Ramsey numbers of fans and large books

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Abstract

For graphs $G$ and $H$, the Ramsey number $R(G, H)$ is the minimum integer $N$ such that any red/blue edge coloring of $K_N$ contains either a red $G$ or a blue $H$. Let $G + H$ be the graph obtained from vertex disjoint $G$ and $H$ by adding new edges connecting $G$ and $H$ completely, $F_m = K_1 + mK_2$ and $B_p(n) = K_p + nK_1$. It is shown $R(F_m, B_p(n)) = 2(n + p - 1) + 1$ for fixed $m, p$ and large $n$.

Mathematics Subject Classifications: 05C55, 05D10

1 Introduction

Our notation is standard. For positive functions $f(n)$ and $g(n)$ we write that $f(n) \leq O(g(n))$ if there exists a positive constant $c$ so that $f(n) \leq cg(n)$ and that $f(n) \geq \Omega(g(n))$ if $g(n) \leq O(f(n))$ for all sufficiently large $n$. We write $f(n) = \Theta(g(n))$ if $\Omega(g(n)) \leq f(n) \leq O(g(n))$.

For graphs $G$ and $H$, the Ramsey number $R(G, H)$ is the minimum $N$ such that any red/blue edge coloring of $K_N$ contains either a red $G$ or a blue $H$. The following is a celebrated result of Chvátal for tree $T_n$ of order $n$ and complete graph $K_m$ of order $m$.

Theorem 1 ([5]). Let $m, n \geq 1$ be integers. Then $R(K_m, T_n) = (m - 1)(n - 1) + 1$.

Let $\chi(F)$ be the chromatic number of $F$, and $s(F)$ the minimum size of a color class over all proper vertex colorings of $F$ with $\chi(F)$ colors. Burr observed following general lower bound.

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Lemma 2 ([3]). Let $H$ be a connected graph of order $|H| \geq s(F)$. Then

$$R(F, H) \geq (\chi(F) - 1)(|H| - 1) + s(F).$$

Burr defined $H$ to be $F$-good if the equality in Lemma 2 holds. Thus Theorem 1 tells us that $T_n$ is $K_m$-good. Many results on Ramsey numbers can be found in the dynamic survey [28].

For vertex disjoint graphs $G_1$ and $G_2$, denote by $G_1 \cup G_2$ the union of $G_1$ and $G_2$, and $G_1 + G_2$ the graph obtained from $G_1 \cup G_2$ by adding new edges to connect $G_1$ and $G_2$ completely. Call $G_1 \cup G_2$ and $G_1 + G_2$ the union and the joint of $G_1$ and $G_2$. Let $mG$ be the union of $m$ disjoint copies of $G$. Call $F_n = K_1 + nK_2$ a fan, and $B_p(n) = K_p + nK_1$ a $p$-book, in which the given $p$-clique is called the base and the $n$ additional vertices are called the pages. The number $n$ of pages of $B_p(n)$ is said to be the size of the $p$-book. The largest book size of a $p$-book in a graph $G$ is denoted by $bs^{(p)}(G)$.

Books and fans play central roles in Ramsey theory, and many important questions and results concern the Ramsey numbers of books and fans versus other natural classes of graphs, see, e.g., [4, 6, 7, 11, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29].

The Ramsey numbers involving only books and fans are pairs of (book,book), (fan,fan) and (book,fan). Rousseau and Sheehan [29] proved that

$$R(B_2(m), B_2(n)) \leq m + n + 2 + 2\sqrt{(m^2 + mn + n^2)}/3.$$

Furthermore, if $4n + 1$ is a prime power, then $R(B_2(n), B_2(n)) = 4n + 2$.

Conlon [6] shown that for fixed $p$

$$R(B_p(n), B_p(n)) = (2^p + o(1))n$$

as $n \to \infty$, for which the small term $o(1)$ is shown to be $O((\log \log n)^{-1/25})$ in Conlon, Fox and Wigderson [7].

For pair of (fan,fan), Chen, Yu and Zhao [4] proved

$$\frac{9n}{2} - 5 \leq R(F_n, F_n) \leq \frac{11n}{2} + 6$$

for all positive $n$.

For different type graphs, a general result on Ramsey goodness [17, 23, 27] is as follows.

Theorem 3 ([17, 23, 27]). Let $G$ and $H$ be graphs. If $m$ is fixed and $n$ is large, then $K_m + nH$ is $(K_2 + G)$-good.

