An Ore-type condition for hamiltonicity in tough graphs

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Abstract

Let G be a t-tough graph on $n \geqslant 3$ vertices for some $t > 0$. It was shown by Bauer et al. in 1995 that if the minimum degree of G is greater than $\frac{n}{t+1} - 1$, then G is hamiltonian. In terms of Ore-type hamiltonicity conditions, the problem was only studied when t is between 1 and 2. In this paper, we show that if the degree sum of any two nonadjacent vertices of G is greater than $\frac{2n}{t+1} + t - 2$, then G is hamiltonian.

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1 Introduction

We consider only simple graphs. Let G be a graph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. Let $v \in V(G)$, $S \subseteq V(G)$, and $H \subseteq G$. Then $N_G(v)$ denotes the set of neighbors of v in G, $d_G(v) := |N_G(v)|$ is the degree of v in G, and $\delta(G) := \min\{d_G(v) : v \in V(G)\}$ is the minimum degree of G. Define $\deg_G(v, H) = |N_G(v) \cap V(H)|$, $N_G(S) = (\bigcup_{x \in S} N_G(x)) \setminus S$, and we write $N_G(H)$ for $N_G(V(H))$. Let $N_H(v) = N_G(v) \cap V(H)$ and $N_H(S) = N_G(S) \cap V(H)$. We use $G[S]$ and $G - S$ to denote the subgraphs of G induced by S and $V(G) \setminus S$, respectively. For notational simplicity we write $G - x$ for $G - \{x\}$. Let $V_1, V_2 \subseteq V(G)$ be two disjoint vertex sets. Then $E_G(V_1, V_2)$ is the set of edges in G with one end in V_1 and the other end in V_2. For two integers a and b, let $[a, b] = \{i \in \mathbb{Z} : a \leqslant i \leqslant b\}$.

Throughout this paper, if not specified, we will assume t to be a nonnegative real number. The number of components of a graph G is denoted by $c(G)$. The graph G is said to be *t-tough* if $|S| \geqslant t \cdot c(G-S)$ for each $S \subseteq V(G)$ with $c(G-S) \geqslant 2$. The *toughness* $\tau(G)$ is the largest real number t for which G is t-tough, or is ∞ if G is complete. This concept, a measure of graph connectivity and "resilience" under removal of vertices, was introduced by Chvátal [6] in 1973. It is easy to see that if G has a hamiltonian cycle then G is 1-tough. Conversely, Chvátal [6] conjectured that there exists a constant t_0 such that every t_0-tough graph is hamiltonian. Bauer, Broersma and Veldman [1] have constructed t-tough graphs that are not hamiltonian for all $t < \frac{9}{4}$, so t_0 must be at least $\frac{9}{4}$ if Chvátal's toughness conjecture is true.

Chvátal's toughness conjecture has been verified when restricted to a number of graph classes [2], including planar graphs, claw-free graphs, co-comparability graphs, and chordal graphs. In general, the conjecture is still wide open. In finding hamiltonian cycles in graphs, sufficient conditions such as Dirac-type and Ore-type conditions are the most classical ones.

Theorem 1 (Dirac's Theorem [7]). *If G is a graph on $n \geqslant 3$ vertices with $\delta(G) \geqslant \frac{n}{2}$, then G is hamiltonian.*

Define $\sigma_2(G) = \min\{d_G(u) + d_G(v) \ : \ u, v \in V(G),\ u \text{ and } v \text{ are nonadjacent}\}$ if G is noncomplete, and define $\sigma_2(G) = \infty$ otherwise. Ore's Theorem, as a generalization of Dirac's Theorem, is stated below.

Theorem 2 (Ore's Theorem [9]). *If G is a graph on $n \geqslant 3$ vertices with $\sigma_2(G) \geqslant n$, then G is hamiltonian.*

Analogous to Dirac's Theorem, Bauer, Broersma, van den Heuvel, and Veldman [4] proved the following result by incorporating the toughness of the graph.

Theorem 3 (Bauer et al. [4]). *Let G be a t-tough graph on $n \geqslant 3$ vertices. If $\delta(G) > \frac{n}{t+1} - 1$, then G is hamiltonian.*

A natural question here is whether we can find an Ore-type condition involving the toughness of G that generalizes Theorem 3. Various theorems were proved prior to Theorem 3 by only taking $\tau(G)$ between 1 and 2. Jung in 1978 [8] showed that if G is a 1-tough graph on $n \geqslant 11$ vertices with $\sigma_2(G) \geqslant n - 4$, then G is hamiltonian. In 1991, Bauer, Chen, and Lasser [3] showed that the degree bound in Jung's Theorem can be slightly lowered if $\tau(G) > 1$. The result states that if G is a graph on $n \geqslant 30$ vertices with $\tau(G) > 1$ and $\sigma_2(G) \geqslant n - 7$, then G is hamiltonian. In 1989/90, Bauer, Veldman, Morgana, and Schmeichel [5] showed that if G is a 2-tough graph on $n \geqslant 3$ vertices with $\sigma_2(G) \geqslant \frac{2n}{3}$, then G is hamiltonian (a consequence of Corollary 16 from [5]). In this paper, we obtain the following result, which provides an Ore-type condition involving $\tau(G)$ that guarantees a hamiltonian cycle in a graph.

Theorem 4. *Let G be a t-tough graph on $n \geqslant 3$ vertices. If $\sigma_2(G) > \frac{2n}{t+1} + t - 2$, then G is hamiltonian.*

In fact, we believe that the following stronger statement might be true.

Conjecture 5. Let G be a t-tough graph on $n \geqslant 3$ vertices. If $\sigma_2(G) > \frac{2n}{t+1} - 2$, then G is hamiltonian.

Considering both toughness and degree sum conditions such as in Theorem 4 and Conjecture 5 is an approach to investigate Chvátal's toughness conjecture while the conjecture remains open. However, in light of the conjecture, those results might only be relevant for some small values of t.

For odd integers $n \geqslant 3$, the complete bipartite graph $G := K_{\frac{n-1}{2},\frac{n+1}{2}}$ is $\frac{n-1}{n+1}$-tough and satisfies $\sigma_2(G) = n-1 = \frac{2n}{1+\frac{n-1}{n+1}} - 2$. However, G is not hamiltonian. Thus, if true, the degree sum condition in Conjecture 5 would be best possible. In fact, for odd integers $n \geqslant 3$, any graph from the family $\mathcal{H} = \{H_{\frac{n-1}{2}} + \overline{K}_{\frac{n+1}{2}} : H_{\frac{n-1}{2}}$ is any graph on $\frac{n-1}{2}$ vertices$\}$ is an extremal graph, where "+" represents the join of two graphs. In light of the results mentioned in the paragraph right above Theorem 4 and Chvátal's toughness conjecture, it suggests that t-tough non-hamiltonian graphs G with $\sigma_2(G) = \frac{2n}{t+1} - 2$ exist only when $t < 1$. Furthermore, by looking at the non-hamiltonian t-tough graphs G with $t < 1$ and $\delta(G) = \frac{n}{t+1} - 1$, which are exactly the graphs in the family $\mathcal{H}$, it suggests that when $t < 1$, any non-hamiltonian t-tough graph G with $t < 1$ and $\sigma_2(G) = \frac{2n}{t+1} - 2$ belongs to the family $\mathcal{H}$. So we propose the following conjecture.

