Ramsey numbers of fans and large books

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Abstract

For graphs G and H, the Ramsey number $R(G, H)$ is the minimum integer N such that any red/blue edge coloring of K_N contains either a red G or a blue H. Let $G + H$ be the graph obtained from vertex disjoint G and H by adding new edges connecting G and H completely, $F_m = K_1 + mK_2$ and $B_p(n) = K_p + nK_1$. It is shown $R(F_m, B_p(n)) = 2(n + p - 1) + 1$ for fixed m, p and large n.

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1 Introduction

Our notation is standard. For positive functions $f(n)$ and $g(n)$ we write that $f(n) \leqslant O(g(n))$ if there exists a positive constant c so that $f(n) \leqslant cg(n)$ and that $f(n) \geqslant \Omega(g(n))$ if $g(n) \leqslant O(f(n))$ for all sufficiently large n. We write $f(n) = \Theta(g(n))$ if $\Omega(g(n)) \leqslant f(n) \leqslant O(g(n))$.

For graphs G and H, the Ramsey number $R(G, H)$ is the minimum N such that any red/blue edge coloring of K_N contains either a red G or a blue H. The following is a celebrated result of Chvátal for tree T_n of order n and complete graph K_m of order m.

Theorem 1 ([5]). *Let $m, n \geqslant 1$ be integers. Then $R(K_m, T_n) = (m - 1)(n - 1) + 1$.*

Let $\chi(F)$ be the chromatic number of F, and $s(F)$ the minimum size of a color class over all proper vertex colorings of F with $\chi(F)$ colors. Burr observed following general lower bound.

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Lemma 2 ([3]). *Let H be a connected graph of order $|H| \geqslant s(F)$. Then*

$$R(F,H) \geqslant (\chi(F)-1)(|H|-1)+s(F).$$

Burr defined H to be *F-good* if the equality in Lemma 2 holds. Thus Theorem 1 tells us that T_n is *K_m-good*. Many results on Ramsey numbers can be found in the dynamic survey [28].

For vertex disjoint graphs G_1 and G_2, denote by $G_1 \cup G_2$ the union of G_1 and G_2, and G_1+G_2 the graph obtained from $G_1 \cup G_2$ by adding new edges to connect G_1 and G_2 completely. Call $G_1 \cup G_2$ and G_1+G_2 the union and the joint of G_1 and G_2. Let mG be the union of m disjoint copies of G. Call $F_n = K_1 + nK_2$ a *fan*, and $B_p(n) = K_p + nK_1$ a *p-book*, in which the given p-clique is called the *base* and the n additional vertices are called the *pages*. The number n of pages of $B_p(n)$ is said to be the *size* of the p-book. The largest book size of a p-book in a graph G is denoted by $bs^{(p)}(G)$.

Books and fans play central roles in Ramsey theory, and many important questions and results concern the Ramsey numbers of books and fans versus other natural classes of graphs, see, e.g., [4, 6, 7, 11, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29].

The Ramsey numbers involving only books and fans are pairs of (book,book), (fan,fan) and (book,fan). Rousseau and Sheehan [29] proved that

$$R(B_2(m), B_2(n)) \leqslant m+n+2+2\sqrt{(m^2+mn+n^2)/3}.$$

Furthermore, if $4n+1$ is a prime power, then $R(B_2(n), B_2(n)) = 4n+2$.

Conlon [6] shown that for fixed p

$$R(B_p(n), B_p(n)) = (2^p + o(1))n$$

as $n \to \infty$, for which the small term $o(1)$ is shown to be $O[(\log\log\log n)^{-1/25}]$ in Conlon, Fox and Wigderson [7].

For pair of (fan,fan), Chen, Yu and Zhao [4] proved

$$\frac{9n}{2} - 5 \leqslant R(F_n, F_n) \leqslant \frac{11n}{2} + 6$$

for all positive n.

For different type graphs, a general result on Ramsey *goodness* [17, 23, 27] is as follows.

Theorem 3 ([17, 23, 27]). *Let G and H be graphs. If m is fixed and n is large, then $K_m + nH$ is (K_2+G)-good.*

The most important corollary of Theorem 3 is the *goodness* of large F_m on fixed $B_p(n)$.

