

On 1-subdivisions of transitive tournaments

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Submitted: Oct 11, 2021; Accepted: Dec 27, 2021; Published: Mar 25, 2022

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Abstract

The oriented Ramsey number $\vec{r}(H)$ for an acyclic digraph H is the minimum integer n such that any n -vertex tournament contains a copy of H as a subgraph. We prove that the 1-subdivision of the k -vertex transitive tournament H_k satisfies $\vec{r}(H_k) = O(k^2 \log \log k)$. This is tight up to multiplicative $\log \log k$ -term.

We also show that if T is an n -vertex tournament with $\Delta^+(T) - \delta^+(T) = O(n/k) - k^2$, then T contains a 1-subdivision of \vec{K}_k , a complete k -vertex digraph with all possible $k(k-1)$ arcs. This is tight up to multiplicative constant.

Mathematics Subject Classifications: 05C20, 05C35, 05D10

1 Introduction

What conditions guarantee certain structures on graphs? This is a central problem in extremal graph theory. Turán [18] proved that $e(G) > (1 - \frac{1}{r-1})\frac{n^2}{2}$ implies that G contains a complete graph K_r on r vertices as a subgraph. For a non-bipartite graph H , an asymptotic threshold $(1 - \frac{1}{\chi(H)-1} + o(1))\binom{n}{2}$ for an n -vertex graph G to contain H as a subgraph was determined by Erdős-Stone [9] and Erdős-Simonovits [8]. This threshold is quadratic in n .

For a (di)graph H , subdividing an edge (arc) uv exactly ℓ times is the operation of replacing the edge (arc) uv by an $(\ell + 1)$ -edge (directed)-path from u to v with newly introduced internal vertices. A *subdivision* of H is a (di)graph obtained by subdividing some edges (arcs) of H . The branch vertices of the subdivision are the vertices which were

*Supported by the POSCO Science Fellowship of POSCO TJ Park Foundation, and by the KAIX Challenge program of KAIST Advanced Institute for Science-X.

†Supported by the KAIST Undergraduate Research Participation program.

already there before subdividing edges. Unlike the above case, if we only want to ensure a subdivision of H within a graph G , a much weaker bound than quadratic number of edges is sufficient. In 1967, Mader [14] proved that for given k , there exists $f(k)$ such that every graph with average degree $f(k)$ contains K_k as a subdivision. Mader [14] and Erdős-Hajnal [6] conjectured that this $f(k)$ can be shown to be $O(k^2)$ and this was verified by Bollobás and Thomason [3] and independently by Komlós and Szemerédi [12].

Another key question in extremal combinatorics is a Ramsey-type question. For a given H , what values of n ensure that any 2-coloring on the edges of K_n contains a monochromatic H ? We write $r(H)$ to denote the smallest such n . In general, such a number $r(H)$ is exponential in $|H|$, as was shown to be

$$(\sqrt{2}/e - o(1))k2^{k/2} \leq r(K_k) \leq e^{-c(\log k)^2} \binom{2k}{k}$$

where $c > 0$ is an absolute constant. The lower bound is proved by Spencer in [16], and the upper bound is by Sah in [15]. See [4] for more recent developments in Ramsey theory. However, Alon [1] in 1994 proved that if H is a subdivision of another graph obtained by subdividing every edge at least once, the Ramsey number $r(H)$ is linear in $|H|$. Note that such a graph H is always 2-degenerate. This result was further improved by the celebrated result of Lee [13] in 2017 proving the Burr-Erdős conjecture stating that any d -degenerate graph has a linear Ramsey number.

There is an analogue considering tournaments instead of complete graphs. A *tournament* is an orientation of a complete graph. For a given oriented graph H , we define the *oriented Ramsey number* $\vec{r}(H)$ to be the smallest n where any n -vertex tournament contains a copy of H . Indeed, $\vec{r}(H)$ exists only when H is acyclic. This is because, no matter how large n is, the n -vertex *transitive tournament* T_n , which is the acyclic tournament of order n , does not contain any digraph with a cycle as a subgraph. More generally, for a collection \mathcal{H} of oriented graphs, we define $\vec{r}(\mathcal{H})$ to be the smallest n where any n -vertex tournament contains a copy of a graph in \mathcal{H} . Again, at least one graph in \mathcal{H} has to be acyclic for the parameter to be defined.