Conjecture 6. Let G be a t-tough graph on $n \geqslant 3$ vertices. If $\sigma_2(G) = \frac{2n}{t+1} - 2$ and G is non-hamiltonian, then $G \in \mathcal{H}$.

In attempting to prove Conjecture 5 by contradiction, the most difficult case to deal with is when G has a cycle C of length $n-1$ and $G - V(C)$ is just a single vertex component H. It seems very hard to deduce any nontrivial property of G using the $\sigma_2(G)$ and toughness conditions. However, by adding t to the $\sigma_2(G)$ bound, vertices in $V(C) \setminus N_C(H)$ can be shown to have degree bigger than $\frac{n}{t+1} + t - 1$. This degree condition allows us to find $|N_C(H)|$ disjoint subgraphs each of order $t+2$ such that there is no edge between any two of them. Then we get to use the toughness condition to give a smaller upper bound on $|N_C(H)|$ ($|N_C(H)| \leqslant \frac{n}{2(t+1)} - \frac{1}{2}$), which plays a key role in the proof of Theorem 4. Therefore, it might require a completely different approach to confirm Conjecture 5.

The remainder of this paper is organized as follows: in Section 2, we introduce some notation and preliminary results, and in Section 3, we prove Theorem 4.

2 Preliminary results

Let G be a graph and λ a positive integer. Following [11], a cycle C of G is a D_λ-*cycle* if every component of $G - V(C)$ has order less than λ. Clearly, a D_1-cycle is just a hamiltonian cycle. We denote by $c_\lambda(G)$ the number of components of G with order at least λ, and write $c_1(G)$ just as $c(G)$. Two subgraphs H_1 and H_2 of G are *remote* if they are disjoint and there is no edge of G joining a vertex of H_1 and a vertex of H_2. For a subgraph H of G, let $d_G(H) = |N_G(H)|$ be the degree of H in G. We denote by $\delta_\lambda(G)$ the minimum degree of a connected subgraph of order λ in G. Again $\delta_1(G)$ is just $\delta(G)$.

Let C be an oriented cycle, and we assume that the orientation is clockwise throughout the rest of this paper. For $x \in V(C)$, denote the immediate successor of x on C by x^+ and the immediate predecessor of x on C by x^-. For $u, v \in V(C)$, $u\overrightarrow{C}v$ denotes the segment of C starting at u, following C in the orientation, and ending at v. Likewise, $u\overleftarrow{C}v$ is the opposite segment of C with endpoints as u and v. Let $\mathrm{dist}_{\overrightarrow{C}}(u, v)$ denote the length of the path $u\overrightarrow{C}v$. For any vertex $u \in V(C)$ and any positive integer k, define

$$L_u^+(k) = \{v \in V(C) \,:\, \mathrm{dist}_{\overrightarrow{C}}(u, v) \in [1, k]\},$$
$$L_u^-(k) = \{v \in V(C) \,:\, \mathrm{dist}_{\overrightarrow{C}}(v, u) \in [1, k]\},$$

to be the set of k consecutive successors of u and the set of k consecutive predecessors of u, respectively. A *chord* of C is an edge uv with $u, v \in V(C)$ and $uv \notin E(C)$. Two chords ux and vy that do not share any endvertices of C are *crossing* if the four vertices u, x, v, y appear along $\overrightarrow{C}$ in the order u, v, x, y or u, y, x, v. Hereafter, all cycles under consideration are oriented.

A path P connecting two vertices u and v is called a (u, v)*-path*, and we write uPv or vPu in order to specify the two endvertices of P. Let uPv and xQy be two paths. If vx is an edge, we write $uPvxQy$ as the concatenation of P and Q through the edge vx.

For an integer $\lambda \geqslant 1$, if a graph G contains a $D_{\lambda+1}$-cycle C but no D_λ-cycle, then $V(G) \setminus V(C) \neq \varnothing$. Furthermore, $G - V(C)$ has a component of order λ. The result below with $d_G(H)$ replaced by $\delta_\lambda(G)$ was proved in [4].

Lemma 7. *Let G be a t-tough 2-connected graph of order n. Suppose G has a $D_{\lambda+1}$-cycle but no D_λ-cycle. Let C be a $D_{\lambda+1}$-cycle of G such that $c_\lambda(G - V(C))$ is minimum. Then $n \geqslant (t + \lambda)(d_G(H) + 1)$ for any component H of $G - V(C)$ with order λ.*

Proof. Let $k = d_G(H)$, which equals the total number of neighbors of vertices of H on C. We assume the k neighbors are $v_1, \ldots, v_k$ and appear in the same order along $\overrightarrow{C}$. It suffices to show that $L_{v_1}^+(\lambda), \ldots, L_{v_k}^+(\lambda)$ and H are pairwise remote. Since in that case, if we let $S = V(G) \setminus \Big((\bigcup_{i=1}^k L_{v_i}^+(\lambda)) \cup V(H) \Big)$, then $|S| = n - (k+1)\lambda$ and $c(G - S) = k + 1$. As G is t-tough, we get

$$n - (k + 1)\lambda = |S| \geqslant t \cdot c(G - S) = t(k + 1),$$

giving $n \geqslant (t + \lambda)(k + 1)$.

Below, we show that $L_{v_1}^+(\lambda), \ldots, L_{v_k}^+(\lambda)$ and H are pairwise remote. It suffices to prove the statement that $\mathrm{dist}_{\overrightarrow{C}}(v_i, v_j) \geqslant \lambda + 1$ and $E_G(L_{v_i}^+(\lambda), L_{v_j}^+(\lambda)) = \varnothing$ for every two distinct $i, j \in [1, k]$. Let $v_i^* \in N_H(v_i), v_j^* \in N_H(v_j)$ and P be a (v_i^*, v_j^*)-path of H.

For the first part of the statement, we only need to show that $\mathrm{dist}_{\overrightarrow{C}}(v_i, v_{i+1}) \geqslant \lambda + 1$ for every $i \in [1, k]$, where $v_{k+1} := v_1$. Suppose to the contrary that $\mathrm{dist}_{\overrightarrow{C}}(v_i, v_{i+1}) \leqslant \lambda$ for some $i \in [1, k]$. Let $C^* = v_i\overleftarrow{C}v_{i+1}v_{i+1}^*Pv_i^*v_i$. Since H has order λ and no vertex of H is adjacent in G to any internal vertex of $v_i\overrightarrow{C}v_{i+1}$, it follows that each component of $H - V(P)$ is a component of $G - V(C^*)$ of order at most $\lambda - 1$ and $v_i^+\overrightarrow{C}v_{i+1}^-$ is a component of $G - V(C^*)$ of order at most $\lambda - 1$. Thus C^* is a D_λ-cycle of G with $c_\lambda(G - V(C^*)) < c_\lambda(G - V(C))$, contradicting the choice of C.

