Edge Isoperimetric Inequalities for Powers of the Hypercube

Cyrus Rashtchian
Dept. of Computer Science & Engineering
UC San Diego
La Jolla, CA, U.S.A.
crashtchian@eng.ucsd.edu

William Raynaud
School of Mathematical Sciences
Queen Mary University of London
London, England, U.K.
wgraynaud@gmail.com

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Abstract

For positive integers n and r, we let Q_n^r denote the rth power of the n-dimensional discrete hypercube graph, i.e., the graph with vertex-set $\{0,1\}^n$, where two 0-1 vectors are joined if they are Hamming distance at most r apart. We study edge isoperimetric inequalities for this graph. Harper, Bernstein, Lindsey and Hart proved a best-possible edge isoperimetric inequality for this graph in the case $r = 1$. For each $r \geqslant 2$, we obtain an edge isoperimetric inequality for Q_n^r; our inequality is tight up to a constant factor depending only upon r. Our techniques also yield an edge isoperimetric inequality for the 'Kleitman-West graph' (the graph whose vertices are all the k-element subsets of $\{1, 2, \ldots, n\}$, where two k-element sets have an edge between them if they have symmetric difference of size two); this inequality is sharp up to a factor of $2 + o(1)$ for sets of size $\binom{n-s}{k-s}$, where $k = o(n)$ and $s \in \mathbb{N}$.

Mathematics Subject Classifications: 05C70, 05D05

1 Introduction

Isoperimetric questions are classical objects of study in mathematics. In general, they ask for the minimum possible 'boundary-size' of a set of a given 'size', where the exact meaning of these words varies according to the problem. A classical example of an isoperimetric problem is to minimise the perimeter among all shapes in the plane with unit area. The solution to this problem was 'known' to the Ancient Greeks, but the first rigorous proof was given by Weierstrass in a series of lectures in Berlin in the 1870s.

In the last fifty years, there has been a great deal of interest in *discrete* isoperimetric inequalities. These deal with the boundaries of sets of vertices in graphs. If $G = (V, E)$ is a graph, and $A \subset V(G)$ is a subset of vertices of G, the *edge boundary* of A consists of the set of edges of G which join a vertex in A to a vertex in $V(G) \setminus A$; it is denoted by

$\partial_G(A)$, or by ∂A when the graph G is understood. The *edge isoperimetric problem for G* asks for a determination of $\min\{|\partial A| : A \subset V(G), |A| = m\}$, for each integer m.

If $G = (V, E)$ is a graph and $A \subset V(G)$, we write $e_G(A)$ for the number of edges of G induced by A, i.e., the number of edges of G that join two vertices in A. We remark that if G is a regular graph, then the edge isoperimetric problem for G is equivalent to finding the maximum possible number of edges induced by a set of given size. Indeed, if G is a d-regular graph, then

$$2e_G(A) + |\partial A| = d|A| \tag{1}$$

for all $A \subset V(G)$.

An important example of a discrete isoperimetric problem is the edge isoperimetric problem for the Hamming graph Q_n of the n-dimensional hypercube. We define Q_n to be the graph with vertex-set $\{0,1\}^n$, where two 0-1 vectors are adjacent if they differ in exactly one coordinate. This isoperimetric problem has numerous applications, both to other problems in mathematics, and in other areas such as distributed algorithms [5, 29], communication complexity [13], network science [7] and game theory [16].

The edge isoperimetric problem for Q_n has been solved by Harper [13], Lindsey [26], Bernstein [6] and Hart [16]. Let us describe the solution. The *binary ordering* on $\{0,1\}^n$ is defined by $x < y$ if and only if $\sum_{i=1}^n 2^{i-1}x_i < \sum_{i=1}^n 2^{i-1}y_i$. If $m \leqslant 2^n$, the *initial segment of the binary ordering on $\{0,1\}^n$ of size m* is simply the subset of $\{0,1\}^n$ consisting of the m smallest elements of $\{0,1\}^n$ with respect to the binary ordering. Note that if $m = 2^d$ for some $d \in \mathbb{N}$, then the initial segment of the binary ordering on $\{0,1\}^n$ of size m is the d-dimensional subcube $\{x \in \{0,1\}^n : x_i = 0 \ \forall i > d\}$.

Harper, Bernstein, Lindsey and Hart proved the following.

Theorem 1 (The edge isoperimetric inequality for $\boldsymbol{Q_n}$)**.** *If $\mathcal{A} \subset \{0,1\}^n$, then $|\partial\mathcal{A}| \geqslant |\partial\mathcal{B}|$, where $\mathcal{B} \subset \{0,1\}^n$ is the initial segment of the binary ordering of size $|\mathcal{A}|$.*

In particular, it follows from Theorem 1 that the minimum edge-boundary of a set of size 2^d is attained by a d-dimensional subcube, for any $d \in \mathbb{N}$. As another consequence, the above theorem implies that $e_{Q_n}(\mathcal{A}) \leqslant \frac{1}{2}|\mathcal{A}|\log_2|\mathcal{A}|$ for all $\mathcal{A} \subset \{0,1\}^n$.

For background on other discrete isoperimetric inequalities, we refer the reader to the surveys of Bezrukov [7] and of Leader [25].

In this paper, we consider the edge isoperimetric problem for *powers* of the hypercube. If $r, n \in \mathbb{N}$, we let Q_n^r denote the rth power of Q_n, that is, the graph with vertex-set $\{0,1\}^n$, where two distinct 0-1 vectors are joined by an edge if they differ in at most r coordinates. Writing $[n] := \{1, 2, \ldots, n\}$, we may identify $\{0,1\}^n$ with the power-set $\mathcal{P}([n])$ via the natural bijection $x \leftrightarrow \{i \in [n] : x_i = 1\}$. By doing so, we may alternatively view Q_n^r as the graph with vertex-set $\mathcal{P}([n])$, where two distinct subsets of $[n]$ are joined if their symmetric difference has size at most r. As usual, the *Hamming weight* of a vector $x \in \{0,1\}^n$ is its number of 1's; if $x, y \in \{0,1\}^n$, the *Hamming distance* between x and y is the number of coordinates on which they differ. Hence, two 0-1 vectors are adjacent in Q_n^r if and only if they are Hamming distance at most r apart.

Note that Q_n^r is a regular graph, so by (1), the edge isoperimetric problem for Q_n^r is equivalent to finding the maximum number of edges of Q_n^r induced by a set of given size.

In other words, it is equivalent to determining

$$D(m,n,r) := \max\{e_{Q_n^r}(\mathcal{A}) :\ \mathcal{A} \subset \{0,1\}^n,\ |\mathcal{A}| = m\},$$

i.e. the maximum possible number of pairs of vectors at Hamming distance r or less, among a set of m vectors in $\{0,1\}^n$, for each $(m,n,r) \in \mathbb{N}^3$. We remark that, since Q_n^r is regular of degree $\sum_{j=1}^{r} \binom{n}{j}$, one has the trivial upper bound

$$D(m,n,r) \leqslant \tfrac{1}{2} m \sum_{j=1}^{r} \binom{n}{j} \qquad \forall m,n,r \in \mathbb{N}. \tag{2}$$

In the light of Theorem 1, which gives a complete answer to the isoperimetric problem for Q_n^r in the case $r = 1$, it is natural to ask whether, for each $n \geqslant r \geqslant 2$, there exists an ordering of the vertices of $\{0,1\}^n$ such that initial segments of this ordering minimize the edge-boundary in Q_n^r, over all sets of the same size. Unfortunately, this is false even for $r = 2$. Indeed, this is easy to check when $r = 2$ and $n = 4$, in which case the optimal isoperimetric sets of size 5 are precisely the Hamming balls of radius 1, whereas an optimal set of size 7 must be a 3-dimensional subcube minus a point, which contains no Hamming ball of radius 1. Hence, the problem for $r \geqslant 2$ is somewhat harder than in the case $r = 1$. Still, as we shall see, reasonably good bounds can be obtained in many cases.

The problem of determining (or bounding) $D(m,n,r)$ was considered by Kahn, Kalai and Linial in [18]. For half-sized sets, they solve the problem completely, proving that

$$D(2^{n-1},n,r) = 2^{n-2} \sum_{j=1}^{r} \binom{n-1}{j} \qquad \forall r,n \in \mathbb{N}. \tag{3}$$

(For odd r, the extremal sets for (3) are precisely the $(n-1)$-dimensional subcubes; for even r, the set of all vectors of even Hamming weight is also extremal.) Kahn, Kalai and Linial also observe that if $(r/n)\log(2^n/m) = o(1)$, then the trivial upper bound (2) is asymptotically sharp, i.e.

$$D(m,n,r) = (1-o(1))\tfrac{1}{2} \cdot m \sum_{j=1}^{r} \binom{n}{j};$$

this can be seen by considering the initial segment of the binary ordering on $\{0,1\}^n$ with size m — for example a subcube, if m is a power of 2. Finally, they observe that Kleitman's diametric theorem [24] implies that if m is 'very' small, then the 'other' trivial upper bound $D(m,n,r) \leqslant \binom{m}{2}$ is sharp. In particular, for even values of r we know that $D(m,n,r) = \binom{m}{2}$ if and only if $m \leqslant \sum_{j=0}^{r/2} \binom{n}{j}$. In this case, one may consider an m-element subset of a Hamming ball of radius $r/2$, which has diameter at most r. A similar result for small sets and odd r holds as well.

It is also natural to consider the edge isoperimetric problem for the subgraph of Q_n^r induced by the binary vectors of Hamming weight k, or equivalently the graph with vertex-set $\binom{[n]}{k}$ where two k-sets are joined if their symmetric difference has size at most r. In the case $r = 2$, this graph is called the 'Kleitman-West graph', and the edge isoperimetric problem has been called the 'Kleitman-West problem' (see e.g. [14]). An elegant conjecture of Kleitman (as to the complete solution of the latter edge isoperimetric problem for all k and all vertex-set sizes) was disproved by Ahlswede and Cai [1]; only for $k \leqslant 2$ is a complete solution known [2, 3]. Related results have been obtained by Ahlswede and Katona [3] and Das, Gan and Sudakov [10] (Theorem 1.8 in the latter paper implies a solution to the Kleitman-West problem for certain large values of n, for each fixed k). Harper attempted to resolve the edge isoperimetric problem in this case via a continuous relaxation [14]. Unfortunately, Harper's argument works only in certain special cases, and he later demoted his claim to a conjecture [15].

1.1 Our results

We obtain the following bounds on $D(m, n, r)$. For brevity, we state our theorems in terms of the function $\ell = \ell(m) = \min\left\{\left\lceil \frac{2\log m}{\log n - \log\log m}\right\rceil, \lfloor \log m \rfloor\right\}$. All logs are base two. Our results are only novel when the minimum for ℓ is achieved by the first term. This is case when m and n satisfy $\frac{2\log m}{\log n - \log\log m} \leqslant \log m$, or in other words, when $m \leqslant 2^{n/4}$. We introduce the ℓ notation here since we use it in several places in our proofs.

Theorem 2. *Let $m, n, t \in \mathbb{N}$ with $2^t \leqslant m \leqslant 2^n$. Then*

$$D(m, n, 2t) \;\leqslant\; \left(\frac{8e}{t}\right)^{2t} \cdot (n \cdot \ell)^t \cdot m.$$

Theorem 3. *Let $m, n, t \in \mathbb{N}$ with $2^t \leqslant m \leqslant 2^n$. Then*

$$D(m, n, 2t+1) \;\leqslant\; \left(\frac{16e}{2t+1}\right)^{2t+1} \cdot (n \cdot \ell)^t \cdot m \cdot \log m.$$

The two theorems above are tight up to a constant factor depending on t, viz., a factor of $\exp(\Theta(t))$; see below for details. In the case $r = 2$, we prove a sharper bound (Theorem 8), which implies a new bound for the Kleitman-West problem (Theorem 11). Determining the optimal solution to the isoperimetric problem for all vertex-set-sizes remains a challenging open problem, one which seems beyond the reach of our techniques. As mentioned above, even the restriction to k-sets and $r = 2$ is open for $k \geqslant 3$, that is, the Kleitman-West problem remains unsolved.

