Treewidth of the generalized Kneser graphs

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Abstract

Let n, k and t be integers with $1 \leqslant t < k \leqslant n$. The *generalized Kneser graph* $K(n,k,t)$ is a graph whose vertices are the k-subsets of a fixed n-set, where two k-subsets A and B are adjacent if $|A \cap B| < t$. The graph $K(n,k,1)$ is the well-known *Kneser graph*. In 2014, Harvey and Wood determined the exact treewidth of the Kneser graphs for large n with respect to k. In this paper, we give the exact treewidth of the generalized Kneser graphs for $t \geqslant 2$ and large n with respect to k and t. In the special case when $t = k-1$, the graph $K(n,k,k-1)$ usually denoted by $\overline{J(n,k)}$ which is the complement of the Johnson graph $J(n,k)$. We give a more precise result for the exact value of the treewidth of $\overline{J(n,k)}$ for any n and k.

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1   Introduction

Throughout this paper graphs are finite and undirected with no loops or multiple edges. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The numbers of vertices and edges of G are denoted by $v(G)$ and $e(G)$, respectively. The degree of a vertex $x \in V(G)$ in G is denoted by $d_G(x)$, and the edge joining vertices u and w are denoted as an unordered pair $\{u,w\}$. Let $\Delta(G)$ and $\delta(G)$ be the maximum and minimum degree of G, respectively. Especially, we call the vertices of the graph T *nodes* when T is a tree. Let n and k be integers with $1 \leqslant k \leqslant n$. Write $[n] = \{1, 2, \ldots, n\}$ and denote by $\binom{[n]}{k}$ the family of all k-subsets of $[n]$. For any positive integer t, a family

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$\mathcal{F} \subseteq \binom{[n]}{k}$ is said to be *t-intersecting* if $|A \cap B| \geqslant t$ for all $A, B \in \mathcal{F}$. The *complement* $\overline{G}$ of a graph G has the same vertex set as G, where two vertices are adjacent in $\overline{G}$ if they are not adjacent in G. A set $S \subseteq V(G)$ is called *independent set* if any pair of vertices in S are non-adjacent in G. The *independence number* $\alpha(G)$ is the cardinality of maximum independent sets in G.

Definition 1. *A* tree decomposition *of a graph G is a pair $(T, (B_t)_{t \in V(T)})$, where T is a tree and $(B_t)_{t \in V(T)}$ is a family of subsets of $V(G)$ satisfying the following properties:*

(i) *For every $v \in V(G)$, the subgraph of T induced by $B^{-1}(v) = \{t \in V(T) \mid v \in B_t\}$ is nonempty and connected.*

(ii) *For every edge $\{u, w\} \in E(G)$, there is a $t \in V(T)$ such that $u, w \in B_t$.*

The *width* of the decomposition $(T, (B_t)_{t \in V(T)})$ is the number $\max\{|B_t| \mid t \in V(T)\} - 1$. The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum width of the tree decompositions of G. By the definition, each graph G has a tree decomposition $(T, (B_t)_{t \in V(T)})$ where T contains only one node t with $B_t = V(G)$. Notice that this kind of decomposition has width $|V(G)| - 1$ which is the maximum width of the graphs on $|V(G)|$ vertices and also the minimum width of the complete graph on $|V(G)|$ vertices. Therefore, the complete graph K_n has treewidth $n - 1$.

Treewidth is a well-studied parameter in modern graph theory that measures how "tree-like" a graph is. It is of fundamental importance in structural graph theory. Robertson and Seymour used it in their famous series of papers proving the Graph Minor Theorem, for example, we refer the reader to [24, 25, 26]. Besides, Treewidth is also of key interest in the field of algorithm design. The problem of deciding whether a graph has tree decomposition of treewidth at most k is NP-complete [1] and the treewidth is regarded as a key parameter in fixed-parameter tractability. There are many NP-complete problems are solvable in polynomial time on graphs of bounded treewidth [3, 4]. In the past few decades, there has been significant research investigating the treewidth of certain graphs, for example, [13, 14, 17, 20, 22, 27]. However, it is difficult to determine the treewidth exactly in most situations, and there are only few papers obtained the exact value of the treewidth of some certain graphs. In 2014, Harvey and Wood determined the exact treewidth of the Kneser graphs which is $\binom{n}{k} - \binom{n-1}{k-1} - 1$ for $n \geqslant 4k^2 - 4k + 3$ [13]. Motivated by this result, we study the exact value of treewidth of the generalized Kneser graphs for $t \geqslant 2$ in this paper.

Let n, k and t be integers with $1 \leqslant t < k \leqslant n$. The *generalized Kneser graph* $K(n, k, t)$ firstly introduced by Denley [9] is a graph whose vertices are the k-subsets of a fixed n-set, where two k-subsets A and B are adjacent if $|A \cap B| < t$. The graph $K(n, k, 1)$ is the well-known *Kneser graph*. Kneser graphs were first investigated by Kneser [18]. There is a famous result of the chromatic number of $K(n, k, 1)$ which was shown to be $n - 2k + 2$ by Lovász [21], as Kneser originally conjectured. The generalized Kneser graphs as the main generalization of the Kneser graphs are also widely studied. The famous Erdős-Ko-Rado Theorem [10] has a well-known relationship to the independent number of the generalized Kneser graphs, since an independent set in the generalized Kneser graph $K(n, k, t)$ is a

t-intersecting family of $\binom{[n]}{k}$. In the special case when $t = k - 1$, the graph $K(n, k, k-1)$, usually denoted by $\overline{J(n,k)}$ is the complement of the Johnson graph $J(n, k)$. The *Johnson graph* $J(n, k)$ is the graph whose vertices are the k-subsets of a fixed n-set as well, where two vertices A and B are adjacent if $|A \cap B| = k - 1$. Over the years several aspects of Johnson graphs such as chromatic number, connectivity, eigenvalues, automorphisms, regular embeddings and some other properties have been widely studied as one can find in, for example, [2, 6, 7, 8, 15, 23]. These graphs are important because they enable us to translate many combinatorial problems about finite sets into graph theory, such as the context of coding theory and design theory. Also, we see that many interesting objects from finite geometry occur encoded as cliques and independent sets in these graphs, and this leads to interesting variants of the Erdős-Ko-Rado Theorem.

The main results in this paper are as follows.

Theorem 2. *Let n, k and t be positive integers with $n \geqslant 2(k-t)(t+1)\binom{k}{t} + k + t + 1$. Let $K(n, k, t)$ be the generalized Kneser graph for $k > t \geqslant 2$. Then*

$$\operatorname{tw}(K(n, k, t)) = \binom{n}{k} - \binom{n-t}{k-t} - 1.$$

From the proof of Theorem 2, one can obtain that $\operatorname{tw}(K(n, k, 1)) = \binom{n}{k} - \binom{n-1}{k-1} - 1$ for $n \geqslant 4k^2 - 3k + 2$, which equals to the value of treewidth of Kneser graphs given by Harvey and Wood [13]. However, the lower bound of n in our result is slightly bigger than that of the result of Harvey and Wood (they let $n \geqslant 4k^2 - 4k + 3$). Therefore, we only consider the situation $t \geqslant 2$.

In the special case when $t = k - 1$, the graph $K(n, k, k-1)$ is the complement of the Johnson graph $J(n, k)$. We give a more precise result for the exact value of the treewidth of $\overline{J(n,k)}$ for any n and k. Note that $\overline{J(n,k)}$ is an empty graph when $n < k + 2$. Thus we only consider the case with $n \geqslant k + 2$.

Theorem 3. *Let n and k be positive integers with $n \geqslant k + 2$ and $k \geqslant 2$. Let $\overline{J(n,k)}$ be the complement of Johnson graph $J(n, k)$. Then*

$$\operatorname{tw}(\overline{J(n,k)}) = \begin{cases} 1, & \text{if } k = 2 \text{ and } n = 4, \\ 14, & \text{if } k = 3 \text{ and } n = 6, \\ 4, & \text{if } k = 3 \text{ and } n = 5, \text{ or } k = 2 \text{ and } n = 5, \\ \binom{n}{k} - n + k - 2, & \text{if } k \geqslant 4 \text{ and } n \geqslant 2k, \text{ or } k \in \{2, 3\} \text{ and } n \geqslant k + 4, \\ \binom{n}{k} - k - 2, & \text{if } k \geqslant 4 \text{ and } n < 2k. \end{cases}$$

The rest of this paper is organized as follows. In the next section we will prove some important inequalities which are useful for the proof in Subsection 3.2. In Section 3, we will give the exact treewidth of the generalized Kneser graphs for $k > t \geqslant 2$ and large n with respect to k and t. After that, we will study the treewidth of the complement of Johnson graphs for any possible n and k in Section 4.