Stearns [17] in 1959 and Erdős and Moser [7] in 1964 initiated the study on the oriented Ramsey number and proved that

$$2^{k/2-1} \leq \vec{r}(T_k) \leq 2^{k-1}.$$

Since then, there has been no improvement to the exponential constants. Note that it is easy to observe $\vec{r}(T_k) \leq r(K_k)$, hence improving the above lower bound is at least as difficult as improving the lower bound on the Ramsey number.

As in the undirected case, we wonder whether a linear bound on $\vec{r}(H)$ holds if H is obtained by subdividing every arc of another digraph, in particular a subdivision of a transitive tournament. Girão, Popielarz, and Snyder [11] proved that any tournament on $\Omega(k^2)$ vertices contains a subdivision of T_k where each arc is subdivided at most twice (hence each directed path in the subdivision has length at most three). In other words, they proved $\vec{r}(\mathcal{H}) = O(k^2)$ when \mathcal{H} is the collection of all graphs which are obtained from T_k by subdividing every edge at most twice.

As subdividing more makes the graph sparser, to consider small number of subdivisions we say a (di)graph is the 1-*subdivision* of H if it is obtained from H by subdividing every edge (arc) exactly once. Let H_k be the 1-subdivision of T_k . Girão, Popielarz, and Snyder [11] proved that $\vec{r}(H_k) = O(k^2 \log^3 k)$ and posed the following conjecture.

Conjecture 1 (Girão, Popielarz, and Snyder). $\vec{r}(H_k) = O(k^2)$.

This conjecture, if true, is an analogue of Alon's result [1] on the subdivision as H_k contains $\binom{k+1}{2}$ vertices, hence $\vec{r}(H_k)$ is linear in $|H_k|$. In this note, we prove the following theorem improving the result of Girão, Popielarz, and Snyder.

Theorem 2. $\vec{r}(H_k) = O(k^2 \log \log k)$.

One natural question is whether an analogue of the Burr-Erdős conjecture for oriented graphs is true. In other words, this asks if degenerate acyclic oriented graphs have linear (or almost linear) Ramsey number. This was disproved by Fox, He, and Wigderson [10]. They showed that for each $\Delta \geq 2$ there exists an acyclic oriented graph H with both maximum outdegree and maximum indegree at most Δ such that $\vec{r}(H) = |H|^{\Omega(\Delta^{2/3}/\log^{5/3}(\Delta))}$. They also proved more general bounds on the oriented Ramsey number of digraphs with bounded maximum degree.

As we mentioned earlier, a transitive tournament does not contain any digraph with a cycle, so $\vec{r}(H)$ only makes sense when H is acyclic digraphs. What if we consider tournaments which are far from being transitive?

Alon, Krivelevich, and Sudakov [2] proved a conjecture of Erdős stating that any n -vertex graph with at least ϵn^2 edges contains a 1-subdivision of a complete graph on $c(\epsilon)\sqrt{n}$ vertices. This states that any graph far from the empty graph contains a subdivision of a large complete graph. In the case of tournaments, we similarly consider an n -vertex tournament T that is far from being transitive, and ask whether it contains a 1-subdivision of a large complete digraph \vec{K}_k . Here \vec{K}_k is the digraph on k vertices having \vec{uv} as an arc for each pair of distinct vertices u and v , and with no parallel arc.

We prove the following theorem stating that it is indeed possible to find a 1-subdivision of $\vec{K}_{\Omega(n^{1/3})}$ in a tournament T if we assume that all vertices in T have outdegree at most $n/2 + O(n^{2/3})$.