Corollary 4. *Let $n, p \geqslant 1$ be fixed integers. Then F_m is $B_p(n)$-good if m is large.*

Among these results, we shall add one that is on *goodness* of large books on fixed fans.

Theorem 5. *Let $m, p \geqslant 1$ be fixed integers. If n is large, then the Ramsey number*

$$R(F_m, B_p(n)) = 2(n+p-1)+1,$$

namely, $B_p(n)$ is F_m-good.

2 Lemmas

The following is Stability Theorem due to Erdős and Simonovits [9, 10, 12, 30], in which $E(H)$ is the edge set of H and $e(H) = |E(H)|$, and $d_U(x) = |N_G(x) \cap U|$ for $U \subseteq V(G)$. The stability theorem is one of tools for the proof of Theorem 5.

Lemma 6. *Let G be a given "forbidden" graph with $\chi(G) = p+1$. For every $\xi > 0$ there exist $\delta = \delta(\xi) > 0$ and $n_0 = n_0(\xi) > 0$ such that if H is a graph of order $N > n_0$ and $e(H) > \frac{1}{2p}(p-1)N^2 - \delta N^2$ that does not contain G, then there exists a partition of $V(H)$ into classes $C_1, C_2, \ldots, C_p$ such that*

- *$N/p - \xi N < |C_i| < N/p + \xi N$ for each $i = 1, 2, \ldots, p$;*
- *all but at most ξN^2 pairs $\{x, y\}$ with $x \in C_i, y \in C_j$ $(i \neq j)$ belong to $E(H)$;*
- *at most ξN^2 pairs $\{x, y\}$ with x, y in the same C_i belong to $E(H)$;*
- *no vertex $x \in C_i$ such that $d_{C_i}(x) > d_{C_j}(x)$ $(i \neq j)$ for each $i = 1, 2, \ldots, p$.*

Lemma 6 describes how a large graph H with forbidden G is similar to $K_k(N/p)$ if $e(H)$ is close to $e(K_k(N/p))$, where $N = |H|$ and $\chi(G) = p + 1$.

Another tool for the proof of Theorem 5 is the regularity lemma of Szemerédi. For the sake of completeness we introduce some notions.

Let G be a graph. If $A, B \subset V(G)$ are nonempty disjoint sets, we write $e(A, B)$ for the number of edges of G between A and B, and call $d(A, B) = \frac{e(A,B)}{|A||B|}$ the *density* of the pair (A, B).

For $\epsilon > 0$, a pair (A, B) of nonempty disjoint sets $A, B \subset V(G)$ is called to be ϵ-regular if

$$|d(A, B) - d(X, Y)| < \epsilon,$$

whenever $X \subset A, Y \subset B$ such that $|X| \geqslant \epsilon|A|$ and $|Y| \geqslant \epsilon|B|$.

We shall use the following version of Szemerédi's Regularity Lemma.

Lemma 7 ([16, 31])**.** *Let $l \geqslant 1$ and $\epsilon > 0$. There exists $M = M(\epsilon, l)$ such that, for every graph G of large order n, there exists a partition $V(G) = \bigcup_{i=0}^{k} V_i$ satisfying $l \leqslant k \leqslant M$ and*

- *$|V_0| < \epsilon n$, $|V_1| = |V_2| = \cdots = |V_k|$;*
- *all but at most ϵk^2 pairs (V_i, V_j), $i, j \in [k]$, are ϵ-regular.*

We also need the following result of Erdős.

Lemma 8 ([8])**.** *Given an integer $p \geqslant 2$ and a graph F of order m, there exists a constant $c_{m,p} > 0$ such that if G is an F-free graph of order $n \geqslant R(F, K_p)$, then G contains at least $c_{m,p}n^p$ independent p-sets.*

The next lemma gives a lower bound on the number of p-cliques in a graph consisting of several dense ϵ-regular pairs sharing a common part.

Lemma 9 ([16]). *Let $0 < \epsilon < d \leqslant 1$ and $(d-\epsilon)^{p-2} > \epsilon$. Suppose H is a graph and $V(H) = A \cup B_1 \cup \cdots \cup B_k$ is a partition with $|A| = |B_1| = \cdots = |B_k|$ such that for every i with $1 \leqslant i \leqslant k$, the pair (A, B_i) is ϵ-regular with $e(A, B_i) \geqslant d|A||B_i|$. If t is the number of the p-cliques in A, then there are at least*

$$k|A|(t - \epsilon p|A|^p)(d-\epsilon)^p$$

$(p+1)$-cliques of H which have exactly p vertices in A.