2 Preliminaries

In this section, we will prove a number of inequalities. In Subsection 3.2, we will use these inequalities to prove a lower bound for the treewidth of the generalized Kneser graphs in Theorem 2.

Lemma 4. *Let n, k and t be positive integers with $1 \leqslant t < k \leqslant n$. If $n \geqslant 2(k-t)(t+1)\binom{k}{t} + k + t + 1$, then*

$$\frac{1}{2\binom{k}{t}}\binom{n-t}{k-t} > \binom{n-t}{k-t} - \binom{n-2t-1}{k-t}.$$

Proof. Firstly, we prove a claim.

Claim 1. For any $i \in \{0, 1, \ldots, k-t-1\}$, let $c := \sqrt[k-t]{1 + \frac{1}{2\binom{k}{t}-1}}$, we have

$$\frac{n-t-i}{n-2t-1-i} < c.$$

Proof of Claim 1 It is clear that $\frac{n-t-i}{n-2t-1-i}$ is increasing as $i \in \{0, 1, \ldots, k-t-1\}$ increases. Therefore, it suffices to prove that $\frac{n-k+1}{n-k-t} < c$.

We have

$$\begin{aligned}
\frac{t+1}{c-1} &= \frac{(t+1)(c^{k-t-1} + c^{k-t-2} + \cdots + c + 1)}{c^{k-t} - 1} \\
&< \frac{(t+1)(k-t)c^{k-t}}{c^{k-t} - 1} \\
&= 2(t+1)(k-t)\binom{k}{t}.
\end{aligned}$$

If $n \geqslant 2(k-t)(t+1)\binom{k}{t} + k + t + 1$, then $n - k - t > 2(k-t)(t+1)\binom{k}{t} > \frac{t+1}{c-1}$, which implies that

$$\begin{aligned}
(c-1)n - (c-1)k - (c-1)t &> t+1, \\
cn - ck - ct &> n - k + 1.
\end{aligned}$$

This yields that $\frac{n-k+1}{n-k-t} < c$. We complete the proof of the claim. $\qquad\square$

By Claim 1, we have

$$\frac{(n-t)(n-t-1)\cdots(n-k+1)}{(n-2t-1)(n-2t-2)\cdots(n-k-t)} < 1 + \frac{1}{2\binom{k}{t}-1}.$$

Thus we get

$$\frac{\binom{n-t}{k-t}}{\binom{n-2t-1}{k-t}} < 1 + \frac{1}{2\binom{k}{t}-1},$$

which implies that

$$\left(\frac{1}{2\binom{k}{t}} - 1\right)\binom{n-t}{k-t} > -\binom{n-2t-1}{k-t},$$

as required. □

Lemma 5. *Let* n, k *and* t *be positive integers with* $1 \leqslant t < k \leqslant n$. *Let* p *be a fixed constant with* $\frac{2}{3} \leqslant p < 1$. *If* $n \geqslant \frac{1}{1-p}(k-t)(k+1) + 2t$, *then*

$$(1-p)\binom{n-t}{k-t} \geqslant (k+1)\sum_{a=0}^{t-2}\binom{n-2t-1}{n-k-t+a} + t(k+1)\binom{n-2t-1}{n-k-1}.$$

Proof. Since $\binom{n-t}{k-t} = \sum_{i=0}^{t}\binom{t}{i}\binom{n-2t}{k-t-i}$, it suffices to prove that

$$(1-p)\sum_{i=0}^{t}\binom{t}{i}\binom{n-2t}{k-t-i} \geqslant (k+1)\sum_{a=0}^{t-2}\binom{n-2t-1}{k-t-1-a} + t(k+1)\binom{n-2t-1}{k-2t}.$$

If $i \neq k-t$, by $\binom{n-2t}{k-t-i} = \frac{n-2t}{k-t-i}\binom{n-2t-1}{k-t-i-1}$ and $n \geqslant \frac{1}{1-p}(k-t)(k+1) + 2t$, we have

$$(1-p)\binom{t}{i}\binom{n-2t}{k-t-i} \geqslant (k+1)\binom{n-2t-1}{k-t-1-i}$$

for $i \in \{0, 1, \ldots, t-2\}$. If $i = k-t$, we can easily see that the above inequality is also true. Furthermore, since $n \geqslant \frac{1}{1-p}(k-t)(k+1) + 2t \geqslant \frac{1}{1-p}(k-2t+1)(k+1) + 2t$, we have

$$(1-p)\binom{t}{i}\binom{n-2t}{k-t-i} \geqslant t(k+1)\binom{n-2t-1}{k-2t}$$

for $i = t-1$.

Therefore, the required result holds. □

3 Treewidth of the generalized Kneser graphs

3.1 Upper bound for treewidth in Theorem 2

In this subsection, we will give an upper bound for the treewidth of the graph $K(n,k,t)$ with the help of the famous Erdős-Ko-Rado Theorem for finite sets.

Theorem 6. (Erdős-Ko-Rado Theorem [10, 28]) *Let* n, k *and* t *be positive integers with* $1 \leqslant t < k \leqslant n$. *If* $n \geqslant (t+1)(k-t+1)$ *and* $\mathcal{F} \subseteq \binom{[n]}{k}$ *is a* t-*intersecting family, then*

$$|\mathcal{F}| \leqslant \binom{n-t}{k-t}.$$

Moreover, if $n > (t+1)(k-t+1)$, *equality holds if and only if* $\mathcal{F}$ *consists of all* k-*subsets that contain a fixed* t-*subset of* $[n]$.

The result of the Erdős-Ko-Rado Theorem for finite sets is clearly equivalent to the independent number of the generalized Kneser graph $K(n, k, t)$. That is,

$$\alpha(K(n, k, t)) = \binom{n-t}{k-t} \tag{1}$$

for $n \geqslant (t+1)(k-t+1)$.

Proposition 7. ([13]) *For any graph G, $\text{tw}(G) \leqslant \max\{\Delta(G), |V(G)| - \alpha(G) - 1\}$.*

Lemma 8. *If n, k and t are integers with $k > t \geqslant 1$ and $n \geqslant 2(k-t)(t+1)\binom{k}{t} + k + t + 1$, then*

$$\text{tw}(K(n, k, t)) \leqslant \binom{n}{k} - \binom{n-t}{k-t} - 1. \tag{2}$$

Proof. According to Proposition 7, to prove an upper bound of $\text{tw}(K(n, k, t))$ we only need to compare the size of $\Delta(K(n, k, t))$ and $|V(K(n, k, t))| - \alpha(K(n, k, t)) - 1$.

For any k-subset $A \in \binom{[n]}{k}$, define $N(A)$ to be the set of all the neighbors of A in $K(n, k, t)$. Thus,

$$N(A) = \left\{ F \in \binom{[n]}{k} \mid |F \cap A| < t \right\} = \bigcup_{i=0}^{t-1} \left\{ F \in \binom{[n]}{k} \mid |F \cap A| = i \right\}.$$

Since $\left| \left\{ F \in \binom{[n]}{k} \mid |F \cap A| = i \right\} \right| = \binom{k}{i}\binom{n-k}{k-i}$, and $K(n, k, t)$ is a regular graph, we have

$$\Delta(K(n, k, t)) = |N(A)| = \sum_{i=0}^{t-1} \binom{k}{i}\binom{n-k}{k-i}.$$

Claim 2. $\frac{(n-k-i)(k-i)}{(n-t-i)(k-t-i)} > 1$ for any $i \in \{0, 1, \ldots, k-t-1\}$.

Proof of Claim 2 Firstly, we have

$$(n-k-i)(k-i) - (n-t-i)(k-t-i) = tn - t^2 + tk - k^2 + (k-2t)i$$

for any $i \in \{0, 1, \ldots, k-t-1\}$. We divide the proof of this claim into the following two cases.

