Maximal sets of k-spaces pairwise intersecting in at least a $(k-2)$-space

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Abstract

In this paper, we analyze the structure of maximal sets of k-dimensional spaces in $\mathrm{PG}(n, q)$ pairwise intersecting in at least a $(k-2)$-dimensional space, for $3 \leqslant k \leqslant n-2$. We give an overview of the largest examples of these sets with size more than $f(k, q) = \max\{3q^4 + 6q^3 + 5q^2 + q + 1, \theta_{k+1} + q^4 + 2q^3 + 3q^2\}$.

Mathematics Subject Classifications: 05B25, 05E20, 51E20

1 Introduction and preliminaries

One of the classical problems in extremal set theory is to determine the size of the largest sets of pairwise non-trivially intersecting subsets. In 1961, it was solved by Erdős, Ko and Rado [11], and their result was improved by Wilson in 1984.

Theorem 1. *[17] Let n, k and t be positive integers and suppose that $k \geqslant t \geqslant 1$ and $n \geqslant (t+1)(k-t+1)$. If $\mathcal{S}$ is a family of subsets of size k in a set Ω with $|\Omega| = n$, such that the elements of $\mathcal{S}$ pairwise intersect in at least t elements, then $|\mathcal{S}| \leqslant \binom{n-t}{k-t}$.*
Moreover, if $n \geqslant (t+1)(k-t+1)+1$, then $|\mathcal{S}| = \binom{n-t}{k-t}$ holds if and only if $\mathcal{S}$ is the set of all the subsets of size k through a fixed subset of Ω of size t.

Note that if $t = 1$, then $\mathcal{S}$ is a collection of subsets of size k of an arbitrary set, which are pairwise not disjoint. In the literature, a family of subsets that are pairwise not disjoint, is called an *Erdős-Ko-Rado set* and the classification of the largest Erdős-Ko-Rado sets is called the *Erdős-Ko-Rado problem*, in short *EKR problem*. Hilton and Milner [13] described the largest Erdős-Ko-Rado sets $\mathcal{S}$ with the property that there is no point contained in all elements of $\mathcal{S}$.

This set-theoretical problem can be generalized in a natural way to many other structures such as designs [16], permutation groups [6] and projective geometries. In this article, we work in the projective setting (for an overview, see [7]); where this problem is known as the *q-analogue* of the Erdős-Ko-Rado problem. More precisely, let q be a prime power and let $\mathrm{PG}(n, q)$ be the projective geometry of the subspaces of the vector space $\mathbb{F}_q^{n+1}$ over the finite field $\mathbb{F}_q$. Clearly, results on families of vector spaces pairwise intersecting in at least a vector space with fixed dimension can be interpreted in projective spaces, and vice versa. Here, in the projective setting, a projective m-dimensional subspace of $\mathrm{PG}(n, q)$ will be called *m-space*. In $\mathrm{PG}(n, q)$, we can consider families of k-spaces pairwise intersecting in at least a t-dimensional subspace for $0 \leqslant t \leqslant k - 1$. In particular for $t = 0$, these sets are the Erdős-Ko-Rado sets of $\mathrm{PG}(n, q)$

Before stating the q-analogue of Theorem 1, we briefly recall the definition of the *Gaussian binomial coefficient*.

Definition 2. Let q be a prime power, let n, k be non-negative integers with $k \leqslant n$. The *Gaussian binomial coefficient* of n and k is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q^n-1)\cdots(q^{n-k+1}-1)}{(q^k-1)\cdots(q-1)} & \text{if } k > 0 \\ 1 & \text{otherwise} \end{cases}$$

We will write $\begin{bmatrix} n \\ k \end{bmatrix}$, if the field size q is clear from the context. The number of k-spaces in $\mathrm{PG}(n, q)$ is $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$ and the number of k-spaces through a fixed t-space in $\mathrm{PG}(n, q)$, with $0 \leqslant t \leqslant k$, is $\begin{bmatrix} n-t \\ k-t \end{bmatrix}$. Moreover, we will denote the number $\begin{bmatrix} n+1 \\ 1 \end{bmatrix}$ by the symbol θ_n.

Theorem 3. *[12, Theorem 1] Let t and k be integers, with $0 \leqslant t \leqslant k$. Let $\mathcal{S}$ be a set of k-spaces in $\mathrm{PG}(n, q)$, pairwise intersecting in at least a t-space.*

(i) *If $n \geqslant 2k + 1$, then $|\mathcal{S}| \leqslant \begin{bmatrix} n-t \\ k-t \end{bmatrix}$. Equality holds if and only if $\mathcal{S}$ is the set of all the k-spaces, containing a fixed t-space of $\mathrm{PG}(n, q)$, or $n = 2k + 1$ and $\mathcal{S}$ is the set of all the k-spaces in a fixed $(2k - t)$-space.*

(ii) *If $2k - t \leqslant n \leqslant 2k$, then $|\mathcal{S}| \leqslant \begin{bmatrix} 2k-t+1 \\ k-t \end{bmatrix}$. Equality holds if and only if $\mathcal{S}$ is the set of all the k-spaces in a fixed $(2k - t)$-space.*

Corollary 4. *Let $\mathcal{S}$ be an Erdős-Ko-Rado set of k-spaces in $\mathrm{PG}(n,q)$. If $n \geqslant 2k+1$, then $|\mathcal{S}| \leqslant \begin{bmatrix} n \\ k \end{bmatrix}$. Equality holds if and only if $\mathcal{S}$ is the set of all the k-spaces, containing a fixed point of $\mathrm{PG}(n,q)$, or $n = 2k+1$ and $\mathcal{S}$ is the set of all the k-spaces in a fixed hyperplane.*

Note that in Theorem 3 the condition $n \geqslant 2k - t$ is not a restriction, since any two k-dimensional subspaces in $\mathrm{PG}(n,q)$, with $n \leqslant 2k - t$, meet in at least a t-dimensional subspace. Furthermore, as new families of any size can be found by deleting elements, the research is focused on *maximal* families: these are sets of k-spaces pairwise intersecting in at least a t-space, not extendable to a larger family of k-spaces with the same property. Related to this question, we report the q-analogue of the Hilton-Milner result on the second-largest maximal Erdős-Ko-Rado sets of subspaces in a finite projective space, due to Blokhuis *et al*. In the context of projective spaces, the set of all subspaces through a fixed t-space will be called a *t-pencil*, and, in particular, a *point-pencil* if $t = 0$ and a *line-pencil* if $t = 1$.

Theorem 5. *[2, Theorem 1.3, Proposition 3.4] Let $\mathcal{S}$ be a maximal set of pairwise intersecting k-spaces in $\mathrm{PG}(n,q)$, with $n \geqslant 2k+2$, $k \geqslant 2$ and $q \geqslant 3$ (or $n \geqslant 2k+4$, $k \geqslant 2$ and $q = 2$). If $\mathcal{S}$ is not a point-pencil, then*

$$|\mathcal{S}| \leqslant \begin{bmatrix} n \\ k \end{bmatrix} - q^{k(k+1)} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} + q^{k+1}.$$

Moreover, if equality holds, then

(i) *either $\mathcal{S}$ consists of all the k-spaces through a fixed point P, meeting a fixed $(k+1)$-space τ, with $P \in \tau$, in a j-space, $j \geqslant 1$, and all the k-spaces in τ;*

(ii) *or else $k = 2$ and $\mathcal{S}$ is the set of all the planes meeting a fixed plane π in at least a line.*

The Erdős-Ko-Rado problem for $k = 1$ has been solved completely. Indeed, in $\mathrm{PG}(n,q)$ with $n \geqslant 3$, a maximal Erdős-Ko-Rado set of lines is either the set of all the lines through a fixed point or the set of all the lines contained in a fixed plane. It is possible to generalize this result for a maximal family $\mathcal{S}$ of k-spaces, pairwise intersecting in a $(k-1)$-space, in a projective space $\mathrm{PG}(n,q)$, $n \geqslant k+2$.