The most important corollary of Theorem 3 is the goodness of large $F_m$ on fixed $B_p(n)$.

Corollary 4. Let $n, p \geq 1$ be fixed integers. Then $F_m$ is $B_p(n)$-good if $m$ is large.

Among these results, we shall add one that is on goodness of large books on fixed fans.

Theorem 5. Let $m, p \geq 1$ be fixed integers. If $n$ is large, then the Ramsey number

$$R(F_m, B_p(n)) = 2(n + p - 1) + 1,$$

namely, $B_p(n)$ is $F_m$-good.

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2 Lemmas

The following is Stability Theorem due to Erdős and Simonovits [9, 10, 12, 30], in which $E(H)$ is the edge set of $H$ and $e(H) = |E(H)|$, and $d_U(x) = |N_G(x) \cap U|$ for $U \subseteq V(G)$. The stability theorem is one of tools for the proof of Theorem 5.

**Lemma 6.** Let $G$ be a given “forbidden” graph with $\chi(G) = p + 1$. For every $\xi > 0$ there exist $\delta = \delta(\xi) > 0$ and $n_0 = n_0(\xi) > 0$ such that if $H$ is a graph of order $N > n_0$ and $e(H) > \frac{1}{2\delta}(p-1)N^2 - \delta N^2$ that does not contain $G$, then there exists a partition of $V(H)$ into classes $C_1, C_2, \ldots, C_p$ such that

- $N/p - \xi N < |C_i| < N/p + \xi N$ for each $i = 1, 2, \ldots, p$;
- all but at most $\xi N^2$ pairs $\{x, y\}$ with $x \in C_i, y \in C_j$ ($i \neq j$) belong to $E(H)$;
- at most $\xi N^2$ pairs $\{x, y\}$ with $x, y$ in the same $C_i$ belong to $E(H)$;
- no vertex $x \in C_i$ such that $d_{C_i}(x) > d_{C_j}(x)$ ($i \neq j$) for each $i = 1, 2, \ldots, p$.

Lemma 6 describes how a large graph $H$ with forbidden $G$ is similar to $K_k(N/p)$ if $e(H)$ is close to $e(K_k(N/p))$, where $N = |H|$ and $\chi(G) = p + 1$.

Another tool for the proof of Theorem 5 is the regularity lemma of Szemerédi. For the sake of completeness we introduce some notions.

Let $G$ be a graph. If $A, B \subseteq V(G)$ are nonempty disjoint sets, we write $e(A, B)$ for the number of edges of $G$ between $A$ and $B$, and call $d(A, B) = \frac{e(A, B)}{|A||B|}$ the density of the pair $(A, B)$.

For $\epsilon > 0$, a pair $(A, B)$ of nonempty disjoint sets $A, B \subseteq V(G)$ is called to be $\epsilon$-regular if

$$|d(A, B) - d(X, Y)| < \epsilon,$$

whenever $X \subseteq A, Y \subseteq B$ such that $|X| \geq \epsilon|A|$ and $|Y| \geq \epsilon|B|$.

We shall use the following version of Szemerédi’s Regularity Lemma.

**Lemma 7** ([16, 31]). Let $l \geq 1$ and $\epsilon > 0$. There exists $M = M(\epsilon, l)$ such that, for every graph $G$ of large order $n$, there exists a partition $V(G) = \bigcup_{i=0}^{k} V_i$ satisfying $l \leq k \leq M$ and

- $|V_0| < \epsilon n, |V_1| = |V_2| = \cdots = |V_k|$;
- all but at most $\epsilon k^2$ pairs $(V_i, V_j), i, j \in [k]$, are $\epsilon$-regular.

We also need the following result of Erdős.

**Lemma 8** ([8]). Given an integer $p \geq 2$ and a graph $F$ of order $m$, there exists a constant $c_{m,p} > 0$ such that if $G$ is an $F$-free graph of order $n \geq R(F, K_p)$, then $G$ contains at least $c_{m,p}n^p$ independent $p$-sets.

The next lemma gives a lower bound on the number of $p$-cliques in a graph consisting of several dense $\epsilon$-regular pairs sharing a common part.
Lemma 9 ([16]). Let $0 < \epsilon < d \leq 1$ and $(d - \epsilon)^{p-2} > \epsilon$. Suppose $H$ is a graph and $V(H) = A \cup B_1 \cup \cdots \cup B_k$ is a partition with $|A| = |B_1| = \cdots = |B_k|$ such that for every $i$ with $1 \leq i \leq k$, the pair $(A, B_i)$ is $\epsilon$-regular with $e(A, B_i) \geq d |A||B_i|$. If $t$ is the number of the $p$-cliques in $A$, then there are at least

$$k|A|(t - c|A|^p)(d - \epsilon)^p$$

$(p + 1)$-cliques of $H$ which have exactly $p$ vertices in $A$.