Case 1. $k \geqslant 2t$.

In this case, since $n \geqslant 2(k-t)(t+1)\binom{k}{t} + k + t + 1$ and $k \geqslant t+1$, we have

$$\begin{aligned}
&(n-k-i)(k-i) - (n-t-i)(k-t-i) \\
\geqslant{}& tn - t^2 + tk - k^2 \\
\geqslant{}& t(2(k-t)(t+1)k + t) - t^2 - k(k-t) \\
>{}& 0.
\end{aligned}$$

Case 2. $k < 2t$.

In this case, similarly, since $n \geqslant 2(k-t)(t+1)\binom{k}{t} + k + t + 1$, we have

$$
\begin{aligned}
&(n-k-i)(k-i) - (n-t-i)(k-t-i) \\
&\geqslant tn - t^2 + tk - k^2 + (k-2t)(k-t-1) \\
&= tn + t^2 + 2t - 2kt - k \\
&> 0.
\end{aligned}
$$

Therefore, we obtain $(n-k-i)(k-i) - (n-t-i)(k-t-i) > 0$ for any $i \in \{0, 1, \ldots, k-t-1\}$, and the result of this claim follows. $\qquad\square$

By Claim 2, we have

$$
\frac{(n-k)(n-k-1)\cdots(n-2k+t+1)k(k-1)\cdots(t+1)}{(n-t)(n-t-1)\cdots(n-k+1)(k-t)(k-t-1)\cdots 1} > 1,
$$

which implies that $\binom{k}{t}\binom{n-k}{k-t} - \binom{n-t}{k-t} \geqslant 0$.

Furthermore, since $\binom{n}{k} = \sum_{i=0}^{k} \binom{k}{i}\binom{n-k}{k-i}$, by (1) and $k \geqslant t+1$, we have

$$
\begin{aligned}
&|V(K(n,k,t))| - \Delta(K(n,k,t)) - \alpha(K(n,k,t)) - 1 \\
&= \sum_{i=t}^{k} \binom{k}{i}\binom{n-k}{k-i} - \binom{n-t}{k-t} - 1 \\
&\geqslant \binom{k}{t}\binom{n-k}{k-t} + \binom{k}{t+1}\binom{n-k}{k-t-1} - \binom{n-t}{k-t} - 1 \\
&\geqslant 0.
\end{aligned}
$$

Therefore, $|V(K(n,k,t))| - \alpha(K(n,k,t)) - 1 \geqslant \Delta(K(n,k,t))$, yielding that

$$
\mathrm{tw}(K(n,k,t)) \leqslant |V(K(n,k,t))| - \alpha(K(n,k,t)) - 1 = \binom{n}{k} - \binom{n-t}{k-t} - 1,
$$

as required. $\qquad\square$

3.2 Lower bound for treewidth in Theorem 2

In this subsection, we will study the lower bound for treewidth of the generalized Kneser graphs.

Let $\mathcal{F}$ be a family of k-subsets of $[n]$, the *t-shadow* of $\mathcal{F}$ is defined as

$$
\partial_t(\mathcal{F}) = \left\{ X \in \binom{[n]}{t} \mid X \subseteq F \text{ for some } F \in \mathcal{F} \right\}.
$$

According to the definition, the t-shadow $\partial_t(\mathcal{F})$ of $\mathcal{F}$ contains all t-subsets that are contained in one of the k-subsets of $\mathcal{F}$. The *complement* of $F \in \binom{[n]}{k}$ is the $(n-k)$-set $\overline{F} := [n] \setminus F$. Define the *complement* of $\mathcal{F} \subseteq \binom{[n]}{k}$ by $\overline{\mathcal{F}} := \{\overline{F} \in \binom{[n]}{n-k} \mid \overline{F} \text{ is the complement of some } F \in \mathcal{F}\}$. Given a t-subset $X \in \binom{[n]}{t}$, denote $\mathcal{F}_X$ the set

of all k-subsets of $\mathcal{F}$ containing X, and let $\mathcal{F}_{-X} := \mathcal{F} \setminus \mathcal{F}_X$. Assume that $\mathcal{F}_{-X} \neq \emptyset$, let $\overline{\mathcal{F}_{-X}} := \{\overline{F} \in \binom{[n]}{n-k} \mid \overline{F} = [n] \setminus F$, where $F \in \mathcal{F}_{-X}\}$ and $\overline{\mathcal{F}_{-X}}^* := \{\overline{F}^* \in \bigcup_{a \in \{0,1,\ldots,t-1\}} \binom{[n]}{n-k-t+a} \mid \overline{F}^* = \overline{F} \setminus X$, where $\overline{F} \in \overline{\mathcal{F}_{-X}}$ such that $|X \cap \overline{F}| = t - a\}$. Similarly, let $\mathcal{F}_X^* := \{F^* \in \binom{[n]}{k-t} \mid F^* = F \setminus X$, where $F \in \mathcal{F}_X\}$. Then $|\mathcal{F}_X^*| = |\mathcal{F}_X|$.

Define the *colexicographic ordering*, colex ordering for short, on the k-subsets of $[n]$ as follows: if $F_1, F_2 \in \binom{[n]}{k}$ are distinct, then $F_1 < F_2$ when $\max\{F_1 \setminus F_2\} < \max\{F_2 \setminus F_1\}$. Thus this is a strict total order. Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $\mathcal{F}$ is *first* if $\mathcal{F}$ consists of the first $|\mathcal{F}|$ k-subsets of $[n]$ in the colex ordering. If we let $\max\{F_1 \setminus F_2\} = 0$ when $F_1 \setminus F_2 = \emptyset$, notice that this is also a strict total order on $\mathcal{F} \subseteq 2^{[n]}$.

Proposition 9. ([16, 19]) *Let $\mathcal{F} \subseteq \binom{[n]}{k}$, and $\partial_t(\mathcal{F})$ be the t-shadow of $\mathcal{F}$. If $|\mathcal{F}|$ is a fixed constant, then $|\partial_t(\mathcal{F})|$ is minimised when $\mathcal{F}$ is first.*

For $\mathcal{F} \subseteq \bigcup_i \binom{[n]}{k+i}$, in the following lemma we can prove a more general result than Proposition 9, where $i \in \{0, 1, \ldots, t-1\}$.

Lemma 10. *Let k, g be positive integers with $g < k$. Let $\mathcal{A} \subseteq \bigcup_i \binom{[n]}{k+i}$ and $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{t-1}$, where $\mathcal{A}_i \subseteq \binom{[n]}{k+i}$ for any $i \in \{0, 1, \ldots, t-1\}$. Let $\mathcal{S}_i = \partial_g(\mathcal{A}_i)$ be the g-shadow of $\mathcal{A}_i$ and $\mathcal{S} = \partial_g(\mathcal{A})$ be the g-shadow of $\mathcal{A}$. If $|\mathcal{A}_i|$ is a fixed constant, then $|\mathcal{S}|$ is minimised when $\mathcal{A}_i$ is first in the colex ordering for any $i \in \{0, 1, \ldots, t-1\}$.*

Proof. Notice that $\mathcal{S} = \partial_g(\mathcal{A}) = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{t-1}$. According to Proposition 9, we have $|\mathcal{S}_i|$ is minimised when $\mathcal{A}_i$ is first in the colex ordering.

We assume that $\mathcal{A}_i$ is first for any $i \in \{0, 1, \ldots, t-1\}$, we prove that there exists $i \in \{0, 1, \ldots, t-1\}$ such that $\mathcal{S}_i \supseteq \bigcup_{j \in \{0,1,\ldots,t-1\} \setminus \{i\}} \mathcal{S}_j$. Let α be the maximum element in $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{n-k}$ in the colex ordering. Let $\mathcal{A}_i$ be the one containing α and let $\max \alpha$ be the maximum element in α. Therefore, it is sufficient to prove that $\mathcal{S}_i \supseteq \mathcal{S}_j$ for any $j \in \{0, 1, \ldots, t-1\}$ and $i \neq j$. Notice that $\max \alpha \geqslant k + i$ and $\max \beta \leqslant \max \alpha$ for any $\beta \in \mathcal{S}_j$ by the choice of α.