Theorem 3. *Suppose that T is an n -vertex tournament with $\Delta^+(T) - \delta^+(T) \leq \frac{n}{10k} - k^2$. Then T contains a 1-subdivision of \vec{K}_k .*

At first glance assuming all vertices have both out-degree and in-degree $n/2 + O(n/k)$ seems like a very strong assumption. However it turns out that this assumption is necessary. The following proposition shows that the above theorem is tight up to a multiplicative constant and an additive k^2 term.

Proposition 4. *For any integers n and k with $2 \leq k \leq \sqrt{n}$, there exists an n -vertex tournament T with $\Delta^+(T) - \delta^+(T) \leq 2n/k$ such that T does not contain 1-subdivision of \vec{K}_k .*

Proof. Consider an odd number $k' \in \{k-2, k-1\}$ and an k' -vertex regular tournament T' . Let $\ell = \lceil n/k' \rceil$. Blow up each vertex of T' into a transitive tournament of size either ℓ or $\ell-1$ to obtain an n -vertex tournament T . Let $V_1, \dots, V_{k'}$ be the vertex sets of those transitive tournaments.

Assume we have a 1-subdivision of \vec{K}_k in T . For those k branch vertices, two vertices u, v of them belong to the same part, say V_i , of T . As those two vertices has exactly same set of out/in-neighbors outside V_i , both paths from u to v and v to u must lie inside V_i . However, $T[V_i]$ is transitive, so one of two paths does not exist, a contradiction. Hence, T does not contain a 1-subdivision of \vec{K}_k and

$$\Delta^+(T) - \delta^+(T) \leq \left(\frac{k'-1}{2} \cdot \ell + (\ell-1) \right) - \frac{k'-1}{2} \cdot (\ell-1) = \frac{k'-1}{2} + \ell - 1 \leq \frac{2n}{k},$$

as desired. □

2 Preliminaries

We write $[n] = \{1, \dots, n\}$ and $\log = \log_2$. For a digraph D , we let $A(D)$ be the arc set of D , and $d_D^-(x)$ and $d_D^+(x)$ refer to the in-degree and out-degree of a vertex $x \in V(D)$, respectively. We denote by $\Delta^+(D)$ the maximum out-degree of D and $\delta^+(D)$ the minimum out-degree of D . We denote by $N_D^-(x)$ and $N_D^+(x)$ the in-neighbourhood and out-neighbourhood of $x \in V(D)$, respectively.

For a (di)graph G and $X \subset V(G)$, we denote by $G[X]$ the sub(di)graph of G induced by X .

Let G be an undirected graph and let u, v be vertices in G . We denote by $\text{dist}(u, v)$ the length (which is the same as the number of edges) of a shortest path between u and v in G . If such a path does not exist, then we let $\text{dist}(u, v) = \infty$. For each $i \in \mathbb{N}$, we let $N_G^i(v)$ be the set of all vertices u with $\text{dist}(v, u) = i$ and let $B_G^i(v) = \bigcup_{0 \leq j < i} N_G^j(v)$. For $X \subseteq V(G)$, let $B_G^1(X) = \bigcup_{v \in X} B_G^1(v)$.

We often omit the subscript D or G when the underlying (di)graph is clear from the context. We will omit floors and ceilings and treat large numbers as integers whenever it does not affect the argument.

To deal with the 1-subdivisions of digraphs, it is useful to count the number of directed paths of length two between two vertices. For a tournament T and $u, v \in V(T)$, let $P_2(u, v)$ be the number of paths of length at most 2 from u to v . As $P_2(u, v) = |N^+(u) \setminus N^+(v)|$, we have

$$P_2(u, v) - P_2(v, u) = d^+(u) - d^+(v). \tag{1}$$

Observe that P_2 satisfies the following triangle inequality.

