Decreasing the maximum average degree by deleting an independent set or a d-degenerate subgraph*

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Abstract

The maximum average degree $\text{mad}(G)$ of a graph G is the maximum over all subgraphs of G, of the average degree of the subgraph. In this paper, we prove that for every G and positive integer k such that $\text{mad}(G) \geqslant k$ there exists $S \subseteq V(G)$ such that $\text{mad}(G - S) \leqslant \text{mad}(G) - k$ and $G[S]$ is $(k-1)$-degenerate. Moreover, such S can be computed in polynomial time. In particular, if G contains at least one edge then there exists an independent set I in G such that $\text{mad}(G - I) \leqslant \text{mad}(G) - 1$ and if G contains a cycle then there exists an induced forest F such that $\text{mad}(G - F) \leqslant \text{mad}(G) - 2$. As a side result, we also obtain a subexponential bound on the diameter of reconfiguration graphs of generalized colourings of graphs with bounded value of their mad.

Mathematics Subject Classifications: 05C07, 05C10, 05C15, 05C20, 05C21, 05C85

1 Introduction

The maximum average degree (abbreviated as mad) of a graph is a heavily studied notion. Multiple results show that a lower or upper bound on mad implies the existence of a particular partition of vertices of G, e.g., [5, 12, 13, 15, 24]. Another class of results considers edge partitions, including [2, 7, 17, 21, 22, 23]. In these directions, particular attention has been paid to planar graphs, where, due to the inequality $(\text{mad}(G)-2)(g(G)-2) < 4$, an upper bound on mad can be inferred from a lower bound on the girth.

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A graph parameter $f(G)$ is called *partitionable* [3, 25, 29] if for every undirected simple graph G and positive real numbers a and b such that $f(G) < a + b$, the vertex set $V(G)$ can be partitioned into A and B so that $f(G[A]) < a$ and $f(G[B]) < b$. It is quite simple to prove that degeneracy [26], maximum degree [25] and treewidth [14] are all partitionable parameters. Hendrey, Norin and Wood asked whether mad is also partitionable as part of the open problems for Barbados workshop [1, Problem #14]. Such a result would agree exactly with or even improve many existing results, for example, the ones mentioned in [15, 24]. We answer this question positively for cases $a = 1$ and $a = 2$. It is a consequence of a following theorem which is the main result of this paper.

Theorem 1. *For every undirected simple graph G and a positive integer k such that $\mathrm{mad}(G) \geqslant k$ there exists $S \subseteq V(G)$ such that $G[S]$ is $(k-1)$-degenerate and $\mathrm{mad}(G-S) \leqslant \mathrm{mad}(G) - k$. Moreover such S can be computed in polynomial time.*

Up to our knowledge this is the first theorem of a kind where we are given a graph with bounded value of its mad where we partition its vertex set into some parts so that their values of mad are smaller, however they need not be bounded by absolute constant. This is opposed to all results where every resulting part induces a forest or is an independent set or has maximum degree 1, etc.

Our results can be applied as a tool for directly deriving many results for some specific sparse graph classes, for example planar graphs with constraints on girth. It seems that our results do not show as much expressive power as it is possible to get on such restrictive graph classes (where arguments specifically adjusted to the researched restricted graph classes can be used), which is a price for deriving them from a more general theorem. However, our results can be seen as a nice way of unifying these results and there are cases where using our results improves the state of the art.

Our results imply a positive answer for the open problem presented in [16] (Problem 2 from the final remarks), which implies a subexponential bound on the diameter of reconfiguration graphs of $(k + 2)$-colourings for graphs G with maximum average degree strictly less than $k + 1$. However, this bound has already been improved in [18] to a polynomial bound depending on the value of $\mathrm{mad}(G)$ in a slightly less general setting. Nevertheless, we are able to get a novel analogous result for reconfiguration graphs of H-colourings, in particular for circular colourings. This line of research is motivated by Cereceda's conjecture [10] that states that reconfiguration graphs of $(k + 2)$-colourings for k-degenerate graphs have a quadratic diameter. Its polynomial version was proved in [8].

The rest of the paper is organized as follows. In Section 2, we introduce a few useful notions. In Section 3, we state our main result and provide its proof. In Section 4, we present our reconfiguration graphs results. In Section 5, we present some direct consequences of our results and conclude this paper. In Appendix A, we present detailed proofs deferred from Section 4.

2 Preliminaries

Theorems proved in this paper will be about simple undirected graphs, however multiple directed graphs will show up throughout the proofs.

An undirected edge between vertices u and v will be denoted as uv. A directed edge from u to v will be denoted as $\overrightarrow{uv}$.

If G is a graph and A is a subset of its vertices, then by $G[A]$ we denote the subgraph of G induced on vertices of A. The length of the shortest cycle in a graph G is called girth and will be denoted as $g(G)$. If G is a forest we set that $g(G) = \infty$. The maximum degree of a vertex in a graph is denoted $\Delta(G)$. The set of neighbours of a vertex v is denoted by $N_G(v)$ (or $N(v)$ if clear from the context) and the closed neighbourhood of v, that is $N_G(v) \cup \{v\}$ is denoted by $N_G[v]$. For $Y \subseteq V(G)$ we additionally denote $N_G[Y] := \bigcup_{v \in Y} N_G[v]$. By $\overline{G}$ we denote the complement of G, that is the graph on the same set of vertices, where for $u, v \in V(G)$, such that $u \neq v$, we have $uv \in E(G) \Leftrightarrow uv \notin E(\overline{G})$.