If $\beta \subseteq \alpha$, then it is clear that $\beta \in \mathcal{S}_i$. Now suppose that $\beta \nsubseteq \alpha$. Since $\beta \in \mathcal{S}_j$, there exists $\delta \in \mathcal{A}_j \subseteq \mathcal{A}$ such that $\beta \subseteq \delta$. By the choice of α, we have that $\delta < \alpha$ in the colex ordering, implying that

$$\max\{\beta \setminus \alpha\} \leqslant \max\{\delta \setminus \alpha\} < \max\{\alpha \setminus \delta\} \leqslant \max\{\alpha \setminus \beta\}.$$

Let $a = \max\{\alpha \setminus \beta\}$, and γ be a $(k+i)$-subset of $(\alpha \cup \beta) \setminus \{a\}$ with $\beta \subseteq \gamma$. Observe that $\gamma \setminus \alpha = \beta \setminus \alpha$ and $\max\{\gamma \setminus \alpha\} = \max\{\beta \setminus \alpha\} < a$. Since $a \in \alpha \setminus \gamma$, we have $a \leqslant \max\{\alpha \setminus \gamma\}$, which implies that $\max\{\gamma \setminus \alpha\} < \max\{\alpha \setminus \gamma\}$. Then $\gamma \in \mathcal{A}_i$ due to that $\mathcal{A}_i$ is first. Thus, we have $\beta \in \mathcal{S}_i$.

Therefore, $\mathcal{S}_i \supseteq \mathcal{S}_j$, as required. Thus we have $|\mathcal{S}| = |\mathcal{S}_i|$ when $\mathcal{A}_i$ is first in the colex ordering for any $i \in \{0, 1, \ldots, t-1\}$, where $\mathcal{S}_i$ is the one containing $\bigcup_{j \in \{0,1,\ldots,t-1\} \setminus \{i\}} \mathcal{S}_j$. $\square$

Let X be a subset of $V(G)$. The graph $G - X$ is a subgraph of G induced by $V(G) \setminus X$. Let $G[X]$ be the subgraph of G induced by X. Let p be a fixed constant with $\frac{2}{3} \leqslant p < 1$.

A *p-separator* of G is a subset $X \subset V(G)$ such that there is no component in $G - X$ that contains more than $p|V(G - X)|$ vertices. If $|X| \leqslant c$, we call X a *p-separator of order* c. There is a well-known relationship between the treewidth and the p-separators of G.

Proposition 11. ((2.5) in [26]) *Every graph G has a p-separator of order $\operatorname{tw}(G) + 1$ for each $\frac{2}{3} \leqslant p < 1$.*

With the help of these important results we give the following lemma.

Lemma 12. *Let n, k, t and p be integers with $k > t \geqslant 1$, $\frac{2}{3} \leqslant p < 1$ and $n \geqslant \max\{2(k - t)(t + 1)\binom{k}{t} + k + t + 1, \frac{1}{1-p}(k - t)(k + 1) + 2t\}$. If X is a p-separator of $K(n, k, t)$, then*

$$|X| \geqslant \binom{n}{k} - \binom{n - t}{k - t}.$$

Proof. Suppose to the contrary that $|X| < \binom{n}{k} - \binom{n-t}{k-t}$. Thus we have

$$|V(K(n, k, t) - X)| > \binom{n - t}{k - t}. \tag{3}$$

Since X is a p-separator of $K(n, k, t)$, we can partition the components of $K(n, k, t) - X$ into two parts such that the components in each part contain at most $p|V(K(n, k, t) - X)|$ vertices. Therefore, $V(K(n, k, t) - X)$ can be partitioned into two parts $\mathcal{A}$ and $\mathcal{B}$ with $|\mathcal{A}| \leqslant |\mathcal{B}|$ such that there is no edge between $\mathcal{A}$ and $\mathcal{B}$. This implies that $|u \cap v| \geqslant t$ for any $u \in \mathcal{A}$ and $v \in \mathcal{B}$. Furthermore, we have

$$(1 - p)|V(K(n, k, t) - X)| \leqslant |\mathcal{A}| \leqslant \frac{1}{2}|V(K(n, k, t) - X)|, \tag{4}$$

$$\frac{1}{2}|V(K(n, k, t) - X)| \leqslant |\mathcal{B}| \leqslant p|V(K(n, k, t) - X)|. \tag{5}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are both non-empty by (4) and (5), respectively, there is a vertex $u \in \mathcal{A}$ such that $|u \cap v| \geqslant t$ for any vertex $v \in \mathcal{B}$. Therefore, according to the Pigeonhole Principle, there exists a t-subset Y of u such that it belongs to at least $\frac{1}{\binom{k}{t}}|\mathcal{B}|$ vertices in $\mathcal{B}$. Thus, we have $|\mathcal{B}_Y| \geqslant \frac{1}{\binom{k}{t}}|\mathcal{B}|$. According to (3) and (5), we have $|\mathcal{B}| \geqslant \frac{1}{2}|V(K(n, k, t) - X)| > \frac{1}{2}\binom{n-t}{k-t}$, which implies that

$$|\mathcal{B}_Y^*| = |\mathcal{B}_Y| \geqslant \frac{1}{\binom{k}{t}}|\mathcal{B}| > \frac{1}{2\binom{k}{t}}\binom{n - t}{k - t}. \tag{6}$$

Notice that $\overline{\mathcal{A}_{-Y}}^*$ is a family that every element of the family has $n - k - t + a$ elements of $[n]$ for some $a \in \{0, 1, \ldots, t-1\}$. Let $\overline{\mathcal{A}_{-Y}}^* = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{t-1}$, where $\mathcal{A}_a \subseteq \binom{[n]}{n-k-t+a}$ and $a \in \{0, 1, \ldots, t - 1\}$. We have the following claim.

Claim 3. $|\overline{\mathcal{A}_{-Y}}^*| = \sum_{a=0}^{t-1}|\mathcal{A}_a| < \sum_{a=0}^{t-1}\binom{n-2t-1}{n-k-t+a}$. *Furthermore, we have $|\mathcal{A}_a| < \binom{n-2t-1}{n-k-t+a}$ for every $a \in \{0, 1, \ldots, t - 1\}$.*

Proof of Claim 3 Clearly, we just need to consider the case $\mathcal{A}_{-Y} \neq \emptyset$. Then $\overline{\mathcal{A}_{-Y}}$ is non-empty since $|\mathcal{A}_{-Y}| = |\overline{\mathcal{A}_{-Y}}|$. Let $w \in \mathcal{B}_Y$ and $\overline{z} \in \overline{\mathcal{A}_{-Y}}$ satisfying $|Y \cap \overline{z}| = t - a$, where $a \in \{0, 1, \ldots, t-1\}$ and $z \in \mathcal{A}_{-Y}$. Then $wz \notin E(K(n,k,t))$. Let $w^* = w \setminus Y$ and $\overline{z}^* = \overline{z} \setminus Y$. If $w^* \subseteq \overline{z}^*$, then

$$
\begin{aligned}
|w \cap \overline{z}| &= |(w^* \cup Y) \cap (\overline{z}^* \cup (Y \cap \overline{z}))| \\
&= |w^* \cup Y| + |\overline{z}^* \cup (Y \cap \overline{z})| - |w^* \cup Y \cup \overline{z}^* \cup (Y \cap \overline{z})| \\
&= |w| + |\overline{z}^*| + |Y \cap \overline{z}| - |\overline{z}^* \cap (Y \cap \overline{z})| - |Y \cup \overline{z}^*| \\
&= k + (n - k - t + a) + (t - a) - (n - k + a) \\
&= k - a,
\end{aligned}
$$

which implies that $|w \cap z| = k - (k - a) = a < t$ and that is $wz \in E(K(n,k,t))$, a contradiction. Thus, we have $w^* \not\subseteq \overline{z}^*$ for any $w^* \in \mathcal{B}_Y^*$ and $\overline{z}^* \in \overline{\mathcal{A}_{-Y}}^*$. Therefore, if we let $\mathcal{S} := \partial_{k-t}(\overline{\mathcal{A}_{-Y}}^*)$ be the $(k-t)$-shadow of $\overline{\mathcal{A}_{-Y}}^*$, then we have $w^* \not\subseteq s$ for any $w^* \in \mathcal{B}_Y^*$ and $s \in \mathcal{S}$, which implies that

$$|\mathcal{B}_Y^*| \leqslant \binom{n-t}{k-t} - |\mathcal{S}|. \tag{7}$$

Hence can obtain an upper bound of $|\mathcal{B}_Y^*|$ by taking $|\mathcal{S}|$ to be minimised.