Proposition 5. *For any $u, v, w \in V(T)$, we have $P_2(u, v) \leq P_2(u, w) + P_2(w, v)$.*

Proof. We have

$$\begin{aligned} P_2(u, w) + P_2(w, v) &= |N^+(u) \setminus N^+(w)| + |N^+(w) \setminus N^+(v)| \\ &\geq |N^+(u)| - |N^+(u) \cap N^+(w) \cap N^+(v)| \\ &\geq |N^+(u) \setminus N^+(v)| = P_2(u, v). \end{aligned} \quad \square$$

3 Proofs of the Theorems

We first prove Theorem 2. Our proof develops the ideas in [11]. Let H_k be the 1-subdivision of T_k . We will show that any tournament with at least $Ck^2 \log \log k$ vertices contains a copy of H_k . We use induction on k with the choice of $C = 2^{30}$. Note an m -vertex tournament contains a transitive tournament $T_{\log m}$ which contains a copy of $H_{\sqrt{\log m}}$. As $\sqrt{\log C} > 5$, we may assume $k \geq 6$. Moreover, as the induction hypothesis, we assume the following.

for each $s < k$, any tournament on at least $Cs^2 \log \log s$ vertices contains a copy of H_s . (2)

Let $n = Ck^2 \log \log k$ and T be an n -vertex tournament. We enumerate the vertices of T into v_1, \dots, v_n so that

$$d^+(v_1) \geq \dots \geq d^+(v_n). \quad (3)$$

We divide $V(T)$ into three sets as follows:

$$V_1 = \{v_1, \dots, v_{n/4}\}, \quad V_2 = \{v_{n/4+1}, \dots, v_{3n/4}\}, \quad \text{and} \quad V_3 = \{v_{3n/4+1}, \dots, v_n\}.$$

By (2), each of $G[V_1]$ and $G[V_3]$ contains a copy of $H_{k/2}$.

If $d_T^+(v_{n/4}) - d_T^+(v_{3n/4}) \geq k^2$, then for any vertex $u \in V_1$ and $v \in V_3$, we have $P_2(u, v) \geq k^2$. Hence, there are at least $k^2 - 1$ paths of length exactly 2 from u to v . As a copy of H_k contains $\binom{k}{2} + k < k^2 - 1$ vertices, we can greedily take internally disjoint paths of length 2 for each pair of vertices between two copies of $H_{k/2}$ until we obtain a copy of H_k . Thus we may assume that, for any $u, v \in V_2$,

$$d_T^+(u) - d_T^+(v) < k^2. \quad (4)$$

Now we consider an auxiliary graph G recording pairs in V_2 with not too many paths of length 2 between. Let

$$V(G) = V_2 \quad \text{and} \quad E(G) = \left\{ uv \in \binom{V_2}{2} : P_2(u, v) < k^2 \text{ or } P_2(v, u) < k^2 \right\}. \quad (5)$$

Then by (1) and (4), for any $uv \in E(G)$ we have

$$P_2(u, v) < 2k^2 \quad \text{and} \quad P_2(v, u) < 2k^2. \quad (6)$$

Note that the definition of G ensures that a large independent set in G yields a 1-subdivision of a large transitive tournament. Moreover, if we have t vertex sets with no edges between them in G , we can obtain a large 1-subdivision as follows.

Claim 6. Let $C_1, \dots, C_t \subseteq V(G)$ be disjoint nonempty vertex sets where G has no edges between C_i and C_j for $i \neq j \in [t]$. For each $i \in [t]$, let $m_i = \sqrt{|C_i|/(C \log \log k)}$. If $\sum_{i=1}^t m_i \geq 2k$, then T contains a copy of H_k .

Proof. Indeed, (2) ensures that each C_i contains a copy of $H_{\lfloor m_i \rfloor}$, and clearly it contains a copy of H_1 which is a vertex. Choose an integer $0 \leq n_i \leq \max\{1, \lfloor m_i \rfloor\}$ for each i so that $\sum_{i=1}^t n_i = k$. If $t \geq k$, then $n_i = 1$ for $1 \leq i \leq k$ and $n_i = 0$ for $i > k$ works; otherwise $\sum_{i=1}^t (m_i - 1) \geq k$, so again such n_i 's exist. By (5), we can greedily connect different copies of H_{n_i} in C_i to obtain a copy of H_k . \square

The above claim states that if G is not well-connected, then we can obtain a large 1-subdivision. On the other hand, the following claim states that our graph G cannot be too well-connected. The proof of this claim was a part of the argument in [11]. We include the proof for completeness.