Theorem 6. *[4, Section 9.3] Let $\mathcal{S}$ be a set of projective k-spaces, pairwise intersecting in a $(k-1)$-space in $\mathrm{PG}(n,q)$, $n \geqslant k+2$, then all the k-spaces of $\mathcal{S}$ go through a fixed $(k-1)$-space or they are contained in a fixed $(k+1)$-space.*

The Erdős-Ko-Rado problem for sets of projective planes is trivial if $n \leqslant 4$. For $n = 5$, Blokhuis, Brouwer and Szőnyi classified the six largest examples [3, Section 6]. De Boeck investigated the maximal Erdős-Ko-Rado sets of planes in $\mathrm{PG}(n,q)$ with $n \geqslant 5$, see [8]. He characterized those sets with sufficiently large size and showed that they belong to one of the 11 known examples, explicitly described in his work.

In [1, 9], a classification of the largest examples of sets of k-spaces in $\mathrm{PG}(n, q)$ pairwise intersecting in precisely a $(k - 2)$-space is given. In [5], Brouwer and Hemmeter investigated sets of generators, pairwise intersecting in at least a space with codimension 2, in quadrics and symplectic polar spaces. In this paper, we will study the projective analogue of this question. Let $f(k, q) = \max\{3q^4 + 6q^3 + 5q^2 + q + 1, \theta_{k+1} + q^4 + 2q^3 + 3q^2\}$, with q a prime power, and so

$$
f(k, q) = \begin{cases} 3q^4 + 6q^3 + 5q^2 + q + 1 & \text{if } k = 3, q \geqslant 2 \text{ or } k = 4, q = 2 \\ \theta_{k+1} + q^4 + 2q^3 + 3q^2 & \text{otherwise.} \end{cases} \tag{1}
$$

We analyze the sets of k-spaces in $\mathrm{PG}(n, q)$ pairwise intersecting in at least a $(k-2)$-space and with more than $f(k, q)$ elements. We will suppose that these sets $\mathcal{S}$ of subspaces are maximal. During the discussion, we will give bounds on the size of the largest examples and we will indicate the order of such families of k-spaces in $\mathrm{PG}(n, q)$, using the *big O notation*.

In [10], families of subspaces pairwise intersecting in at least a t-space were investigated. More specifically, the author investigates the largest non-trivial example of a set of k-spaces, pairwise intersecting in at least a t-space in $\mathrm{PG}(n, q)$. The main theorem of this article is consistent with Theorem 1 from [10].

Theorem 7. *[10, **Theorem 1**] Let $\mathcal{F}$ be a set of k-spaces pairwise intersecting in at least a t-space in $\mathrm{PG}(n, q)$, $n \geqslant 2k + 5 + \frac{t(t+5)}{2}$, of maximum size, with $\mathcal{F}$ not a t-pencil, then $\mathcal{F}$ is one of the following examples:*

  i) *the set of k-spaces, meeting a fixed $(t + 2)$-space in at least a $(t + 1)$-space,*

  ii) *the set of k-spaces in a fixed $(k+1)$-space Y together with the set of k-spaces through a t-space $\pi \subset Y$, that have at least a $(t + 1)$-space in common with Y.*

Note that the two examples in the previous theorem correspond to Example 9(ii) and (iii) for $t = k - 2$ respectively. While, in [10], David Ellis classifies the largest non-trivial example for all values of t, here we specify this problem classifying the ten largest examples when $t = k - 2$. More precisely, we will show the following theorem.

Main Theorem 8. Let $\mathcal{S}$ be a maximal set of k-spaces pairwise intersecting in at least a $(k - 2)$-space in $\mathrm{PG}(n, q)$, $n \geqslant 2k$, $k \geqslant 3$. Let $f(k, q)$ be defined as in (1).
   If $|\mathcal{S}| > f(k, q)$, then $\mathcal{S}$ is one of the families described in Example 9.

To prove this result, the possible configurations of a maximal family $\mathcal{S}$ of k-spaces meeting in at least a $(k - 2)$-space will be analyzed. Firstly, in Section 2, we will suppose there is no point contained in all elements of $\mathcal{S}$ and the family contains three k-spaces A, B, C with $\dim(A \cap B \cap C) = k - 4$. Later, we will distinguish the possibilities for $\mathcal{S}$ depending on the dimension of the subspace $\alpha = \langle D \cap \langle A, B \rangle \mid D \in \mathcal{S}' \rangle$, where $\mathcal{S}' = \{D \in \mathcal{S} \mid D \not\subset \langle A, B \rangle\}$. Accordingly, it will follow that if $|\mathcal{S}| > f(k, q)$, then $\mathcal{S}$ belongs to one of the ten examples listed below.

In Section 3, we will suppose there is no point contained in all elements of $\mathcal{S}$ and every three elements of $\mathcal{S}$ meet in at least a $(k-3)$-space. Again we will conclude that $\mathcal{S}$ is one of the families listed below, see Theorem 34.

Finally, in Section 4, we will investigate a maximal family $\mathcal{S}$ admitting the existence of at least a point contained in all k-spaces of $\mathcal{S}$. Let γ be the maximal subspace contained in all k-spaces of $\mathcal{S}$, with $\dim \gamma = g$. By working in the quotient projective space $\mathrm{PG}(n,q)/\gamma \cong \mathrm{PG}(n-g-1,q)$, we will obtain that the only examples of maximal families of this type are possible when $g = k-3$ and they are still listed in Example 9.

Thus, we end this section listing some examples of maximal sets $\mathcal{S}$ of k-spaces in $\mathrm{PG}(n,q)$ pairwise intersecting in at least a $(k-2)$-space, $n \geqslant k+2$ and $k \geqslant 3$ and we add a proof of maximality.

Example 9.   *(i)* $(k-2)$-*pencil:* the set $\mathcal{S}$ is the set of all k-spaces that contain a fixed $(k-2)$-space. Then $|\mathcal{S}| = \begin{bmatrix} n-k+2 \\ 2 \end{bmatrix}$.

Lemma 10. *The set $\mathcal{S}$ from Example 9(i) is maximal for $n \geqslant k+3$.*

Proof. Suppose there is a k-space $E \notin \mathcal{S}$, meeting all elements of $\mathcal{S}$ in at least a $(k-2)$-space. Then $\dim(E \cap \alpha) = k - t$ with $t \geqslant 3$, where α is the center of the pencil $\mathcal{S}$. Clearly, $n \geqslant \dim\langle E, \alpha \rangle = k + t - 2$. We will prove there exists an element $A \in \mathcal{S}$ such that $\dim(A \cap E) \leqslant k - 3$ for all values of $t \geqslant 3$. If $t = 3$, since $n \geqslant k+t = k+3$, there exists a line l in $\mathrm{PG}(n,q)$ disjoint from $\langle E, \alpha \rangle$. Then, let A be the unique element of $\mathcal{S}$ containing l and α, one gets indeed that $\dim(A \cap E) = k-3$. For $t = 4$, there is a line l that meets $\langle E, \alpha \rangle$ in at most a point. Let A be again the k-space containing l and α, then we have that $\dim(A \cap E) \leqslant k - t + 1 = k - 3$. Finally, if $t \geqslant 5$, let $l \subset E$ be a line disjoint to α and consider A the k-space containing l and α. Then $\dim(A \cap E) = k - t + 2 \leqslant k - 3$. So, all three cases give a contradiction. This proves the lemma. $\square$

   (ii) *Star:* there is a k-space ζ such that $\mathcal{S}$ contains all k-spaces that have at least a $(k-1)$-space in common with ζ. Then $|\mathcal{S}| = q\theta_k\theta_{n-k-1} + 1$.