For the second part of the statement, assume to the contrary that $E_G(L^+_{v_i}(\lambda), L^+_{v_j}(\lambda)) \neq \varnothing$ for some distinct $i, j \in [1, k]$. Applying the first part, we know that $\mathrm{dist}_{\overrightarrow{C}}(v_i, v_j) \geqslant \lambda + 1$ and $\mathrm{dist}_{\overrightarrow{C}}(v_j, v_i) \geqslant \lambda + 1$ (exchanging the role of v_i and v_j). Thus $L^+_{v_i}(\lambda) \cap L^+_{v_j}(\lambda) = \varnothing$. We choose $x \in L^+_{v_i}(\lambda)$ with $\mathrm{dist}_{\overrightarrow{C}}(v_i, x)$ minimum and $y \in L^+_{v_j}(\lambda)$ with $\mathrm{dist}_{\overrightarrow{C}}(v_j, y)$ minimum such that $xy \in E(G)$. By this choice of x and y, it follows that $E_G(V(v_i^+\overrightarrow{C}x^-), V(v_j^+\overrightarrow{C}y^-)) = \varnothing$. Let $C^* = v_i\overleftarrow{C}yx\overrightarrow{C}v_jv_j^*Pv_i^*v_i$. Since H has order λ and no vertex of H is adjacent in G to any vertex of $v_i^+\overrightarrow{C}x^-$ or $v_j^+\overrightarrow{C}y^-$ by the first part of the statement, it follows that each component of $H - V(P)$ is a component of $G - V(C^*)$ of order at most $\lambda - 1$. Also $v_i^+\overrightarrow{C}x^-$ and $v_j^+\overrightarrow{C}y^-$ are components of $G - V(C^*)$ of order at most $\lambda - 1$. Thus C^* is a D_λ-cycle of G with $c_\lambda(G - V(C^*)) < c_\lambda(G - V(C))$, contradicting the choice of C. □

Lemma 8 ([10])**.** *Let $t > 0$ and G be a non-complete n-vertex t-tough graph. Then $|W| \leqslant \frac{n}{t+1}$ for every independent set W in G.*

The following lemma provides a way of extending a cycle C provided that the vertices outside C have many neighbors on C. The proof follows from Lemma 8 and is very similar to the proof of Lemma 10 in [10].

Lemma 9. *Let $t \geqslant 1$ and G be an n-vertex t-tough graph, and let C be a non-hamiltonian cycle of G. If $x \in V(G) \setminus V(C)$ satisfies $\deg_G(x, C) > \frac{n}{t+1} - 1$, then G has a cycle C' such that $V(C') = V(C) \cup \{x\}$.*

3 Proof of Theorem 4

We may assume that G is not a complete graph. Thus G is $2\lceil t \rceil$-connected as it is t-tough. Suppose to the contrary that G is not hamiltonian. By Theorem 3, we have $\delta(G) \leqslant \frac{n}{t+1} - 1$. Since $\delta(G) \geqslant 2\lceil t \rceil$, we get

$$n \geqslant (t + 1)(2\lceil t \rceil + 1).$$

Claim 10. *$t > 1$.*

Proof. Assume to the contrary that $t \leqslant 1$. By Ore's result, Theorem 2, and the assumption that G is not hamiltonian, we get $\sigma_2(G) \leqslant n - 1$. Thus

$$\frac{2n}{t+1} + t - 2 < \sigma_2(G) \leqslant n - 1.$$

This gives $\frac{2n}{t+1} + t < n + 1$. Let $f(t) = \frac{2n}{t+1} + t$. Since $t \leqslant 1$ and $n \geqslant (t + 1)(2\lceil t \rceil + 1) > (t + 1)^2$, $f'(t) = \frac{(t+1)^2 - 2n}{(t+1)^2} < 0$. Thus the minimum value of $f(t)$ is achieved at $t = 1$ and $f(1) = n + 1$, showing a contradiction to $\frac{2n}{t+1} + t < n + 1$. □

Since $t > 1$ and G is not complete, G is 2-connected and so contains cycles. We choose $\lambda \geqslant 0$ to be a smallest integer such that G admits no D_λ-cycle but a $D_{\lambda+1}$-cycle. Then we choose C to be a longest $D_{\lambda+1}$-cycle such that $c_\lambda(G - V(C))$ is minimum. As G is not hamiltonian, we have $\lambda \geqslant 1$. Thus $V(G) \setminus V(C) \neq \varnothing$. Since λ is taken to be minimum, $G - V(C)$ has a component H of order λ. Let

$$W = N_C(V(H)) \quad \text{and} \quad \omega = |W|.$$

Since G is a connected t-tough graph, it follows that $\omega \geqslant 2\lceil t \rceil$. On the other hand, Lemma 7 implies that $\omega \leqslant \frac{n}{t+\lambda} - 1$.

Claim 11. $\lambda + \omega \leqslant \frac{n}{t+1}$.

Proof. Assume to the contrary that $\lambda + \omega > \frac{n}{t+1}$. If $\lambda = 1$, then H has only one vertex and $\omega > \frac{n}{t+1} - 1$. By Lemma 9, we can find a cycle C' with $V(C') = V(C) \cup V(H)$, contradicting the choice of C. Thus $\lambda \geqslant 2$. Since $2t \leqslant \omega \leqslant \frac{n}{t+\lambda} - 1 \leqslant \frac{n}{t+2} - 1$, we have $n \geqslant (t+2)(2t+1)$. By Lemma 7, we have

$$\begin{aligned}
n &\geqslant (\lambda + t)(\omega + 1) \\
&> \left(\frac{n}{t+1} - \omega + t\right)(\omega + 1) \quad \left(\lambda + \omega > \frac{n}{t+1} \text{ by assumption.}\right) \\
&\geqslant \begin{cases} (\frac{n}{t+1} - 2t + t)(2t+1), & \text{if } f(\omega) = (\frac{n}{t+1} - \omega + t)(\omega+1) \text{ is increasing;} \\ (\frac{n}{t+1} - \frac{n}{t+2} + 1 + t)\frac{n}{t+2}, & \text{if } f(\omega) = (\frac{n}{t+1} - \omega + t)(\omega+1) \text{ is decreasing;} \end{cases} \\
&\geqslant \begin{cases} n + \frac{tn}{t+1} - 2t^2 - t \geqslant n + \frac{t(t+2)(2t+1)}{t+1} - 2t^2 - t > n + t(2t+1) - 2t^2 - t = n, \\ \frac{n}{(t+1)(t+2)}\frac{n}{t+2} + \frac{(t+1)n}{t+2} \geqslant \frac{n}{(t+1)(t+2)}\frac{(t+2)(2t+1)}{t+2} + \frac{(t+1)n}{t+2} > \frac{n}{t+2} + \frac{(t+1)n}{t+2} = n, \end{cases}
\end{aligned}$$

reaching a contradiction. □

Claim 12. *H is the only component of $G - V(C)$ and H is a complete subgraph of G.*

Proof. Suppose $H^* \neq H$ is another component of $G - V(C)$. Since $\sigma_2(G) > \frac{2n}{t+1} + t - 2$, Claim 11 implies that $|V(H^*)| + |N_C(V(H^*))| > \frac{n}{t+1} + t - 1 > \frac{n}{t+1}$. Repeating exactly the same argument for $|V(H^*)| + |N_C(V(H^*))|$ as in the proof of Claim 11 leads to a contradiction. Thus H is the only component of $G - V(C)$. Since $\lambda + \omega \leqslant \frac{n}{t+1}$ by Claim 11 and $\sigma_2(G) > \frac{2n}{t+1} + t - 2$, every two distinct vertices of H are adjacent. Thus H is a complete subgraph of G. □

Since H is the only component of $G - V(C)$, every vertex $v \in V(C) \setminus W$ is only adjacent in G to vertices on C. As $d_G(u) \leqslant \frac{n}{t+1} - 1$ for any $u \in V(H)$ by Claim 11, using $\sigma_2(G) > \frac{2n}{t+1} + t - 2$, we have

$$\deg_G(v, C) > \frac{n}{t+1} + t - 1 \quad \text{for any } v \in V(C) \setminus W. \tag{1}$$

Equation (**1**) allows us to construct the vertex sets $L_u^+(t+2)$ for each $u \in W$. For notation simplicity, we use L_u^+ for $L_u^+(t+2)$.