Tightness. For fixed $t \in \mathbb{N}$, Theorem 2 is sharp up to a factor of $\exp(\Theta(t))$, as can be seen by taking $\mathcal{A} = [n]^{(\leqslant k)}$, i.e., a Hamming ball. In fact, this example motivates our definition of ℓ above, capturing the way $e_{Q_n^{2t}}(\mathcal{A})$ scales as a function of $|\mathcal{A}|$.

Theorem 3 is also sharp up to a factor of $\exp(\Theta(t))$, as can be seen by considering

$$\mathcal{A}_{k,t} = \{x \subseteq [n] \;:\; |x \cap \{k - t + 1, \dots, n\}| \leqslant t\}.$$

When $n \geqslant k \geqslant t$, we have

$$|\mathcal{A}_{k,t}| = 2^{k-t} \sum_{i=0}^{t} \binom{n-k+t}{i}.$$

Denoting $\mathcal{A} = \mathcal{A}_{k,t}$, we sketch the calculations for $k = \Theta(\log n)$ and for fixed $t \in \mathbb{N}$. Note that for this parameter range, $\log |\mathcal{A}| = \Theta_t(\log n) = \Theta_t(k)$, and hence, $\ell = \Theta_t(1)$. We claim that there are $\Omega_t(1)n^t|\mathcal{A}|\log|\mathcal{A}|$ pairs with Hamming distance $2t+1$. Given our assumptions on k, t, this implies that $D(m, n, 2t+1) \geqslant \Omega_t(1)(n\ell)^t m \log m$ for sets of size $m = 2^{\Theta(\log n)}$. We count pairs $x, y \in \mathcal{A}$ with $|x\Delta y| = 2t+1$ and $|(x\setminus y)\cap\{k-t+1, \ldots, n\}| = |(y \setminus x) \cap \{k-t+1, \ldots, n\}| = t$. There are $\binom{n-k}{t}\binom{n-k+t}{t}$ ways to satisfy this equality, and doing so incurs a Hamming distance of $2t$ restricted to $\{k-t+1, \ldots, n\}$. Then, if x has j ones in positions $\{1, \ldots, k-t\}$, changing one of these ones to a zero, i.e., $|(x \setminus y) \cap [k-t]| = 1$, leads to $|x\Delta y| = 2t+1$. Hence, the number of such $\{x, y\}$ pairs is

$$\begin{aligned}\binom{n-k}{t}\binom{n-k+t}{t} \cdot \sum_{j=0}^{k-t}\binom{k-t}{j} j &= \binom{n-k}{t}\binom{n-k+t}{t} \cdot 2^{k-t-1}(k-t) \\ &= \Omega_t(1) \cdot (n-k)^t|\mathcal{A}|(k-t) \\ &= \Omega_t(1) \cdot n^t|\mathcal{A}|k \\ &= \Omega_t(1) \cdot n^t|\mathcal{A}|\log|\mathcal{A}|.\end{aligned}$$

Independent Work. Kirshner and Samorodnitsky [23] independently obtained isoperimetric results similar to those proved in this paper. They use very different methods, and we briefly sketch their results here. For any function $f : \{0,1\}^n \to \mathbb{R}$ and $p \geqslant 1$, as usual we define the *p-norm* of f by $\|f\|_p = (\mathbb{E}_x[|f(x)|^p])^{1/p}$, where the expectation is over a uniformly random $x \in \{0,1\}^n$. Let $H(\cdot)$ be the binary entropy function (i.e., for $q \in (0,1)$ we let $H(q) := -q\log_2(q) - (1-q)\log_2(1-q)$), and let $\psi(p, t)$ be the function on $[2, \infty) \times [0, 1/2]$ defined by

$$\psi(p,t) = (p-1) + \log_2((1-\delta)^p + \delta^p) - \frac{p}{2}H(t) - pt\log_2(1-2\delta),$$

where δ is determined by $t = (\frac{1}{2} - \delta) \cdot \frac{(1-\delta)^{p-1} - \delta^{p-1}}{(1-\delta)^p + \delta^p}$. Kirshner and Samorodnitsky show that for $p \geqslant 2$ and $0 \leqslant s \leqslant \frac{n}{2}$, and for a homogeneous polynomial f of degree s on $\{0,1\}^n$ we have

$$\frac{\|f\|_p}{\|f\|_2} \leqslant 2^{\psi(p,s/n)\cdot\frac{n}{p}}.$$

Furthermore, they show that in a well-defined sense this inequality is 'nearly tight' if f is the Krawchouk polynomial (the Fourier transform of the characteristic function of a Hamming sphere). Kirshner and Samorodnitsky then show that these results imply for each $0 \leqslant s \leqslant n/2$ and $1 \leqslant r \leqslant 2s(1-\frac{s}{n})$ that

$$D\left(\sum_{t=0}^{s}\binom{n}{t}, n, r\right) \leqslant \left(\sum_{t=0}^{s}\binom{n}{t}\right) \cdot 2^{H\left(\frac{r}{2s}\right)\cdot s + H\left(\frac{r}{2(n-s)}\right)\cdot(n-s)}. \tag{4}$$

For odd r this upper bound is tight up to a factor of $O\left(\sqrt{\frac{n-s}{s}} \cdot r\right)$. We can compare this to one of the main theorems of this paper, Theorem 3, which is tight up to a factor

$\exp(\Theta(r))$. For fixed s, Theorem 3 is stronger for $r < \frac{1}{2}\log n$ and n sufficiently large. However, the isoperimetric bounds achieved by Kirshner and Samorodnitsky for even r, which are tight up to a factor of $O(r)$, improve upon the second main theorem of the paper, Theorem 2, which is only tight up to $\exp(\Theta(r))$. Applying the upper bound Eq. (4) to the Kleitman-West graph ($r = 2$), we see that their result implies a bound that is tight up to a factor of $2e^2 \approx 14.778$. This is weaker than our upper bound for $r = 2$ in Theorem 11, which is tight up to $2 + o(1)$; see Section 2, where Theorem 11 follows from Theorem 8.

Subsequent Work. In the time since the submission of this paper, Theorem 2 and 3 have been used to obtain new model counting results [27]. Improved bounds on $D(m, n, r)$ have also been proven for large set sizes, $m = \alpha 2^n$ with $\alpha \in (0, 1)$, using probabilistic techniques [30]. In general, there has been much work on discrete isoperimetric inequalities in the Hamming cube [17, 20, 21, 22, 28] and related studies [11, 19].

1.2 Notation and Preliminaries

For subsets $\mathcal{A} \subseteq \{0,1\}^n$, we let $\mathsf{E}_{\leqslant r}(\mathcal{A})$ denote the set of edges in the subgraph of Q_n^r induced by vertices in $\mathcal{A}$, and we write $\mathsf{e}_{\leqslant r}(\mathcal{A}) := |\mathsf{E}_{\leqslant r}(\mathcal{A})|$. In this notation, notice that $D(m, n, r) = \max_{\mathcal{A}:|\mathcal{A}|=m} \mathsf{e}_{\leqslant r}(\mathcal{A})$. Abusing notation slightly, we move freely between $\{0,1\}^n$ and $\mathcal{P}([n])$ via the bijection $x \leftrightarrow \{i \in [n] : x_i = 1\}$. We say $\mathcal{A} \subseteq \{0,1\}^n$ is a *down-set* if $(x \in \mathcal{A},\, y \subseteq x) \Rightarrow y \in \mathcal{A}$. We say $\mathcal{A}$ is *left-compressed* if whenever $1 \leqslant i < j \leqslant n$ and $x \in \mathcal{A}$ with $x \cap \{i, j\} = \{j\}$, we have $(x \cup \{i\}) \setminus \{j\} \in \mathcal{A}$.

Standard compression arguments (cf. [2, 4, 15]) imply the following.

Proposition 4. *Let n, m be positive integers with $m \leqslant 2^n$. Among all subsets $\mathcal{A}$ of $\{0,1\}^n$ of size m, the maximum of $\mathsf{e}_{\leqslant r}(\mathcal{A})$ is attained where $\mathcal{A}$ is a left-compressed down-set.*

Proposition 5. *Let $\mathcal{A} \subseteq \{0,1\}^n$ be a down-set. For every $x \in \mathcal{A}$, we have $|x| \leqslant \lfloor \log |\mathcal{A}| \rfloor$.*

Proof. As $x \in \mathcal{A}$, we also have $y \in \mathcal{A}$ for all $y \subseteq x$. The number of such y is $2^{|x|} \leqslant |\mathcal{A}|$. □

Remark 6. Proposition 4 and Proposition 5 imply $\mathsf{e}_{\leqslant 1}(\mathcal{A}) \leqslant \lfloor \log |\mathcal{A}| \rfloor \cdot |\mathcal{A}|$. Indeed, for a down-set $\mathcal{A}$, we have $\mathsf{e}_{\leqslant 1}(\mathcal{A}) = \sum_{x \in \mathcal{A}} |x| \leqslant |\mathcal{A}| \cdot \lfloor \log |\mathcal{A}| \rfloor$. This approximates, up to a factor of two, the optimal bound $\mathsf{e}_{\leqslant 1}(\mathcal{A}) \leqslant (1/2) \cdot |\mathcal{A}| \cdot \lfloor \log |\mathcal{A}| \rfloor$ mentioned above [6, 13, 16, 26].

We also make use of the following technical result to bound sums of binomial coefficients. The proof of this proposition can be found in the appendix.

Proposition 7. *For all $m \in \mathbb{N} \cup \{0\}, \lambda \in [0, 1), K \in \mathbb{R}^+$ we have for $m \neq 0$*

$$\left(\frac{K}{m}\right)^m + \left(\frac{K}{m+1}\right)^{m+1} \geqslant \left(\frac{K}{m+\lambda}\right)^{m+\lambda},$$

and for $m = 0$, we have $1 + K \geqslant \left(\frac{K}{\lambda}\right)^{\lambda}$.

2 The distance two case

The special case of our theorem for $r = 2$ has a fairly simple proof and a tighter bound.

Theorem 8. *Let $\mathcal{A} \subset \{0,1\}^n$ satisfy $1 \leqslant \log|\mathcal{A}| < n$. Then*

$$\mathsf{e}_{\leqslant 2}(\mathcal{A}) \leqslant n \cdot \ell' \cdot |\mathcal{A}|,$$

where $\ell' := \min\left\{\left\lceil \frac{\log|\mathcal{A}|}{\log n - \log\log|\mathcal{A}|}\right\rceil, \lfloor \log|\mathcal{A}| \rfloor\right\}$.

Using an observation of Ahlswede and Cai [2], we reduce the problem to bounding the "sum of ranks" of elements in $\mathcal{A}$. We provide a proof for completeness. Define the rank of x as

$$\|x\| := \sum_{j\in[n]} jx_j = \sum_{j\in x} j.$$

Lemma 9. *Let $\mathcal{A}$ be a left-compressed down-set. Then, $\mathsf{e}_{\leqslant 2}(\mathcal{A}) = \sum_{x\in\mathcal{A}} \|x\|$.*

Proof. Notice that $\{x, y\} \in \mathsf{E}_{\leqslant 2}(\mathcal{A})$ implies that either $\|y\| < \|x\|$ or vice versa. We fix $x \in \mathcal{A}$ and count y such that $\|y\| < \|x\|$. Assume that $x \neq \emptyset, \{1\}$, or the bound is trivial. We separate the cases $|y| = |x|$ and $|y| < |x|$. In the first case, we count y of the form $y = x \cup \{i\} \setminus \{j\}$, where $i < j$, $j \in x$ and $i \notin x$. The number of such y is exactly

$$\sum_{j\in x}\Big(j - 1 - |\{i \in x : \ i < j\}|\Big) \ = \ \|x\| - \binom{|x|+1}{2}.$$

For the second case, with $|y| < |x|$, there are $\binom{|x|+1}{2}$ choices for y of the form $y = x \setminus \{i, j\}$ or $y = x \setminus \{i\}$, where $i, j \in x$. As we have assumed that $\mathcal{A}$ is a left-compressed down-set, the counted pairs in both cases are in $\mathsf{E}_{\leqslant 2}(\mathcal{A})$. Summing over $x \in \mathcal{A}$ completes the proof. □

To obtain Theorem 8 we use the left-compressedness and down-set conditions on $\mathcal{A}$ to find an upper bound of $\|x\|$ for each $x \in \mathcal{A}$ which depends only on $|\mathcal{A}|$ and n. The theorem then follows from summing these upper bounds over $x \in \mathcal{A}$.