The following is a special form of blow-up lemma [15] due to Komlós, Sárközy and Szemerédi, in which G is a kind of blow-up of F. The "blow-up" in Lemma 10 means to replace each vertex of F by additional m vertices and each edge of F by an edge set of edge density d instead of $K_{m,m}$.

Lemma 10 ([16]). *Let $0 < \epsilon < d < 1$ be real numbers and F be a graph. For a positive integer m, let us construct a graph G by replacing every vertex of F by m vertices, and replacing the edges of F with ϵ-regular pairs of density at least d. Let H be a subgraph of F with h vertices and maximum degree $\Delta > 0$. If $\epsilon \leqslant (d-\epsilon)^{\Delta}/(\Delta+2)$, then H is a subgraph of G.*

The following is Erdős-Stone theorem [13], a fundamental theorem in extremal graph theory. Sharpening this result, Bollobás and Erdős [2] proved that $t \geqslant \Omega(\log n)$.

Lemma 11. *Let $k \geqslant 2$ be an integer and $\epsilon > 0$. Then there is $n_0 = n_0(k, \epsilon)$ such that if F is a graph of order $n \geqslant n_0$ with*

$$e(F) \geqslant \left(\frac{k-2}{k-1} + \epsilon\right)\binom{n}{2},$$

then F contains $K_k(t)$ for some $t \geqslant \Omega(\log n)$.

3 Proof of Theorem 5

This section is devoted to the proof of Theorem 5. To do so, we need another lemma [1] as follows.

Lemma 12. *If $m \geqslant 1$ and $n \geqslant 2$, then*

$$R(K_n, mK_2) = n + 2m - 2.$$

Proof of Theorem 5. Let $\zeta > 0$ be sufficiently small, and let $\xi \ll \zeta$ be as in Lemma 6 and $c_{m,p}$ be as in Lemma 8. Let $N = 2(n+p-1)+1$. It suffices to show $R(F_m, K_p + nK_1) \leqslant N$ by Lemma 2. Thus we shall show that any red/blue edge coloring of K_N yields either a red F_m or a blue $K_p + nK_1$ if n is large.

Suppose that there is a coloring that contains neither a red F_m nor a blue $K_p + nK_1$. We shall find a contradiction which proves Theorem 5.

Let R and B denote the red graph and the blue graph, respectively. Let $d_R(v)$ and $d_B(v)$ be the red degree and the blue degree of v, respectively.

Claim 1. $e(R) \geqslant (\frac{1}{4} - o(1))N^2$.

Proof of Claim 1. The proof of Claim 1 is very similar to that of Nikiforov and Rousseau [23] although there are some difference when using Lemma 10 to embed F_m instead of K_m in [23], so we shall briefly outline it as follows. For some properly selected ϵ, applying Lemma 7, we partition all but ϵN vertices of R into k sets $V_1, V_2, \ldots, V_k$ of equal cardinality such that almost all pairs (V_i, V_j) are ϵ-regular. We shall admit a pair (V_i, V_j) as dense ϵ-regular if the pair is ϵ-regular with edge density at least c for a constant $c > 0$. We may assume that the number of dense ϵ-regular pairs (V_i, V_j) in R is no more than $\frac{k^2}{4}$, since otherwise, from Lemma 10 and Lemma 11, R will contain a F_m. Therefore, there are at least $(\frac{1}{4} + o(1))k^2$ dense ϵ-regular pairs (V_i, V_j) in B. From Lemma 8, it follows that the number of blue p-cliques in any of the sets $V_1, \ldots, V_k$ is $\Theta(N^p)$. Consider the size of the p-books in B, each of which has its base in a single V_i. From Lemma 9, for every dense ϵ-regular pair (V_i, V_j) in B, almost every vertex in V_j is a page of such a book. Also each ϵ-regular pair (V_i, V_j) whose density is not very close to 1 in R contributes substantially many additional pages to such books. Precise estimations show that either $bs^{(p)}(B) > N/2$ or else the number of all ϵ-regular pairs (V_i, V_j) with density close to 1 in R is at least $(\frac{1}{4} - o(1))k^2$. Thus the size of R is at least $(\frac{1}{4} - o(1))N^2$.