Claim 13. *For any two distinct vertices $u, v \in W$, $\mathrm{dist}_{\overrightarrow{C}}(u,v) \geqslant t+3$ and $E_G(L_u^+, L_v^+) = \varnothing$.*

Proof. Let $u^* \in N_H(u), v^* \in N_H(v)$ and P be a (u^*, v^*)-path of H. For the first part of the statement, it suffices to show that when we arrange the vertices of W along $\overrightarrow{C}$, for any two consecutive vertices u and v from the arrangement, we have $\mathrm{dist}_{\overrightarrow{C}}(u,v) \geqslant t+3$. Note that $V(u^+\overrightarrow{C}v^-) \cap W = \varnothing$ for such pairs of u and v. Assume to the contrary that there are distinct $u, v \in W$ with $V(u^+\overrightarrow{C}v^-) \cap W = \varnothing$ and $\mathrm{dist}_{\overrightarrow{C}}(u,v) \leqslant t+2$. Let $C^* = u\overleftarrow{C}vv^*Pu^*u$. Since H is complete and $V(u^+\overrightarrow{C}v^-) \cap W = \varnothing$, $H - V(P)$ is a component of $G - V(C^*)$ of order at most $\lambda - 1$ and $u^+\overrightarrow{C}v^-$ is a component of $G - V(C^*)$ of order at most $t+1$. By (**1**), for each vertex $x \in V(u^+\overrightarrow{C}v^-)$, $\deg_G(x, C^*) > \frac{n}{t+1} - 1$. Applying Lemma 9, we find a cycle C^{**} of G with $V(C^{**}) = V(C^*) \cup V(u^+\overrightarrow{C}v^-)$. Since $V(G) \setminus V(C^{**}) = V(H) \setminus V(P)$, C^{**} is a D_λ-cycle of G, contradicting the choice of C.

For the second part of the statement, we assume to the contrary that $E_G(L_u^+, L_v^+) \neq \varnothing$. Applying the first part, we know that $\mathrm{dist}_{\overrightarrow{C}}(u,v) \geqslant t+3$ and $\mathrm{dist}_{\overrightarrow{C}}(v,u) \geqslant t+3$ (exchanging the role of u and v). Thus $L_u^+ \cap L_v^+ = \varnothing$. We choose $x \in L_u^+$ with $\mathrm{dist}_{\overrightarrow{C}}(u,x)$ minimum and $y \in L_v^+$ with $\mathrm{dist}_{\overrightarrow{C}}(v,y)$ minimum such that $xy \in E(G)$. By this choice of x and y, it follows that $E_G(V(u^+\overrightarrow{C}x^-), V(v^+\overrightarrow{C}y^-)) = \varnothing$. Let $C^* = u\overleftarrow{C}yx\overrightarrow{C}vv^*Pu^*u$. Since H is complete of order λ and no vertex of H is adjacent in G to any vertex of $u^+\overrightarrow{C}x^-$ or $v^+\overrightarrow{C}y^-$ by the first part of the statement, $H - V(P)$ is a component of $G - V(C^*)$ of order at most $\lambda - 1$. Also $u^+\overrightarrow{C}x^-$ and $v^+\overrightarrow{C}y^-$ are components of $G - V(C^*)$ of order at most $t+1$. Since $E_G(V(u^+\overrightarrow{C}x^-), V(v^+\overrightarrow{C}y^-)) = \varnothing$, by (**1**), for each vertex $w \in V(u^+\overrightarrow{C}x^-) \cup V(v^+\overrightarrow{C}y^-)$, $\deg_G(w, C^*) > \frac{n}{t+1} - 1$. Applying Lemma 9, we find a cycle C^{**} of G with $V(C^{**}) = V(C^*) \cup V(u^+\overrightarrow{C}x^-) \cup V(v^+\overrightarrow{C}y^-)$. Since $V(G) \setminus V(C^{**}) = V(H) \setminus V(P)$, C^{**} is a D_λ-cycle of G, contradicting the choice of C. □

Claim 14. $\omega \leqslant \frac{n}{2(t+1)} - \frac{1}{2}$.

Proof. Assume otherwise that $\omega > \frac{n}{2(t+1)} - \frac{1}{2}$. By Claim 13, for any two distinct $u, v \in W$, $G[L_u^+]$ and $G[L_v^+]$ are remote, and $G[L_u^+]$ and H are remote. Thus in G, there are $\omega + 1$ pairwise remote subgraphs. By the definition, $G[L_u^+]$ has order $t+2$ for each $u \in W$. Let $S = (V(G) \setminus V(H)) \setminus (\bigcup_{u \in W} L_u^+)$. Then $|S| \leqslant n - \lambda - \frac{n(t+2)}{2(t+1)} + \frac{t+2}{2} \leqslant \frac{tn}{2(t+1)} + \frac{t}{2}$. Thus

$$\frac{|S|}{c(G-S)} \leqslant \frac{\frac{tn}{2(t+1)} + \frac{t}{2}}{\omega + 1} < \frac{\frac{tn}{2(t+1)} + \frac{t}{2}}{\frac{n}{2(t+1)} + \frac{1}{2}} = t,$$

contradicting the toughness of G. □

Since $\omega \geqslant 2t$, by Claim 14, we have

$$n \geqslant 4t(t+1). \tag{2}$$

Claim 15. $\lambda+\omega \leqslant \frac{3n}{4(t+1)}+t$.

Proof. Assume to the contrary that $\lambda+\omega > \frac{3n}{4(t+1)}+t$. By Claim 14, we know that $\omega \leqslant \frac{n}{2(t+1)}-\frac{1}{2}$. Since $\omega \geqslant 2t$, Lemma 7 implies that $\lambda \leqslant \frac{n}{2t+1}-t$. Thus $\omega > \frac{3n}{4(t+1)}-\frac{n}{2t+1}+2t$. By Lemma 7, we have