The following is a special form of blow-up lemma [15] due to Komlós, Sárközy and Szemerédi, in which $G$ is a kind of blow-up of $F$. The “blow-up” in Lemma 10 means to replace each vertex of $F$ by additional $m$ vertices and each edge of $F$ by an edge set of edge density $d$ instead of $K_{m,m}$.

Lemma 10 ([16]). Let $0 < \epsilon < d < 1$ be real numbers and $F$ be a graph. For a positive integer $m$, let us construct a graph $G$ by replacing every vertex of $F$ by $m$ vertices, and replacing the edges of $F$ with $\epsilon$-regular pairs of density at least $d$. Let $H$ be a subgraph of $F$ with $h$ vertices and maximum degree $\Delta > 0$. If $\epsilon \leq (d - \epsilon)^\Delta/2$, then $H$ is a subgraph of $G$.

The following is Erdős-Stone theorem [13], a fundamental theorem in extremal graph theory. Sharpening this result, Bollobás and Erdős [2] proved that $t \geq \Omega(\log n)$.

Lemma 11. Let $k \geq 2$ be an integer and $\epsilon > 0$. Then there is $n_0 = n_0(k, \epsilon)$ such that if $F$ is a graph of order $n \geq n_0$ with

$$e(F) \geq \binom{k - 2}{k - 1} + \epsilon \binom{n}{2},$$

then $F$ contains $K_k(t)$ for some $t \geq \Omega(\log n)$.

3 Proof of Theorem 5

This section is devoted to the proof of Theorem 5. To do so, we need another lemma [1] as follows.

Lemma 12. If $m \geq 1$ and $n \geq 2$, then

$$R(K_n, mK_2) = n + 2m - 2.$$

Proof of Theorem 5. Let $\zeta > 0$ be sufficiently small, and let $\xi \ll \zeta$ be as in Lemma 6 and $c_{m,p}$ be as in Lemma 8. Let $N = 2(n + p - 1) + 1$. It suffices to show $R(F_m, K_p + nK_1) \leq N$ by Lemma 2. Thus we shall show that any red/blue edge coloring of $K_N$ yields either a red $F_m$ or a blue $K_p + nK_1$ if $n$ is large.
Suppose that there is a coloring that contains neither a red $F_m$ nor a blue $K_p + nK_1$. We shall find a contradiction which proves Theorem 5.

Let $R$ and $B$ denote the red graph and the blue graph, respectively. Let $d_R(v)$ and $d_B(v)$ be the red degree and the blue degree of $v$, respectively.

Claim 1. $e(R) \geq (\frac{1}{4} - o(1))N^2$.

Proof of Claim 1. The proof of Claim 1 is very similar to that of Nikiforov and Rousseau [23] although there are some difference when using Lemma 10 to embed $F_m$ instead of $K_m$ in [23], so we shall briefly outline it as follows. For some properly selected $\epsilon$, applying Lemma 7, we partition all but $\epsilon N$ vertices of $R$ into $k$ sets $V_1, V_2, \ldots, V_k$ of equal cardinality such that almost all pairs $(V_i, V_j)$ are $\epsilon$-regular. We shall admit a pair $(V_i, V_j)$ as dense $\epsilon$-regular if the pair is $\epsilon$-regular with edge density at least $c$ for a constant $c > 0$. We may assume that the number of dense $\epsilon$-regular pairs $(V_i, V_j)$ in $R$ is no more than $\frac{k^2}{4}$, since otherwise, from Lemma 10 and Lemma 11, $R$ will contain a $F_m$. Therefore, there are at least $(\frac{1}{4} + o(1))k^2$ dense $\epsilon$-regular pairs $(V_i, V_j)$ in $B$. From Lemma 8, it follows that the number of blue $p$-cliques in any of the sets $V_1, \ldots, V_k$ is $\Theta(N^p)$. Consider the size of the $p$-books in $B$, each of which has its base in a single $V_i$. From Lemma 9, for every dense $\epsilon$-regular pair $(V_i, V_j)$ in $B$, almost every vertex in $V_j$ is a page of such a book. Also each $\epsilon$-regular pair $(V_i, V_j)$ whose density is not very close to 1 in $R$ contributes substantially many additional pages to such books. Precise estimations show that either $bs^{(p)}(B) > N/2$ or else the number of all $\epsilon$-regular pairs $(V_i, V_j)$ with density close to 1 in $R$ is at least $(\frac{1}{4} - o(1))k^2$. Thus the size of $R$ is at least $(\frac{1}{4} - o(1))N^2$.