Lemma 11. *The set $\mathcal{S}$ from Example 9(ii) is maximal for $n \geqslant k+3$.*

Proof. Suppose that $E \notin \mathcal{S}$ is a k-space that meets every element of $\mathcal{S}$ in at least a $(k-2)$-space. Since $E \notin \mathcal{S}$, we know that $\dim(E \cap \zeta) = k - 2$. Let γ be a $(k-1)$-space in ζ such that $\dim(E \cap \gamma) = k - 3$. Then E must meet all elements of $\mathcal{S}$ through γ in a subspace outside of ζ. Since $n \geqslant k+3$, this is not possible. Hence, $\mathcal{S}$ is maximal. $\square$

   (iii) *Generalized Hilton-Milner example:* there is a $(k+1)$-space ν and a $(k-2)$-space $\pi \subset \nu$ such that $\mathcal{S}$ consists of all k-spaces in ν (type 1), together with all k-spaces of $\mathrm{PG}(n,q)$, not in ν, through π that intersect ν in a $(k-1)$-space (type 2). Then $|\mathcal{S}| = \theta_{k+1} + q^2(q^2 + q + 1)\theta_{n-k-2}$.

Lemma 12. *The set $\mathcal{S}$ from Example 9(iii) is maximal for $n \geqslant k+3$.*

Proof. Suppose that $E \notin \mathcal{S}$ is a k-space that meets every element of $\mathcal{S}$ in at least a $(k-2)$-space. If $\pi \subset E$, then, since $E \notin \mathcal{S}$, E meets ν only in π. Hence, there is an element A of type 1 in $\mathcal{S}$ such that $\dim(A \cap \pi) = k-3$, and so, $\dim(A \cap E) < k-2$. This gives a contradiction with the fact that E meets all elements of $\mathcal{S}$ in at least a $(k-2)$-space. Hence, we may suppose that $\pi \nsubseteq E$. So, E meets π in a d-space with $d \leqslant k-3$. Note that $\dim(E \cap \nu) = k-1$ since E meets all k-spaces in ν in at least a $(k-2)$-space and $E \nsubseteq \nu$. So, $\dim(E \cap \pi) = k-3$. Let P be a point in $\mathrm{PG}(n,q)$, not contained in $\langle \nu, E \rangle$ and let Q be a point in $\nu \setminus E$, not contained in π. Then the k-space $B = \langle \pi, P, Q \rangle$ is an element of $\mathcal{S}$, which meets E in a subspace with dimension at most $k-3$. This gives the contradiction. $\qquad\square$

(iv) There is a $(k+2)$-space ρ, a k-space $\alpha \subset \rho$ and a $(k-2)$-space $\pi \subset \alpha$ so that $\mathcal{S}$ contains all k-spaces in ρ that meet α in a $(k-1)$-space not through π (type 1), all k-spaces in ρ through π (type 2), and all k-spaces in $\mathrm{PG}(n,q)$, not in ρ, that contain a $(k-1)$-space of α through π (type 3). Then $|\mathcal{S}| = (q+1)\theta_{n-k} + q^3(q+1)\theta_{k-2} + q^4 - q$.

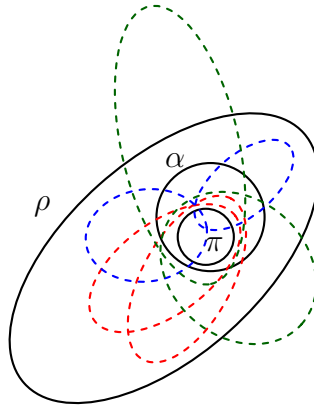This example will be discussed in Proposition 2.5.

Figure 1: Example (iv): The blue, red and green k-spaces correspond to the k-spaces of type 1, 2 and 3, respectively.

Lemma 13. *The set $\mathcal{S}$ from Example 9(iv) is maximal.*

Proof. Suppose there is a k-space $E \notin \mathcal{S}$, meeting all elements of $\mathcal{S}$ in at least a $(k-2)$-space. We start with the case $\pi \not\subset E$. If $\dim(E \cap \pi) \leqslant k-2$, then there is a $(k-1)$-space μ in α with $\dim((E \cap \pi) \cap \mu) \leqslant k-3$. The elements of type 3 through μ meet E in a subspace of dimension at most $k-3$, which gives a contradiction. Hence, E contains a $(k-1)$-space $\sigma_E \subset \alpha$. Let G be an element of $\mathcal{S}$ of type 2 such that $\langle G, \alpha \rangle = \rho$, and so $G \cap \alpha = \pi$. We have

$$\dim(E \cap \rho) \geqslant \dim(\langle E \cap G, E \cap \alpha \rangle) \geqslant \dim(E \cap \alpha) + \dim(E \cap G) - \dim(E \cap G \cap \alpha)$$

$$\geqslant (k-1) + (k-2) - (k-3) \geqslant k.$$

So, $E \subset \rho$, which implies that $E \in \mathcal{S}$ (type 1), a contradiction. Now, we suppose that $\pi \subset E$. Let F_1 and F_2 be two elements of $\mathcal{S}$ of type 1, with $\langle F_1, F_2 \rangle = \rho$ and $\dim(\pi \cap F_1 \cap F_2) = k - 4$. First we show that their existence is assured. Indeed, let π_1 and π_2 be two different $(k-3)$-spaces in π and let α_i be a $(k-1)$-space in α through π_i, $i = 1, 2$. Let P_1 be a point in $\rho \setminus \alpha$ and let $F_1 = \langle P_1, \alpha_1 \rangle$. Finally, consider P_2 be a point in $\rho \setminus \langle \alpha, F_1 \rangle$ and let $F_2 = \langle P_2, \alpha_2 \rangle$. Since $E \notin \mathcal{S}$ and $\pi \subset E$, we know that E cannot contain a $(k-1)$-space of α, and so, $E \cap \alpha = \pi$. Hence, from $F_1 \cap F_2 \subset \alpha$, it follows that $\dim(E \cap F_1 \cap F_2) = \dim(\pi \cap F_1 \cap F_2)$. Then

$$\begin{aligned}
\dim(E \cap \rho) &= \dim(E \cap \langle F_1, F_2 \rangle) \\
&\geqslant \dim(E \cap F_1) + \dim(E \cap F_2) - \dim(E \cap F_1 \cap F_2) \\
&\geqslant (k-2) + (k-2) - (k-4) \geqslant k.
\end{aligned}$$

Hence, $E \subset \rho$ which implies that $E \in \mathcal{S}$, type 2, again a contradiction. $\qquad\square$

(v) There is a $(k+2)$-space ρ, and a $(k-1)$-space $\alpha \subset \rho$ such that $\mathcal{S}$ contains all k-spaces in ρ that meet α in at least a $(k-2)$-space (type 1), and all k-spaces in $\mathrm{PG}(n, q)$, not in ρ, through α (type 2). Note that all k-spaces in $\mathrm{PG}(n, q)$ through α are contained in $\mathcal{S}$.

Then $|\mathcal{S}| = \theta_{n-k} + q^2(q^2 + q + 1)\theta_{k-1}$.

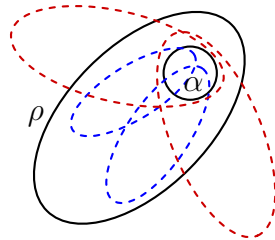This example will be discussed in Proposition 19.

Figure 2: Example(v): The blue and red k-spaces correspond to the k-spaces of type 1, 2, respectively.