Claim 1 says that the edge density condition for Lemma 6 is satisfied by the red graph for arbitrary $\xi > 0$. In particular, with the forbidden graph F_m, Lemma 6 shows that if n is sufficiently large, then there is a partition of the vertex set of colored K_N into two classes C_1, C_2 such that

- $N/2 - \xi N < |C_i| < N/2 + \xi N$ for each $i = 1, 2$;
- all but at most ξN^2 pairs $\{x, y\}$ with $x \in C_1, y \in C_2$ belong to $E(R)$;
- at most ξN^2 pairs $\{x, y\}$ with x, y in the same C_i belong to $E(R)$;
- no vertex $x \in C_i$ such that $d_{C_i}(x) > d_{C_j}(x)$ $(i \neq j)$ for each $i = 1, 2$.

We define a subset C'_i of C_i as

$$C'_i = \{x \in C_i \mid d_R(x, C_{3-i}) \geqq (1 - 2\sqrt{\xi})|C_{3-i}|\},$$

where $d_R(x, C_{3-i}) = |N_R(x) \cap C_{3-i}|$, which is the number of red neighbors of x in the other class.

Claim 2. $|C'_i| \geqq (1 - 4\sqrt{\xi})|C_i|$ for $i = 1, 2$.

Proof of Claim 2. We suppose that $|C'_i| < (1 - 4\sqrt{\xi})|C_i|$, then we have that

$$|C_i \backslash C'_i| > 4\sqrt{\xi}|C_i| \geqq 4\sqrt{\xi}(1/2 - \xi)N.$$

Any $x \in C_i \backslash C_i'$ satisfies

$$d_B(x, C_{3-i}) \geqq 2\sqrt{\xi}|C_{3-i}| \geqq 2\sqrt{\xi}(1/2-\xi)N.$$

Thus the number of blue edges between C_1 and C_2 is at least

$$4\sqrt{\xi}(1/2-\xi)N \cdot 2\sqrt{\xi}(1/2-\xi)N = 8\xi(1/2-\xi)^2N^2 > \xi N^2,$$

which is a contradiction. $\square$

We now partition $C_i \backslash C_i'$ into two subsets Z_{i1} and Z_{i2} for $i = 1, 2$ such that

$$Z_{i1} = \{w \in C_i \backslash C_i' : d_R(w, C_{3-i}') \geqslant \zeta |C_{3-i}'|\}.$$

Claim 3. Any vertex $w \in Z_{i1}$ satisfies $d_R(w, C_i') \leqslant m - 1$.

Proof of Claim 3. By symmetry, we will prove the assertion for $i = 1$. Suppose to the contrary, there exists a vertex $w \in Z_{11}$ such that $d_R(w, C_1') \geqslant m$. Choose m vertices from $N_R(w, C_1')$, denote them by $M_1 = \{y_{11}, \ldots, y_{1m}\}$. Note that $d_R(y, C_2) \geqslant (1-2\sqrt{\xi})|C_2|$ for each $y \in M_1$, so we obtain $|\cap_{y\in M_1} N_R(y, C_2)| \geqslant (1-2m\sqrt{\xi})|C_2|$. Since $d_R(w, C_2') \geqslant \zeta|C_2'|$ and $0 < \xi \ll \zeta < 1$ are sufficiently small numbers, it follows that

$$|N_R(w, C_2') \cap (\cap_{y\in M_1} N_R(y, C_2))| \geqslant \zeta|C_2'| + (1-2m\sqrt{\xi})|C_2| - |C_2'| \geqslant m$$

for large n. So we can choose m vertices from $N_R(w, C_2') \cap (\cap_{y\in M_1} N_R(y, C_2))$ to get a red $K_{1,m,m}$, thus we find a red F_m, which is a contradiction. $\square$

Claim 4. $|Z_{i2}| \leqslant a$, where $a = R(F_m, K_p) - 1$.