Suppose to the contrary that

$$|\overline{\mathcal{A}_{-Y}}^*| \geqslant \sum_{a=0}^{t-1} \binom{n-2t-1}{n-k-t+a}.$$

First we prove that $|\mathcal{S}| \geqslant \binom{n-2t-1}{k-t}$. Let $\mathcal{S} = \partial_{k-t}(\overline{\mathcal{A}_{-Y}}^*) = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{t-1}$, where $\mathcal{S}_a$ is the $(k-t)$-shadow of $\mathcal{A}_a$. According to Lemma 10, we have $|\mathcal{S}|$ is minimised when $\mathcal{A}_a$ is first in the colex ordering for any $a \in \{0, 1, \ldots, t-1\}$. Therefore, to obtain a lower bound of $|\mathcal{S}|$, we assume that $\mathcal{A}_a$ is first for any $a \in \{0, 1, \ldots, t-1\}$. As $|\overline{\mathcal{A}_{-Y}}^*| \geqslant \sum_{a=0}^{t-1} \binom{n-2t-1}{n-k-t+a}$, there exists at least one $\mathcal{A}_a$ such that $|\mathcal{A}_a| \geqslant \binom{n-2t-1}{n-k-t+a}$, implying that $\mathcal{A}_a$ contains the first $\binom{n-2t-1}{n-k-t+a}$ $(n-k-t+a)$-sets in the colex ordering. This implies that $\mathcal{A}_a$ contains all $(n-k-t+a)$-subsets of $[n-2t-1]$. Thus $\mathcal{S}_a$ contains all $(k-t)$-subsets of $[n-2t-1]$, which implies that $|\mathcal{S}_a| \geqslant \binom{n-2t-1}{k-t}$. Therefore, we have $|\mathcal{S}| \geqslant \binom{n-2t-1}{k-t}$, as required.

Next, by the lower bound of $|\mathcal{S}|$ and (7), we have the following upper bound for $|\mathcal{B}_Y^*|$.

$$|\mathcal{B}_Y^*| \leqslant \binom{n-t}{k-t} - \binom{n-2t-1}{k-t}. \tag{8}$$

However, combining with (6) and (8), by $n \geqslant 2(k-t)(t+1)\binom{k}{t} + k + t + 1$ and Lemma 4, we have a contradiction. Thus, we have the claim holds. $\qquad\square$

Claim 4. $|\mathcal{A}_{-Y}| < \sum_{a=0}^{t-2} \binom{n-2t-1}{n-k-t+a} + t\binom{n-2t-1}{n-k-1}$.

Proof of Claim 4 If $|\mathcal{A}_{-Y}| = |\overline{\mathcal{A}_{-Y}}^*|$, we have the conclusion. If $|\mathcal{A}_{-Y}| \neq |\overline{\mathcal{A}_{-Y}}^*|$, there must be distinct $v_1, v_2 \in \mathcal{A}_{-Y}$ such that $\overline{v_1}^* = \overline{v_2}^*$, which means $v_1 \setminus Y = v_2 \setminus Y$.

Let $|v_1 \cap Y| = |v_2 \cap Y| = r$, where $r \in \{1, 2, \ldots, t-1\}$. We say $r = t-1$. Suppose to the contrary that $r \leqslant t-2$.

For any $u \in B_Y$, since u is not adjacent to v_1, v_2, we have $|u \cap v_1| \geqslant t$ and $|u \cap v_2| \geqslant t$, which implies $|u \cap (v_1 \setminus Y)| \geqslant t - r$. Therefore, according to the Pigeonhole Principle, there exists a $(t-r)$-subset Z such that it belongs to at least $\frac{1}{\binom{k-r}{t-r}}|B_Y|$ vertices in $\mathcal{B}_Y$. Thus we have

$$|\mathcal{B}_{Y \cup Z}| \geqslant \frac{1}{\binom{k-r}{t-r}}|B_Y| > \frac{1}{2\binom{k-r}{t-r}\binom{k}{t}}\binom{n-t}{k-t}, \tag{9}$$

by (6).

On the other hand, we know that each element of $\mathcal{B}_{Y \cup Z}$ contains $2t - r$ elements of $Y \cup Z$, thus

$$|\mathcal{B}_{Y \cup Z}| \leqslant \binom{n - 2t + r}{k - 2t + r}. \tag{10}$$

Comparing with the lower bound and the upper bound of $\mathcal{B}_{Y \cup Z}$ in (9) and (10), we have

$$\frac{1}{2\binom{k}{t}}\binom{n-t}{k-t} \leqslant \binom{k-r}{t-r}\binom{n-2t+r}{k-2t+r}.$$

Let $f(r) = \binom{k-r}{t-r}\binom{n-2t+r}{k-2t+r}$ for $r \in \{1, 2, \ldots, t-2\}$. Since $n \geqslant 2(k-t)(t+1)\binom{k}{t}+k+t+1$, we have

$$\frac{n-2t+r}{k-2t+r} = \frac{n-k}{k-2t+r} + 1 \geqslant \frac{n-k}{k-t} + 1 \geqslant 2(t+1)\binom{k}{t} + 1.$$

Then, for $r \in \{2, 3, \ldots, t-2\}$,

$$\begin{aligned}
f(r) - f(r-1) &= \binom{k-r}{t-r}\binom{n-2t+r-1}{k-2t+r-1}\left(\frac{n-2t+r}{k-2t+r} - \frac{k-r+1}{t-r+1}\right) \\
&\geqslant \binom{k-r}{t-r}\binom{n-2t+r-1}{k-2t+r-1}\left(2(t+1)\binom{k}{t} + 1 - k - 1\right) \\
&\geqslant 0.
\end{aligned}$$

Therefore, $\binom{k-r}{t-r}\binom{n-2t+r}{k-2t+r}$ is increasing as $r \in \{1, 2, \ldots, t-2\}$ increases, and we get

$$\frac{1}{2\binom{k}{t}}\binom{n-t}{k-t} \leqslant \binom{k-t+2}{2}\binom{n-t-2}{k-t-2},$$

a contradiction with $n \geqslant 2(k-t)(t+1)\binom{k}{t} + k + t + 1$. Then, we have $r = t-1$. Recall that $\overline{\mathcal{A}_{-Y}}^* = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{t-1}$, where $\mathcal{A}_a \subseteq \binom{[n]}{n-k-t+a}$ and $a \in \{0, 1, \ldots, t-1\}$. Then we know for every $v \in \mathcal{A}_b$, there exists exactly one $w \in \mathcal{A}_{-Y}$ such that $\overline{w}^* = v$, where $b \in \{0, 1, \ldots, t-2\}$. And for each $v \in \mathcal{A}_{t-1}$, there exists at most $\binom{t}{t-1}$ $w \in \mathcal{A}_{-Y}$ such that $\overline{w}^* = v$, which implies $|\mathcal{A}_{-Y}| \leqslant \sum_{a=0}^{t-2}|\mathcal{A}_a| + t|\mathcal{A}_{t-1}|$. By Claim 3, we know $|\mathcal{A}_{t-1}| < \binom{n-2t-1}{n-k-1}$ and $|\mathcal{A}_{-Y}| < \sum_{a=0}^{t-2}\binom{n-2t-1}{n-k-t+a} + t\binom{n-2t-1}{n-k-1}$. $\square$

Claim 5. $|\mathcal{A}_Y| \geqslant \frac{k}{k+1}|\mathcal{A}|$.