Lemma 10. *Let $\mathcal{A} \subset \{0,1\}^n$ be a left-compressed down-set with $|\mathcal{A}| \geqslant 2$. For any $x \in \mathcal{A}$,*

$$\|x\| \leqslant n \cdot \ell',$$

where $\ell' = \min\left\{\left\lceil \frac{\log|\mathcal{A}|}{\log n - \log\log|\mathcal{A}|}\right\rceil, \lfloor \log|\mathcal{A}| \rfloor\right\}$

Assuming this lemma, we now complete the proof of Theorem 8.

Proof of Theorem 8. Applying Proposition 4, we may assume that $\mathcal{A}$ is a left-compressed down-set. Then, Lemma 9 and Lemma 10 together imply the desired bound:

$$\mathsf{e}_{\leqslant 2}(\mathcal{A}) = \sum_{x\in\mathcal{A}} \|x\| \leqslant n \cdot \ell' \cdot |\mathcal{A}|.$$ □

Approximately solving the Kleitman-West problem. Theorem 8 has the following immediate corollary for the isoperimetric problem on the Kleitman-West graph, i.e., the graph on $\binom{[n]}{k}$ where two k-element sets are joined if they have symmetric difference of size two. For $\mathcal{A} \subset \binom{[n]}{k}$, let $\mathsf{e}(\mathcal{A})$ denote the number of edges of this graph induced by $\mathcal{A}$.

Theorem 11. *Let $\mathcal{A} \subset \binom{[n]}{k}$ be nonempty. Then*

$$\mathsf{e}(\mathcal{A}) \leqslant n \cdot \ell' \cdot |\mathcal{A}|,$$

where $\ell' := \min\left\{ \left\lceil \frac{\log|\mathcal{A}|}{\log n - \log\log|\mathcal{A}|} \right\rceil, \lfloor \log|\mathcal{A}| \rfloor \right\}$.

We remark that Theorem 11 is sharp up to a factor of $2 + o(1)$. This is evidenced by the families defined by $\left\{ x \in \binom{[n]}{k} : \ [s] \subset x \right\}$ for $k = o(n)$ and $s \in \mathbb{N}$.

2.1 Proof of Lemma 10

Proposition 5 implies that $|x| \leqslant \lfloor \log|\mathcal{A}| \rfloor$, and thus, $\|x\| \leqslant n|x| \leqslant n\lfloor \log|\mathcal{A}| \rfloor$. Therefore, we may assume that we are in the case where $\ell' = \lceil \frac{\log|\mathcal{A}|}{\log n - \log\log|\mathcal{A}|} \rceil < \lfloor \log|\mathcal{A}| \rfloor$. We note for later use that since $\ell' = \lceil \frac{\log|\mathcal{A}|}{\log n - \log\log|\mathcal{A}|} \rceil < \lfloor \log|\mathcal{A}| \rfloor$, we have

$$2 < \frac{n}{\log|\mathcal{A}|}. \tag{5}$$

We use the fact that $\mathcal{A}$ is a left-compressed down-set to lower bound the number of $y \in \mathcal{A}$ that are guaranteed in $\mathcal{A}$ by the existence of $x \in \mathcal{A}$. To this end, define $\beta' := \left\lfloor \frac{n\ell'}{\log|\mathcal{A}|} \right\rfloor$, and let $x = x' \cup x''$, where $x' \subseteq \{1, \ldots, \beta'\}$ and $x'' \subseteq \{\beta' + 1, \ldots, n\}$ correspond to the integers in x with values at most β' and at least $\beta' + 1$, respectively (so that $|x| = |x'| + |x''|$). We will show that

$$\|x\| \leqslant \beta'|x'| + n|x''| \leqslant n\ell'.$$

Notice that if $|x''| = 0$, then $\|x\| = \beta'|x'| = \beta'|x| \leqslant \beta' \log|\mathcal{A}| \leqslant n\ell'$, where the inequalities use Proposition 5 and the definition of β'. Thus, we may assume that $|x'| \leqslant |x| - 1$ and $|x''| \geqslant 1$.

Consider $y \in \{0,1\}^n$ of the form $y = y' \cup y''$, where $y' \subseteq x'$, $y'' \subseteq ([\beta'] \setminus x') \cup x''$, and $|y''| \leqslant |x''|$. We claim every y of this form is in $\mathcal{A}$. Indeed, this follows directly from the left-compressed down-set assumption. To count such $y \in \mathcal{A}$, first define $\varepsilon_x \in [0,1)$ as the real number satisfying $2^{|x'|} = |\mathcal{A}|^{\varepsilon_x}$. We will show $|x''| \leqslant (1 - \varepsilon_x)\ell'$. Clearly, there are $2^{|x'|} = |\mathcal{A}|^{\varepsilon_x}$ choices for $y' \subseteq x'$ and

$$\#\text{ of choices for } y'' = \sum_{j=0}^{|x''|} \binom{\beta' + |x''| - |x'|}{j},$$

where the j^{th} term counts y'' with $|y''| = j$. Since the choice of y' is independent of y'', we know that the sum above must be at most $|\mathcal{A}|^{1-\varepsilon_x}$, otherwise we would have guaranteed more than $|\mathcal{A}|$ distinct y in $\mathcal{A}$.

Aiming for a contradiction, we suppose that $|x''| \geqslant \lceil (1-\varepsilon_x)\ell' \rceil$ and $\varepsilon_x \leqslant 1 - 1/\ell'$. It is a standard fact that for $a, b \in \mathbb{N}$ where $a \geqslant b \geqslant 1$ we have $\binom{a}{b} \geqslant \left(\frac{a}{b}\right)^b$. This fact and the assumption $|x''| \geqslant \lceil (1-\varepsilon_x)\ell' \rceil$ imply the lower bound

$$\sum_{j=0}^{|x''|} \binom{\beta' + |x''| - |x'|}{j} \geqslant \left(\frac{\beta' + |x''| - |x'|}{\lceil (1-\varepsilon_x)\ell' \rceil} \right)^{\lceil (1-\varepsilon_x)\ell' \rceil} + \left(\frac{\beta' + |x''| - |x'|}{\lceil (1-\varepsilon_x)\ell' \rceil - 1} \right)^{\lceil (1-\varepsilon_x)\ell' \rceil - 1} \tag{6}$$

$$\geqslant \left(\frac{\beta' + |x''| - |x'|}{(1-\varepsilon_x)\ell'} \right)^{(1-\varepsilon_x)\ell'}, \tag{7}$$

where the final inequality follows by applying Proposition 7.

We note that if $a > 2$ then $\frac{\lfloor a \rfloor}{\log a} \geqslant \frac{2}{\log(3)}$. Using our observation in equation (5) we apply this fact to the definition of β' to see

$$\beta' = \left\lfloor \frac{n\ell'}{\log |\mathcal{A}|} \right\rfloor \geqslant \left\lfloor \frac{n}{\log |\mathcal{A}|} \right\rfloor \left\lceil \frac{\log |\mathcal{A}|}{\log n - \log \log |\mathcal{A}|} \right\rceil \geqslant \frac{2}{\log(3)} \log |\mathcal{A}|. \tag{8}$$

Observe that (8) and the fact $|x'| = \varepsilon_x \log |\mathcal{A}|$ together imply $\beta' - |x'| \geqslant (1 - \frac{\log 3}{2}\varepsilon_x)\beta'$.

We now split into the following cases:

(1) $|x'| \geqslant 4$,

(2) $2 \leqslant |x'| \leqslant 3$,

(3) $|x'| \leqslant 1$.

Case (1): $|x'| \geqslant 4$. We note that $|x'| \geqslant 4$ is equivalent to $\varepsilon_x \log |\mathcal{A}| \geqslant 4$ and this implies $\varepsilon_x > \frac{\log 3}{(2-\log 3)\log |\mathcal{A}|}$, which after rearranging is equivalent to $\frac{2-\log 3}{2}\varepsilon_x > \frac{\log 3}{2 \log |\mathcal{A}|}$. Using inequality (8), and that $1/(1-\varepsilon_x) \geqslant 1$, we see $\frac{2-\log 3}{2(1-\varepsilon_x)}\varepsilon_x > \frac{1}{\beta'}$. Now, by the definition of β', the right hand side of this inequality trivially satisfies

$$\frac{1}{\beta'} \geqslant \frac{\frac{n\ell'}{\log |\mathcal{A}|} - \beta'}{\beta'}, \tag{9}$$

so rearranging we see that

$$\left(\frac{(1 - \frac{\log 3}{2}\varepsilon_x)\beta'}{(1-\varepsilon_x)\ell'} \right) = \left(1 + \frac{2 - \log 3}{2(1-\varepsilon_x)}\varepsilon_x \right) \frac{\beta'}{\ell'} > \frac{n}{\log |\mathcal{A}|}.$$

Using our observation that $\beta' - |x'| \geqslant (1 - \frac{\log 3}{2}\varepsilon_x)\beta'$ we arrive at

$$\frac{\beta' + |x''| - |x'|}{(1-\varepsilon_x)\ell'} > \frac{n}{\log |\mathcal{A}|}.$$

Substituting this into the lower bound (7) we see

$$\sum_{j=0}^{|x''|} \binom{\beta' + |x''| - |x'|}{j} > \left(\frac{n}{\log |\mathcal{A}|} \right)^{(1-\varepsilon_x)\ell'} \geqslant |\mathcal{A}|^{1-\varepsilon_x},$$

giving the required contradiction.

Case (2): $2 \leqslant |x'| \leqslant 3$. As $|x'| \leqslant 3$ we have $|x''| \geqslant 1 \geqslant |x'|/3$, and so

$$\beta' + |x''| - |x'| \geqslant \beta' - 2|x'|/3.$$

We combine this with fact (8) to get $\beta' + |x''| - |x'| \geqslant (1 - \frac{\log 3}{3}\varepsilon_x)\beta'$. Therefore

$$\frac{\beta' + |x''| - |x'|}{(1-\varepsilon_x)\ell'} \geqslant \left(\frac{1 - \frac{\log 3}{3}\varepsilon_x}{1-\varepsilon_x}\right)\frac{\beta'}{\ell'} = \left(1 + \frac{3-\log 3}{3(1-\varepsilon_x)}\varepsilon_x\right)\frac{\beta'}{\ell'}. \tag{10}$$

Now, since $|x'| \geqslant 2$ is equivalent to $\varepsilon_x \log|\mathcal{A}| \geqslant 2$ we see $\varepsilon_x > \frac{3\log 3}{2(3-\log 3)\log|\mathcal{A}|}$ which after rearranging is equivalent to $\frac{3-\log 3}{3}\varepsilon_x > \frac{\log 3}{2\log|\mathcal{A}|}$. Using inequality (8), and that $1/(1-\varepsilon_x) \geqslant 1$, we see $\frac{3-\log 3}{3(1-\varepsilon_x)}\varepsilon_x > \frac{1}{\beta'}$. Now, as in the previous case, we appeal to equation (9) and rearrange to find

$$\left(1 + \frac{3-\log 3}{3(1-\varepsilon_x)}\varepsilon_x\right)\frac{\beta'}{\ell'} > \frac{n}{\log|\mathcal{A}|}.$$

Combining this with the inequality (10) we find again $\frac{\beta'+|x''|-|x'|}{(1-\varepsilon_x)\ell'} > \frac{n}{\log|\mathcal{A}|}$. Substituting this into the lower bound (7) gives the required contradiction.