The maximum average degree of a given graph G is defined as follows:

$$\operatorname{mad}(G) := \max_{H \subseteq G, H \neq \emptyset} \frac{2|E(H)|}{|V(H)|},$$

where $E(H)$ and $V(H)$ are respectively the set of edges in H and the set of vertices of H. We assume that mad of a graph with an empty vertex set is $-\infty$.

We say that undirected graph G is k-degenerate if each of its subgraphs contains a vertex of degree at most k. Degeneracy of a graph is the smallest value of k such that this graph is k-degenerate.

Let us note that class of 0-degenerate graphs is exactly the same class of graphs as graphs with $\operatorname{mad}(G) < 1$, because both are just edgeless graphs. Moreover class of 1-degenerate graphs is exactly the same class of graphs as graphs with $\operatorname{mad}(G) < 2$, because both are just forests.

3 Proof of Theorem 1

In order to prove Theorem 1 we are going to investigate a flow network that allows us to determine the value of mad in polynomial time. An example of such network can be found in [20], however we are going to use one adjusted to our own use.

Let us define a flow network $F(G, c)$ for given undirected graph G and any nonnegative real number c. The network will consist of one node for each $v \in V(G)$, one node for

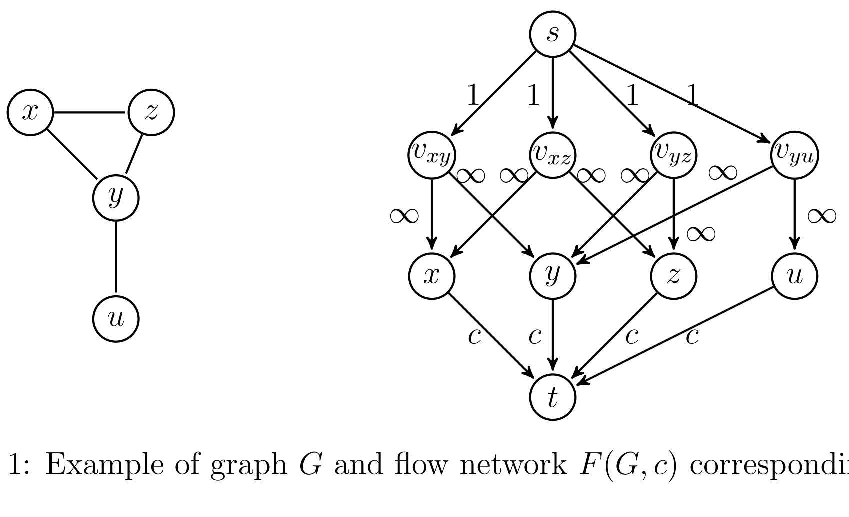

Figure 1: Example of graph G and flow network $F(G, c)$ corresponding to it.

each $e \in E(G)$ denoted as v_e and two special nodes s and t, respectively source and sink. There will be three layers of directed edges in $F(G,c)$:

- The first layer – Edges of capacity one from s to each node v_e.
- The second layer – Edges of infinite capacity from each v_e where $e = uw \in E(G)$ to u and to w.
- The third layer – Edges of capacity c from each $v \in V(G)$ to t.

Lemma 2. *For any graph G and any real number c, maximum flow between s and t in $F(G,c)$ is equal to $|E(G)|$ if and only if $2c \geqslant \mathrm{mad}(G)$.*

Proof. By the max-flow min-cut theorem we know that maximum flow in a graph G is equal to the minimum cut, so we are going to investigate structure of $s-t$ cuts in this graph. We refer to cuts as sets of edges. The set of all edges from first layer form an inclusion-wise minimal cut of weight $|E(G)|$. Since edges in the second layer have infinite capacities they surely do not belong to any minimum cut, so if maximum flow is smaller than $|E(G)|$ then there exists a minimum cut with some edges in third layer. Let us fix some minimal cut $C \subsetneq E(F(G,c))$ and let W be the nonempty subset of $V(G)$ of all vertices w such that $\overrightarrow{wt}$ belongs to C. Let $H = G[W]$. If $e \notin E(H)$ then $\overrightarrow{sv_e}$ has to belong to C. Observe that all mentioned edges, that is $\overrightarrow{wt}$ for $w \in W$ and $\overrightarrow{sv_e}$ for $e \notin E(H)$ already form an $s-t$ cut. Its weight is $c|V(H)| + |E(G)| - |E(H)|$. If this value is less than $|E(G)|$ then we know that maximum flow in this graph is less than $|E(G)|$. However, if for any H this value is not smaller than $|E(G)|$ then we know that maxflow in this graph is $|E(G)|$.

We get that maxflow in this graph is smaller than $|E(G)|$ if and only if there exists $H \subseteq G$ such that $c|V(H)|+|E(G)|-|E(H)| < |E(G)| \Leftrightarrow c|V(H)| < |E(H)| \Leftrightarrow c < \frac{|E(H)|}{|V(H)|}$. The maximum value of $\frac{|E(H)|}{|V(H)|}$ equals $\frac{\mathrm{mad}(G)}{2}$, so we get that maxflow in $F(G,c)$ is equal to $|E(G)|$ if and only if $c \geqslant \frac{\mathrm{mad}(G)}{2}$, as desired. $\square$

Let us note that by using Lemma 2, observing that $\mathrm{mad}(G) = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $a \leqslant n^2, b \leqslant n$ and knowing that we can compute maximum flow in polynomial time, we can conclude that $\mathrm{mad}(G)$ can be computed in polynomial time.