$$\begin{aligned}
n &\geqslant (\lambda+t)(\omega+1)\\
&> \left(\frac{3n}{4(t+1)}+t-\omega+t\right)(\omega+1) \qquad \left(\lambda+\omega > \frac{3n}{4(t+1)}+t \text{ by the assumption.}\right)\\
&\geqslant \begin{cases} \frac{n}{2t+1}\left(\frac{3n}{4(t+1)}+2t-\frac{n}{2t+1}+1\right), & \text{if } f(\omega)=\left(\frac{3n}{4(t+1)}+t-\omega+t\right)(\omega+1) \text{ is increasing;}\\ \left(\frac{n}{4(t+1)}+2t+\frac{1}{2}\right)\left(\frac{n}{2(t+1)}+\frac{1}{2}\right), & \text{if } f(\omega)=\left(\frac{3n}{4(t+1)}+t-\omega+t\right)(\omega+1) \text{ is decreasing;}\end{cases}\\
&> \begin{cases} \frac{n}{2t+1}(2t+1)=n, & \text{if } f(\omega) \text{ is increasing;}\\ \left(\frac{n}{4(t+1)}+2t+\frac{1}{2}\right)\frac{n}{2(t+1)}+\frac{n}{8(t+1)}+t & \text{if } f(\omega) \text{ is decreasing;}\\ > \begin{cases} \left(2t+2t+\frac{1}{2}\right)\frac{n}{2(t+1)} > n, & \text{if } n \geqslant 8t(t+1);\\ \left(3t+\frac{1}{2}\right)\frac{n}{2(t+1)}+\frac{n}{8(t+1)}+t = \frac{3tn}{2(t+1)}+\frac{3n+8t(t+1)}{8(t+1)} > n, & \text{if } n < 8t(t+1);\end{cases}\end{cases}
\end{aligned}$$

achieving a contradiction, where $n \geqslant 4t(t+1)$ was used to obtain $\frac{n}{4(t+1)} \geqslant t$ in the last inequality when $f(\omega)$ is decreasing. □

By Claim 12 and Claim 15, we have

$$\deg_G(v,C) > \frac{1.25n}{t+1}-1 \quad \text{for any } v \in V(C)\setminus W. \tag{3}$$

We will now explore the neighborhood of vertices from $W^+ := \{w \in V(C) : w^- \in W\}$, and show that some vertices from the neighborhood have similar properties as those in W^+. By Claim 10, we know that $|W| \geqslant 3$ and so $|W^+| \geqslant 3$. Equation (**3**) allows us to construct the vertex sets $L_x^-(\frac{0.25n}{t+1}+2)$ for each $x \in N_C(W^+)$. For notation simplicity, we use L_x^- for $L_x^-(\frac{0.25n}{t+1}+2)$. Note that the statement below is not true in general if we replace $L_x^-(\frac{0.25n}{t+1}+2)$ by $L_x^+(\frac{0.25n}{t+1}+2)$.

Claim 16. *Let $u \in W^+$ and $x \in N_C(u)$. Then*

(1) $L_x^- \cap W = \varnothing$.

(2) Let $v \in W^+$ and $y \in N_C(v)$ such that ux and vy are two crossing chords of C. Then $\mathrm{dist}_{\overrightarrow{C}}(x,y) \geqslant \frac{0.25n}{t+1}+3$.

Proof. For Statement (1), suppose to the contrary that there exists $z \in W$ such that $z \in L_x^-$. Then $\mathrm{dist}_{\overrightarrow{C}}(z,x) \leqslant \frac{0.25n}{t+1}+2$. We choose $z \in W$ with $\mathrm{dist}_{\overrightarrow{C}}(z,x)$ minimum. Then $V(z^+\overrightarrow{C}x^-)\cap W = \varnothing$ and $\mathrm{dist}_{\overrightarrow{C}}(z,x) \leqslant \frac{0.25n}{t+1}+2$. Let $u^* \in N_H(u^-)$, $z^* \in N_H(z)$, and P^* be a (u^*,z^*)-path of H. Then $C^* = z\overleftarrow{C}ux\overrightarrow{C}u^-u^*P^*z^*z$ is a cycle. As H is complete of order λ and $V(z^+\overrightarrow{C}x^-)\cap W = \varnothing$, we know that $H-V(P^*)$ is a component of $G-V(C^*)$ of order at most $\lambda-1$. Also, $z^+\overrightarrow{C}x^-$ is a component of $G-V(C^*)$ of order at most $\frac{0.25n}{t+1}+1$. By (**3**), for each vertex $w \in V(z^+\overrightarrow{C}x^-)$, $\deg_G(w, C^*) > \frac{n}{t+1}-1$. Applying Lemma 9, we find a cycle C^{**} of G with $V(C^{**}) = V(C^*) \cup V(z^+\overrightarrow{C}x^-)$. Since $V(G) \setminus V(C^{**}) = V(H) \setminus V(P^*)$, C^{**} is a D_λ-cycle of G, contradicting the choice of C.

Let $u^* \in N_H(u^-), v^* \in N_H(v^-)$ and P be a (u^*, v^*)-path of H. For Statement (2), suppose to the contrary that $\mathrm{dist}_{\overrightarrow{C}}(x, y) \leqslant \frac{0.25n}{t+1}+2$. We assume without loss of generality that u, v, x, y appear in this order along $\overrightarrow{C}$. Let $C^* = u\overrightarrow{C}v^-v^*Pu^*u^-\overleftarrow{C}yv\overrightarrow{C}xu$. Since H is complete of order λ and $V(x^+\overrightarrow{C}y^-) \cap W = \varnothing$ by Statement (1) (note $V(x^+\overrightarrow{C}y^-) \subseteq L_y^-$), $H-V(P)$ is a component of $G-V(C^*)$ of order at most $\lambda-1$. Also, $x^+\overrightarrow{C}y^-$ is a component of $G-V(C^*)$ of order at most $\frac{0.25n}{t+1}+1$. By (**3**), for each vertex $w \in V(x^+\overrightarrow{C}y^-)$, $\deg_G(w, C^*) > \frac{n}{t+1}-1$. Applying Lemma 9, we find a cycle C^{**} of G with $V(C^{**}) = V(C^*) \cup V(x^+\overrightarrow{C}y^-)$. Since $V(G) \setminus V(C^{**}) = V(H) \setminus V(P)$, C^{**} is a D_λ-cycle of G, contradicting the choice of C. □

For two distinct vertices $x, y \in N_C(W^+)$, we say x and y form a *crossing* if there exist distinct vertices $u, v \in W^+$ such that ux and vy are crossing chords of C. By Claim 16(2), there are at least $\frac{0.5n}{t+1}+2$ vertices between x and y along $\overrightarrow{C}$ for any two $x, y \in N_C(W^+)$ such that x and y form a crossing. Our goal below is to find at least $\frac{n}{2(t+1)}$ vertices from $N_C(W^+)$ such that there are at least $\frac{0.5n}{t+1}+2$ vertices between any two of them along $\overrightarrow{C}$. Then we will reach a contradiction by showing that $|V(C)| \geqslant n$. Define

$$A = \{u \in V(C) : \deg_G(u, W^+) = 1\} \quad \text{and} \quad B = \{u \in V(C) : \deg_G(u, W^+) \geqslant 2\}.$$