Proof of Claim 5 Assume for the sake of contradiction that $|\mathcal{A}_Y| < \frac{k}{k+1}|\mathcal{A}|$. Then we have $|\mathcal{A}| < (k+1)|\mathcal{A}_{-Y}|$ since $|\mathcal{A}| = |\mathcal{A}_Y| + |\mathcal{A}_{-Y}|$. By Claim 4, we have $|\mathcal{A}| < (k+1)\sum_{a=0}^{t-2}\binom{n-2t-1}{n-k-t+a} + (k+1)t\binom{n-2t-1}{n-k-1}$. On the other hand, by (4), we have $|\mathcal{A}| \geqslant (1-p)|K(n,k,t) - X|$. Therefore, it follows that

$$(1-p)\binom{n-t}{k-t} < (k+1)\sum_{a=0}^{t-2}\binom{n-2t-1}{n-k-t+a} + (k+1)t\binom{n-2t-1}{n-k-1}.$$

However, by $n \geqslant \frac{1}{1-p}(k-t)(k+1) + 2t$ and Lemma 5, we get a contradiction. $\qquad\square$

Claim 6. $\mathcal{B}_Y = \mathcal{B}$.

Proof of Claim 6 We suppose to the contrary that $\mathcal{B}_Y \neq \mathcal{B}$. Therefore, there exists some $v \in \mathcal{B}$ such that $Y \not\subseteq v$. For any $u \in \mathcal{A}_Y$, since u is not adjacent to v in $K(n,k,t) - X$, we have $|u \cap v| \geqslant t$. Let $|Y \cap v| = a$, $a \in \{0, 1, \ldots, t-1\}$. Thus u contains at least $t - a$ elements of $v \setminus Y$ as $|u \cap v| \geqslant t$. It follows that

$$|\mathcal{A}_Y| \leqslant \binom{k-a}{t-a}\binom{n-t-(t-a)}{k-t-(t-a)}. \tag{11}$$

On the other hand, combining the result in Claim 5 with (3) and (4), we get

$$|\mathcal{A}_Y| \geqslant \frac{k}{k+1}|\mathcal{A}| > \frac{(1-p)k}{k+1}\binom{n-t}{k-t}. \tag{12}$$

Comparing the upper with the lower bound for $|\mathcal{A}_Y|$ in (11) and (12), we have

$$\frac{(1-p)k}{k+1}\binom{n-t}{k-t} < \binom{k-a}{t-a}\binom{n-2t+a}{k-2t+a}.$$

Since $\binom{k-a}{t-a}\binom{n-2t+a}{k-2t+a}$ is increasing as $a \in \{0, 1, \ldots, t-1\}$ increases, we get

$$\frac{(1-p)k}{k+1}\binom{n-t}{k-t} < (k-t+1)\binom{n-t-1}{k-t-1}.$$

By $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, we have $n < \frac{k-t+1}{(1-p)k}(k-t)(k+1) + t < \frac{1}{1-p}(k-t)(k+1) + t$, a contradiction with $n \geqslant \frac{1}{1-p}(k-t)(k+1) + 2t$. Thus, we obtain $\mathcal{B}_Y = \mathcal{B}$, as required. $\qquad\square$

Claim 7. $\mathcal{A}_Y = \mathcal{A}$.

Proof of Claim 7 This claim follows by essentially the same argument as Claim 6 above. We suppose to the contrary that $\mathcal{A}_Y \neq \mathcal{A}$. If $\mathcal{A}_{-Y} = \emptyset$, then we have the claim holds. In the following, let $\mathcal{A}_{-Y} \neq \emptyset$. Therefore, there exists some $w \in \mathcal{A}$ such that $Y \not\subseteq w$, and for any $z \in \mathcal{B}_Y$, $|w \cap z| \geqslant t$. By Claim 6, (3) and (5), we have $|\mathcal{B}_Y| = |\mathcal{B}| > \frac{1}{2}\binom{n-t}{k-t}$. On the other hand, let $|Y \cap w| = a$, $a \in \{0, 1, \ldots, t-1\}$. Thus z contains at least $t - a$ elements of $w \setminus Y$ since $|w \cap z| \geqslant t$. It follows that $|\mathcal{B}_Y| \leqslant \binom{k-a}{t-a}\binom{n-t-(t-a)}{k-t-(t-a)}$, this equals to

the upper bound of $|\mathcal{A}_Y|$ in the previous proof of Claim 6. Therefore, combining with the upper bound and the lower bound of $|\mathcal{B}_Y|$, we have

$$\frac{1}{2}\binom{n-t}{k-t} < \binom{k-a}{t-a}\binom{n-2t+a}{k-2t+a}.$$

Since $\binom{k-a}{t-a}\binom{n-2t+a}{k-2t+a}$ is increasing as $a \in \{0, 1, \ldots, t-1\}$ increases, we obtain $n < 2k^2 - 4kt + 2t^2 + 2k - t$, a contradiction. Therefore, we get $\mathcal{A}_Y = \mathcal{A}$, as required. $\square$

From Claims 6 and 7, we have every vertex in $\mathcal{A} \cup \mathcal{B} = \mathcal{A}_Y \cup \mathcal{B}_Y$ contains Y. Since $V(K(n,k,t)-X)$ can be partitioned into $\mathcal{A}$ and $\mathcal{B}$, we get $|K(n,k,t)-X| = |\mathcal{A}_Y| + |\mathcal{B}_Y| \leqslant \binom{n-t}{k-t}$, which implies that $|X| \geqslant \binom{n}{k} - \binom{n-t}{k-t}$, a contradiction. $\square$

Proof of Theorem 2. By Lemma 12, if we let X be a $\frac{2}{3}$-separator of $K(n,k,t)$, then $|X| \geqslant \binom{n}{k} - \binom{n-t}{k-t}$. Since $2(k-t)(t+1)\binom{k}{t} + k + t + 1 \geqslant \frac{1}{1-p}t(k-t)(k+1) + t + 1$ for $k > t \geqslant 2$, we have $\mathrm{tw}(K(n,k,t)) \geqslant \binom{n}{k} - \binom{n-t}{k-t} - 1$ for $n \geqslant 2(k-t)(t+1)\binom{k}{t} + k + t + 1$ by Proposition 11. Next, combining with the upper bound of $\mathrm{tw}(K(n,k,t))$ in Lemma 8, we obtain the result directly. $\square$

In the special case when $t = 1$, by Lemma 12, if we let X be a $\frac{2}{3}$-separator of $K(n,k,1)$, then $|X| \geqslant \binom{n}{k} - \binom{n-1}{k-1}$ for $n \geqslant 4k^2 - 3k + 2$. Therefore, we have $\mathrm{tw}(K(n,k,1)) = \binom{n}{k} - \binom{n-1}{k-1} - 1$ for $n \geqslant 4k^2 - 3k + 2$ by Proposition 11 and Lemma 8.

4   Treewidth of the complement of Johnson graphs

In this section, we study the treewidth of the complement of Johnson graphs, and give the exact value of the treewidth of $\overline{J(n,k)}$ for $n \geqslant k+2$. Note that $\overline{J(n,k)}$ is an empty graph when $n < k+2$. Firstly, we can easily get the upper bound of $\mathrm{tw}(\overline{J(n,k)})$ as follow according to Theorem 6 and Proposition 7.