Lemma 14. *The set $\mathcal{S}$ from Example 9(v) is maximal.*

Proof. Suppose there is a k-space $E \notin \mathcal{S}$, meeting all elements of $\mathcal{S}$ in at least a $(k-2)$-space. Then E contains a $(k-2)$-space σ_E in α, since E meets all elements of $\mathcal{S}$ of type 2. Note that E cannot contain α, since then, E would be a k-space in $\mathcal{S}$. Let σ_1 and σ_2 be two distinct $(k-2)$-spaces in α with $\dim(\sigma_1 \cap \sigma_2 \cap \sigma_E) = k - 4$. Consider F_1 and F_2, two elements of $\mathcal{S}$ of type 1 through σ_1 and σ_2, respectively, with $\dim(F_1 \cap F_2) = k - 2$. Note that $\dim(E \cap F_1 \cap F_2) = k - 4$. Indeed,

$$k - 4 \leqslant \dim(E \cap F_1 \cap F_2) \leqslant k - 2.$$

(a) If $\dim(E \cap F_1 \cap F_2) = k - 2$, then $E \cap F_1 \cap F_2 \cap \alpha = F_1 \cap F_2 \cap \alpha$, a contradiction.

(b) If $\dim(E \cap F_1 \cap F_2) = k - 3$, there exists a point $P \in F_1 \cap F_2 \cap E$ not in α and $\dim(E \cap \rho) \geqslant k - 1$. Since $E \notin \mathcal{S}$, then $E \not\subset \rho$. The only possibility is $\dim(E \cap \rho) = k - 1$, but then we can find a k-space F of type 1 such that $E \cap F$ is a $(k-3)$-space, again a contradiction.

Hence, $\dim(E \cap F_1 \cap F_2) = k - 4$ and

$$\dim(E \cap \rho) = \dim(E \cap \langle F_1, F_2 \rangle) \geqslant \dim(E \cap F_1) + \dim(E \cap F_2) - \dim(E \cap F_1 \cap F_2)$$
$$\geqslant (k-2) + (k-2) - (k-4) \geqslant k.$$

And so, $E \subset \rho$, which implies that $E \in \mathcal{S}$, a contradiction. $\square$

(vi) There are two $(k+2)$-spaces ρ_1, ρ_2 intersecting in a $(k+1)$-space $\alpha = \rho_1 \cap \rho_2$. There are two $(k-1)$-spaces $\pi_A, \pi_B \subset \alpha$ with $\pi_A \cap \pi_B$ the $(k-2)$-space λ, there is a point $P_{AB} \in \alpha \setminus \langle \pi_A, \pi_B \rangle$, and let $\lambda_A, \lambda_B \subset \lambda$ be two different $(k-3)$-spaces. Then $\mathcal{S}$ contains

type 1. all k-spaces in α,

type 2. all k-spaces of $\mathrm{PG}(n, q)$ through $\langle P_{AB}, \lambda \rangle$, not in ρ_1 and not in ρ_2.

type 3. all k-spaces in ρ_1, not in α, through P_{AB} and a $(k-2)$-space in π_A through λ_A,

type 4. all k-spaces in ρ_1, not in α, through P_{AB} and a $(k-2)$-space in π_B through λ_B,

type 5. all k-spaces in ρ_2, not in α, through P_{AB} and a $(k-2)$-space in π_A through λ_B,

type 6. all k-spaces in ρ_2, not in α, through P_{AB} and a $(k-2)$-space in π_B through λ_A.

Then $|\mathcal{S}| = \theta_{n-k} + q^2 \theta_{k-1} + 4q^3$.

This example will be discussed in Proposition 22 for $k = 3$ and in Proposition 25 for $k > 3$.

Lemma 15. *The set $\mathcal{S}$ from Example 9(vi) is maximal.*

Proof. Suppose there is a k-space $E \notin \mathcal{S}$, meeting all elements of $\mathcal{S}$ in at least a $(k-2)$-space. Suppose first that $P_{AB} \notin E$. As E contains at least a $(k-2)$-space of all elements of $\mathcal{S}$, type 1 and 2, E contains a $(k-1)$-space β in α such that β contains a $(k-2)$-space of $\langle P_{AB}, \lambda \rangle$, not through P_{AB}. Consider now the elements F and G of $\mathcal{S}$, type 3 and 4 respectively, with $F \cap G \cap \alpha = \langle P_{AB}, \lambda_A \cap \lambda_B \rangle$. If $E \not\subset \rho_1$, then $\dim(E \cap F \cap G) \leqslant k - 4$ and

$$k - 1 = \dim(E \cap \alpha) = \dim(E \cap \rho_1) = \dim(E \cap \langle F, G \rangle)$$
$$\geqslant \dim(E \cap F) + \dim(E \cap G) - \dim(E \cap F \cap G)$$
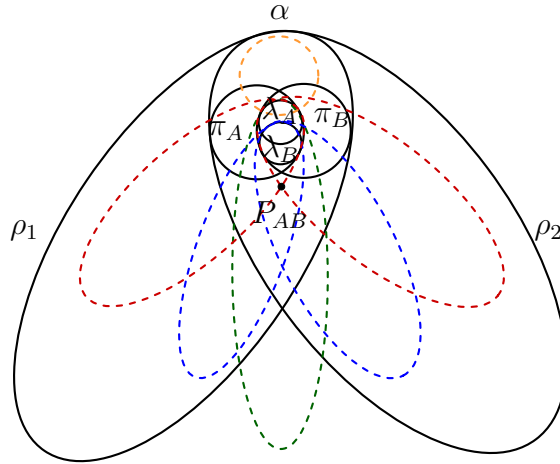$$\geqslant (k-2) + (k-2) - (k-4) \geqslant k,$$

Figure 3: Example(vi): The orange k-space is of type 1, the green one of type 2, the red ones of type 3 and 6, and the blue ones of type 4 and 5.

a contradiction. Hence, $E \subset \rho_1$. Analogously, we find that $E \subset \rho_2$, using two elements of $\mathcal{S}$ of type 5 and 6. And so, $E \subset \rho_1 \cap \rho_2 = \alpha$, which implies that $E \in \mathcal{S}$, type 1, a contradiction. So now we can suppose that $P_{AB} \in E$. Then E contains a $(k-1)$-space of α that meets λ in a $(k-3)$-space. This follows since E meets the elements of $\mathcal{S}$ of type 1 and 2 in at least a $(k-2)$-space. Note that the dimension of $E \cap \pi_A$ and $E \cap \pi_B$ is $k-2$ or $k-3$ as $E \cap \lambda$ is a $(k-3)$-space. Moreover, the latter spaces do not both have the same dimension. Indeed, if $\dim(E \cap \pi_A) = \dim(E \cap \pi_B) = k-2$, then $E \subset \alpha$, type 1, a contradiction. Moreover, since E contains P_{AB}, and since $\dim(E \cap \alpha) = k-1$, we know that $\dim(E \cap \langle \pi_A, \pi_B \rangle) = k-2$. If $\dim(E \cap \pi_A) = \dim(E \cap \pi_B) = k-3$, then w.l.o.g. we can suppose that $E \cap \lambda \neq \lambda_A$. Consider now an element X of type 3 such that $\lambda \nsubseteq X$. Then $\dim(X \cap E \cap \alpha) = k-3$, and so, $E \cap X \nsubseteq \alpha$. Hence, E and X also share points (also a point) in $\rho_1 \setminus \alpha$ and so, $E \subset \rho_1$. Similarly, $E \subset \rho_2$ and so $E \subset \rho_1 \cap \rho_2 = \alpha$' which cannot occur.