Let $u \in W^+$ and $p = \deg_G(u, B)$ for some positive integer p, and let

$$N_G(u) \cap B = \{x_1, x_2, x_3, \ldots, x_p\}.$$

We may assume that $x_1, x_2, \ldots, x_p$ appear in the same order along $\overrightarrow{C}$. We separate those vertices according to vertices of W. By Claim 16(1), we have $L_{x_i}^- \cap W = \varnothing$ for each $i \in [1, p]$. Therefore for some integer $q \geqslant 1$, we assume that $x_1, \ldots, x_p$ are grouped into q sets

$$B_1 = \{x_{b_0+1}, \ldots, x_{b_1}\}, \quad B_2 = \{x_{b_1+1}, \ldots, x_{b_2}\}, \quad \ldots, \quad B_q = \{x_{b_{q-1}+1}, \ldots, x_{b_q}\},$$

where $b_0 = 0$ and $b_q = p$, such that $V(x_{b_j+1}^+\overrightarrow{C}x_{b_{j+1}}^-) \cap W = \varnothing$ for each $j \in [0, q-1]$. Furthermore, we may assume that the number q of sets with the property above is minimum. As $W \neq \varnothing$, the minimality of q in turn implies $V(x_{b_j}\overrightarrow{C}x_{b_j+1}) \cap W \neq \varnothing$ for each $j \in [1, q]$, where $x_{b_q+1} := x_1$. Hence, by Claim 16(1), we have

$$\mathrm{dist}_{\overrightarrow{C}}(x_{b_j}, x_{b_j+1}) \geqslant \frac{0.5n}{t+1}+3. \tag{4}$$

Claim 17. *For each $i \in [1,q]$, B_i has at least $|B_i|/2$ vertices such that the distance between any two of them on C is at least $\frac{0.5n}{t+1}+3$.*

Proof. We partition B_i into two subsets according to whether or not vertices in $N_G(x) \cap (W^+ \setminus \{u\})$ fall into $x_{b_i}\overrightarrow{C}u$ for $x \in B_i$. Define

$$B_{i1} = \{x \in B_i \,:\, (N_G(x) \cap (W^+ \setminus \{u\})) \cap V(x_{b_i}\overrightarrow{C}u) \neq \varnothing\}, \quad B_{i2} = B_i \setminus B_{i1}.$$

By the Pigeonhole Principle, we have $|B_{i1}| \geqslant |B_i|/2$ or $|B_{i2}| \geqslant |B_i|/2$. We show that any two distinct vertices from B_{i1} or from B_{i2} have distance at least $\frac{0.5n}{t+1}+3$ between them on C. Let $x_a, x_b \in B_{i1}$ or $x_a, x_b \in B_{i2}$ be distinct, where $a, b \in [b_{i-1}+1, b_i]$. If $a > b$, then $\mathrm{dist}_{\overrightarrow{C}}(x_a, x_b) \geqslant \mathrm{dist}_{\overrightarrow{C}}(x_{b_i}, x_{b_{i-1}+1}) \geqslant \mathrm{dist}_{\overrightarrow{C}}(x_{b_i}, x_{b_i+1})$. As $\mathrm{dist}_{\overrightarrow{C}}(x_{b_i}, x_{b_i+1}) \geqslant \frac{0.5n}{t+1}+3$ by (**4**), we have $\mathrm{dist}_{\overrightarrow{C}}(x_a, x_b) \geqslant \frac{0.5n}{t+1}+3$. Thus we assume $a < b$.

If $x_a, x_b \in B_{i1}$, since $V(x^+_{b_{i-1}+1}\overrightarrow{C}x^-_{b_{i+1}}) \cap W = \varnothing$, then we know $x_{b_i} \notin W^+$. Thus by the definition of B_{i1}, there exists $v \in (N_G(x_a) \cap (W^+ \setminus \{u\})) \cap V(x^+_{b_i}\overrightarrow{C}u)$. Then the four vertices u, v, x_a, x_b appear in the order x_a, x_b, v, u along $\overrightarrow{C}$ and so vx_a and ux_b are crossing chords of C. By Claim 16(2), we have $\mathrm{dist}_{\overrightarrow{C}}(x_a, x_b) \geqslant \frac{0.5n}{t+1}+3$.

Suppose now that $x_a, x_b \in B_{i2}$. If $a > b_{i-1}+1$ or $a = b_{i-1}+1$ but $x_a \notin W$, then $W^+ \cap V(x_a\overrightarrow{C}x_b) = \varnothing$ by the property of B_i that $V(x^+_{b_{i-1}+1}\overrightarrow{C}x^-_{b_{i+1}}) \cap W = \varnothing$. By the definition of B_{i2}, there exists $v \in (N_G(x_b) \cap (W^+ \setminus \{u\})) \cap V(u\overrightarrow{C}x_{b_{i-1}+1})$. Then the four vertices u, v, x_a, x_b appear in the order x_a, x_b, u, v along $\overrightarrow{C}$ and so ux_a and vx_b are crossing chords of C. By Claim 16(2), we have $\mathrm{dist}_{\overrightarrow{C}}(x_a, x_b) \geqslant \frac{0.5n}{t+1}+3$. Thus we assume $a = b_{i-1}+1$ and $x_a \in W$. By Claim 16(1), we know that $\mathrm{dist}_{\overrightarrow{C}}(x_a, x_{a+1}) \geqslant \frac{0.5n}{t+1}+3$ and so $\mathrm{dist}_{\overrightarrow{C}}(x_a, x_b) \geqslant \frac{0.5n}{t+1}+3$.

By Claim 17, for each $i \in [1,q]$, we take a subset of at least $|B_i|/2$ vertices from B_i such that the distance between any two of them on C is at least $\frac{0.5n}{t+1}+3$. We let $\{y_1, y_2, \ldots, y_k\}$ be the union of all these q subsets of vertices. Then $k \geqslant \lceil \frac{p}{2} \rceil$. We further assume those vertices appear in the order $y_1, \ldots, y_k$ along $\overrightarrow{C}$. For any two distinct y_j, y_ℓ with $j, \ell \in [1,k]$, if y_j, y_ℓ are from the same B_i for some $i \in [1,q]$, then we have $\mathrm{dist}_{\overrightarrow{C}}(y_j, y_\ell) \geqslant \frac{0.5n}{t+1}+3$ by Claim 17. Otherwise, by (**4**), we also have $\mathrm{dist}_{\overrightarrow{C}}(y_j, y_\ell) \geqslant \frac{0.5n}{t+1}+3$. Thus by (**2**) that $n \geqslant 4t(t+1)$, we have $n > |V(C)| \geqslant k\left(\frac{0.5n}{t+1}+2\right) \geqslant k(2t+2)$. This inequality implies $k \leqslant \frac{n}{2(t+1)}$. Therefore

$$\deg_G(u, B) \leqslant 2k \leqslant \frac{n}{t+1} \quad \text{for any } u \in W^+. \tag{5}$$

Let $s = \sum_{v \in W^+} \deg_G(v, B)$ for some positive integer s. Then there exists $u \in W^+$ such that $\deg_G(u, B) \geqslant \frac{s}{|W^+|} = \frac{s}{\omega}$. Following the notation defined above, we let $\{y_1, \ldots, y_k\} \subseteq N_G(u) \cap B$ such that $\mathrm{dist}_{\overrightarrow{C}}(y_i, y_j) \geqslant \frac{0.5n}{t+1}+3$ for any distinct $i, j \in [1,k]$, where $k \geqslant \frac{1}{2}\deg_G(u, B)$. By Claim 16(1), we have $L^-_{y_i} \cap W = \varnothing$ for any $i \in [1,k]$. Thus, as