Lemma 13. *Let n and k be positive integers with $k \geqslant 2$ and $n \geqslant 2k$. Then*

$$\mathrm{tw}(\overline{J(n,k)}) \leqslant \binom{n}{k} - n + k - 2.$$

Lemma 14. *Let n and k be positive integers with $k \geqslant 2$ and $n \geqslant \max\{k+4, 2k\}$. Then*

$$\mathrm{tw}(\overline{J(n,k)}) \geqslant \binom{n}{k} - n + k - 2.$$

Proof. We suppose to the contrary that $\mathrm{tw}(\overline{J(n,k)}) < \binom{n}{k} - n + k - 2$. By Proposition 11, there exists a $\frac{2}{3}$-separator X such that $|X| < \binom{n}{k} - n + k - 1$. Therefore, $|V(\overline{J(n,k)}-X)| > n - k + 1$. Furthermore, since $n \geqslant \max\{k+4, 2k\} \geqslant k + 4$, we have

$$|V(\overline{J(n,k)} - X)| \geqslant 6.$$

By the similar analysis of the proof of Lemma 12, it is easy to see that $V(\overline{J(n,k)} - X)$ can be partitioned into two parts $\mathcal{A}$ and $\mathcal{B}$ such that there is no edge between $\mathcal{A}$ and $\mathcal{B}$, and the equations

$$\frac{1}{3}|V(\overline{J(n,k)} - X)| \leqslant |\mathcal{A}|, |\mathcal{B}| \leqslant \frac{2}{3}|V(\overline{J(n,k)} - X)| \tag{13}$$

holds. Thus, $|\mathcal{A}|, |\mathcal{B}| \geqslant 2$. By Theorem 6 and $n \geqslant \max\{k+4, 2k\} \geqslant 2k$, we have $\alpha(\overline{J(n,k)}) = n - k + 1$. Thus $V(\overline{J(n,k)} - X)$ is too large to be an independent set which implies that there exists an edge in the subgraph induced by $\mathcal{A}$ or the subgraph induced by $\mathcal{B}$. Without loss of generality, assume that this edge is in $\mathcal{A}$ and let the two endpoints of the edge are

$$v_1 = \{1, 2, \ldots, x, a_1, \ldots, a_{k-x}\} \text{ and } v_2 = \{1, 2, \ldots, x, b_1, \ldots, b_{k-x}\},$$

where $v_1 \cap v_2 = \{1, 2, \ldots, x\}$ and $a_i \neq b_j$ for any $1 \leqslant i, j \leqslant k - x$ and $x \leqslant k - 2$.

We claim that $x = k - 2$. Since for any vertex $w \in \mathcal{B}$ there is no vertex in $\mathcal{A}$ is adjacent to w in $\overline{J(n,k)} - X$, we have $|w \cap v_1|, |w \cap v_2| \geqslant k - 1$. This implies that $|w \cap v_1|, |w \cap v_2| = k - 1$ and then $|v_1 \cap v_2| \geqslant k - 2$, as w is a k-set. On the other hand, $x \leqslant k - 2$ from above. Thus we have $x = k - 2$, as required.

Next, we prove that $v_1 \cap v_2 = \{1, 2, \ldots, k - 2\} \subseteq w$ for any vertex $w \in \mathcal{B}$. As $w \cap v_1 \cap v_2 \subseteq \{1, 2, \ldots, k - 2\}$, we only need to prove $|w \cap v_1 \cap v_2| \geqslant k - 2$. We have

$$|w \cap v_1 \cap v_2| = |w \cap v_1| + |w \cap v_2| - |w \cap (v_1 \cup v_2)| \geqslant 2(k-1) - k = k - 2,$$

as required.

Therefore, we obtain

$$\mathcal{A} \supseteq \big\{v_1 = \{1, 2, \ldots, k - 2, a_1, a_2\}, v_2 = \{1, 2, \ldots, k - 2, b_1, b_2\}\big\},$$
$$\mathcal{B} \subseteq \big\{\{1, 2, \ldots, k - 2, a_1, b_1\}, \{1, 2, \ldots, k - 2, a_1, b_2\},$$
$$\{1, 2, \ldots, k - 2, a_2, b_1\}, \{1, 2, \ldots, k - 2, a_2, b_2\}\big\}.$$

We first have the following two claims. Let $G = \overline{J(n,k)} - X$ for short.

Claim 8. If $G[\mathcal{B}]$ contains an edge, then $n = k + 4$, $|V(G)| = 6$ and one of $G[\mathcal{A}]$ and $G[\mathcal{B}]$ is connected.

Proof of Claim 8 According to the above analysis, in this situation

$$V(G) \subseteq \{\{1, 2, \ldots, k - 2, i, j\} \mid i, j \in \{a_1, a_2, b_1, b_2\}\}.$$

Thus, $|V(G)| \leqslant 6$. Since $|V(G)| \geqslant n - k + 2 \geqslant 6$, we have $|V(G)| = 6$ and then $n = k + 4$. Therefore, there exists three edges in G in total. If there is one edge in $G[\mathcal{A}]$ (that is (v_1, v_2)), and the other two edges in $G[\mathcal{B}]$, then $G[\mathcal{A}]$ is connected. Otherwise, $G[\mathcal{B}]$ contains exactly one edge and it is connected. □

Claim 9. If $\mathcal{B}$ is an independent set in G, then $n = k + 4$, $|V(G)| = 6$ and $G[\mathcal{A}]$ is connected.

$$\boxed{\{\{1,2,5\}\} \cup (X \setminus \{\{2,4,5\}\})}$$

$$\boxed{X}$$

$$\boxed{\{\{1,2,3\}\} \cup (X \setminus \{\{1,3,4\}\})} \qquad \boxed{\{\{1,2,4\}\} \cup (X \setminus \{\{1,3,4\}\})}$$

$$\boxed{\{\{1,2,3\},\{1,4,5\}\} \cup (X \setminus \{\{1,3,4\},\{2,4,5\}\})} \quad \boxed{\{\{1,2,4\},\{1,3,5\}\} \cup (X \setminus \{\{1,3,4\},\{2,3,5\}\})}$$
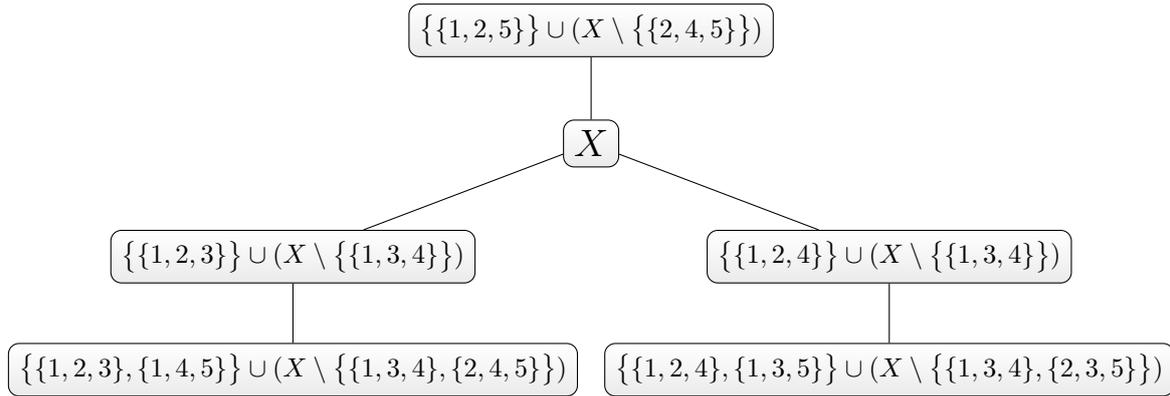
Figure 1: Tree decomposition

Proof of Claim 9 Without loss of generality, let

$$\mathcal{B} = \big\{\{1,2,\ldots,k-2,a_1,b_1\},\{1,2,\ldots,k-2,a_1,b_2\}\big\}.$$

Thus, we have

$$\mathcal{A} \subseteq \{\{1,2,\ldots,k-2,a_1,i\} \mid i \in [n] \setminus \{1,2,\ldots,k-2,a_1,b_1,b_2\}\}$$
$$\cup \{\{1,2,\ldots,k-2,a_1,b_1,b_2\} - j \mid j \in \{1,\ldots,k-2,a_1\}\}, \tag{14}$$

and $|V(G)| = |\mathcal{A}| + |\mathcal{B}| = |\mathcal{A}| + 2$. On the other hand, since $|\mathcal{A}| \leqslant \frac{2}{3}|V(G)| = \frac{2}{3}(|\mathcal{A}|+2)$ by (13), we have $|\mathcal{A}| \leqslant 4$ and $|V(G)| \leqslant 6$. Therefore, $|\mathcal{A}| = 4$, $|V(G)| = 6$ and $n = k+4$ since $|V(G)| \geqslant 6$. As $G[\mathcal{A}]$ is connected for arbitrary 4 elements chosen from (14) when $n = k+4$, we have the claim holds. $\qquad\square$

