (except the zero subspace and the subspace W) of $\mathcal{F}$ exactly once. We will show that $|\mathcal{F}^{j,k}| \leqslant \begin{bmatrix} n \\ 1 \end{bmatrix}_q + 1$, which when multiplied with the number of values j and k can take will immediately imply the theorem.

Let $m^{j,k} = |\mathcal{F}^{j,k}|$. Let $M^{j,k}$ be an $m^{j,k} \times \begin{bmatrix} n \\ 1 \end{bmatrix}_q$ 0-1 matrix whose rows correspond to the subspaces of $\mathcal{F}^{j,k}$ in any given order, whose columns correspond to the 1-dimensional subspaces of V in any given order, and the $(i\text{-}l)^{th}$ entry is 1 if and only if the i^{th} subspace of $\mathcal{F}^{j,k}$ contains the l^{th} 1-dimensional subspace. Let $N^{j,k} = M^{j,k} \cdot (M^{j,k})^T$. Any diagonal entry $N^{j,k}_{i,i}$ is the number of 1-dimensional subspaces in the i^{th} subspace in $\mathcal{F}^{j,k}$, and an off-diagonal entry $N^{j,k}_{i,l}$ is number of 1-dimensional subspaces common to the i^{th} and l^{th} subspaces of $\mathcal{F}^{j,k}$. In the rest of the proof, to reduce notational clutter, we shall use $G(x, y, z)$ to denote the Gaussian binomial coefficient $\begin{bmatrix} x \\ y \end{bmatrix}_z$. We have

$$
\begin{aligned}
N^{j,k}_{i,i} &= G(r_1 b^{k+1} + jb^k, 1, q) = G(b^{k-1}, 1, q)G(r_1 b^2 + jb, 1, q^{b^{k-1}}), \\
N^{j,k}_{i,l} &= G(r_2 ab^k + jab^{k-1}, 1, q) = G(b^{k-1}, 1, q)G(r_2 ab + ja, 1, q^{b^{k-1}}),
\end{aligned}
$$

for some r_1, r_2 (may be different for different values of i, l). Let $P^{j,k}$ be the matrix over $\mathbb{R}$ obtained by dividing each entry of $N^{j,k}$ by $G(b^{k-1}, 1, q)$.

$$\det(N^{j,k}) = G(b^{k-1}, 1, q)^{m^{j,k}} \det(P^{j,k})$$

We will show that $\det(P^{j,k})$ is non-zero, thereby implying $\det(N^{j,k})$ is also non-zero. Consider $\det(P^{j,k}) \pmod{G(b, 1, q^{b^{k-1}})}$.

$$
\begin{aligned}
P^{j,k}_{i,i} &\equiv G(r_1 b^2 + jb, 1, q^{b^{k-1}}) \pmod{G(b, 1, q^{b^{k-1}})} \equiv 0 \pmod{G(b, 1, q^{b^{k-1}})}, \\
P^{j,k}_{i,l} &\equiv G(r_2 ab + ja, 1, q^{b^{k-1}}) \pmod{G(b, 1, q^{b^{k-1}})} \equiv G(r_3, 1, q^{b^{k-1}}) \pmod{G(b, 1, q^{b^{k-1}})},
\end{aligned}
$$

where $r_3 = ja \mod b$ and $1 \leqslant r_3 \leqslant b-1$ (since $j < b, a < b$, and b is a prime, we have $1 \leqslant r_3 \leqslant b-1$). We know that the determinant of an $r \times r$ matrix where diagonal entries are 0 and off-diagonal entries are all 1 is $(-1)^{r-1}(r-1)$.

$$\det(P^{j,k}) \equiv (G(r_3, 1, q^{b^{k-1}}))^{m^{j,k}} (-1)^{m^{j,k}-1} (m^{j,k} - 1) \pmod{G(b, 1, q^{b^{k-1}})}$$

Let $Q^{j,k}$ be the matrix formed by taking all but the last row and the last column of $P^{j,k}$.

$$\det(Q^{j,k}) \equiv (G(r_3, 1, q^{b^{k-1}}))^{m^{j,k}-1} (-1)^{m^{j,k}-2} (m^{j,k} - 2) \pmod{G(b, 1, q^{b^{k-1}})}$$

We will now show that one of $\det(P^{j,k})$ or $\det(Q^{j,k})$ is non-zero $\pmod{G(b, 1, q^{b^{k-1}})}$ and therefore non-zero in $\mathbb{R}$. First, we show that $G(r_3, 1, q^{b^{k-1}})^{m^{j,k}}$ is not divisible by $G(b, 1, q^{b^{k-1}})$. Suppose $s_3 \equiv r_3^{-1} \pmod{b}$.