Let us fix any graph G and denote $F := F(G, \frac{\mathrm{mad}(G)}{2})$. Let us define a directed graph G_f for a given $s-t$ flow f in F of capacity $|E(G)|$ by directing some of edges from G and discarding the rest. Flow f routes one unit of flow through each v_{uw}. Node v_{uw} has two outgoing edges to u and to w. If f sends more than $\frac{1}{2}$ unit of flow to w then in G_f we put directed edge $\overrightarrow{uw}$, similarly if f sends more than $\frac{1}{2}$ unit of flow to u we put edge $\overrightarrow{wu}$. Otherwise if f sends exactly $\frac{1}{2}$ unit to both u and w we simply discard this edge.

Lemma 3. *There exists flow f of capacity $|E(G)|$ in F such that G_f is acyclic. Moreover, it can be determined in polynomial time.*

Proof. From Lemma 2 we know that there exists at least one flow f between s and t of capacity $|E(G)|$. Let us take f such that number of edges in G_f is as small as possible. Suppose there is a cycle in G_f on vertices $c_1, c_2, \ldots, c_k$ respectively. Denote $c_{k+1} := c_1$ as we are dealing with a cycle. Let x be the minimum amount of flow that f sends through some edge $\overrightarrow{v_{c_ic_{i+1}}c_{i+1}}$ for some valid i. From definition of G_f we deduce that $x > \frac{1}{2}$. Let us define f' by decreasing flow f on edges $\overrightarrow{v_{c_ic_{i+1}}c_{i+1}}$ and increasing it on edges $\overrightarrow{v_{c_ic_{i+1}}c_i}$ by $x - \frac{1}{2}$. The amount of flow leaving and entering each vertex remains unchanged hence f' is also a flow. Moreover, f' still satisfies the capacity constraints. Flow f' for at least one vertex $v_{c_ic_{i+1}}$ sends exactly $\frac{1}{2}$ unit of flow through both edges outgoing from it, so at least one edge on the cycle is no longer present in $G_{f'}$ and edges outside the cycle remain unchanged when compared to G_f. This contradicts the assumption that G_f has the smallest possible number of edges, which implies the existence of such an f.

In order to compute such f in polynomial time let us take any f of capacity $|E(G)|$ in $F(G, \frac{\mathrm{mad}(G)}{2})$ (let us remind the reader that we can determine the value of $\mathrm{mad}(G)$ in the polynomial time). If G_f contains a cycle, we can detect one, determine the corresponding value of x and adjust f in the manner described in previous paragraph to remove this cycle. The number of edges in $G_{f'}$ is strictly smaller than in G_f, so we will not do this more than $|E(G)|$ times, which gives us an algorithm performing a polynomial number of operations. In order to omit dealing with rational numbers we can multiply all capacities in F by $2b$, where $\mathrm{mad}(G) = \frac{a}{b}$ for some coprime integers a, b. That concludes the description of a polynomial time algorithm determining the desired f. □

Let us fix f from the above lemma. We will present an algorithm in which:

- the routine $NoInEdges(H_f)$ returns any vertex from directed acyclic graph H_f which has no incoming edges (as the graph is acyclic there always exists at least one such vertex)
- the routine $KNeighborhood(H, S, k)$ takes as input a given graph H, a subset of its vertices S and an integer k, and returns the set of all vertices from H outside of S adjacent to at least k vertices from S.

Algorithm 1

```
function SOLVE(H, H_f, k)
    S ← ∅
    while H_f ≠ ∅ do
        x ← NoInEdges(H_f)
        S ← S ∪ {x}
        H_f ← H_f − {x} − KNeighborhood(H, S, k)
return S
```

Theorem 4. *For positive integer k such that* $\mathrm{mad}(G) \geqslant k$ *algorithm $Solve(G, G_f, k)$ returns a set $S \subseteq V(G)$ such that $G[S]$ is a $(k-1)$-degenerate and that* $\mathrm{mad}(G - S) \leqslant \mathrm{mad}(G) - k$.

Proof. First we argue that the graph induced on the set of vertices returned by the algorithm is $(k-1)$-degenerate. In each iteration the vertex x picked by the algorithm is adjacent to at most $k-1$ already picked vertices. So $G[S]$ is $(k-1)$-degenerate indeed.

To show that $\mathrm{mad}(G - S) \leqslant \mathrm{mad}(G) - k$ we just have to find a flow f' in graph $F' := F(G - S, \frac{\mathrm{mad}(G)}{2} - \frac{k}{2})$ of value $|E(G - S)|$ thanks to Lemma 2. Observe that F'