By a similar argument, we find that the dimension of $E \cap \lambda_A$ and $E \cap \lambda_B$ is $k-3$ or $k-4$, both not the same dimension. Then E contains a $(k-2)$-space of π_A or π_B, and E contains λ_A or λ_B. W.l.o.g. we can suppose that E contains λ_A and a $(k-2)$-space of π_A, and meets π_B in λ_A.

Let H be an element of type 1 of $\mathcal{S}$, and let G be an element of type 4 of $\mathcal{S}$ through a $(k-2)$-space $\sigma \neq \lambda$ in π_B with $H \cap G = \sigma$. Then, since $\dim(E \cap G \cap H) = k-4$,

$$\dim(E \cap \rho_1) = \dim(E \cap \langle G, H \rangle) \geqslant \dim(E \cap G) + \dim(E \cap H) - \dim(E \cap G \cap H)$$
$$\geqslant (k-2) + (k-2) - (k-4) \geqslant k,$$

and so $E \subset \rho_1$. Hence, $E \in \mathcal{S}$, type 3, a contradiction. $\qquad \square$
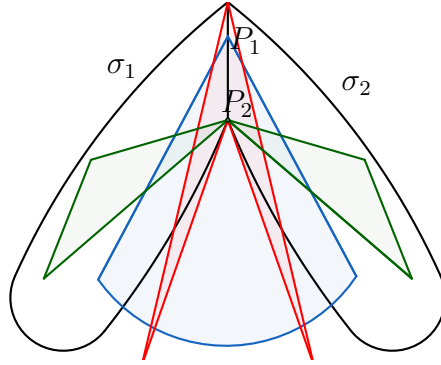
Figure 4: Example(vii): The red, blue and green planes correspond to the k-spaces of type 1, 2 and 3 in $\mathrm{PG}(n,q)/\gamma$, respectively.

(vii) There is a $(k-3)$-space γ contained in all k-spaces of $\mathcal{S}$. In the quotient space $\mathrm{PG}(n,q)/\gamma$, the set of planes corresponding to the elements of $\mathcal{S}$ is the set of planes of example $VIII$ in [8]: Let Ψ be an $(n-k+2)$-space, disjoint to γ, in $\mathrm{PG}(n,q)$. Consider two solids σ_1 and σ_2 in Ψ, intersecting in a line l. Take the points P_1 and P_2 on l. Then $\mathcal{S}$ is the set containing all k-spaces through $\langle \gamma, l \rangle$ (type 1), all k-spaces through $\langle \gamma, P_1 \rangle$ that contain a line in σ_1 and a line in σ_2 (type 2), and all k-spaces through $\langle \gamma, P_2 \rangle$ in $\langle \gamma, \sigma_1 \rangle$ or in $\langle \gamma, \sigma_2 \rangle$ (type 3). Then $|\mathcal{S}| = \theta_{n-k} + q^4 + 2q^3 + 3q^2$. In Lemma 36, we prove that the set $\mathcal{S}$ is maximal.

($viii$) There is a $(k-3)$-space γ contained in all k-spaces of $\mathcal{S}$. In the quotient space $\mathrm{PG}(n,q)/\gamma$, the set of planes corresponding to the elements of $\mathcal{S}$ is the set of planes of example IX in [8]: Let Ψ be an $(n-k+2)$-space, disjoint to γ, in $\mathrm{PG}(n,q)$, and let l be a line and σ a solid skew to l, both in Ψ. Denote $\langle l, \sigma \rangle$ by ρ. Let P_1 and P_2 be two points on l and let $\mathcal{R}_1$ and $\mathcal{R}_2$ be a regulus and its opposite regulus in σ. Then $\mathcal{S}$ is the set containing all k-spaces through $\langle \gamma, l \rangle$ (type 1), all k-spaces through $\langle \gamma, P_1 \rangle$ in the $(k+1)$-space generated by γ, l and a fixed line of $\mathcal{R}_1$ (type 2), and all k-spaces through $\langle \gamma, P_2 \rangle$ in the $(k+1)$-space generated by γ, l and a fixed line of $\mathcal{R}_2$ (type 3). Then $|\mathcal{S}| = \theta_{n-k} + 2q^3 + 2q^2$.

In Lemma 37, we prove that the set $\mathcal{S}$ is maximal.

(ix) There is a $(k-3)$-space γ contained in all k-spaces of $\mathcal{S}$. In the quotient space $\mathrm{PG}(n,q)/\gamma$, the set of planes corresponding to the elements of $\mathcal{S}$ is the set of planes of example VII in [8]: Let Ψ be an $(n-k+2)$-space, disjoint to γ in $\mathrm{PG}(n,q)$ and let ρ be a 5-space in Ψ. Consider a line l and a 3-space σ disjoint to l. Choose three points P_1, P_2, P_3 on l and choose four non-coplanar points Q_1, Q_2, Q_3, Q_4 in σ. Denote $l_1 = Q_1 Q_2$, $\bar{l}_1 = Q_3 Q_4$, $l_2 = Q_1 Q_3$, $\bar{l}_2 = Q_2 Q_4$, $l_3 = Q_1 Q_4$, and $\bar{l}_3 = Q_2 Q_3$. Then $\mathcal{S}$ is the set containing all k-spaces through $\langle \gamma, l \rangle$ (type 0) and all k-spaces through $\langle \gamma, P_i \rangle$ in $\langle \gamma, l, l_i \rangle$ or in $\langle \gamma, l, \bar{l}_i \rangle$, $i = 1, 2, 3$ (type i). Then $|\mathcal{S}| = \theta_{n-k} + 6q^2$.
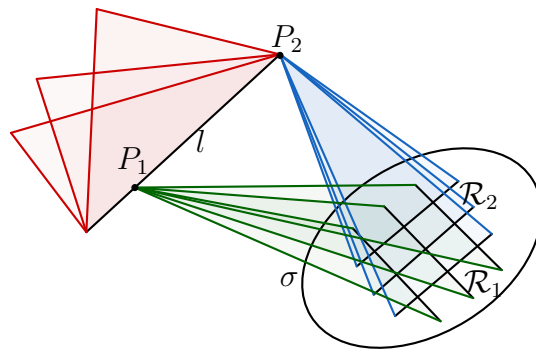
Figure 5: Example(*viii*): the red, green and blue planes correspond to the k-spaces of type 1, 2, 3 in $\mathrm{PG}(n,q)/\gamma$, respectively.

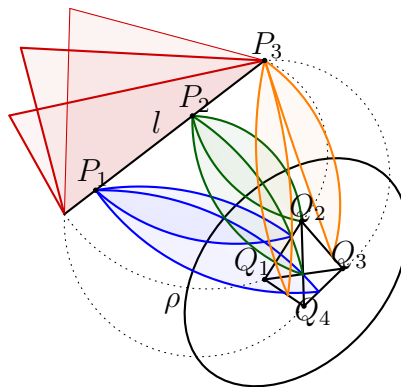In Lemma 35, we prove that the set $\mathcal{S}$ is maximal.

Figure 6: Example(*ix*): The red, blue, green and orange planes correspond to the k-spaces of type $0, 1, 2$ and 3 respectively.

(x) $\mathcal{S}$ is the set of all k-spaces contained in a fixed $(k+2)$-space ρ. Then $|\mathcal{S}| = \left[\begin{smallmatrix} k+3 \\ 2 \end{smallmatrix} \right]$.

From now on, let $\mathcal{S}$ be a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space in the projective space $\mathrm{PG}(n,q)$ with $n \geqslant k+2$.