Claim 7. For any $v \in V(G)$ and $r \in \mathbb{N}$, we have $|B_G^r(v)| \leq 20rk^2$.

Proof. Let $B = B_G^r(v)$ and consider the subtournament $T' = T[B]$. Choose a vertex x in T' with out-degree $d_{T'}^+(x) \geq (|T'| - 1)/2$. Again $T[N_{T'}^+(x)]$ forms another tournament, hence there exists $y \in N_{T'}^+(x)$ having at least $(d_{T'}^+(x) - 1)/2$ in-neighbors in $N_{T'}^+(x)$. Thus we have

$$P_2(x, y) \geq |N_T^+(x) \cap N_T^-(y)| \geq \frac{1}{4}|B| - 1.$$

However, as $x, y \in B$, there exists a path $(x = z_1, z_2, \dots, z_s = y)$ of length at most $2r$ between x and y in G . Hence Proposition 5 together with (6) implies that

$$P_2(x, y) \leq \sum_{i \in [s-1]} P_2(z_i, z_{i+1}) \leq 4rk^2.$$

This yields $|B| \leq 16rk^2 + 4 \leq 20rk^2$. \square

In order to utilize the above two claims in a right way, we need to quantify the expansion of graphs so that the claims give what we want.

Claim 8. Let G' be a subgraph of G . For any $v \in V(G')$, there exists $1 \leq r \leq 1 + \log \log k$ such that the following holds where $B^i = B_{G'}^i(v)$:

$$|B^r| \leq \frac{1}{10}n^{1/2}|B^{r-1}|^{1/2}.$$

Proof. If $|B^1| \leq \frac{1}{10}n^{1/2}$, then $r = 1$ suffices, so we assume $|B^1| \geq \frac{1}{10}n^{1/2} > 1$. Suppose that such an r does not exist, meaning that for each $r \in [1 + \log \log k]$, we have

$$|B^r| > \frac{1}{10}n^{1/2}|B^{r-1}|^{1/2}. \tag{7}$$

For each $i \in [1 + \log \log k]$, let α be a real number satisfying $|B^i| = n^{1-\alpha_i}$. Then we have $0 \leq \alpha_1 < 1$.

By (7), for each $i \in [1 + \log \log k]$, we have $n^{1-\alpha_i} > \frac{1}{10}n^{1-\frac{1}{2}\alpha_{i-1}}$, i.e., $\alpha_i < \frac{1}{2}\alpha_{i-1} + \frac{4}{\log n}$. Inductively applying this yields

$$\begin{aligned} \alpha_i &< \frac{1}{2}\alpha_{i-1} + \frac{4}{\log n} < \frac{1}{4}\alpha_{i-2} + \frac{4}{\log n} \left(1 + \frac{1}{2}\right) \\ &< \dots < 2^{1-i}\alpha_1 + \frac{4}{\log n} \sum_{j=0}^{i-2} 2^{-j} < 2^{1-i} + \frac{8}{\log n}. \end{aligned}$$

Hence, for $s = 1 + \log \log k$, we have $\alpha_s < \frac{1}{\log k} + \frac{8}{\log n}$. However, as $k \geq 6$, we have $n = 2^{30}k^2 \log \log k < k^{16}$, whence $\alpha_s < \frac{24}{\log n}$. Thus

$$|B^s| = n^{1-\alpha_s} \leq 2^{\log n - 24} = 2^{-24}n = 2^{-24}Ck^2 \log \log k.$$

However, Claim 7 implies $|B^s| \leq |B_G^s(v)| \leq 20sk^2 \leq 20k^2(1 + \log \log k) < 2^{-24}Ck^2 \log \log k$ as $C = 2^{30}$, a contradiction. Hence there exists $r \in [1 + \log \log k]$ such that $|B^{r+1}| > \frac{1}{10}n^{1/2}|B^r|$. \square

Now we are ready to prove Theorem 2. We take distinct vertices v_1, \dots, v_s in $V(G)$, positive integers r_1, \dots, r_s , vertex sets $X_1, \dots, X_s, Y_1, \dots, Y_s$, and subgraphs G_1, \dots, G_{s+1} of G satisfying the following with the maximum possible s .