is a subgraph of F with capacities of edges on the third layer reduced by $\frac{k}{2}$. The flow f' has to saturate all edges from the first layer in order to have value $|E(G-S)|$. On the second layer we define f' using f, for each edge from the second layer of F' flow f' will send exactly the same amount of flow as f on corresponding edge in F. Now we just have to argue that the amount of flow sent by f' to any node between the second and third layer in F' is bounded by $\frac{\mathrm{mad}(G)}{2} - \frac{k}{2}$ i.e. capacity of edge going from that node to sink. Each such node corresponds to vertex from $G-S$, so let us take arbitrary vertex $u \in V(G-S)$. During execution of the algorithm vertex u has been removed from H_f as incident to some k vertices already picked to S. Denote them $x_1, \ldots, x_k$ and let us consider arbitrary x_i. When the algorithm picked x_i from H_f, there were no incoming edges to x_i. In particular, in H_f there was no edge $\overrightarrow{ux_i}$. At that time u still belonged to H_f, so there was no edge $\overrightarrow{ux_i}$ even in G_f. Since u and x_i are adjacent in G, there was either an edge $\overrightarrow{x_iu}$ in G_f which means that flow f sends more than $\frac{1}{2}$ unit of flow from v_{ux_i} to u in F or there was no $\overrightarrow{x_iu}$ and $\overrightarrow{ux_i}$ which means that flow f sends exactly $\frac{1}{2}$ unit of flow from v_{ux_i} to u in F. Through node u in F flow f sends at most $\frac{\mathrm{mad}(G)}{2}$ units of flow and for every $1 \leqslant i \leqslant k$ at least $\frac{1}{2}$ unit of flow comes from v_{ux_i} to u. Therefore flow going through u is decreased by at least $\frac{1}{2}$ unit per each x_i in F' what implies that f' sends at most $\frac{\mathrm{mad}(G)}{2} - \frac{k}{2}$ units of flow to vertex u in F'. $\square$

What is more, procedure $Solve(G, G_f, k)$ can be trivially implemented in a polynomial time. Theorem 1 directly follows from Theorem 4. As two notable special cases we mention following corollaries:

Theorem 5. *For every undirected simple graph G there exists $I \subseteq V(G)$ such that I is an independent set and $\mathrm{mad}(G-I) \leqslant \mathrm{mad}(G) - 1$. Moreover such I can be computed in polynomial time.*

Theorem 6. *For every undirected simple graph G there exists $F \subseteq V(G)$ such that $G[F]$ is a forest and $\mathrm{mad}(G-F) \leqslant \mathrm{mad}(G) - 2$. Moreover such F can be computed in polynomial time.*

4 Reconfiguration graphs results

The reconfiguration graph $R_k(G)$ for a positive integer k and a graph G is the graph whose vertex set is the set of k-colourings of G and there is an edge between two colourings if and only if they differ by colour of exactly one vertex.

Let us recall the Cereceda's conjecture [10].

Conjecture 7. Let k and $l \geqslant k+2$ be positive integers and let G be a k-degenerate graph on n vertices. Then $R_l(G)$ has diameter $\mathcal{O}(n^2)$.

As we have already mentioned, our results imply the positive answer to the last remaining step in the outline of the proof by Eiben and Feghali [16] of the following fact:

Theorem 8. *Let $k \geqslant 2$ and $l \geqslant k+2$ be integers and let G be a graph on n vertices such that $\mathrm{mad}(G) < k+1$. Then $R_l(G)$ has diameter $k^{\mathcal{O}(k^2\sqrt{n})}$.*

Let us remind that any graph with $\mathrm{mad}(G) < k+1$ is k-degenerate, hence the setting of Theorem 8 can be seen as an easier setting of Conjecture 7. The obtained bound on the diameter is worse as well. Nonetheless, Theorem 8 seems interesting as previous proofs regarding the connectivity of such reconfiguration graphs yield only exponential bounds. However, this bound under a slightly less general assumption has already been improved by Feghali [18] to the polynomial one. Nevertheless, the idea of the proof of Theorem 8 transfers over to a novel analogous result for reconfiguration graphs of colouring viewed as homomorphism to a given graph H, which we shall present now.

For a given graph H, the *H-colouring* of a graph G is any homomorphism $f : V(G) \to V(H)$, that is, a function f such that if $uv \in E(G)$, then $f(u)f(v) \in E(H)$. We call $f(v)$ the colour of v. In particular, if H is loopless and $uv \in E(H)$, then it must hold that $f(u) \neq f(v)$. The *H-reconfiguration graph* of G is the graph whose vertex set consists of all H-colourings of G and two colourings are adjacent if and only if they differ by colour of exactly one vertex.

The main result of this section is the following theorem:

Theorem 9. *Let $k \geqslant 2$ be a positive integer and let $G = (V, E)$ be a graph on n vertices with* $\mathrm{mad}(G) < k+1$ *and let H be a graph such that $\Delta(\overline{H}) \leqslant d$ and $|V(H)| \geqslant (d+1)(k+1)+1$. Let α and β be two H-colourings of G. It is possible to get β from α by a sequence of $k^{\mathcal{O}(k^2\sqrt{n})}$ recolourings.*

Note that if we set $H = K_l$ and $d = 0$ we get the exact statement of Theorem 8, as regular colouring is exactly H-colouring for clique H. Moreover, to prove that fact we mainly follow the outline of the proof by Eiben and Feghali [16] of Theorem 8. Since the proof requires following closely proofs of all the intermediate lemmas and generalizing them to H-colourings, we defer this proof to the Appendix.

As a motivation for such generalization, we use the notion of circular colourings. We focus on the following description of the circular colourings [9], also known as (p, q)-colouring [30].

Definition 10. Let a and b, where $a \geqslant 2b$, be positive integers. The *circular clique* $G_{a,b}$ has the vertex set $\{0, 1, \dots, a-1\}$ where ij is an edge when $b \leqslant |i-j| \leqslant a-b$. A homomorphism $\phi : G \to G_{a,b}$ is called a *circular colouring* in general, and an (a, b)-colouring of G for the specific pair (a, b).

We remark that $G_{a,1}$ is isomorphic to K_a and so an $(a, 1)$-colouring is simply an a-colouring.

Brewster and Noel [9] have proved that for a given graph G and positive integers a, b if $\frac{a}{b} \geqslant 2l + 2$, where l is the degeneracy of G, then the reconfiguration graph of (a, b)-colourings of G is connected. However, the bound on its diameter that follows from their proof is exponential. Based on our H-colourings result, we are able to deduce the following bound on this diameter.