We will focus on the sets $\mathcal{S}$ such that $|\mathcal{S}| > f(k,q)$. In Section 2, we investigate the sets $\mathcal{S}$ of k-spaces in $\mathrm{PG}(n,q)$ such that there is no point contained in all elements of $\mathcal{S}$ and such that $\mathcal{S}$ contains a set of three k-spaces that meet in a $(k-4)$-space. In Section 3, we assume again that there is no point contained in all elements of $\mathcal{S}$ and that for any three k-spaces X, Y, Z in $\mathcal{S}$, $\dim(X \cap Y \cap Z) \geqslant k-3$. In Section 4, we investigate the maximal sets $\mathcal{S}$ of k-spaces such that there is at least a point contained in all elements of $\mathcal{S}$. We end this article with the Main Theorem 38 that classifies all sets of k-spaces pairwise intersecting in at least a $(k-2)$-space with size larger than $f(k,q)$.

2   There are three elements of $\mathcal{S}$ that meet in a $(k-4)$-space

Suppose there exist three k-spaces A, B, C in $\mathcal{S}$ with $\dim(A \cap B \cap C) = k-4$, and suppose that there is no point contained in all elements of $\mathcal{S}$. By the existence of Example 9(x), we may assume that the elements of $\mathcal{S}$ span at least a $(k+3)$-space. In this subsection, we will use the following notation.

Notation 16. Let $\mathcal{S}$ be a maximal set of k-spaces in $\mathrm{PG}(n,q)$ pairwise intersecting in at least a $(k-2)$-space. Let A, B and C in $\mathcal{S}$ be three k-spaces with $\pi_{ABC} = A \cap B \cap C$ a $(k-4)$-space. Let $\pi_{AB} = A \cap B$, $\pi_{AC} = A \cap C$ and $\pi_{BC} = B \cap C$. Let $\mathcal{S}'$ be the set of k-spaces of $\mathcal{S}$ not contained in $\langle A, B \rangle$, and let α be the span of all subspaces $D' := D \cap \langle A, B \rangle$, $D \in \mathcal{S}'$.

Note explicitly, by Grassmann's dimension property, that π_{AB}, π_{BC} and π_{AC} are $(k-2)$-spaces and $\langle A, B \rangle = \langle B, C \rangle = \langle A, C \rangle$.

We first present a lemma that will be useful for the later classification results in this section.

Lemma 17. [Using Notation 16] *If there exist three k-spaces A, B and C in $\mathcal{S}$, with $\dim(A \cap B \cap C) = k-4$, then a k-space of $\mathcal{S}'$ meets $\langle A, B \rangle$ in a $(k-1)$-space. More specifically, it contains π_{ABC} and meets π_{AB}, π_{AC} and π_{BC}, each in a $(k-3)$-space through π_{ABC}.*

Proof. Consider a k-space E of $\mathcal{S}'$. Clearly,

$$k - 2 \leqslant \dim(E \cap \langle A, B \rangle) \leqslant k - 1.$$

If $\dim(E \cap \langle A, B \rangle) = k - 2$, then this $(k-2)$-space has to lie in A, B and C, and so in the $(k-4)$-space π_{ABC}, a contradiction. Hence, we know that $\dim(E \cap \langle A, B \rangle) = k - 1$. By the symmetry of the k-spaces A, B and C, it suffices to prove that E contains π_{ABC} and meets π_{AB} in a $(k-3)$-space through π_{ABC}. Using Grassmann's dimension property we find that

$$\dim(E \cap \pi_{AB}) \geqslant \dim(E \cap A) + \dim(E \cap B) - \dim(E \cap \langle A, B \rangle),$$

and so, $\dim(E \cap \pi_{AB})$ is $k - 2$ or $k - 3$. If $\dim(E \cap \pi_{AB}) = k - 2$, then

$$\begin{aligned}
\dim(E \cap C) &\leqslant \dim(E \cap \pi_{ABC}) + \dim(E \cap \langle C, \pi_{AB} \rangle) - \dim(E \cap \pi_{AB}) \\
&\leqslant (k-4) + (k-1) - (k-2) = k - 3,
\end{aligned}$$

a contradiction since any two elements of $\mathcal{S}$ meet in at least a $(k-2)$-space. Hence, $\dim(E \cap \pi_{AB}) = k - 3$, and so

$$\begin{aligned}
\dim(E \cap \pi_{ABC}) &\geqslant \dim(E \cap C) + \dim(E \cap \pi_{AB}) - \dim(E \cap \langle C, \pi_{AB} \rangle) \\
&\geqslant (k-2) + (k-3) - (k-1) = k - 4.
\end{aligned}$$

This implies that the $(k-4)$-space π_{ABC} is contained in E. $\qquad\square$

Therefore, let D be a k-space of $\mathcal{S}'$. By Lemma 17, for the remaining part of this section, we will denote by D' the $(k-1)$-space $D \cap \langle A, B \rangle$.

Corollary 18. [Using Notation 16] *Suppose $\mathcal{S}$ contains three elements A, B and C, meeting in a $(k-4)$-space, and α is a $(k+i)$-space. Up to a suitable labelling of A, B and C, we have the following results.*

    a) *If $i = -1$, then $\alpha = D \cap \langle A, B \rangle$ for every $D \in \mathcal{S}'$.*

    b) *If $i = 0$, then $\alpha = \langle \rho_1, \rho_2, \rho_3 \rangle$, with ρ_1 a $(k-3)$-space in π_{AB}, ρ_2 a $(k-3)$-space in π_{BC}, $\rho_3 = \pi_{AC}$ and $\pi_{ABC} \subset \rho_j, j = 1, 2, 3$. In this case, all elements of $\mathcal{S}'$ contain the $(k-2)$-space $\langle \rho_1, \rho_2 \rangle$.*

    c) *If $i = 1$, then $\alpha = \langle \rho_1, \rho_2, \rho_3 \rangle$, with ρ_1 a $(k-3)$-space in π_{AB}, $\rho_2 = \pi_{BC}$, $\rho_3 = \pi_{AC}$ and $\pi_{ABC} \subset \rho_j, j = 1, 2, 3$. In this case, all elements of $\mathcal{S}'$ contain the $(k-3)$-space ρ_1.*

    d) *If $i = 2$, then $\alpha = \langle A, B \rangle$.*

Proof. For $i = -1$ and $i = 2$, the corollary follows immediately from Lemma 17. Hence we start with the case that α is a k-space. Consider two elements of $\mathcal{S}'$, say D_1, D_2, meeting $\langle A, B \rangle$ in two different $(k-1)$-spaces D_1', D_2'. These two elements of $\mathcal{S}'$ exist, as otherwise $\dim(\alpha) = k - 1$. Since D_1' and D_2' span the k-space α, they meet in a $(k-2)$-space. By Lemma 17, this $(k-2)$-space $D_1' \cap D_2'$ contains π_{ABC}, together with a $(k-3)$-space ρ_1 through π_{ABC} in π_{XY} and a $(k-3)$-space ρ_2 through π_{ABC} in π_{YZ}, with $\{X, Y, Z\} = \{A, B, C\}$. By Lemma 17, every other element of $\mathcal{S}'$ will meet $\langle A, B \rangle$ in a $(k-1)$-space through this $(k-2)$-space $D_1' \cap D_2' = \langle \rho_1, \rho_2 \rangle$, which proves the statement.