Claim 1 says that the edge density condition for Lemma 6 is satisfied by the red graph for arbitrary $\xi > 0$. In particular, with the forbidden graph $F_m$, Lemma 6 shows that if $n$ is sufficiently large, then there is a partition of the vertex set of colored $K_N$ into two classes $C_1, C_2$ such that

- $N/2 - \xi N < |C_i| < N/2 + \xi N$ for each $i = 1, 2$;
- all but at most $\xi N^2$ pairs $\{x, y\}$ with $x \in C_1, y \in C_2$ belong to $E(R)$;
- at most $\xi N^2$ pairs $\{x, y\}$ with $x, y$ in the same $C_i$ belong to $E(R)$;
- no vertex $x \in C_i$ such that $d_{C_i}(x) > d_{C_j}(x)$ ($i \neq j$) for each $i = 1, 2$.

We define a subset $C_i'$ of $C_i$ as

$$C_i' = \{x \in C_i | d_R(x, C_{3-i}) \geq (1 - 2\sqrt{\xi})|C_{3-i}|\},$$

where $d_R(x, C_{3-i}) = |N_R(x) \cap C_{3-i}|$, which is the number of red neighbors of $x$ in the other class.

Claim 2. $|C_i'| \geq (1 - 4\sqrt{\xi})|C_i|$ for $i = 1, 2$.

Proof of Claim 2. We suppose that $|C_i'| < (1 - 4\sqrt{\xi})|C_i|$, then we have that

$$|C_i \setminus C_i'| > 4\sqrt{\xi}|C_i| \geq 4\sqrt{\xi}(1/2 - \xi)N.$$
Any $x \in C_i \setminus C'_i$ satisfies
\[d_B(x, C_{3-i}) \geq 2\sqrt{\xi} |C_{3-i}| \geq 2\sqrt{\xi} (1/2 - \xi) N.\]

Thus the number of blue edges between $C_1$ and $C_2$ is at least
\[4\sqrt{\xi} (1/2 - \xi) N \cdot 2\sqrt{\xi} (1/2 - \xi) N = 8\xi (1/2 - \xi)^2 N^2 > \xi N^2,
\]
which is a contradiction. \hfill \square

We now partition $C_i \setminus C'_i$ into two subsets $Z_{i_1}$ and $Z_{i_2}$ for $i = 1, 2$ such that
\[Z_{i_1} = \{ w \in C_i \setminus C'_i : d_R(w, C'_{3-i}) \geq \xi |C'_{3-i}| \} \]

Claim 3. Any vertex $w \in Z_{i_1}$ satisfies $d_R(w, C'_i) \leq m - 1$.

Proof of Claim 3. By symmetry, we will prove the assertion for $i = 1$. Suppose to the contrary, there exists a vertex $w \in Z_{i_1}$ such that $d_R(w, C'_1) \geq m$. Choose $m$ vertices from $N_R(w, C'_1)$, denote by $M_1 = \{ y_1, \ldots, y_{1m} \}$. Note that $d_R(y, C_2) \geq (1 - 2\sqrt{\xi}) |C_2|$ for each $y \in M_1$, so we obtain $|\cap_{y \in M_1} N_R(y, C_2)| \geq (1 - 2m\sqrt{\xi}) |C_2|$. Since $d_R(w, C'_2) \geq \xi |C'_2|$ and $0 < \xi \ll \zeta < 1$ are sufficiently small numbers, it follows that
\[|N_R(w, C'_2) \cap (\cap_{y \in M_1} N_R(y, C_2))| \geq \zeta |C'_2| + (1 - 2m\sqrt{\xi}) |C_2| - |C'_2| \geq m\]
for large $n$. So we can choose $m$ vertices from $N_R(w, C'_2) \cap (\cap_{y \in M_1} N_R(y, C_2))$ to get a red $K_{1,m,m}$, thus we find a red $F_m$, which is a contradiction. \hfill \square

Claim 4. $|Z_{i_2}| \leq a$, where $a = R(F_m, K_p) - 1$.