Case (3): $|x'| \leqslant 1$. Suppose first that $|x'| = 0$, and so $\varepsilon_x = 0$. Then by assumption $|x''| \geqslant \lceil \ell' \rceil$. Hence

$$\begin{aligned}\sum_{j=0}^{|x''|}\binom{\beta' + |x''| - |x'|}{j} &\geqslant \binom{\beta' + |x''|}{\ell'} + \binom{\beta' + |x''|}{\ell' - 1} = \binom{\beta' + |x''| + 1}{\ell'} \\ &\geqslant \left(\frac{\beta' + |x''| + 1}{\ell'}\right)^{\ell'},\end{aligned}$$

and since $\beta' + |x''| + 1 = \left\lfloor \frac{n\ell'}{\log|\mathcal{A}|}\right\rfloor + |x''| + 1 > \frac{n\ell'}{\log|\mathcal{A}|}$ we see that

$$\sum_{j=0}^{|x''|}\binom{\beta' + |x''| - |x'|}{j} > \left(\frac{n}{\log|\mathcal{A}|}\right)^{\ell'} \geqslant |\mathcal{A}|,$$

providing the required contradiction.

Secondly, we suppose that $|x'| = 1 \leqslant |x''|$. In this case, we have

$$\sum_{j=0}^{|x''|}\binom{\beta' + |x''| - |x'|}{j} \geqslant \left(\frac{\beta' + |x''| - |x'|}{(1-\varepsilon_x)\ell'}\right)^{(1-\varepsilon_x)\ell'} \geqslant \left(\frac{\beta'}{(1-\varepsilon_x)\ell'}\right)^{(1-\varepsilon_x)\ell'}.$$

Now $|x'| \geqslant 1$ is equivalent to $\varepsilon_x \log|\mathcal{A}| \geqslant 1$ which implies $\varepsilon_x > \frac{\log 3}{2\log|\mathcal{A}|}$. Using inequality (8) we see $\varepsilon_x > 1/\beta'$, which implies $\frac{\beta'}{1-\varepsilon_x} > \frac{n\ell'}{\log|\mathcal{A}|}$. Thus, if $|x'| = 1 \leqslant |x''|$ then

$$\sum_{j=0}^{|x''|}\binom{\beta' + |x''| - |x'|}{j} > \left(\frac{n}{\log|\mathcal{A}|}\right)^{(1-\varepsilon_x)\ell'} = |\mathcal{A}|^{(1-\varepsilon_x)},$$

again giving a contradiction.

Since in every case we arrive at a contradiction, the assumption $|x''| \geqslant \lceil (1-\varepsilon_x)\ell' \rceil$ is false and so we must have $|x''| \leqslant \lceil (1-\varepsilon_x)\ell' \rceil - 1 < (1-\varepsilon_x)\ell'$, and thus we conclude that

$$\|x\| \;\leqslant\; \beta'|x'| + n|x''| \;=\; \beta'\varepsilon_x \log|\mathcal{A}| + n|x''| \;\leqslant\; \varepsilon_x n\ell' + (1-\varepsilon_x)n\ell' \;=\; n\ell'.$$

3 The general case for even distances

In this section, we prove Theorem 2, which, using the notation defined in Section 1.2, is equivalent to the statement that if $\mathcal{A} \subset \{0,1\}^n$ and $t \in \mathbb{N}$ with $t \leqslant \log|\mathcal{A}|$, then

$$|\mathsf{E}_{\leqslant 2t}(\mathcal{A})| := \mathsf{e}_{\leqslant 2t}(\mathcal{A}) \;\leqslant\; \left(\frac{8e}{t}\right)^{2t} \cdot (n \cdot \ell)^t \cdot |\mathcal{A}|,$$

where

$$\ell = \ell(\mathcal{A}) := \min\left\{ \left\lceil \frac{2\log|\mathcal{A}|}{\log n - \log\log|\mathcal{A}|} \right\rceil, \lfloor \log|\mathcal{A}| \rfloor \right\}.$$

We start with some more notation. For $(b,a) \in \mathbb{Z}^2_{\geqslant 0}$, let

$$\mathsf{E}_{(b,a)}(\mathcal{A}) := \{\{x,y\} \in \mathsf{E}_{\leqslant 2t}(\mathcal{A}) :\; |x \setminus y| = b,\; |y \setminus x| = a\}.$$

and define $\mathsf{e}_{(b,a)}(\mathcal{A}) := |\mathsf{E}_{(b,a)}(\mathcal{A})|$. Letting

$$\mathcal{U} = \{(b,a) \in \mathbb{Z}^2_{\geqslant 0} :\; b \geqslant a \text{ and } b + a \leqslant 2t\},$$

observe that we can decompose $\mathsf{E}_{\leqslant 2t}(\mathcal{A})$ as a disjoint union

$$\mathsf{E}_{\leqslant 2t}(\mathcal{A}) = \bigcup_{(b,a)\in\mathcal{U}} \mathsf{E}_{(b,a)}(\mathcal{A}),$$

and in particular, this implies,

$$\mathsf{e}_{\leqslant 2t}(\mathcal{A}) = \sum_{(b,a)\in\mathcal{U}} \mathsf{e}_{(b,a)}(\mathcal{A}). \tag{11}$$

Our strategy will be to prove upper bounds on $\mathsf{e}_{(b,a)}(\mathcal{A})$, and then combine these to obtain the theorem. We will need a variant of the bound on $|x''|$ from the proof of Lemma 10. In what follows, we express our results using integers $\ell := \ell(\mathcal{A})$ and $\beta := \beta(\mathcal{A})$, defined in the next proposition. We also define $\ell_x := |x \cap \{\beta+1,\ldots,n\}|$ for $x \in \mathcal{A}$. Intuitively, β is the threshold for 'big' elements; ℓ_x is the number of these 'big' elements; and, we will show that $\ell_x \leqslant \ell$.

Proposition 12. *Let $n \geqslant 2$ and $\mathcal{A} \subset \{0,1\}^n$ be a down-set with $|\mathcal{A}| \geqslant 2$. Let*

$$\ell = \min\left\{ \left\lceil \frac{2\log|\mathcal{A}|}{\log n - \log\log|\mathcal{A}|} \right\rceil, \lfloor \log|\mathcal{A}| \rfloor \right\}, \quad \beta = \left\lfloor \left(\frac{n}{\log|\mathcal{A}|}\right)^{1/2} \ell \right\rfloor.$$

For any $x \in \mathcal{A}$, we have the following:
(i) $|x| \cdot \beta \leqslant n\ell$,
(ii) $\beta^2 \leqslant n\ell$,
(iii) $\log^2 |\mathcal{A}| \leqslant \frac{n}{n-1} n\ell$,
(iv) $|x|^2 \leqslant n\ell$,
(v) $\lfloor \log |\mathcal{A}| \rfloor \log |\mathcal{A}| \leqslant n\ell$.

Proof. Parts (i) and (ii) follow immediately from Proposition 5, the fact that $\log |\mathcal{A}| \leqslant n$ and the definitions of β and ℓ.

For part (iii), since $\log(n/\log |\mathcal{A}|) \leqslant n/\log |\mathcal{A}|$ we see that

$$\log^2 |\mathcal{A}| \leqslant \frac{n \log |\mathcal{A}|}{\log(n/\log |\mathcal{A}|)}.$$

Hence, if $\ell = \left\lceil \frac{2\log |\mathcal{A}|}{\log n - \log\log |\mathcal{A}|} \right\rceil$ then $\ell \geqslant \frac{\log |\mathcal{A}|}{\log(n/\log |\mathcal{A}|)}$ and we see the stronger statement $\log^2 |\mathcal{A}| \leqslant n\ell$ holds, and we note this for later. On the other hand, if $\ell = \lfloor \log |\mathcal{A}| \rfloor < \left\lceil \frac{2\log |\mathcal{A}|}{\log n - \log\log |\mathcal{A}|} \right\rceil$, then $n\ell \geqslant n(\log |\mathcal{A}| - 1)$, so it is sufficient to show $\frac{n}{n-1} n(\log |\mathcal{A}| - 1) \geqslant \log^2 |\mathcal{A}|$, which is true if and only if $\frac{n}{n-1} \leqslant \log |\mathcal{A}| \leqslant n$.

Therefore, the only remaining cases to check are when $1 \leqslant \log |\mathcal{A}| < \frac{n}{n-1}$. Under this assumption, $\ell = 1$ and $\log^2 |\mathcal{A}| < \left(\frac{n}{n-1}\right)^2$, so as $n \geqslant 2$ we see that $\frac{n^2}{n-1} \geqslant \left(\frac{n}{n-1}\right)^2$ which in turn shows $\frac{n}{n-1} n\ell \geqslant \log^2 |\mathcal{A}|$ as required.

For part (iv) let $x \in \mathcal{A}$. We have already seen $|x| \leqslant \lfloor \log |\mathcal{A}| \rfloor$ and $|x| \leqslant n$ is trivial. If $\ell = \left\lceil \frac{2\log |\mathcal{A}|}{\log n - \log\log |\mathcal{A}|} \right\rceil$, we recall that $\log^2 |\mathcal{A}| \leqslant n\ell$, and so $|x|^2 \leqslant n\ell$. On the other hand, if $\ell = \lfloor \log |\mathcal{A}| \rfloor$, then $|x|^2 \leqslant n\ell$. This proves (iv).

Finally, for part (v), again recall that if $\ell = \left\lceil \frac{2\log |\mathcal{A}|}{\log n - \log\log |\mathcal{A}|} \right\rceil$ then $\lfloor \log |\mathcal{A}| \rfloor \log |\mathcal{A}| \leqslant \log^2 |\mathcal{A}| \leqslant n\ell$ and so $\lfloor \log |\mathcal{A}| \rfloor \log |\mathcal{A}| \leqslant n\ell$ follows. On the other hand if $\ell = \lfloor \log |\mathcal{A}| \rfloor$, then as $\log |\mathcal{A}| \leqslant n$ we see $\lfloor \log |\mathcal{A}| \rfloor \log |\mathcal{A}| \leqslant n\ell$, completing the proof of (v). ☐

Lemma 13. *Let $\mathcal{A} \subset \{0,1\}^n$, $|\mathcal{A}| \geqslant 2$ be a left-compressed down-set. If $x \in \mathcal{A}$, then $\ell_x \leqslant \ell$.*

Proof. Proposition 5 implies $|x| \leqslant \lfloor \log |\mathcal{A}| \rfloor$, and clearly $\ell_x \leqslant |x|$, so we may assume that we are in the case when $\ell = \lceil \frac{2\log |\mathcal{A}|}{\log n - \log\log |\mathcal{A}|} \rceil$. Let $x = x' \cup x''$ where $x' \subseteq \{1, \ldots, \beta\}$ and $x'' \subseteq \{\beta+1, \ldots, n\}$. By definition, $|x''| = \ell_x$, and since $\mathcal{A}$ is a down-set, we know that $x'' \in \mathcal{A}$. Suppose $y \subseteq [\beta] \cup x''$ with $|y| \leqslant \ell_x$. As $\mathcal{A}$ is left-compressed and a down-set $y \in \mathcal{A}$. Counting such y we have

$$|\mathcal{A}| \geqslant \sum_{j=0}^{\ell_x} \binom{\beta + \ell_x}{j}. \tag{12}$$