Suppose now that α is a $(k+1)$-space. Then, we consider two elements D_3, D_4 of $\mathcal{S}'$ meeting $\langle A, B \rangle$ in two $(k-1)$-spaces D_3', D_4' such that $\dim(D_3' \cap D_4') = k - 3$. These elements of $\mathcal{S}'$ exist as otherwise all elements of $\mathcal{S}'$ correspond to $(k-1)$-spaces pairwise intersecting in a $(k-2)$-space. But then, since these $(k-1)$-spaces span a $(k+1)$-space, they form a $(k-2)$-pencil (see Theorem 6). Using Lemma 17, and the proof above of the case $\dim(\alpha) = k$ or $i = 0$, it follows that α would be a k-space. Now, again by Lemma 17, we see that $D_3' \cap D_4'$ contains π_{ABC} and a $(k-3)$-space ρ_1 through π_{ABC} in π_{XY}, with $\{X, Y, Z\} = \{A, B, C\}$. Using dimension properties and the fact that $D_3' \cap D_4' = \rho_1$, we see that every other element of $\mathcal{S}'$ will contain ρ_1, which proves the statement. $\square$

We distinguish between several cases depending on the dimension of $\alpha = \langle D \cap \langle A, B \rangle \,|\, D \in \mathcal{S}' \rangle$.

2.1 α is a $(k-1)$-space

Proposition 19. [Using Notation 16] *If $\mathcal{S}$ contains three k-spaces that meet in a $(k-4)$-space and $\dim(\alpha) = k - 1$, then $\mathcal{S}$ is Example 9(v).*

Proof. From Corollary 18, we have that for all $D \in \mathcal{S}', D \cap \langle A, B \rangle = \alpha$, so all the k-spaces in $\mathcal{S}'$ meet $\langle A, B \rangle$ in α. As a k-space of $\mathcal{S}$ in $\langle A, B \rangle$ needs to have at least a $(k-2)$-space in common with every $D \in \mathcal{S}'$, we find that every k-space of $\mathcal{S}$ in $\langle A, B \rangle$ meets α in at least a $(k-2)$-space. Note that the condition that every two k-spaces of $\mathcal{S}$ in $\langle A, B \rangle$ meet in at least a $(k-2)$-space is fulfilled. Hence, $\mathcal{S}$ is Example 9(v) with $\rho = \langle A, B \rangle$. $\qquad\square$

2.2 α is a k-space

Proposition 20. [Using Notation 16] *If $\mathcal{S}$ contains three k-spaces that meet in a $(k-4)$-space and $\dim(\alpha) = k$, then $\mathcal{S}$ is Example 9(iv).*

Proof. If α is a k-space, we can suppose w.l.o.g., by Corollary 18, that $\alpha = \langle \pi_{AB}, P_{AC}, P_{BC} \rangle$ with P_{AC} and P_{BC} points in $\pi_{AC} \setminus \pi_{ABC}$ and $\pi_{BC} \setminus \pi_{ABC}$, respectively. We also know that all the k-spaces $D \in \mathcal{S}'$ have a $(k-1)$-space D' in common with α and they contain the $(k-2)$-space $\pi = \langle \pi_{ABC}, P_{AC}P_{BC} \rangle$. So every pair of k-spaces in $\mathcal{S}'$ meets in a $(k-2)$-space inside $\langle A, B \rangle$. Consider a k-space E of $\mathcal{S}$ in $\langle A, B \rangle$, not having a $(k-1)$-space in common with α, and let D_1 and D_2 be k-spaces of $\mathcal{S}'$ with $D_1' \cap D_2' = \pi$, and so $\langle D_1', D_2' \rangle = \alpha$. If E does not contain π, then

$$\dim(E \cap \alpha) \geqslant \dim\langle E \cap D_1', E \cap D_2' \rangle \geqslant k - 2 + k - 2 - \dim(E \cap \pi) \geqslant k - 1.$$

This is a contradiction. Hence, every k-space of $\mathcal{S} \setminus \mathcal{S}'$ contains π or has a $(k-1)$-space in common with α. From the maximality of $\mathcal{S}$, it follows that $\mathcal{S}$ is Example 9(iv) with $\rho = \langle A, B \rangle$ and $\pi = \langle \pi_{ABC}, P_{AC}P_{BC} \rangle$. $\qquad\square$

2.3 α is a $(k+1)$-space

To understand the structure of these sets of k-spaces, we will first investigate the case $k = 3$ and then we will generalize our results to $k \geqslant 3$.

2.3.1 $k = 3$ and α is a 4-space

Note that for $k = 3$, the spaces π_{AB}, π_{BC} and π_{AC} are pairwise disjoint lines and π_{ABC} is the empty space. By Corollary 18, we can suppose w.l.o.g. that $\alpha = \langle P_{AB}, \pi_{AC}, \pi_{BC} \rangle$, with P_{AB} a point in $\pi_{AB} \setminus \pi_{ABC}$. Hence, each of the planes $D' = D \cap \langle A, B \rangle$, $D \in \mathcal{S}'$, contain P_{AB} and the set of all these planes D' span the 4-space α.

From now on, let $\mathcal{L}$ be the set of lines $D \cap C$, $D \in \mathcal{S}'$.

Proposition 21. [Using Notation 16] *If $\mathcal{S}$ contains three solids such that there is no point contained in the three of them, and if $\dim(\alpha) = 4$, then a solid of $\mathcal{S}$ in $\langle A, B \rangle$ either*

  i) is contained in α, or

  ii) contains P_{AB} and a line r of C, intersecting all lines of $\mathcal{L}$.
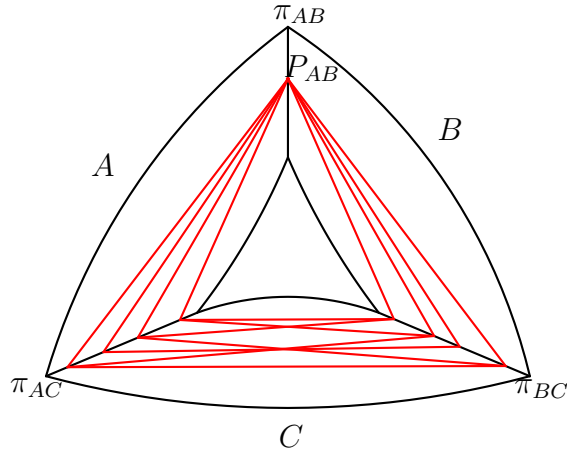
Figure 7: There are three solids A, B, C in $\mathcal{S}$, with $A \cap B \cap C = \emptyset$ and $\dim(\alpha) = 4$

Proof. Recall that each of the intersection planes $D \cap \langle A, B \rangle$ contain P_{AB} and that the set of all these planes span the $(k+1)$-space α. Hence, we can see that there exist solids $D_1, D_2 \in \mathcal{S}'$, such that their intersection planes D_1' and D_2' with α, meet exactly in the point P_{AB}. Indeed, by Theorem 6, if all the planes $D \cap \langle A, B \rangle$, $D \in \mathcal{S}'$, would pairwise intersect in a line, then these planes lie in a fixed solid or contain a fixed line. Neither possibility can occur since α is a 4-space, and P_{AB} is the only point contained in all intersection planes.

Suppose first that E is a solid of $\mathcal{S}$ in $\langle A, B \rangle$, not containing P_{AB}. As E needs to contain at least a line of every plane $D' = D \cap \langle A, B \rangle$, $D \in \mathcal{S}'$, E contains at least a line $l_1 \subset D_1' \subset \alpha$ and a line $l_2 \subset D_2' \subset \alpha$. Note that l_1 and l_2 are disjoint as they do not contain the point P_{AB}. Hence, $E = \langle l_1, l_2 \rangle \subset \alpha$.

So now we can suppose that E contains the point P_{AB} and meets α in precisely the plane γ. The plane γ is the span of P_{AB} and the line $r = \gamma \cap C$. As $E \cap D$ is at least a line of the plane $D' = D \cap \langle A, B \rangle$ for every $D \in \mathcal{S}'$, and since every two lines in the plane γ meet each other, we have that r has to intersect all the lines of $\mathcal{L}$. Hence, we find the second possibility. $\qquad \square$

In the previous proposition, we proved that there are two types of solids of $\mathcal{S}$ contained in $\langle A, B \rangle$. One of them are the solids containing P_{AB} and a line $r \subset C$, intersecting all lines of $\mathcal{L}$. The number of these solids depends on the number of lines r meeting all lines of $\mathcal{L}$.