Corollary 11. *Let $k \geqslant 2$ and $G = (V, E)$ be a graph on n vertices with* $\mathrm{mad}(G) < k+1$. *Let $a \geqslant 2b$ be positive integers. If $\frac{a}{b} \geqslant 2k+2$, then the (a, b)-reconfiguration graph of G is connected and has diameter $k^{\mathcal{O}(k^2\sqrt{n})}$.*

Proof. We use Theorem 9 for $H := G_{a,b}$. The complement of $G_{a,b}$ is $(2b-2)$-regular, hence we set $d := 2b-2$. If $\frac{a}{b} \geqslant 2k+2$, then $V(H) = a \geqslant b(2k+2) = (d+2)(k+1) \geqslant$

$(d+1)(k+1)+1$, hence all assumptions of that theorem are satisfied and the conclusion follows. $\square$

5 Conclusions and open problems

Our main results imply many results for some specific classes of graphs as a direct consequence and here we mention a few of them.

Following folklore fact will come in handy in deriving some of the consequences:

Fact 12. *For every planar graph G we have* $(\mathrm{mad}(G)-2)(g(G)-2)<4$.

Based on Theorems 5, we are able to improve Theorem 1 from [15] and one of its consequences.

Theorem 13 ([15])**.** *Let M be a real number such that $M<3$. Let $d\geqslant 0$ be an integer and let G be a graph with $\mathrm{mad}(G)<M$. If $d\geqslant\frac{2}{3-M}-2$, then $V(G)$ can be partitioned into $A\uplus B$ such that $G[A]$ is an independent set and $G[B]$ is a forest with maximum degree at most d.*

We are able to strengthen this to the following theorem:

Theorem 14. *Let M be a real number such that $M<3$. Let $d\geqslant 0$ be an integer and let G be a graph with $\mathrm{mad}(G)<M$. If $d\geqslant\frac{2}{3-M}-2$, then $V(G)$ can be partitioned into $A\uplus B$ such that $G[A]$ is an independent set and $G[B]$ is a forest whose connected components have size at most $d+1$.*

Proof. By using Theorem 5 one can partition $V(G)$ into $A\uplus B$ such that $G[A]$ is an independent set and $\mathrm{mad}(G[B])<M-1$. Let us bound $M-1$ from above using assumed inequalities:

$$d\geqslant\frac{2}{3-M}-2\;\Rightarrow\;d+2\geqslant\frac{2}{3-M}\;\Rightarrow\;3-M\geqslant\frac{2}{d+2}\;\Rightarrow\;M-1\leqslant 2-\frac{2}{d+2}.$$

$G[B]$ does not contain a cycle because $\mathrm{mad}(G[B])<M-1\leqslant 2-\frac{2}{d+2}<2$. Assume that $G[B]$ contains a tree T on $d+2$ vertices as a subgraph. Then $\mathrm{mad}(G[B])\geqslant\frac{2|E(T)|}{|V(T)|}=\frac{2(d+1)}{d+2}=2-\frac{2}{d+2}\geqslant M-1>\mathrm{mad}(G[B])$. Note that if $G[B]$ contains a component which is a tree on at least $d+2$ vertices then it contains tree on exactly $d+2$ vertices as a subgraph, hence shown contradiction finishes the proof that connected components of $G[B]$ are trees on at most $d+1$ vertices. $\square$

Dross et al. [15] use their theorem and Fact 12 to deduce the following corollary:

Corollary 15. *For every planar graph G with $g(G)\geqslant 10$, the vertex set $V(G)$ can be partitioned into $A\uplus B$ such that A is an independent set and $G[B]$ is a forest with maximum degree 2.*

We are able to strengthen this to the following corollary:

Corollary 16. *For every planar graph G with $g(G)\geqslant 10$, the vertex set $V(G)$ can be partitioned into $A\uplus B$ such that A is an independent set and $G[B]$ is a forest whose connected components have size at most 3.*

Borodin et al. proved in [6] that the vertex set of any planar graph with $g(G) \geqslant 7$ admits a partition into an independent set and a set that induces graph with maximum degree at most 4. Dross et al. proved in [15] that the vertex set of any planar graph with $g(G) \geqslant 7$ admits a partition into an independent set and a set that induces forest of max degree at most 5. In Corollary 17 we add another partition result for the class of planar graphs with girth at least 7.

Corollary 17. *For every planar graph G with $g(G) \geqslant 7$, the vertex set $V(G)$ can be partitioned into $A \uplus B$ such that A is an independent set and $G[B]$ is a forest where every connected component has at most 9 vertices.*

Proof. Since $g(G) \geqslant 7$ we deduce that $\mathrm{mad}(G) < 1 + \frac{9}{5}$, so based on Theorem 5 we get that there exist A and B such that $V(G) = A \uplus B$, A is an independent set and $\mathrm{mad}(G[B]) < \frac{9}{5}$. It can be readily verified that class of graphs with value of their mad smaller than $\frac{9}{5}$ is class of graphs which are forests with connected components of size at most 9. $\square$

Recently, independently of our work, Cranston and Yancey [13] improved Corollaries 16 and 17. Namely, they claim that every planar graph G of girth at least 9 (resp. 8, 7) has a partition of $V(G)$ into an independent set I and a set F such that $G[F]$ is a forest with each component of order at most 3 (resp. 4, 6).