$n \geqslant 4t(t+1)$ by (**2**) and $t > 1$, we have $\mathrm{dist}_{\overrightarrow{C}}(y_i, w^+) \geqslant \frac{0.5n}{t+1} + 2 \geqslant 2t + 2 > t + 3$ for any $w \in W$. Thus for any $i \in [1, k]$, $y_i^+ \overrightarrow{C} y_{i+1}^-$ has at least $t + 2$ vertices that are nonadjacent in G to any vertex of W^+, where $y_{k+1} := y_1$. Hence

$$\begin{aligned} |A| &\leqslant n - |B| - \frac{1}{2}\deg_G(u, B)(t+2) \\ &\leqslant n - \frac{s}{\omega} - \frac{s}{2\omega}(t+2), \end{aligned}$$

since $|B| \geqslant \frac{s}{\omega}$. By (**3**), we get

$$\omega\left(\frac{1.25n}{t+1} - 1\right) < \sum_{u \in W^+} d_G(u) = |A| + s \leqslant n - \frac{s}{\omega} - \frac{s}{2\omega}(t+2) + s. \tag{6}$$

Since $\omega \geqslant 2t$, $n \geqslant 4t(t+1)$ by (**2**) and $t > 1$, it follows that $\omega(\frac{1.25n}{t+1} - 1) \geqslant \frac{2tn}{t+1} > n$. Thus, $-\frac{s}{\omega} - \frac{s}{2\omega}(t+2) + s > 0$ and so $2\omega - t - 4 > 0$. Thus by (**6**),

$$s > \frac{2\omega^2(\frac{1.25n}{t+1} - 1) - 2n\omega}{2\omega - t - 4}.$$

Next, we claim

$$\frac{2\omega^2(\frac{1.25n}{t+1} - 1) - 2n\omega}{2\omega - t - 4} > \frac{n\omega}{t+1}, \tag{7}$$

which will in turn give $s > \frac{n\omega}{t+1}$. As $s = \sum_{v \in W^+} d_G(v, B)$ and $|W^+| = \omega$, it then will follow that there exists $u \in W^+$ with $\deg_G(u, B) > \frac{n}{t+1}$, and so will give a contradiction to (**5**). To prove (**7**), it suffices to show that $2\omega(\frac{1.25n}{t+1} - 1) - 2n > \frac{n(2\omega - t - 4)}{t+1}$, which is true as shown below:

$$\begin{aligned} & \frac{2n}{t+1} - 4t > 0 && \text{(since } n \geqslant 4t(t+1) \text{ by } (\mathbf{2})) \\ \Longrightarrow\ & \frac{0.5\omega n}{t+1} + \frac{(t+4)n}{t+1} - 2\omega - 2n > 0 && \text{(the left-hand side increases in } \omega,\ \omega \geqslant 2t) \\ \Longleftrightarrow\ & 2\omega(\frac{1.25n}{t+1} - 1) - 2n > \frac{n(2\omega - t - 4)}{t+1}. && \square \end{aligned}$$

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Corrigendum – Added March 23, 2022

Thanks to Dr. Lingjuan Shi who pointed out to me that the proof of Lemma 7 misses the case when $G-V(C)$ has components other than H. The new proof, following the same idea as in the original one, combines the two cases by defining the "remote subgraphs" differently. Furthermore, the result is now stated more generally by choosing the cycle C with a different constraint and so the component H can be taken to have any order.

Lemma 7. *Let G be a t-tough 2-connected graph of order n. Suppose G has a D_{s+1}-cycle but no D_s-cycle for some integer $s \geqslant 1$. Let C be a D_{s+1}-cycle of G such that C minimizes $c_p(G-V(C))$ prior to minimizing $c_q(G-V(C))$ for any $p,q \in [1,s]$ with $p > q$. Then $n \geqslant (t+|V(H)|)(d_G(H)+1)$ for any component H of $G-V(C)$.*

Proof. Let $\lambda = |V(H)|$ and $k = d_G(H)$, where note that k is the total number of neighbors of vertices of H on C. We assume the k neighbors are $v_1,\ldots,v_k$ and appear in the same order along $\overrightarrow{C}$. For each $i \in [1,k]$, and each $v \in V(v_i^+\overrightarrow{C}v_{i+1}^-)$, where $v_{k+1} := v_1$, we let $\mathcal{H}(v)$ be the set of components of $G-V(C)$ that have a vertex joining to v by an edge in G. Note that $H \notin \mathcal{H}(v)$. Let $w_i^* \in V(v_i^+\overrightarrow{C}v_{i+1}^-)$ be the vertex with $\text{dist}_{\overrightarrow{C}}(v_i, w_i^*)$ minimum such that

$$\sum_{D \in \bigcup_{v \in V(v_i^+\overrightarrow{C}w_i^*)} \mathcal{H}(v)} |V(D)| + |V(v_i^+\overrightarrow{C}w_i^*)| \geqslant \lambda.$$

If such a vertex w_i^* exists, let $L^*_{v_i}(\lambda)$ be the union of the vertex set $V(v_i^+\overrightarrow{C}w_i^*)$ and all those vertex sets of graphs in $\bigcup_{v \in V(v_i^+\overrightarrow{C}w_i^*)} \mathcal{H}(v)$; if such a vertex w_i^* does not exist, let $L^*_{v_i}(\lambda) = L^+_{v_i}(\lambda)$. Note that when w_i^* exists, by its definition, $w_i^* \in V(v_i^+\overrightarrow{C}v_{i+1}^-)$. Thus $V(v_i^+\overrightarrow{C}w_i^*) \cap V(v_j^+\overrightarrow{C}w_j^*) = \emptyset$ if both w_i^* and w_j^* exist for distinct $i,j \in [1,k]$.

To prove the lemma, it suffices to show that $L^*_{v_1}(\lambda),\ldots,L^*_{v_k}(\lambda)$ and H are pairwise remote. Since in that case, if we let $S = V(G) \setminus \Big((\bigcup_{i=1}^k L^*_{v_i}(\lambda)) \cup V(H)\Big)$, then $|S| \leqslant n-(k+1)\lambda$ and $c(G-S) = k+1$. As G is t-tough, we get

$$n-(k+1)\lambda \geqslant |S| \geqslant t \cdot c(G-S) = t(k+1),$$

giving $n \geqslant (t+\lambda)(k+1)$.

Below, we show that $L^*_{v_1}(\lambda),\ldots,L^*_{v_k}(\lambda)$ and H are pairwise remote. It suffices to prove Statement (a): $\text{dist}_{\overrightarrow{C}}(v_i, v_{i+1}) \geqslant \lambda+1$ if for some $i \in [1,k]$ it holds that $L^*_{v_i}(\lambda) = L^+_{v_i}(\lambda)$, where $v_{k+1} := v_1$ when $i = k$ (this implies that each $L^*_{v_i}(\lambda)$ and H are remote), and $L^*_{v_i}(\lambda) \cap L^*_{v_j}(\lambda) = \emptyset$ for every two distinct $i,j \in [1,k]$; and Statement (b): $E_G(L^*_{v_i}(\lambda), L^*_{v_j}(\lambda)) = \emptyset$ for every two distinct $i,j \in [1,k]$. Let $v_i^* \in N_H(v_i), v_j^* \in N_H(v_j)$ and P be a (v_i^*, v_j^*)-path of H for any distinct $i,j \in [1,k]$.