Suppose now, for a contradiction, that $\ell_x \geqslant \ell + 1$. Then clearly

$$\sum_{j=0}^{\ell_x} \binom{\beta + \ell_x}{j} \geqslant \binom{\beta + \ell_x}{\ell} + \binom{\beta + \ell_x}{\ell - 1}.$$

Applying Proposition 7 to this inequality and combining with the lower bound (12) we find that

$$|\mathcal{A}| \geqslant \left(\frac{\beta+\ell_x}{2\log|\mathcal{A}|/\log(n/\log|\mathcal{A}|)}\right)^{2\log|\mathcal{A}|/\log(n/\log|\mathcal{A}|)}. \tag{13}$$

Now, since $\ell_x \geqslant \ell+1$ it is clear that

$$\frac{\beta+\ell_x}{2\log|\mathcal{A}|/\log(n/\log|\mathcal{A}|)} \geqslant \frac{\beta+1+\ell}{2\log|\mathcal{A}|}\cdot\log\left(\frac{n}{\log|\mathcal{A}|}\right),$$

and so by substituting the definition of β into this inequality, we see that

$$\frac{\beta+\ell_x}{2\log|\mathcal{A}|/\log(n/\log|\mathcal{A}|)} \geqslant \frac{\left(\left(\frac{n}{\log|\mathcal{A}|}\right)^{1/2}+1\right)\cdot\ell}{2\log|\mathcal{A}|}\cdot\log\left(\frac{n}{\log|\mathcal{A}|}\right) > \left(\frac{n}{\log|\mathcal{A}|}\right)^{1/2}.$$

From this, and equation (13) we see that

$$|\mathcal{A}| > \left(\frac{n}{\log|\mathcal{A}|}\right)^{\log|\mathcal{A}|/\log(n/\log|\mathcal{A}|)} = |\mathcal{A}|,$$

which is a contradiction. We therefore deduce that $\ell_x \leqslant \ell$. □

In what follows, let $\mathcal{A} \subseteq \{0,1\}^n$ be a left-compressed down-set with $1 \leqslant \log|\mathcal{A}| < n$. Let ℓ, β be defined as in Proposition 12. Recall that $\ell_x = |x \cap \{\beta+1,\dots,n\}|$ equals the number of large elements in $x \in \mathcal{A}$. In our proofs, it will be helpful to order $\{0,1\}^n$ based on ℓ_x. In particular, we upper bound $\mathsf{e}_{(b,a)}(\mathcal{A})$ by partitioning the pairs $\{x,y\} \in \mathsf{E}_{(b,a)}(\mathcal{A})$ into two sets, based on the cases $\ell_y \leqslant \ell_x$ and $\ell_y > \ell_x$. By the definition of $\mathsf{E}_{(b,a)}(\mathcal{A})$, with $b \geqslant a$, we always have $|x| \geqslant |y|$. Ordering based on ℓ_x and ℓ_y enables us to use different arguments in the two cases: when $\ell_y \leqslant \ell_x$, we count pairs based on x, and when $\ell_y > \ell_x$, we count pairs based on y.

3.1 The case $\ell_y \leqslant \ell_x$

Lemma 14. *Let b, a be nonnegative integers with $b \geqslant a$ and $1 \leqslant b+a \leqslant 2\log|\mathcal{A}|$.*

- *If $b+a$ is even, then*

$$|\{\{x,y\} \in \mathsf{E}_{(b,a)}(\mathcal{A}) :\ \ell_y \leqslant \ell_x\}| \leqslant \left(\frac{4\sqrt{2}e}{b+a}\right)^{(b+a)} \cdot (n\cdot\ell)^{(b+a)/2}\cdot|\mathcal{A}|.$$

- *If $b+a$ is odd, then*

$$|\{\{x,y\} \in \mathsf{E}_{(b,a)}(\mathcal{A}) :\ \ell_y \leqslant \ell_x\}| \leqslant \left(\frac{4\sqrt{2}e}{b+a}\right)^{b+a} \cdot (n\cdot\ell)^{(b+a-1)/2}\cdot\log|\mathcal{A}|\cdot|\mathcal{A}|.$$

Proof. Fix $x \in \mathcal{A}$. For each $p \in [a] \cup \{0\}$, we will bound the number of $y \in \{0,1\}^n$ such that $\{x, y\} \in \mathsf{E}_{(b,a)}(\mathcal{A})$ and $\ell_y \leqslant \ell_x$ and $|(y \setminus x) \cap \{\beta+1, \ldots, n\}| = p$. We claim that the number of such y is at most

$$\binom{n-\beta-\ell_x}{p}\binom{\ell_x}{p}\binom{\beta-|x|+\ell_x}{a-p}\binom{|x|}{b-p}. \tag{14}$$

Indeed, the first two factors count the ways to replace p elements in x with p new elements that are larger than β, and the final two factors count the ways to replace $b-p$ elements in x with $a-p$ new elements that are at most β.

Recall that Lemma 13 implies that $\ell_x \leqslant \ell$. Therefore, the quantity in (14) is at most

$$\binom{n}{p}\binom{\ell}{p}\binom{\beta}{a-p}\binom{|x|}{b-p} \leqslant \frac{(n\ell)^p \cdot \beta^{a-p}|x|^{b-p}}{(p!)^2 \cdot (a-p)! \cdot (b-p)!}. \tag{15}$$

We note that for $i, j \geqslant 0$ we have $i^i j^j \geqslant \left(\frac{i+j}{2}\right)^{i+j}$. Indeed, taking logs and dividing by 2, this is equivalent to

$$\tfrac{1}{2}(i \log i + j \log j) \geqslant \tfrac{i+j}{2} \log\left(\tfrac{i+j}{2}\right),$$

which follows from the convexity of the function $z \mapsto z \log z$. Hence, we may bound from below the denominator of the right-hand side of equation (15) as follows:

$$(p!)^2 \cdot (a-p)! \cdot (b-p)! \geqslant \frac{p^{2p} \cdot (a-p)^{a-p} \cdot (b-p)^{b-p}}{e^{b+a}} \quad \text{(by Stirling's approximation)} \tag{16}$$

$$\geqslant \left(\frac{b+a}{4e}\right)^{b+a} \quad \text{(by two applications of } i^i j^j \geqslant \left(\frac{i+j}{2}\right)^{i+j}\text{)}. \tag{17}$$

We now break the bounding of (15) into two cases, based on the parity of $b+a$. For both cases, recall that Proposition 12 implies that $\beta|x| \leqslant n\ell$ and $\beta^2 \leqslant n\ell$ and $|x|^2 \leqslant n\ell$.

The case where $b+a$ is even. We bound the numerator of the RHS of (15) by

$$(n\ell)^p \cdot \beta^{a-p}|x|^{b-p} \leqslant (n\ell)^p \cdot (n\ell)^{(a-p)/2} \cdot (n\ell)^{(b-p)/2} = (n\ell)^{(b+a)/2}.$$

Summing the above bound on (15) over $p \in [a] \cup \{0\}$ and employing (17), we obtain

$$\begin{aligned} |\{y \in \mathcal{A} : \{x,y\} \in \mathsf{E}_{(b,a)}(\mathcal{A}),\ \ell_y \leqslant \ell_x\}| &\leqslant \sum_{p=0}^{a} \frac{(n\ell)^{(b+a)/2}}{(p!)^2 \cdot (b-p)! \cdot (a-p)!} \\ &\leqslant (a+1) \cdot \frac{(n\ell)^{(b+a)/2}(4e)^{(b+a)}}{(b+a)^{b+a}} \\ &\leqslant \frac{(n\ell)^{(b+a)/2}(4\sqrt{2}e)^{(b+a)}}{(b+a)^{b+a}}, \end{aligned}$$

where the last inequality uses the fact that $(a+1) \leqslant (\sqrt{2})^{b+a}$, leading to the factor $(4\sqrt{2}e)^{(b+a)}$.

The case where $b+a$ is odd. In this case, we have $b \geqslant a+1 \geqslant p+1$. We recall that $|x| \leqslant \log|\mathcal{A}|$, and we upper bound the numerator of the RHS of (15) by

$$(n\ell)^p \cdot \beta^{a-p}|x|^{b-p} \leqslant (n\ell)^p \cdot (n\ell)^{(a-p)/2} \cdot (n\ell)^{(b-p-1)/2} \cdot \log|\mathcal{A}| = (n\ell)^{(b+a-1)/2} \cdot \log|\mathcal{A}|.$$

Summing the above bound on (15) over $p \in [a] \cup \{0\}$ and employing (17), we obtain

$$\begin{aligned} |\{y \in \mathcal{A} : \{x,y\} \in \mathsf{E}_{(b,a)}(\mathcal{A}),\ \ell_y \leqslant \ell_x\}| &\leqslant \sum_{p=0}^{a} \frac{(n\ell)^{(b+a-1)/2} \cdot \log|\mathcal{A}|}{(p!)^2 \cdot (b-p)! \cdot (a-p)!} \\ &\leqslant \frac{(n\ell)^{(b+a-1)/2}(4\sqrt{2}e)^{(b+a)} \cdot \log|\mathcal{A}|}{(b+a)^{b+a}}. \end{aligned}$$

In both even and odd cases, summing over $x \in \mathcal{A}$ completes the proof. □

3.2 The case $\ell_y > \ell_x$

Lemma 15. *Let b, a be nonnegative integers with $b \geqslant a$ and $1 \leqslant b+a \leqslant 2\log|\mathcal{A}|$.*

- *If $b+a$ is even, then*

$$|\{\{x,y\} \in \mathsf{E}_{(b,a)}(\mathcal{A}) :\ \ell_y > \ell_x\}| \leqslant \left(\frac{4\sqrt{2}e}{b+a}\right)^{(b+a)} \cdot (n \cdot \ell)^{(b+a-2)/2} \cdot \ell\beta \cdot |\mathcal{A}|.$$

- *If $b+a$ is odd, then*

$$|\{\{x,y\} \in \mathsf{E}_{(b,a)}(\mathcal{A}) :\ \ell_y > \ell_x\}| \leqslant \left(\frac{4\sqrt{2}e}{b+a}\right)^{b+a} \cdot (n \cdot \ell)^{(b+a-1)/2} \cdot \ell \cdot |\mathcal{A}|.$$

Proof. Fix $y \in \mathcal{A}$. For each $p \in [a]$, we will bound the number of $x \in \{0,1\}^n$ such that $\{x,y\} \in \mathsf{E}_{(b,a)}(\mathcal{A})$ and $\ell_y > \ell_x$ and $|(x \setminus y) \cap \{\beta+1,\ldots,n\}| = p-1$. We claim that the number of such x is at most

$$\binom{n-\beta-\ell_y}{p-1}\binom{\ell_y}{p}\binom{\beta-|x|+\ell_y}{b-p+1}\binom{|y|}{a-p}. \tag{18}$$

Indeed, the first two factors count the ways to replace p elements in y with $p-1$ new elements that are larger than β, and the final two factors count the ways to replace $a-p$ elements in y with $b-p+1$ new elements that are at most β.

Recall that Lemma 13 implies that $\ell_y \leqslant \ell$. Thus, the quantity in (18) is at most

$$\binom{n}{p-1}\binom{\ell}{p}\binom{\beta}{b-p+1}\binom{|y|}{a-p} \leqslant \frac{(n\ell)^{p-1} \cdot \ell \cdot \beta^{b-p+1} \cdot |y|^{a-p}}{(p-1)! \cdot p! \cdot (b-p+1)! \cdot (a-p)!}. \tag{19}$$

Similarly to in the proof of Lemma 14 (i.e., by applying Stirling's approximation and the fact $i^i j^j \geqslant (\frac{i+j}{2})^{i+j}$), we lower bound the denominator of the right hand side of (19) as follows.

$$(p-1)! \cdot p! \cdot (b-p+1)! \cdot (a-p)! \geqslant \frac{(p-1)^{p-1} \cdot p^p \cdot (a-p)^{a-p} \cdot (b-p+1)^{b-p+1}}{e^{b+a}} \tag{20}$$

$$\geqslant \left(\frac{b+a}{4e}\right)^{b+a}. \tag{21}$$

Recall that Proposition 12 implies that $\beta^2 \leqslant n\ell$ and $|y|^2 \leqslant n\ell$. We now break into two cases, based on the parity of $b+a$.