Apart from that, based on Theorem 5 and 6 and Fact 12 we are able to deduce following corollaries:

Corollary 18. *For every planar graph G, the vertex set $V(G)$ can be partitioned into $A \uplus B \uplus C$ such that $G[A], G[B], G[C]$ are forests.*

Proof. Every planar graph satisfies $\mathrm{mad}(G) < 6$, so using Theorem 6 we can partition $V(G)$ into A and D such that $\mathrm{mad}(G[A]) < 2$ and $\mathrm{mad}(G[D]) < 4$ and then using Theorem 6 again we can partition D into B and C such that $\mathrm{mad}(G[B]) < 2$ and $\mathrm{mad}(G[C]) < 2$. Hence $G[A], G[B], G[C]$ are forests. $\square$

Corollary 19. *For every planar graph G without triangles, the vertex set $V(G)$ can be partitioned into $A \uplus B$ such that $G[A], G[B]$ are forests.*

Proof. Based on Fact 12 we know that if G has no triangles then $g(G) \geqslant 4 \Rightarrow g(G) - 2 \geqslant 2 \Rightarrow \mathrm{mad}(G) < 4$. Therefore using Theorem 6 we deduce that there exist A, B such that $V(G) = A \uplus B$ and $G[A], G[B]$ are forests. $\square$

Corollary 20. *For every planar graph G without cycles of length 3 and 4, the vertex set $V(G)$ can be partitioned into $A \uplus B$ such that $G[A]$ is a forest and $\Delta(G[B]) \leqslant 1$.*

Proof. Since $g(G) \geqslant 5$ we deduce that $\mathrm{mad}(G) < 2 + \frac{4}{3}$, so based on Theorem 6 we get that there exist A and B such that $V(G) = A \uplus B$ and $\mathrm{mad}(G[A]) < 2$ and $\mathrm{mad}(G[B]) < \frac{4}{3}$. Therefore $G[A]$ is a forest and $\Delta(G[B]) \leqslant 1$, because if $G[B]$ contains a vertex with degree $\geqslant 2$ then this vertex together with its two neighbours induce a graph with mad at least $\frac{4}{3}$. $\square$

Corollary 21. *For every planar graph G with $g(G) \geqslant 6$ its vertex set $V(G)$ can be partitioned into $A \uplus B$ such that $G[A]$ is a forest and B is an independent set.*

Proof. Since $g(G) \geqslant 6$ we deduce that $\mathrm{mad}(G) < 3$, so based on either Theorem 6 or Theorem 5 we get that there exist A and B such that $V(G) = A \uplus B$ and $\mathrm{mad}(G[A]) < 2$ and $\mathrm{mad}(G[B]) < 1$. Therefore $G[A]$ is a forest and B is an independent set. □

However, Corollaries 18, 19, 20 and 21 have already been proven and even improved before. Corollary 18 was proven in [11] and later improved in [27]. An improved version of Corollary 19 was proven in [28]. Theorem improving both Corollaries 20 and 21 was proven in [4].

5.1 Open problem

As the main open problem in the area of partitionability of graphs with bounded *mad*, we recall the following conjecture.

Conjecture 22. For every graph G and positive real numbers c_1, c_2 if $\mathrm{mad}(G) < c_1 + c_2$ then there exists a partition of the vertex set $V(G) = A \uplus B$ such that $\mathrm{mad}(G[A]) < c_1$ and $\mathrm{mad}(G[B]) < c_2$.

Our main result shows that this conjecture is true for $c_2 \in \{1, 2\}$. Moreover, since for positive k we have that k-degenerate graphs fulfill $\mathrm{mad}(G) < 2k$ we can deduce that for every integer $k \geqslant 2$ and a graph G that satisfies $\mathrm{mad}(G) < c_1 + k$ there exists a partition of the vertex set $V(G) = A \uplus B$ such that $\mathrm{mad}(G[A]) < c_1$ and $\mathrm{mad}(G[B]) < 2k - 2$.

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A Proof of the bound on the diameter of reconfiguration graphs of H-colourings of graphs with bounded maximum average degree

The overall structure of proof of Theorem 9 consists of combining the generalization of Lemma 2 from [16] with Theorem 5 in the same way that Lemmas 8, 9 and 10 in [19] are combined to obtain Theorem 6 in [19]. It is the generalization of the outline of the analogous result for standard colourings from [16].

Lemma 23. *Let $k, d \geqslant 0$, and let G be the graph on n vertices such that* $\mathrm{mad}(G) < k+1$. *Let $\{u_1, \ldots, u_s\}$ be the set of vertices of G of degree at least $k+2$. Let H be a graph such that $\Delta(\overline{H}) \leqslant d$ and $|V(H)| \geqslant (d+1)(k+1)+1$ and let α be a H-colouring of G. Let F be any subset of $V(H)$ of size at most $d+1$. It is possible to reconfigure α to some H-colouring α' of G such that $\alpha'(V(G)) \subseteq V(H) - F$ by using at most $n^2 \prod_{i=1}^{s} deg(u_i)$ recolourings.*

Let us note that the case $d = 0$ coincides with Lemma 1 from [16] with the slight change in the assumption on G. The proof presented here is a generalization of its proof.