Proof of Claim 4. Suppose to the contrary that $|Z_{i2}| \geqslant a+1$. As we know, each vertex $w \in Z_{i2}$ satisfies

$$d_R(w, C_{3-i}') < \zeta|C_{3-i}'|.$$

Then we have that

$$d_R(w, C_i') \leqslant d_R(w, C_i) \leqslant d_R(w, C_{3-i}) \leqslant d_R(w, C_{3-i}') + |C_{3-i} \backslash C_{3-i}'| < 2\zeta|C_{3-i}'|$$

for large n. Thus

$$\begin{aligned} d_B(w, C_i' \cup C_{3-i}') &\geqslant |C_i'| + |C_{3-i}'| - 1 - (d_R(w, C_i') + d_R(w, C_{3-i}')) \\ &\geqslant |C_i'| + |C_{3-i}'| - 3\zeta|C_{3-i}'| \\ &> (1-3\zeta)(|C_i'| + |C_{3-i}'|). \end{aligned}$$

Since $|Z_{i2}| \geqslant R(F_m, K_p)$, and there is no red F_m from the assumption, we can find a blue clique of order p in Z_{i2}, these p vertices have at least

$$(1-3p\zeta)(|C_i'| + |C_{3-i}'|) \geqslant n$$

blue neighbors in common in $C_i' \cup C_{3-i}'$ for large n. This forces a blue $K_p + nK_1$, which is a contradiction. $\square$

Since there is no red F_m, we have that $N_R(x, C_1')$ contains no red mK_2, where $x \in C_2'$. By Lemma 12 and Claim 2, we can find a blue clique on $X \subset N_R(x, C_1')$ such that

$$|X| \geqslant (1-6\sqrt{\xi})|C_1| - 2(m-1) \geqslant (1-7\sqrt{\xi})|C_1|.$$

for large n. Thus we can take a blue clique on $X \subset C_1'$ with $|X| = (1-7\sqrt{\xi})|C_1|$ and put the other vertices in C_1' into Z_{11}. With a similar proof to that in Claim 3, we know that any vertex $u \in C_1'$ satisfies $d_R(u, C_1') \leqslant m-1$, which is a basic fact that we shall use in the following proof. Similarly, we can find a blue clique on $Y \subset N_R(y, C_2')$ where $y \in C_1'$, such that

$$|Y| \geqslant (1-6\sqrt{\xi})|C_2| - 2(m-1) \geqslant (1-7\sqrt{\xi})|C_2|$$

for large n. Thus we can take a blue clique on $Y \subset C_2'$ with $|Y| = (1-7\sqrt{\xi})|C_2|$ and put the other vertices in C_2' into Z_{21}. With a similar proof to that in Claim 3, we know that any vertex $v \in C_2'$ satisfies $d_R(v, C_2') \leqslant m-1$, which is a basic fact that we shall use in the following proof.

Thus we get a new partition

$$\begin{aligned} V(K_N) &= C_1 \cup C_2 = (C_1' \cup Z_{11} \cup Z_{12}) \cup (C_2' \cup Z_{21} \cup Z_{22}) \\ &= (X \cup Z_{11} \cup Z_{12}) \cup (Y \cup Z_{21} \cup Z_{22}) \\ &= (X \cup Z_{11} \cup Z_{22}) \cup (Y \cup Z_{12} \cup Z_{21}). \end{aligned}$$

Without loss of generality, we may assume $|X \cup Z_{11} \cup Z_{22}| \geqslant p+n$. We shall construct a blue $K_p + nK_1$ as follows.

Noting that any vertex $x \in Z_{11}$ satisfies $d_R(x, X) \leqslant m$, and that any vertex $y \in Z_{22}$ satisfies $d_R(y, X) \leqslant \zeta|C_1'|$, we can find a subset $X' \subset X$ such that any vertex of X' is blue-adjacent to all vertices of $Z_{11} \cup Z_{22}$ with

$$\begin{aligned} |X'| &\geqslant |X| - m|Z_{11}| - \zeta|C_1'| \cdot |Z_{22}| \\ &\geqslant (1-7\sqrt{\xi})|C_1| - m \cdot 7\sqrt{\xi}|C_1| - a \cdot \zeta|C_1| \geqslant p \end{aligned}$$

for small ζ and large n. Then X' is blue adjacent to all vertices of $(X \backslash X') \cup Z_{11} \cup Z_{22}$. Hence we can find a blue $K_p + nK_1$, which is a contradiction. □

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