The case where $b+a$ is even. Notice that $\ell_y > \ell_x$ and $|x| \geqslant |y|$ implies $a \geqslant 1$ and $b+a \geqslant 2$. We upper bound the numerator of the RHS of (19) by

$$(n\ell)^{p-1} \cdot \ell \cdot \beta^{b-p+1} \cdot |y|^{a-p} \leqslant (n\ell)^{p-1} \cdot \ell \cdot \beta \cdot (n\ell)^{(b-p)/2} \cdot (n\ell)^{(a-p)/2} = (n\ell)^{(b+a-2)/2} \cdot \ell\beta.$$

Summing our bound on (19) over $p \in [a]$, employing (21), and using that $a \leqslant (\sqrt{2})^{b+a}$,

$$\begin{aligned} |\{x \in \mathcal{A} : \{x,y\} \in \mathsf{E}_{(b,a)}(\mathcal{A}),\ \ell_y > \ell_x\}| &\leqslant \sum_{p=1}^{a} \frac{(n\ell)^{(b+a-2)/2} \cdot \ell\beta}{p! \cdot (p-1)! \cdot (b-p+1)! \cdot (a-p)!} \\ &\leqslant \frac{(n\ell)^{(b+a-2)/2} (4\sqrt{2}e)^{(b+a)} \cdot \beta\ell}{(b+a)^{b+a}}. \end{aligned}$$

The case where $b+a$ is odd. Notice that $\ell_y > \ell_x$ and $|x| \geqslant |y|$ implies $a \geqslant 1$, and in this case, $b \geqslant a+1 \geqslant p+1$. We upper bound the RHS of (19) by

$$(n\ell)^{p-1} \cdot \ell \cdot \beta^{b-p+1} \cdot |y|^{a-p} \leqslant (n\ell)^{p-1} \cdot \ell \cdot (n\ell)^{(b-p+1)/2} \cdot (n\ell)^{(a-p)/2} = (n\ell)^{(b+a-1)/2} \cdot \ell.$$

Summing our bound on (19) over $p \in [a]$, employing (21), and using that $a \leqslant (\sqrt{2})^{b+a}$,

$$\begin{aligned} |\{x \in \mathcal{A} : \{x,y\} \in \mathsf{E}_{(b,a)}(\mathcal{A}),\ \ell_y > \ell_x\}| &\leqslant \sum_{p=1}^{a} \frac{(n\ell)^{(b+a-1)/2} \cdot \ell}{p! \cdot (p-1)! \cdot (b-p+1)! \cdot (a-p)!} \\ &\leqslant \frac{(n\ell)^{(b+a-1)/2} (4\sqrt{2}e)^{(b+a)} \cdot \ell}{(b+a)^{b+a}}. \end{aligned}$$

In both even and odd cases, summing over $y \in \mathcal{A}$ completes the proof. □

3.3 Finishing the proof

Proof of Theorem 2. Recall that $\mathcal{U} := \{(b,a) \in \mathbb{Z}^2_{\geqslant 0} :\ b \geqslant a \text{ and } b+a \leqslant 2t\}$. Invoking (11) and using Lemma 14 and Lemma 15, we will upper bound each term in

$$\mathsf{e}_{\leqslant 2t}(\mathcal{A}) = \sum_{(b,a) \in \mathcal{U}} \mathsf{e}_{(b,a)}(\mathcal{A}).$$

For all $(b, a) \in \mathcal{U}$, we claim that

$$\frac{\mathsf{e}_{(b,a)}(\mathcal{A})}{|\mathcal{A}|} \leqslant \left(\frac{4e}{t}\right)^{2t} (n\ell)^t. \tag{22}$$

Assuming that (22) holds, and using that $|\mathcal{U}| \leqslant 2^{2t}$, we have

$$\sum_{(b,a)\in\mathcal{U}} \frac{\mathsf{e}_{(b,a)}(\mathcal{A})}{|\mathcal{A}|} \leqslant |\mathcal{U}| \cdot \left(\frac{4e}{t}\right)^{2t} (n\ell)^t \leqslant \left(\frac{8e}{t}\right)^{2t} (n\ell)^t,$$

which implies the bound in the theorem statement. To prove (22), we will use Proposition 12 and the fact that $t \leqslant \lfloor \log |\mathcal{A}| \rfloor$. When $b + a$ is even, then combining Lemma 14 and Lemma 15 (using $\beta\ell \leqslant n\ell$), we have

$$\begin{aligned}
e_{(b,a)}(\mathcal{A}) &\leqslant \left(\frac{4\sqrt{2}e}{b+a}\right)^{(b+a)} \cdot (n\ell)^{(b+a)/2} \cdot |\mathcal{A}| + \left(\frac{4\sqrt{2}e}{b+a}\right)^{(b+a)} \cdot (n\ell)^{(b+a-2)/2} \cdot \ell\beta \cdot |\mathcal{A}| \\
&= \left(\frac{4\sqrt{2}e}{b+a}\right)^{(b+a)} \cdot |\mathcal{A}| \cdot (n\ell)^{(b+a-2)/2} \cdot (n\ell + \ell\beta) \\
&\leqslant 2 \cdot \left(\frac{4\sqrt{2}e}{b+a}\right)^{(b+a)} \cdot |\mathcal{A}| \cdot (n\ell)^{(b+a)/2} \quad (\text{as } \ell\beta \leqslant n\ell) \\
&\leqslant \left(\frac{8e}{b+a}\right)^{(b+a)} \cdot |\mathcal{A}| \cdot (n\ell)^{(b+a)/2} \quad (\text{as } 2 \leqslant \sqrt{2}^{(b+a)}).
\end{aligned}$$

To verify (22), it suffices to show that the RHS of the above inequality increases with $b+a$ (i.e. that it is maximized over $\mathcal{U}$ at $b + a = 2t$). Indeed, let $k = b + a \geqslant 2$. Then, it suffices to show that

$$\left(\frac{8e}{k-1}\right)^{k-1} \cdot (n \cdot \ell)^{k/2-1/2} \leqslant \left(\frac{8e}{k}\right)^{k} \cdot (n \cdot \ell)^{k/2}. \tag{23}$$

After rearranging, we have

$$\frac{k}{8e}\left(\frac{k}{k-1}\right)^{k-1} \leqslant \frac{k}{8} \leqslant (n\ell)^{1/2},$$

where the first inequality uses that $(\frac{k}{k-1})^{k-1} \leqslant e$, and the second inequality uses that $(k/8)^2 \leqslant t^2 \leqslant \lfloor \log |\mathcal{A}| \rfloor^2 \leqslant n\ell$, which holds by Proposition 12 (v).

Similarly, when $b+a$ is odd, Lemma 14 and Lemma 15 (using $\ell \leqslant \log |\mathcal{A}|$) imply that

$$e_{(b,a)}(\mathcal{A}) \leqslant \left(\frac{4\sqrt{2}e}{b+a}\right)^{(b+a)} (n\ell)^{(b+a-1)/2} \log |\mathcal{A}| \cdot |\mathcal{A}| + \left(\frac{4\sqrt{2}e}{b+a}\right)^{(b+a)} \cdot (n\ell)^{(b+a-1)/2} \ell |\mathcal{A}|$$

$$
\begin{aligned}
&= \left(\frac{4\sqrt{2}e}{b+a}\right)^{(b+a)} \cdot |\mathcal{A}| \cdot (n\ell)^{(b+a-1)/2} \cdot (\log|\mathcal{A}| + \ell) \\
&\leqslant 2 \cdot \left(\frac{4\sqrt{2}e}{b+a}\right)^{(b+a)} \cdot |\mathcal{A}| \cdot (n\ell)^{(b+a-1)/2} \cdot \log|\mathcal{A}| \quad (\text{as } \ell \leqslant \log|\mathcal{A}|) \\
&\leqslant \left(\frac{8e}{b+a}\right)^{(b+a)} \cdot |\mathcal{A}| \cdot (n\ell)^{(b+a-1)/2} \cdot \log|\mathcal{A}| \quad (\text{as } 2 \leqslant \sqrt{2}^{(b+a)}).
\end{aligned}
$$

We claim that $\left(\frac{8e}{b+a}\right)^{(b+a)} \cdot |\mathcal{A}| \cdot (n\ell)^{(b+a-1)/2} \cdot \log|\mathcal{A}|$ is maximised over $\mathcal{U}$ when $b+a = 2t-1$. Indeed, letting $k = b+a \geqslant 2$, we have

$$
\begin{aligned}
&\left(\frac{8e}{k-1}\right)^{k-1} \cdot |\mathcal{A}| \cdot (n\ell)^{(k-2)/2} \cdot \log|\mathcal{A}| \leqslant \left(\frac{8e}{k}\right)^{k} \cdot |\mathcal{A}| \cdot (n\ell)^{(k-1)/2} \cdot \log|\mathcal{A}| \\
\iff &\left(\frac{k}{k-1}\right)^{k-1} \frac{k}{8e} \leqslant (n\ell)^{1/2},
\end{aligned}
$$

where the last inequality holds since $(k/8)^2 \leqslant t^2 \leqslant \lfloor \log|\mathcal{A}| \rfloor^2 \leqslant n\ell$, by Proposition 12 (v) and $\left(\frac{k}{k-1}\right)^{k-1} \leqslant e$. It follows that

$$
\begin{aligned}
e_{(b,a)}(\mathcal{A}) &\leqslant \left(\frac{8e}{2t-1}\right)^{(2t-1)} \cdot |\mathcal{A}| \cdot (n\ell)^{t-1} \cdot \log|\mathcal{A}| \\
&= \left(\frac{4e}{t}\right)^{2t} \cdot |\mathcal{A}| \cdot (n\ell)^{t} \cdot \log|\mathcal{A}| \cdot \left(\frac{2t}{2t-1}\right)^{(2t-1)} \cdot \frac{t}{4e} \cdot \frac{1}{n\ell} \\
&\leqslant \left(\frac{4e}{t}\right)^{2t} \cdot |\mathcal{A}| \cdot (n\ell)^{t} \cdot \log|\mathcal{A}| \cdot \frac{t}{4} \cdot \frac{1}{n\ell} \\
&\leqslant \left(\frac{4e}{t}\right)^{2t} \cdot |\mathcal{A}| \cdot (n\ell)^{t},
\end{aligned}
$$

where the last inequality follows from noting that $\frac{t \log|\mathcal{A}|}{4} \leqslant \lfloor \log|\mathcal{A}| \rfloor \log|\mathcal{A}| \leqslant n\ell$ (which follows from Proposition 12 (v)). □

4 The general case for odd distances

Proof of Theorem 3. The following proof has very similar structure to the proof of Theorem 2, so we omit detailed calculations.