Proof of Claim 4. Suppose to the contrary that $|Z_{i_2}| \geq a + 1$. As we know, each vertex
\[w \in Z_{i_2}\]
satisfies
\[d_R(w, C'_{3-i}) < \xi |C'_{3-i}|.\]

Then we have that
\[d_R(w, C'_i) \leq d_R(w, C_i) \leq d_R(w, C_{3-i}) \leq d_R(w, C'_{3-i}) + |C_{3-i} \setminus C'_{3-i}| < 2\xi |C'_{3-i}|\]
for large $n$. Thus
\[d_B(w, C'_i \cup C'_{3-i}) \geq |C'_i| + |C'_{3-i}| - 1 - (d_R(w, C'_i) + d_R(w, C'_{3-i}))\]
\[\geq |C'_i| + |C'_{3-i}| - 3\xi |C'_{3-i}|\]
\[> (1 - 3\xi) (|C'_i| + |C'_{3-i}|).\]

Since $|Z_{i_2}| \geq R(F_m, K_p)$, and there is no red $F_m$ from the assumption, we can find a blue clique of order $p$ in $Z_{i_2}$, these $p$ vertices have at least
\[(1 - 3p\xi) (|C'_i| + |C'_{3-i}|) \geq n\]
blue neighbors in common in $C'_i \cup C'_{3-i}$ for large $n$. This forces a blue $K_p + nK_1$, which is a contradiction. \hfill \square
Since there is no red $F_m$, we have that $N_R(x, C'_1)$ contains no red $mK_2$, where $x \in C'_2$. By Lemma 12 and Claim 2, we can find a blue clique on $X \subset N_R(x, C'_1)$ such that

$$|X| \geq (1 - 6\sqrt{\xi})|C_1| - 2(m - 1) \geq (1 - 7\sqrt{\xi})|C_1|.$$ 

for large $n$. Thus we can take a blue clique on $X \subset C'_1$ with $|X| = (1 - 7\sqrt{\xi})|C_1|$ and put the other vertices in $C'_1$ into $Z_{11}$. With a similar proof to that in Claim 3, we know that any vertex $u \in C'_1$ satisfies $d_R(u, C'_1) \leq m - 1$, which is a basic fact that we shall use in the following proof. Similarly, we can find a blue clique on $Y \subset N_R(y, C'_2)$ where $y \in C'_1$, such that

$$|Y| \geq (1 - 6\sqrt{\xi})|C_2| - 2(m - 1) \geq (1 - 7\sqrt{\xi})|C_2|$$

for large $n$. Thus we can take a blue clique on $Y \subset C'_2$ with $|Y| = (1 - 7\sqrt{\xi})|C_2|$ and put the other vertices in $C'_2$ into $Z_{21}$. With a similar proof to that in Claim 3, we know that any vertex $v \in C'_2$ satisfies $d_R(v, C'_2) \leq m - 1$, which is a basic fact that we shall use in the following proof.

Thus we get a new partition

$$V(K_N) = C_1 \cup C_2 = (C'_1 \cup Z_{11} \cup Z_{12}) \cup (C'_2 \cup Z_{21} \cup Z_{22}) = (X \cup Z_{11} \cup Z_{12}) \cup (Y \cup Z_{21} \cup Z_{22}) = (X \cup Z_{11} \cup Z_{22}) \cup (Y \cup Z_{12} \cup Z_{21}).$$

Without loss of generality, we may assume $|X \cup Z_{11} \cup Z_{22}| \geq p + n$. We shall construct a blue $K_p + nK_1$ as follows.

Noting that any vertex $x \in Z_{11}$ satisfies $d_R(x, X) \leq m$, and that any vertex $y \in Z_{22}$ satisfies $d_R(y, X) \leq \zeta|C'_1|$, we can find a subset $X' \subset X$ such that any vertex of $X'$ is blue-adjacent to all vertices of $Z_{11} \cup Z_{22}$ with

$$|X'| \geq |X| - m|Z_{11}| - \zeta|C'_1| \cdot |Z_{22}| \geq (1 - 7\sqrt{\xi})|C_1| - m \cdot 7\sqrt{\xi}|C_1| - \zeta|C_1| \geq p$$

for small $\zeta$ and large $n$. Then $X'$ is blue adjacent to all vertices of $(X \setminus X') \cup Z_{11} \cup Z_{22}$. Hence we can find a blue $K_p + nK_1$, which is a contradiction. \hfill $\Box$

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**References**