Let $(T,(B_t)_{t \in V(T)})$ be a minimum width tree decomposition for $\overline{J(n,k)}$, such that if $t_1 t_2 \in V(T)$, then $B_{t_1} \nsubseteq B_{t_2}$. According to the assumption that $\mathrm{tw}(\overline{J(n,k)}) < \binom{n}{k} - n + k - 2$, and $n = k+4$ by Claims 8 and 9, we have $|B_t| \leqslant \binom{n}{k} - 6$ for all $t \in V(T)$. Since there is a fact that $X \subset V(\overline{J(n,k)})$ is a subset of some bag B_t and $|V(G)| = 6$, by Claims 8 and 9, it follows that $|X| = \binom{n}{k} - 6$ and X is a bag with maximum order, that is $X = B_t$. By Claims 8 and 9, we have $G[\mathcal{A}]$ (resp. $G[\mathcal{B}]$) is a component of G if $G[\mathcal{A}]$ is connected (resp. if $G[\mathcal{B}]$ is connected). Then there is a subtree in $T - t$ contains all vertices of $\mathcal{A}$ (resp. $\mathcal{B}$). Notice that every vertex in X has a neighbor in $\mathcal{A}$ (resp. $\mathcal{B}$) if $G[\mathcal{A}]$ is connected (resp. if $G[\mathcal{B}]$ is connected) since the vertices which are non-adjacent with the vertices in $\mathcal{A}$ are all in $\mathcal{B}$. Let t' be the node of this subtree adjacent to t. Thus we have $B_t \subseteq B_{t'}$, a contradiction. $\qquad\square$

Proof of Theorem 3. We divide the proof of this theorem into the following two cases.

Case 1. $k \geqslant 4$.

In this case we have $2k \geqslant k+4$. If $n \geqslant 2k$, by Lemmas 13 and 14, we have $\mathrm{tw}(\overline{J(n,k)}) = \binom{n}{k} - n + k - 2$. Note that $\overline{J(n,k)} \cong \overline{J(n,n-k)}$. If $n < 2k$ then $n > 2(n-k)$. By Lemmas 13 and 14, $\mathrm{tw}(\overline{J(n,k)}) = \mathrm{tw}(\overline{J(n,n-k)}) = \binom{n}{k} - k - 2$.

Case 2. $1 < k < 4$.

We have $k + 4 > 2k$ in this case.

Subcase 1. $k = 2$.

If $n \geqslant 6 = k+4$, by Lemmas 13 and 14, we get $\mathrm{tw}(\overline{J(n,2)}) = \binom{n}{k} - n + k - 2 = \binom{n}{2} - n$. If $n = 4$ or 5, then $\mathrm{tw}(\overline{J(4,2)}) = 1$ and $\mathrm{tw}(\overline{J(5,2)}) = 4$ [13], respectively.

Subcase 2. $k = 3$.

If $n \geqslant 7 = k+4$, by Lemmas 13 and 14 again, we obtain $\mathrm{tw}(\overline{J(n,2)}) = \binom{n}{3} - n + 1$. If $n = 5$, by $\overline{J(5,3)} \simeq \overline{J(5,2)}$, we have $\mathrm{tw}(\overline{J(5,3)}) = \mathrm{tw}(\overline{J(5,2)}) = 4$ [13]. Notice that $\overline{J(5,3)}$ is the Peterson graph and it is easy to construct a minimal width tree decomposition for it shown in Figure 1, where

$$X = \big\{\{1,3,4\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\big\}.$$

If $n = 6$, we can easily construct a tree decomposition for $\overline{J(6,3)}$, and that also satisfies the structure in Figure 1, where

$$X = \binom{[6]}{3} \setminus \big\{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,5\},\{1,4,5\}\big\}.$$

This implies that $\mathrm{tw}(\overline{J(6,3)}) \leqslant 14$.

Suppose to the contrary that $\mathrm{tw}(\overline{J(6,3)}) < 14$. By Proposition 11, there exists a $\frac{2}{3}$-separator X such that $|X| < 15$. Therefore, $|V(\overline{J(6,3)} - X)| \geqslant 6$ and $V(\overline{J(6,3)} - X)$ can be partitioned into two parts $\mathcal{A}$ and $\mathcal{B}$ such that there is no edge between $\mathcal{A}$ and $\mathcal{B}$, and the equation (13) holds. Thus, $|\mathcal{A}|, |\mathcal{B}| \geqslant 2$. Denote $G = \overline{J(6,3)} - X$ for short. By Theorem 6, $\alpha(\overline{J(6,3)}) = 4$. Thus there exists an edge (v_1, v_2) in $G[\mathcal{A}]$ or $G[\mathcal{B}]$. Without loss of generality, assume that (v_1, v_2) is in $G[\mathcal{A}]$. Since for any vertex $w \in \mathcal{B}$ there is no vertex in $\mathcal{A}$ is adjacent to w in G, we have $|w \cap v_1|, |w \cap v_2| \geqslant 2$. This implies that $|w \cap v_1| = |w \cap v_2| = 2$ and then $|v_1 \cap v_2| \geqslant 1$, as w is a 3-set. Note that $|v_1 \cap v_2| \leqslant 1$. Therefore, we have $|v_1 \cap v_2| = 1$. Without loss of generality, let

$$v_1 = \{1, a_1, a_2\} \text{ and } v_2 = \{1, b_1, b_2\},$$

where $a_i \neq b_j$ for any $1 \leqslant i, j \leqslant 2$.

Since for any vertex $w \in \mathcal{B}$, $|w \cap v_1 \cap v_2| = |w \cap v_1| + |w \cap v_2| - |w \cap (v_1 \cup v_2)| \geqslant 1$ and $w \cap v_1 \cap v_2 \subseteq \{1\}$, we have $v_1 \cap v_2 = \{1\} \subseteq w$. Therefore, we obtain

$$\mathcal{A} \supseteq \big\{v_1 = \{1, a_1, a_2\}, v_2 = \{1, b_1, b_2\}\big\},$$
$$\mathcal{B} \subseteq \big\{\{1, a_1, b_1\}, \{1, a_1, b_2\}, \{1, a_2, b_1\}, \{1, a_2, b_2\}\big\}.$$

If $G[\mathcal{B}]$ contains an edge, then $V(G) \subseteq \{\{1, i, j\} \mid i, j \in \{a_1, a_2, b_1, b_2\}\}$. Thus, $|V(G)| \leqslant 6$. Since $|V(G)| \geqslant 6$, we have $|V(G)| = 6$. Therefore, there exists three edges in G in total. Without loss of generality, assume that there is one edge in $G[\mathcal{A}]$

(that is (v_1, v_2)), and the other two edges are in $G[\mathcal{B}]$. Then $G[\mathcal{A}]$ is connected. If $\mathcal{B}$ is an independent set in G, without loss of generality, let

$$\mathcal{B} = \big\{\{1, a_1, b_1\}, \{1, a_1, b_2\}\big\}.$$

Thus, we have

$$\mathcal{A} \subseteq \big\{\{1, a_1, i\} \mid i \in [6] \setminus \{1, a_1, b_1, b_2\}\big\} \cup \big\{\{1, b_1, b_2\}, \{a_1, b_1, b_2\}\big\}.$$

Therefore, $|V(G)| = |\mathcal{A}| + |\mathcal{B}| \leqslant 6$. Since $|V(G)| \geqslant 6$, we have $|V(G)| = 6$. Therefore, $\mathcal{A}$ is exactly that set above and $G[\mathcal{A}]$ is connected.

Let $(T, (B_t)_{t \in V(T)})$ be a minimum width tree decomposition for $\overline{J(6,3)}$, such that if $t_1 t_2 \in V(T)$, then $B_{t_1} \nsubseteq B_{t_2}$. Since $\mathrm{tw}(\overline{J(6,3)}) < 14$, we have $|B_t| \leqslant 14$ for all $t \in V(T)$. Since there is a fact that $X \subset V(\overline{J(6,3)})$ is a subset of some bag B_t and $|V(G)| = 6$, it follows that $|X| = 14$ and X is a bag with maximum order, that is $X = B_t$. Since $G[\mathcal{A}]$ is a component of G, there is a subtree of $T - t$ contains all vertices of $\mathcal{A}$. Notice that every vertex in X has a neighbor in $\mathcal{A}$ since the vertices which are non-adjacent with the vertices in $\mathcal{A}$ are all in $\mathcal{B}$. Let t' be the node of this subtree adjacent to t. Thus we have $B_t \subseteq B_{t'}$, a contradiction.

Consequently, we complete the proof of this theorem. $\qquad\square$

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