Using the notation defined above, it is required to prove that if $\mathcal{A} \subset \{0,1\}^n$ and $t \in \mathbb{N}$ with $t \leqslant \log|\mathcal{A}|$, then

$$
|\mathsf{E}_{\leqslant 2t+1}(\mathcal{A})| := \mathsf{e}_{\leqslant 2t+1}(\mathcal{A}) \;\leqslant\; \left(\frac{16e}{2t+1}\right)^{2t+1} \cdot (n \cdot \ell)^{t} \cdot |\mathcal{A}| \cdot \log|\mathcal{A}|.
$$

Letting $\mathcal{U}' = \{(b,a) \in \mathbb{Z}^2_{\geqslant 0} : \; b \geqslant a \text{ and } b+a \leqslant 2t+1\}$, observe that

$$\mathsf{e}_{\leqslant 2t+1}(\mathcal{A}) = \sum_{(b,a)\in\mathcal{U}'} \mathsf{e}_{(b,a)}(\mathcal{A}).$$

We will upper bound each term in the above sum. For $(b,a) \in \mathcal{U}'$, we claim that

$$\frac{\mathsf{e}_{(b,a)}(\mathcal{A})}{|\mathcal{A}|} \leqslant 2\left(\frac{4\sqrt{2}e}{2t+1}\right)^{2t+1} (n\ell)^t \cdot \log|\mathcal{A}| \leqslant \left(\frac{8e}{2t+1}\right)^{2t+1} (n\ell)^t \cdot \log|\mathcal{A}|. \tag{24}$$

Assuming that (24) holds, and using that $|\mathcal{U}'| \leqslant 2^{2t+1}$, we have

$$\sum_{(b,a)\in\mathcal{U}} \frac{\mathsf{e}_{(b,a)}(\mathcal{A})}{|\mathcal{A}|} \leqslant |\mathcal{U}'| \cdot \left(\frac{8e}{2t+1}\right)^{2t+1} (n\ell)^t \cdot \log|\mathcal{A}| \;\; \leqslant \;\; \left(\frac{16e}{2t+1}\right)^{2t+1} (n\ell)^t \cdot \log|\mathcal{A}|,$$

which establishes the bound in the theorem statement.

We now prove (24). When $b+a$ is even, then $b+a \leqslant 2t$ and (24) follows from (22). When $b+a$ is odd, then Lemma 14 and Lemma 15 (using $\ell \leqslant \log|\mathcal{A}|$) imply that

$$\frac{\mathsf{e}_{(b,a)}(\mathcal{A})}{|\mathcal{A}|} \leqslant \left(\frac{8e}{b+a}\right)^{b+a} \cdot (n\cdot\ell)^{(b+a-1)/2} \cdot \log|\mathcal{A}| \leqslant \left(\frac{8e}{2t+1}\right)^{2t+1} \cdot (n\cdot\ell)^t \cdot \log|\mathcal{A}|,$$

where we use that the quantity $\left(\frac{8e}{b+a}\right)^{b+a} \cdot (n\cdot\ell)^{(b+a-1)/2}$ increases with $b+a$ (and is maximized over $\mathcal{U}'$ at $b+a = 2t+1$), analogous to the proof of (23). $\square$

5 Some open questions

An immediate open problem is to prove exact edge isoperimetric inequalities for the graphs we consider, i.e., to precisely determine $D(m,n,r)$ for all $(m,n,r) \in \mathbb{N}^3$. Another direction is to prove stability results for Q_n^r with $r \geqslant 2$, generalizing prior results for sets with small edge boundary in the hypercube [12, 21]. It would also be interesting to study graphs on $[k]^n$ with $k \geqslant 3$ with edges induced by other metrics. For example, is it possible to prove edge isoperimetric inequalities for the families of graphs connecting pairs in $[k]^n$ with either ℓ_1-distance at most r or Hamming distance at most r? Bollobás and Leader [8] and Clements and Lindström [9] have solved the respective distance one cases.

References

[1] Rudolf Ahlswede and Ning Cai. A Counterexample to Kleitman's Conjecture Concerning an Edge-Isoperimetric Problem. *Combinatorics, Probability and Computing*, 8(04):301–305, 1999.

[2] Rudolf Ahlswede and Ning Cai. Appendix: On Edge-Isoperimetric Theorems for Uniform Hypergraphs. In *General Theory of Information Transfer and Combinatorics*, pages 979–1005. Springer, 2006.

[3] Rudolf Ahlswede and Gyula Katona. Graphs with Maximal Number of Adjacent Pairs of Edges. *Acta Mathematica Hungarica*, 32(1-2):97–120, 1978.

[4] Ian Anderson. *Combinatorics of Finite Sets*. Courier Corporation, 1987.

[5] Paul Beame and Cyrus Rashtchian. Massively-Parallel Similarity Join, Edge-Isoperimetry, and Distance Correlations on the Hypercube. In *Proc. Twenty-Eighth ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 289–306, 2017.

[6] Arthur J Bernstein. Maximally Connected Arrays on the n-cube. *SIAM J. Applied Mathematics*, 15(6):1485–1489, 1967.

[7] Sergei Bezrukov. Edge Isoperimetric Problems on Graphs. *Graph Theory and Combinatorial Biology*, 7:157–197, 1999.

[8] Béla Bollobás and Imre Leader. Edge-Isoperimetric Inequalities in the Grid. *Combinatorica*, 11(4):299–314, 1991.

[9] George F Clements and Bernt Lindström. A Generalization of a Combinatorial Theorem of Macaulay. *Journal of Combinatorial Theory*, 7(3):230–238, 1969.

[10] Shagnik Das, Wenying Gan, and Benny Sudakov. The Minimum Number of Disjoint Pairs in Set Systems and Related Problems. *Combinatorica*, 36(6):623–660, 2016.

[11] Ronen Eldan and Renan Gross. Concentration on the boolean hypercube via pathwise stochastic analysis. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, pages 208–221, 2020.

[12] David C Ellis, Nathan Keller, and Noam Lifshitz. On the structure of subsets of the discrete cube with small edge boundary. *Discrete Analysis*, 2018.

[13] L. H. Harper. Optimal Assignments of Numbers to Vertices. *Journal of the Society for Industrial and Applied Mathematics*, 12(1):131–135, 1964.

[14] L. H. Harper. On a Problem of Kleitman and West. *Discrete Mathematics*, 93(2):169–182, 1991.

[15] L. H. Harper. *Global Methods for Combinatorial Isoperimetric Problems*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2004.

[16] Sergiu Hart. A Note on the Edges of the n-cube. *Discrete Mathematics*, 14(2):157–163, 1976.

[17] Zilin Jiang and Amir Yehudayoff. An isoperimetric inequality for hamming balls and local expansion in hypercubes. *The Electronic Journal of Combinatorics*, #P1.15, 2022.

[18] J. Kahn, G. Kalai, and N. Linial. The Influence of Variables on Boolean Functions. In *Proceedings of the 29th Annual Symposium on Foundations of Computer Science*, SFCS '88, pages 68–80, Washington, DC, USA, 1988. IEEE Computer Society.

[19] Jeff Kahn and Jinyoung Park. The number of maximal independent sets in the hamming cube. arXiv:1909.04283, 2019.

[20] Jeff Kahn and Jinyoung Park. An isoperimetric inequality for the hamming cube and some consequences. *Proceedings of the American Mathematical Society*, 148(10):4213–4224, 2020.

[21] Peter Keevash and Eoin Long. A stability result for the cube edge isoperimetric inequality. *Journal of Combinatorial Theory, Series A*, 155:360–375, 2018.

[22] Peter Keevash and Eoin Long. Stability for vertex isoperimetry in the cube. *Journal of Combinatorial Theory, Series B*, 145:113–144, 2020.

[23] Naomi Kirshner and Alex Samorodnitsky. A Moment Ratio Bound for Polynomials and Some Extremal Properties of Krawchouk Polynomials and Hamming Spheres. *IEEE Transactions on Information Theory*, 67(6):3509–3541, 2021.

[24] Daniel J. Kleitman. Families of Non-disjoint Subsets. *Journal of Combinatorial Theory*, 1(1):153–155, 1966.

[25] Imre Leader. Discrete Isoperimetric Inequalities. In B. Bollobás and F.K.R. Chung, editors, *Probabilistic Combinatorics and its Applications*. AMS, 1991.

[26] John H Lindsey. Assignment of Numbers to Vertices. *The American Mathematical Monthly*, 71(5):508–516, 1964.

[27] Kuldeep S Meel and S Akshay. Sparse hashing for scalable approximate model counting: theory and practice. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 728–741, 2020.

[28] Michał Przykucki and Alexander Roberts. Vertex-isoperimetric stability in the hypercube. *Journal of Combinatorial Theory, Series A*, 172:105186, 2020.

[29] Anish Das Sarma, Foto N Afrati, Semih Salihoglu, and Jeffrey D Ullman. Upper and Lower Bounds on the Cost of a Map-Reduce Computation. *Proceedings of the VLDB Endowment*, 6(4):277–288, 2013.

[30] Lei Yu. Edge-isoperimetric inequalities and ball-noise stability: Linear programming and probabilistic approaches. *Journal of Combinatorial Theory, Series A*, 188:105583, 2022.

Appendix

Here we provide proof of the technical proposition, Proposition 7. For this we need the following tool.

Proposition 16. *Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ be defined as follows*

$$f(x) = \begin{cases} \left(\frac{x}{m}\right)^m + \left(\frac{x}{m+1}\right)^{m+1} - e^{x/e} & \text{if } x \in [me, (m+1)e), \text{ for some } m \in \mathbb{N}, m \geqslant 1 \\ 1 + x - e^{x/e} & \text{if } x \in [0, e) \end{cases}$$

Then the following hold.

(1) For $x \in [0, e)$, $f(x) \geqslant x/e \geqslant 0$.

(2) For $x \in [e, 2e)$, $f(x) \geqslant \frac{e^2}{4} + (2 - \frac{e}{4})(x - e) \geqslant 0$.

(3) For $m \geqslant 2$ and $x \in [me, (m+1)e)$, we have

$$e^{x/e} - \min\left\{\left(\frac{x}{m}\right)^m, \left(\frac{x}{m+1}\right)^{m+1}\right\} \leqslant \frac{1}{m}\min\left\{\left(\frac{x}{m}\right)^m, \left(\frac{x}{m+1}\right)^{m+1}\right\},$$

from which it immediately follows that

$$f(x) \geqslant \max\left\{\left(\frac{x}{m}\right)^m, \left(\frac{x}{m+1}\right)^{m+1}\right\} - \frac{1}{m}\min\left\{\left(\frac{x}{m}\right)^m, \left(\frac{x}{m+1}\right)^{m+1}\right\} \geqslant 0.$$

Proof. We split our proof into parts for each of the statements.

Part (1). Suppose first that $x \in [0, e)$, so $f(x) = 1 + x - e^{x/e}$. Then $\frac{d^2 f}{dx^2} = -e^{x/e-2} < 0$ and so f is concave in this range. Hence, we have

$$f(x) \geqslant f(0) + \frac{f(e) - f(0)}{e - 0}x = \frac{x}{e},$$

as required.

Part (2). Suppose next that $x \in [e, 2e)$, so that $f(x) = x + \frac{x^2}{4} - e^{x/e}$. We let

$$g(x) = f(x) - (\frac{e^2}{4} + (2 - \frac{e}{4})(x - e)) = (2e - \frac{e^2}{2}) + (-1 + \frac{e}{4})x + \frac{x^2}{4} - e^{x/e},$$

and note the following:

$$g'(x) = (-1 + \frac{e}{4}) + \frac{x}{2} - e^{x/e-1}$$

$$
\begin{aligned}
g''(x) &= \frac{1}{2} - e^{x/e-2} \\
g(e) &= g(2e) = 0.
\end{aligned}
$$

Clearly, $g''(x)$ is decreasing in x and has a unique root at $x = e(2 - \ln(2))$. Therefore $g''(x) > 0$ for $x \in [e, e(2 - \ln(2)))$ and $g''(x) < 0$ for $x \in (e(2 - \ln(2)), 2e)$. We also note that $g'(e) = \frac{3e}{4} - 2 > 0$, $g'(e(2 - \ln(2))) = -1 + \frac{3-2\ln(2)}{4}e > 0$ and $g'(2e) = -1 + \frac{e}{4} < 0$.