Proof. Since $\mathrm{mad}(G) < k+1$, we know that G is k-degenerate, hence we can fix its k-degenerate ordering $\sigma = v_1, \ldots, v_n$ and without loss of generality, let u_i appear before u_j in σ whenever $i < j$. In the following, we will describe an algorithm $Recolour(h, F_h)$, which given an index $h \in [n]$ and the subset $F_h \subseteq V(H)$ of forbidden colours for v_h such that $|F_h| \leqslant d+1$, outputs a sequence of recolourings with the following properties:

- for $i < h$, v_i is not recoloured,
- for $i \geqslant h$, v_i is recoloured at most $\prod_{j=l}^{s} deg(u_j)$ times, where u_l is the first vertex of degree at least $k+2$ with index at least h in σ
- v_h ends up with a colour from $V(H) - F_h$ (in particular, if $f(v_h) \in V(H) - F_h$, then an empty sequence is a feasible one, where $f(v_h)$ denotes the current colour of v_h)

Notice that the algorithm takes at most $n \prod_{i=1}^{s} deg(u_i)$ recolourings to recolour v_h. Hence, by repeatedly calling $Recolour(i, F)$ for $i = 1, \ldots, n$, we obtain the colouring α' in which colours from F do not appear by using at most $n^2 \prod_{i=1}^{s} deg(u_i)$ recolourings, as required.

Given $h \in [n]$, k-degenerate ordering $\sigma = v_1, \ldots, v_n$ of G and the set F_h of forbidden colours, the algorithm $Recolour(h, F_h)$ works as follows:

1. If $f(v_h) \notin F_h$, then terminate.

2. If v_h has degree at most k, then

    - Let c be a colour not belonging to $N_{\overline{H}}[f(N(v_h))] \cup F_h$. Such colour exists since $\Delta(\overline{H}) \leqslant d$, so $|N_{\overline{H}}[f(N(v_h))] \cup F_h| \leqslant (d+1)k + (d+1) < |V(H)|$.
    - Recolour v_h to c.

3. If v_h has degree at least $k+1$, then

    - Let c be a colour not belonging to $N_{\overline{H}}[f(Z)] \cup F_h$, where Z is the set consisting of first k neighbours of v_h in ordering σ. Such colour exists since $\Delta(\overline{H}) \leqslant d$, so $|N_{\overline{H}}[f(Z)] \cup F_h| \leqslant (d+1)k + (d+1) < |V(H)|$.
    - Let $v_{i_1}, \ldots, v_{i_t}$ be the neighbours of v_h outside Z with $i_1 < i_2 < \ldots < i_t$. Note that $h < i_1$, since G is k-degenerate.
    - For each $j \in [t]$ in the ascending order call $Recolour(i_j, N_{\overline{H}}[c])$.
    - Recolour v_h to c.

It is clear that this algorithm is correct. To estimate the number of used recolourings, it is sufficient to observe that the recursion branches only on vertices of degree at least $k+2$. $\square$

Lemma 24. *Let $k \geqslant 1, d \geqslant 0$ and let G be a graph on n vertices and with* $\operatorname{mad}(G) < k+1$. *Let H be a graph such that $\Delta(\overline{H}) \leqslant d$ and $|V(H)| \geqslant (d+1)(k+1)+1$ and let α be a H-colouring of G. Let F be a subset of $V(H)$ of size at most $d+1$. It is possible to reconfigure α to some H-colouring α' of G such that $\alpha'(V(G)) \subseteq V(H) - F$ by* $\max(k,2)^{\mathcal{O}(k^2\sqrt{n})}$ *recolourings.*

Let us note again that the case $d = 0$ coincides with Lemma 2 from [16] and that the proof presented here is a generalization of its proof.

Proof. We will call H-colourings of G *small* if they do not use colours from F. We will denote $k' = \max(k, 2)$.

We shall prove by the induction on the size $n \coloneqq |V(G)|$ that we can reconfigure α to a small H-colouring α', such that each vertex in G is recoloured at most $n^2 \cdot k'^{9k'^2\sqrt{n}}$ times, which implies the lemma.

As the base case we distinguish graphs P on p vertices such that $\operatorname{mad}(P) < k+1$ that contain at most $2(k+1)\sqrt{p}$ vertices of degree at most k. Let $\{u_1, \ldots, u_s\}$ be a set of vertices of P of degree at least $k+2$. In the case of such graphs $\prod_{i=1}^{s} deg(u_i)$ is bounded from above by $(2(k+1))^{2k(k+1)\sqrt{p}} \leqslant (k'^3)^{3k'^2\sqrt{p}} = k'^{9k'^2\sqrt{p}}$ as proven by Eiben and Feghali as part of the proof of Lemma 2 in [16]. Thus, in this case we can use algorithm from Lemma 23 and prove that we can get α' using at most $p^2 \prod_{i=1}^{s} deg(u_i) \leqslant p^2 k'^{9k'^2\sqrt{p}}$ recolourings in total, in particular the number of recolourings of each particular vertex is bounded by $p^2 k'^{9k'^2\sqrt{p}}$.