For Statement (a), it suffices to show that if for some $i \in [1,k]$ it holds that $L^*_{v_i}(\lambda) = L^+_{v_i}(\lambda)$, then $\mathrm{dist}_{\overrightarrow{C}}(v_i, v_{i+1}) \geqslant \lambda + 1$, where $v_{k+1} := v_1$ when $i = k$; and that for distinct $i, j \in [1,k]$, $v \in L^*_{v_i}(\lambda) \cap V(v_i^+ \overrightarrow{C} v_{i+1}^-)$ and $u \in L^*_{v_j}(\lambda) \cap V(v_j^+ \overrightarrow{C} v_{j+1}^-)$, we have $\mathcal{H}(v) \cap \mathcal{H}(u) = \emptyset$. We prove the statement by contradiction. If $L^*_{v_i}(\lambda) = L^+_{v_i}(\lambda)$ for some $i \in [1,k]$ but $\mathrm{dist}_{\overrightarrow{C}}(v_i, v_{i+1}) \leqslant \lambda$, we then let $C^* = v_i \overleftarrow{C} v_{i+1} v^*_{i+1} P v^*_i v_i$. Since H has order λ and no vertex of H is adjacent in G to any internal vertex of $v_i \overrightarrow{C} v_{i+1}$, it follows that each component of $H - V(P)$ is a component of $G - V(C^*)$ of order at most $\lambda - 1$ and $v_i^+ \overrightarrow{C} v_{i+1}^-$ is contained in a component of $G - V(C^*)$ with order at most $\lambda - 1$ since $L^*_{v_i}(\lambda) = L^+_{v_i}(\lambda)$. Thus C^* is a D_{s+1}-cycle of G with $c_p(G - V(C^*)) = c_p(G - V(C))$ for any $p \in [1,s]$ and $p > \lambda$ but $c_\lambda(G - V(C^*)) < c_\lambda(G - V(C))$, contradicting the choice of C. If for some distinct $i, j \in [1,k]$, $v \in L^*_{v_i}(\lambda) \cap V(v_i^+ \overrightarrow{C} v_{i+1}^-)$ and $u \in L^*_{v_j}(\lambda) \cap V(v_j^+ \overrightarrow{C} v_{j+1}^-)$, we have $\mathcal{H}(v) \cap \mathcal{H}(u) \neq \emptyset$, we then further choose v closest to v_i and u closest to v_j along $\overrightarrow{C}$ with the property. Thus for any $w_i \in V(v_i^+ \overrightarrow{C} v^-)$ and any $w_j \in V(v_j^+ \overrightarrow{C} u^-)$, it holds that $\mathcal{H}(w_i) \cap \mathcal{H}(w_j) = \emptyset$. Let $D \in \mathcal{H}(v) \cap \mathcal{H}(u)$ and $v', u' \in V(D)$ such that $vv', uu' \in E(G)$. Let P' be a (v', u')-path of D and $C^* = v_i v^*_i P v^*_j v_j \overleftarrow{C} v v' P' u' u \overrightarrow{C} v_i$. Since H has order λ and no vertex of H is adjacent in G to any vertex in $v_i^+ \overrightarrow{C} v^-$ or any vertex in $v_j^+ \overrightarrow{C} u^-$, it follows that each component of $H - V(P)$ is a component of $G - V(C^*)$ of order at most $\lambda - 1$. Furthermore, by the choices of v and u, the components of $G - V(C^*)$ that respectively contain $v_i^+ \overrightarrow{C} v^-$ and $v_j^+ \overrightarrow{C} u^-$ are disjoint. Since $V(v_i^+ \overrightarrow{C} v^-)$ is a proper subset of $L^*_{v_i}(\lambda) \cap V(v_i^+ \overrightarrow{C} v_{i+1}^-)$ and $V(v_j^+ \overrightarrow{C} u^-)$ is a proper subset of $L^*_{v_j}(\lambda) \cap V(v_j^+ \overrightarrow{C} v_{j+1}^-)$, it follows by the definitions of $L^*_{v_i}$ and $L^*_{v_j}$ that the components of $G - V(C^*)$ that respectively contain $v_i^+ \overrightarrow{C} v^-$ and $v_j^+ \overrightarrow{C} u^-$ have order at most $\lambda - 1$. Thus C^* is a D_{s+1}-cycle of G with $c_p(G - V(C^*)) = c_p(G - V(C))$ for any $p \in [1,s]$ and $p > \lambda$ but $c_\lambda(G - V(C^*)) < c_\lambda(G - V(C))$, contradicting the choice of C. The argument above verifies Statement (a).

For Statement (b), assume to the contrary that $E_G(L^*_{v_i}(\lambda), L^*_{v_j}(\lambda)) \neq \emptyset$ for some distinct $i, j \in [1,k]$. Applying Statement (a), we know that $L^*_{v_i}(\lambda) \cap L^*_{v_j}(\lambda) = \emptyset$. Since there is no edge between any two components of $G - V(C)$, $E_G(L^*_{v_i}(\lambda), L^*_{v_j}(\lambda)) \neq \emptyset$ implies that there exist $x \in L^*_{v_i}(\lambda) \cap V(v_i^+ \overrightarrow{C} v_{i+1}^-)$ and $y \in L^*_{v_j}(\lambda) \cap V(v_j^+ \overrightarrow{C} v_{j+1}^-)$ such that $xy \in E(G)$. We choose $x \in L^*_{v_i}(\lambda) \cap V(v_i^+ \overrightarrow{C} v_{i+1}^-)$ with $\mathrm{dist}_{\overrightarrow{C}}(v_i, x)$ minimum and $y \in L^*_{v_j}(\lambda) \cap V(v_j^+ \overrightarrow{C} v_{j+1}^-)$ with $\mathrm{dist}_{\overrightarrow{C}}(v_j, y)$ minimum such that $xy \in E(G)$. By this choice of x and y, it follows that $E_G(V(v_i^+ \overrightarrow{C} x^-), V(v_j^+ \overrightarrow{C} y^-)) = \emptyset$. Let $C^* = v_i \overleftarrow{C} y x \overrightarrow{C} v_j v^*_j P v^*_i v_i$. Since H has order λ and no vertex of H is adjacent in G to any vertex of $v_i^+ \overrightarrow{C} x^-$ or $v_j^+ \overrightarrow{C} y^-$ by the fact that $v_i^+ \overrightarrow{C} x^- \subseteq v_i^+ \overrightarrow{C} v_{i+1}^-$ and $v_j^+ \overrightarrow{C} y^- \subseteq v_j^+ \overrightarrow{C} v_{j+1}^-$ from Statement (a), it follows that each component of $H - V(P)$ is a component of $G - V(C^*)$ of order at most $\lambda - 1$. Also $v_i^+ \overrightarrow{C} x^-$ and $v_j^+ \overrightarrow{C} y^-$ are contained in distinct components of $G - V(C^*)$ each of order at most $\lambda - 1$. Thus C^* is a D_{s+1}-cycle of G with $c_p(G - V(C^*)) = c_p(G - V(C))$ for any $p \in [1, s]$ and $p > \lambda$ but $c_\lambda(G - V(C^*)) < c_\lambda(G - V(C))$, contradicting the choice of C. This verifies Statement (b) and completes the proof. □

Remark: Reflecting the change in the statement of Lemma 7, in the proof of Theorem 4 in the original paper, we need to choose the cycle C to be a longest $D_{\lambda+1}$-cycle that minimizes $c_p(G - V(C))$ prior to minimizing $c_q(G - V(C))$ for any $p, q \in [1, \lambda]$ with $p > q$.