As $g''(x) < 0$ for $x \in (e(2-\ln(2)), 2e)$ and $g'(e(2-\ln(2)))g'(2e) < 0$ we see that $g'(x) = 0$ has a unique root in $(e(2 - \ln(2)), 2e)$. In addition, $g''(x) > 0$ for $x \in [e, e(2 - \ln(2)))$ and $g'(e)g'(e(2 - \ln(2))) > 0$ so we see that $g'(x) = 0$ has no solutions in $[e, e(2 - \ln(2))]$. Hence $g(x)$ has a unique maximum in $[e, 2e)$, and no other stationary points. From this, and the fact that $g(e) = g(2e) = 0$ we deduce that $g(x) \geqslant 0$ for all $x \in [e, 2e)$. This shows that

$$
f(x) \geqslant \frac{e^2}{4} + (2 - \frac{e}{4})(x - e)
$$

for $x \in [e, 2e)$, as claimed.

Part (3). Suppose finally that $x \in [me, (m + 1)e)$ for some $2 \leqslant m \in \mathbb{N}$. We now split into two cases: the case $\left(\frac{x}{m}\right)^m \geqslant \left(\frac{x}{m+1}\right)^{m+1}$, and the case $\left(\frac{x}{m}\right)^m < \left(\frac{x}{m+1}\right)^{m+1}$.

Case 1: Suppose first that the former case holds. Then

$$
\begin{aligned}
e^{x/e} - \min\left\{\left(\frac{x}{m}\right)^m, \left(\frac{x}{m+1}\right)^{m+1}\right\} &= e^{x/e} - \left(\frac{x}{m+1}\right)^{m+1} \\
&= -\int_{t=x/e}^{m+1} \left(\frac{x}{t}\right)^t (\ln\left(\frac{x}{t}\right) - 1)dt \\
&= \int_{t=x/e}^{m+1} \left(\frac{x}{t}\right)^t \ln\left(\frac{t}{x/e}\right) dt \\
&\leqslant (m + 1 - x/e) \max_{t\in[x/e,m+1]} \left\{\left(\frac{x}{t}\right)^t \ln\left(\frac{t}{x/e}\right)\right\}.
\end{aligned}
$$

To bound $\max_{t\in[x/e,m+1]} \left\{\left(\frac{x}{t}\right)^t \ln\left(\frac{t}{x/e}\right)\right\}$ we show the maximum is attained at $t = m + 1$. Indeed, differentiating with respect to t we get:

$$
\begin{aligned}
\frac{d}{dt}\left(\left(\frac{x}{t}\right)^t \ln\left(\frac{t}{x/e}\right)\right) &= \left(\frac{x}{t}\right)^t \left(\frac{1}{t} - \ln\left(\frac{t}{x/e}\right)^2\right) \\
&\geqslant \left(\frac{x}{t}\right)^t \left(\frac{1}{m+1} - \left(\ln\left(\frac{m+1}{x/e}\right)\right)^2\right).
\end{aligned}
$$

It is a standard fact that for $y > 0$ we have $\frac{y-1}{y} \leqslant \ln(y) \leqslant y - 1$. Noting that $\frac{m+1}{x/e} > 0$, we apply this fact to see:

$$
\ln\left(\frac{m+1}{x/e}\right) \leqslant \frac{m+1}{x/e} - 1 = \frac{(m+1) - x/e}{x/e} \leqslant e/x.
$$

Hence, we have

$$\frac{d}{dt}\left(\left(\frac{x}{t}\right)^t \ln\left(\frac{t}{x/e}\right)\right) \geqslant \left(\frac{x}{t}\right)^t \left(\frac{1}{m+1} - (e/x)^2\right)$$
$$= \left(\frac{x}{t}\right)^t \left(\frac{(x/e)^2 - (m+1)}{(m+1)(x/e)^2}\right)$$
$$\geqslant \left(\frac{x}{t}\right)^t \left(\frac{m^2 - m - 1}{(m+1)(x/e)^2}\right) \geqslant 0,$$

where the final inequality holds since $m \geqslant 2$. Thus $\left(\frac{x}{t}\right)^t \ln\left(\frac{t}{x/e}\right)$ is increasing on the interval $t \in [x/e, m+1]$, and attains its maximum at $t = m+1$. Therefore, we may bound the integral as follows:

$$\int_{t=x/e}^{m+1} \left(\frac{x}{t}\right)^t \ln\left(\frac{t}{x/e}\right) dt \leqslant (m+1-x/e)\left(\frac{x}{m+1}\right)^{m+1} \ln\left(\frac{m+1}{x/e}\right) \leqslant \left(\frac{x}{m+1}\right)^{m+1} \frac{1}{m}.$$

The final inequality holds as $(m+1-x/e) \leqslant 1$ and $\ln\left(\frac{m+1}{x/e}\right) \leqslant \frac{1}{m}$. The first of these is trivial, and the second can be seen as follows. We define $\varepsilon \in [0,1)$ by $x = (m+\varepsilon)e$, then

$$\ln\left(\frac{m+1}{x/e}\right) = \ln\left(\frac{m+1}{m+\varepsilon}\right) \leqslant \frac{1-\varepsilon}{m+\varepsilon} \leqslant \frac{1}{m}.$$

Hence, we have shown that

$$e^{x/e} - \min\left\{\left(\frac{x}{m}\right)^m, \left(\frac{x}{m+1}\right)^{m+1}\right\} \leqslant \left(\frac{x}{m+1}\right)^{m+1} \frac{1}{m},$$

i.e. that the claim holds in the former case.

Case 2: Suppose secondly that the latter case holds. Then we have

$$e^{x/e} - \min\left\{\left(\frac{x}{m}\right)^m, \left(\frac{x}{m+1}\right)^{m+1}\right\} = e^{x/e} - \left(\frac{x}{m}\right)^m$$
$$= \int_{t=m}^{x/e} \left(\frac{x}{t}\right)^t \ln\left(\frac{x/e}{t}\right) dt$$
$$\leqslant (x/e - m) \max_{t\in[m,x/e]} \left\{\left(\frac{x}{t}\right)^t \ln\left(\frac{x/e}{t}\right)\right\}.$$

To bound $\max_{t\in[m,x/e]}\left\{\left(\frac{x}{t}\right)^t \ln\left(\frac{x/e}{t}\right)\right\}$ we show that the maximum is attained at $t = m$. Differentiating with respect to t we get:

$$\frac{d}{dt}\left(\left(\frac{x}{t}\right)^t \ln\left(\frac{x/e}{t}\right)\right) = \left(\frac{x}{t}\right)^t \left(\ln\left(\frac{x/e}{t}\right)^2 - \frac{1}{t}\right)$$

$$\leqslant \left(\frac{x}{t}\right)^t \left(\left(\ln\left(\frac{x/e}{m}\right)\right)^2 - \frac{1}{x/e}\right).$$

Observe that

$$\ln\left(\frac{x/e}{m}\right) \leqslant \frac{x/e}{m} - 1 = \frac{(x/e)-m}{m} \leqslant \frac{1}{m}.$$

Substituting this bound into the previous equation gives

$$\begin{aligned} \frac{d}{dt}\left(\left(\frac{x}{t}\right)^t \ln\left(\frac{x/e}{t}\right)\right) &\leqslant \left(\frac{x}{t}\right)^t \left(\left(\frac{1}{m}\right)^2 - \frac{1}{x/e}\right) \\ &= \left(\frac{x}{t}\right)^t \left(\frac{x/e - m^2}{m^2(x/e)}\right) \\ &\leqslant \left(\frac{x}{t}\right)^t \left(\frac{m+1-m^2}{m^2(x/e)}\right) \leqslant 0. \end{aligned}$$

(Note that the final inequality holds as $m \geqslant 2$.) Hence, $\left(\frac{x}{t}\right)^t \ln\left(\frac{x/e}{t}\right)$ is non-increasing on the interval $t \in [m, x/e]$, and so attains its maximum at $t = m$. We may bound the integral as follows:

$$\int_{t=m}^{x/e} \left(\frac{x}{t}\right)^t \ln\left(\frac{x/e}{t}\right) dt \leqslant (x/e - m)\left(\frac{x}{m}\right)^m \ln\left(\frac{x/e}{m}\right) \leqslant \left(\frac{x}{m}\right)^m \frac{1}{m}.$$

(Note that the final inequality holds as $((x/e)-m) \leqslant 1$ and $\ln\left(\frac{x/e}{m}\right) \leqslant \frac{1}{m}$. The first of these is trivial, and the second can be seen as follows. We define $\varepsilon \in [0,1)$ by $x = (m+\varepsilon)e$. Then

$$\ln\left(\frac{x/e}{m}\right) = \ln\left(\frac{m+\varepsilon}{m}\right) \leqslant \frac{\varepsilon}{m} \leqslant \frac{1}{m}.)$$

Hence, we have shown that

$$e^{x/e} - \min\left\{\left(\frac{x}{m}\right)^m, \left(\frac{x}{m+1}\right)^{m+1}\right\} \leqslant \left(\frac{x}{m}\right)^m \frac{1}{m},$$

i.e. that the claim holds in the latter case. This completes the proof of the claim. □

We now prove Proposition 7.

Proof of Proposition 7. Fix $m \in \mathbb{N}, K \in \mathbb{R}^+$ and consider $\left(\frac{K}{m+\lambda}\right)^{m+\lambda}$. Differentiating this with respect to λ we find:

$$\frac{d}{d\lambda}\left(\left(\frac{K}{m+\lambda}\right)^{m+\lambda}\right) = \left(\frac{K}{m+\lambda}\right)^{m+\lambda}\left(\ln\left(\frac{K/e}{m+\lambda}\right)\right).$$

The only solution to $\frac{d}{d\lambda}\left(\left(\frac{K}{m+\lambda}\right)^{m+\lambda}\right) = 0$ is $\lambda = \frac{K}{e} - m$.

If $\frac{K}{e} - m < 0$, then for all $\lambda \in [0,1)$ we have $\frac{K/e}{m+\lambda} < \frac{m}{m+\lambda} \leqslant 1$, so the derivative is negative, and the maximum is attained by $\left(\frac{K}{m}\right)^m$, so the claim holds in this case.

If $\frac{K}{e} - m \geqslant 1$, then for all $\lambda \in [0,1)$ we have $\frac{K/e}{m+\lambda} \geqslant \frac{m+1}{m+\lambda} > 1$, so the derivative is positive, and the maximum is attained by $\left(\frac{K}{m+1}\right)^{m+1}$, so the claim holds in this case also.

Finally, suppose that $\frac{K}{e} - m \in [0,1)$. Then the maximum is at $\lambda = \frac{K}{e} - m$, but we appeal to Proposition 16 to get

$$\left(\frac{K}{m}\right)^m + \left(\frac{K}{m+1}\right)^{m+1} - \left(\frac{K}{m+\lambda}\right)^{m+\lambda} = \left(\frac{K}{m}\right)^m + \left(\frac{K}{m+1}\right)^{m+1} - e^{K/e} = f(K) \geqslant 0.$$

This leaves the case $m = 0$, which we resolve similarly. First, we differentiate $(K/\lambda)^\lambda$ with respect to λ to get

$$\frac{d}{d\lambda}\left(\left(\frac{K}{\lambda}\right)^\lambda\right) = \left(\frac{K}{\lambda}\right)^\lambda \left(\ln\left(\frac{K/e}{\lambda}\right)\right),$$

and note that

(1) the derivative has a unique root at $\lambda = K/e$,

(2) the derivative is strictly positive if $\lambda < K/e$,

(3) the derivative is strictly negative if $\lambda > K/e$.

Consequently, if $K/e \geqslant 1$, then $\left(\frac{K}{\lambda}\right)^\lambda \leqslant K$ for all $\lambda \in [0,1)$, so the claim holds. If $0 < K/e < 1$ then $\left(\frac{K}{\lambda}\right)^\lambda \leqslant e^{K/e}$, so by Proposition 16

$$1 + K - \left(\frac{K}{\lambda}\right)^\lambda \geqslant 1 + K - e^{K/e} = f(K) \geqslant 0.$$

This completes the proof. □