For the inductive step, suppose that G contains more than $2(k+1)\sqrt{n}$ vertices of degree at most k and that we can reconfigure any subgraph P of G with $p < n$ vertices to some small H-colouring α_P such that each vertex gets recoloured at most $p^2 k'^{9k'^2\sqrt{p}}$ times. Let S be an independent set in G of size at least $2\sqrt{n}$ containing only vertices of degree at most k. Note that G is k-degenerate, so it can be partitioned into $k+1$ independent sets and one of them has to contain at least $2\sqrt{n}$ vertices of degree at most k, so such S exists. Using an inductive argument, we can recolour the graph $P = G - S$ to some small H-colouring through a recolouring sequence R. We can extend this sequence of recolourings in P to a sequence in G in the following way. Let a recolouring of $v \in V(P)$ from c_1 to c_2 be one of the recolouring operations from R and let $u \in S$ be a neighbour of v. Let C be the sum of $\{c_1, c_2\}$ and colours present in the neighbours of u in P other than v. We have that $|C| \leqslant k+1$, so $N_{\overline{H}}[C] \leqslant (d+1)(k+1) < |V(H)|$, so there exists a colour $c \in V(H) - N_{\overline{H}}[C]$. In the reconfiguration sequence extended to G we put an operation of recolouring u to c before recolouring v. This way we get a valid recolouring sequence for G. Finally, we recolour each vertex $u \in S$ to any colour not belonging to $F \cup N_{\overline{H}}[f(N_G(u))]$. It is possible as $|F \cup N_{\overline{H}}[f(N_G(u))]| \leqslant (d+1)(k+1) < |V(H)|$. For each vertex outside S we recoloured it exactly as many times as in R, whereas for each vertex in S we recoloured it at most as many times as all its neighbours were recoloured in total in R plus one.

Let $g(n)$ be the maximum number of times that a vertex of a graph on n vertices is recoloured in this process. In the base case we require at most $n^2 k'^{9k'^2\sqrt{n}}$ recolourings, while in the inductive step case we require at most $k \cdot g(\lfloor n - 2\sqrt{n} \rfloor) + 1$ recolourings, hence we have $g(n) \leqslant \max(n^2 \cdot k'^{9k'^2\sqrt{n}}, k \cdot g(\lfloor n - 2\sqrt{n} \rfloor) + 1)$. However, simple calculations show that $k \cdot k'^{9k'^2\sqrt{n-2\sqrt{n}}} + 1 \leqslant k'^{9k'^2\sqrt{n}}$ (as $\sqrt{n} > \sqrt{n - 2\sqrt{n}} + 1$), so $n^2 \cdot k \cdot k'^{9k'^2\sqrt{n-2\sqrt{n}}} + 1 \leqslant n^2 \cdot k'^{9k'^2\sqrt{n}}$, hence $g(n) \leqslant n^2 \cdot k'^{9k'^2\sqrt{n}}$, which concludes this proof that each vertex can be recoloured at most $n^2 \cdot k'^{9k'^2\sqrt{n}}$ times. Thus, in total we get a sequence of at most $n^3 \cdot k'^{9k'^2\sqrt{n}} = \max(2,k)^{\mathcal{O}(k^2\sqrt{n})}$ recolourings. $\square$

Remark. If we were a bit more meticulous, then in Lemma 23 it would be possible to get the bound of $\mathcal{O}(n^2 \prod_{i=1}^{s} (deg(u_i) - k))$ instead of $\mathcal{O}(n^2 \prod_{i=1}^{s} deg(u_i))$. In consequence, it would be possible to improve the bound from Lemma 24 to $2^{\mathcal{O}(k^2\sqrt{n})}$, but proving that would lead to significantly longer exposition and the improvement would not be that significant, hence we decided not to present it.

Now we are ready to prove Theorem 9.

Proof. Let $k \geqslant 0$ be a positive integer and let $G = (V, E)$ be a graph on n vertices with $\operatorname{mad}(G) < k+1$ and let H be a graph such that $\Delta(\overline{H}) \leqslant d$ and $|V(H)| \geqslant (d+1)(k+1)+1$. We will define a H-colouring of G called γ and then prove that any H-colouring δ can be reconfigured to γ by a sequence of at most $\max(2,k)^{\mathcal{O}(\max(k^2,1)\sqrt{n})}$ recolourings, which clearly implies the thesis by using twice this statement for $\delta = \alpha$ and for $\delta = \beta$.

We will prove this inductively on k. The base case $k = 0$ is trivial: if $\operatorname{mad}(G) < 1$, then G is an edgeless graph and every colouring is valid. We can set γ as an arbitrary colouring of G. Now we can recolour each vertex directly from colour from δ to colour from γ using at most n recolourings in total.

Now we present the inductive step. Let us consider an independent set I from Theorem 5 such that $mad(G - I) < k$ and any colour $u \in V(H)$. For each $v \in I$ we set $\gamma(v) = u$. Thanks to Lemma 24 for $F := N_{\overline{H}}[u]$, we are able to reconfigure δ to a colouring δ' that does not use colours from $N_{\overline{H}}[u]$. Afterwards, we reconfigure δ' to δ'' by recolouring each vertex from I to the colour u. We define $G' = G - I$ and $H' = H - N_{\overline{H}}[u]$. Graphs G' and H' meet the assumptions of the inductive hypothesis for $k' = k - 1$. So there exists γ' independent of δ'' and a sequence of recolourings R' configuring δ'' to γ'. Now we just need to concatenate all those reconfiguration sequences and set $\gamma(v) = \gamma'(v)$ for $v \in G'$. This concatenation yields a valid reconfiguration sequence because colour c is connected to all colours from H'.

To build the final reconfiguration sequence we used the construction from Lemma 24 exactly k times. Apart from that, each vertex is recoloured at most once, to its final colour, so in total we used at most $n + n \cdot max(2,k)^{\mathcal{O}(k^2\sqrt{n})} \leqslant \max(2,k)^{\mathcal{O}(\max(k^2,1)\sqrt{n})}$ recolourings, as desired. $\square$

It is worth noting that all parts of this proof were constructive, hence we can compute such sequence of recolourings in $k^{\mathcal{O}(k^2\sqrt{n})}$ time and polynomial space.