(G1) For each $i \in [s]$, we have $r_i \leq 1 + \log \log k$.

(G2) For each $i \in [s]$, we have $G_1 = G$ and $G_{i+1} = G - \bigcup_{j=1}^i Y_j$ and $X_i = B_{G_i}^{r_i-1}(v_i)$ and $Y_i = B_{G_i}^{r_i}(v_i)$.

(G3) For each $i \in [s]$, we have $|Y_i| \leq \frac{1}{10}n^{1/2}|X_i|^{1/2}$.

Indeed, such a collection exists as the trivial collection with $s = 0$ vacuously satisfies all three conditions.

We claim that with such a choice with the maximum s , we have

$$\sum_{i=1}^s |Y_i| = |V(G)| = \frac{n}{2}.$$

Suppose not. Then G_{s+1} is a nonempty graph, so it contains a vertex. Let v_{s+1} be an arbitrary vertex in G_{s+1} . By applying Claim 8 with G_{s+1} and v_{s+1} , we can find $r_{s+1} \in [1 + \log \log k]$ such that $B_{G_{s+1}}^{r_{s+1}-1}(v_{s+1}) = X_{s+1}$ and $B_{G_{s+1}}^{r_{s+1}}(v_{s+1}) = Y_{s+1}$ satisfying (G1)–(G3), which contradicts to the maximality of s .

Furthermore, for each $i \in [s]$, as $r_i \leq 1 + \log \log k$, Claim 7 implies that

$$|X_i| \leq |B_G^{r_i-1}(v_i)| \leq 20k^2 \log \log k \leq \frac{n}{100}.$$

Note that X_1, \dots, X_s are disjoint vertex sets, and for $i < j$, we have $B_G^1(X_i) \cap X_j = Y_i \cap X_j = \emptyset$, so there are no edges between X_i and X_j in G . In addition, we have

$$\sum_{i \in [s]} \sqrt{\frac{|X_i|}{C \log \log k}} \geq \sum_{i \in [s]} \frac{10|Y_i|}{n^{1/2}(C \log \log k)^{1/2}} = \frac{5n}{n^{1/2}(C \log \log k)^{1/2}} \geq 2k.$$

Hence Claim 6 applies so that T contains a copy of H_k . This finishes the proof of Theorem 2.

Now we prove Theorem 3. Let $\ell = \Delta^+(T) - \delta^+(T)$. As in the previous section, let $P_2(u, v)$ be the number of paths from u to v of length at most two, and we construct an auxiliary graph G with $V(G) = V(T)$ and

$$E(G) = \{uv \in \binom{V(G)}{2} : P_2(u, v) < k^2 \text{ or } P_2(v, u) < k^2\}.$$

As in (6), we can use the above to obtain that for $uv \in E(G)$,

$$P_2(u, v) < k^2 + \ell \quad \text{and} \quad P_2(v, u) < k^2 + \ell. \quad (8)$$

Moreover, as in the proof of Claim 7, we can show that any vertex v satisfies $|B_G^r(v)| \leq 10r(k^2 + \ell)$ for any $r \in \mathbb{N}$. In particular, with $r = 1$, this implies that any vertex v has degree at most $10(k^2 + \ell) - 1$ in G . By Turán's theorem, G contains an independent set of size at least $\frac{n}{10(k^2 + \ell)} \geq \frac{n}{10(k^2 + n/(10k) - k^2)} = k$. Take such an independent set $\{v_1, \dots, v_k\}$ and greedily connect all pairs with internally disjoint paths. Then we obtain a 1-subdivision of \vec{K}_k . This finishes the proof of Theorem 3.

Acknowledgement

Shortly after we uploaded our draft to the arXiv, Draganic, Correia, Sudakov, and Yuster [5] confirmed Conjecture 1 by proving a stronger bound $r(H_k) \leq (2 + o(1))k^2$.

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