$$
\begin{aligned}
G(r_3, 1, q^{b^{k-1}})^{m^{j,k}} G(s_3, 1, q^{r_3 b^{k-1}})^{m^{j,k}} &= G(r_3 s_3, 1, q^{b^{k-1}})^{m^{j,k}} \\
G(r_3 s_3, 1, q^{b^{k-1}})^{m^{j,k}} \equiv G(1, 1, q^{b^{k-1}})^{m^{j,k}} &\pmod{G(b, 1, q^{b^{k-1}})} \equiv 1 \pmod{G(b, 1, q^{b^{k-1}})}
\end{aligned}
$$

Therefore, $G(r_3, 1, q^{b^{k-1}})^{m^{j,k}}$ is invertible modulo $G(b, 1, q^{b^{k-1}})$, and hence the former is not divisible by the latter. Suppose $G(r_3, 1, q^{b^{k-1}})^{m^{j,k}}(-1)^{m^{j,k}-1}(m^{j,k} - 1)$ is divisible by $G(b, 1, q^{b^{k-1}})$. We may ignore $(-1)^{m^{j,k}-1}$ for divisibility purpose. Then, there must be a product of prime powers that is equal to $(m^{j,k} - 1)$ multiplied by $G(r_3, 1, q^{b^{k-1}})^{m^{j,k}}$ such that this product is divisible by $G(b, 1, q^{b^{k-1}})$. Observe that, $G(r_3, 1, q^{b^{k-1}})^{m^{j,k}-1}$ has only lesser powers of the same primes, and $m^{j,k} - 1$ and $m^{j,k} - 2$ cannot have any prime in common. So, the product $G(r_3, 1, q^{b^{k-1}})^{m^{j,k}-1}(m^{j,k} - 2)$ cannot be divisible by $G(b, 1, q^{b^{k-1}})$, which is what we wanted to prove.

Therefore, either $P^{j,k}$ or $Q^{j,k}$ is a full rank matrix, or $rank(P^{j,k}) \geqslant m^{j,k} - 1$. Being a non-zero multiple of $P^{j,k}$, $rank(N^{j,k}) \geqslant m^{j,k} - 1$. But we know that $rank(AB) \leqslant \min(rank(A), rank(B))$, for any two matrices A, B.

$$\begin{aligned} m^{j,k} - 1 \leqslant rank(N^{j,k}) &\leqslant \min(rank(M^{j,k}), rank((M^{j,k})^T)) \\ &= rank(M^{j,k}) \\ &\leqslant G(n, 1, q) \end{aligned}$$

Or, $m^{j,k} \leqslant G(n, 1, q) + 1$, as required. It follows that,

$$|\mathcal{F}| = m \leqslant 2 + \sum_{\substack{1 \leqslant k \leqslant \lceil \frac{\ln n}{\ln b} \rceil \\ 1 \leqslant j < b}} m^{j,k} \leqslant (b-1)(G(n, 1, q) + 1) \left\lceil \frac{\ln n}{\ln b} \right\rceil + 2. \qquad \square$$

4 Concluding remarks

In Theorem 3, for $|\mathcal{F}|$ to be at most $N(n, s, r, q)$, one of the necessary conditions is $r(s - r + 1) \leqslant b - 1$. When $r = 1$, this condition is always true as $L \subseteq \{0, 1, \ldots, b - 1\}$. However, when $r \geqslant 2$, it is not the case. Would it be possible to get the same upper bound for $|\mathcal{F}|$ without having to satisfy such a strong necessary condition? Another interesting question concerning Theorem 3 is regarding its tightness. From Example 2, we know that Theorem 3 is tight when $r = 1$. However, since Theorem 3 requires the sets K and L to be disjoint it is not possible to extend the construction in Example 2 to obtain a tight example for the case $r \geqslant 2$. Further, we know of no other tight example for this case. Therefore, we are not clear whether Theorem 3 is tight when $r \geqslant 2$.

We believe that the upper bounds given by Theorems 5 and 7 are not tight. Proving tight upper bounds in both the scenarios is a question that is obviously interesting. One possible approach to try would be to answer the following simpler question. Consider the case when $L = \{\frac{1}{2}\}$. We call such a family a *bisection-closed family* of subspaces. Let $\mathcal{F}$ be a bisection closed family of subspaces of a vector space V of dimension n over a finite field of size q. From Theorem 7, we know that $|\mathcal{F}| \leqslant (\begin{bmatrix} n \\ 1 \end{bmatrix}_q + 1) \log_2 n + 2$. We believe that $|\mathcal{F}| \leqslant c \begin{bmatrix} n \\ 1 \end{bmatrix}_q$, where c is a constant. Example 8 gives a 'trivial' bisection-closed family of size $\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q$ where every subspace contains the vector v_1. It would be interesting to look for non-trivial examples of large bisection-closed families.

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