Case 1. There is a line $l \in \mathcal{L}$ that intersects all the lines of $\mathcal{L}$

Note that there cannot be two lines in $\mathcal{L}$ intersecting all the lines of $\mathcal{L}$, since then all lines of $\mathcal{L}$ would lie in a plane or go through a fixed point in C. This gives a contradiction as the lines of $\mathcal{L}$ span C and at least two points of both π_{AB} and π_{BC} are covered by the lines of $\mathcal{L}$.

Proposition 22. $\mathcal{S}$ *is Example 9(vi) for* $k = 3$.

Proof. Let $P_A = l \cap \pi_{AC}$, $P_B = l \cap \pi_{BC}$, $\pi_A = \langle \pi_{AC}, l \rangle$ and $\pi_B = \langle \pi_{BC}, l \rangle$. Since every line $m \neq l$ of $\mathcal{L}$ intersects the lines π_{AC}, π_{BC} and l, it follows that m contains the point P_A and is contained in π_B, or m contains the point P_B and is contained in π_A. Note that since $\dim(\alpha) = 4$, there is at least one line $m_1 \neq l$ in $\mathcal{L}$ through P_A and there is at least one line $m_2 \neq l$ in $\mathcal{L}$ through P_B. As a consequence of Proposition 21, we have that a solid of $\mathcal{S}$ in $\langle A, B \rangle$, not contained in α, contains P_{AB} and it meets C in a line r that meets all lines of $\mathcal{L}$. Hence, r is a line of the plane π_A through P_A or in a line of π_B through P_B. Consider now the set $\mathcal{F}$ of solids of $\mathcal{S}'$, not through $\langle P_{AB}, l \rangle$. We will prove that these solids lie in a 5-space that meets $\langle A, B \rangle$ in α. Let $E_A, E_B \in \mathcal{F}$ be two solids through $m_1 \ni P_A$ and $m_2 \ni P_B$ respectively. Since the planes $E_A \cap \alpha$ and $E_B \cap \alpha$ meet in precisely the point P_{AB}, the solids E_A and E_B have precisely a line in common, and so, they span a 5-space ρ_2 through α. Then every other solid $F \in \mathcal{F}$ is contained in ρ_2 as it meets $E_A \cap \alpha$, or $E_B \cap \alpha$, precisely in one point, namely P_{AB}, and so it must contain at least a point of E_A, or E_B respectively, in $\rho_2 \setminus \alpha$. This point, together with the plane $F \cap \alpha$, spans F and so $F \subset \rho_2$. Hence, $\mathcal{S}$ is Example 9(*vi*), with $\rho_1 = \langle A, B \rangle$, $\pi_A = \langle \pi_{AC}, l \rangle$, $\pi_B = \langle \pi_{BC}, l \rangle$, $\lambda_A = P_A$, $\lambda_B = P_B$ and $\lambda = l$. $\qquad\square$

Case 2. For every line in $\mathcal{L}$, there exists another line in $\mathcal{L}$ disjoint to the given line

Depending on the structure of the set of lines $\mathcal{L}$, we discuss the set of the solids of $\mathcal{S}$ in $\langle A, B \rangle$ not contained in α. We have different possibilities for a line r of C, meeting all lines of $\mathcal{L}$ (see Proposition 21):

i) Suppose there are three pairwise disjoint lines in $\mathcal{L}$, then these three lines belong to a unique regulus $\mathcal{R}$.

   a) If $\mathcal{L}$ is contained in $\mathcal{R}$, then $|\mathcal{L}| \leqslant q + 1$ and r is a line of the opposite regulus $\mathcal{R}^c$. Hence, there are $q + 1$ possibilities for r.

   b) If $\mathcal{L}$ is not contained in $\mathcal{R}$, then there are exactly two lines, intersecting all the lines of $\mathcal{L}$, namely π_{AC} and π_{BC}. Let $l \in \mathcal{L} \setminus \mathcal{R}$. If there were a third line r meeting all lines of $\mathcal{L}$, then $r \in \mathcal{R}^c$. But then there would be three lines, namely r, π_{AC} and π_{BC}, in $\mathcal{R}^c$, all of them intersecting l. Hence, l also has to lie in $\mathcal{R}$, a contradiction. In this case there are at most 2 possibilities for r and $|\mathcal{L}| \leqslant (q + 1)^2$.
   Note that in this case, $\mathcal{L}$ is not contained in any regulus. This follows since the three pairwise disjoint lines in $\mathcal{L}$ define a unique regulus.

ii) Suppose there are no three pairwise disjoint lines in $\mathcal{L}$. In this case, we can prove the following lemma.

   Lemma 23. *The set $\mathcal{L}$ is contained in the union of two point-pencils such that their vertices are contained either in π_{AC} or in π_{BC}.*

Proof. We can suppose that $\mathcal{L}$ contains at least two disjoint lines l_1, l_2, since the lines of $\mathcal{L}$ span the solid C. Let $P_i = \pi_{AC} \cap l_i$ and $Q_i = \pi_{BC} \cap l_i$, for $i = 1, 2$. As there are no three pairwise disjoint lines in $\mathcal{L}$ we see that every line $l \in \mathcal{L}$ contains at least one of the points P_i and Q_i, with $i = 1, 2$, and so $\mathcal{L}$ is contained in the union of 4 point-pencils with vertices P_1, P_2, Q_1, Q_2. If $|\mathcal{L}| \leqslant 4$, then it is easy to see that $\mathcal{L}$ is contained in the union of two point-pencils. Suppose now that $|\mathcal{L}| \geqslant 5$ and that $\mathcal{L}$ is not contained in the union of two of these point-pencils. Due to the symmetry, we can suppose w.l.o.g. that $\mathcal{L}$ contains a line l_3 different from l_1 and P_1Q_2 that contains P_1, a line l_4 different from l_2 and P_1Q_2 that contains Q_2 and a line l_5 not equal to l_2 or P_2Q_1 that contains P_2. Let $Q_3 = l_3 \cap \pi_{BC}$ and $P_4 = l_4 \cap \pi_{AC}$. Then l_5 contains the point Q_3 as otherwise l_3, l_4 and l_5 would be pairwise disjoint. So $l_5 = P_2Q_3$, but then we see that l_1, l_4 and l_5 are three pairwise disjoint lines, a contradiction. Hence, $\mathcal{L}$ is contained in the union of two point-pencils. $\qquad\square$

By using the notations in the lemma above, since $\mathcal{L}$ contains no 3 pairwise disjoint lines, we know that if $|\mathcal{L}| \geqslant 3$, then every line $l_0 \in \mathcal{L} \setminus \{l_1, l_2\}$ contains at least one of the points P_1, P_2, Q_1, Q_2. From Lemma 23, we find the following possibilities for the set $\mathcal{L}$.

a) If $\mathcal{L}$ only contains two lines l_1, l_2, then l_1 and l_2 are disjoint and we find $(q+1)^2$ possibilities for r, as every such line is defined by a point of l_1 and a point of l_2.

b) If $\mathcal{L}$ contains at least 3 elements and is contained in the union of a line l_0 and a point-pencil through a point P, then $|\mathcal{L}| \leqslant q + 2$. Let $P_0 = l_0 \cap \pi_{AC}$, $Q_0 = l_0 \cap \pi_{BC}$ and suppose w.l.o.g. that $P \in \pi_{AC}$. A line r that meets all lines of $\mathcal{L}$ is a line that contains P and a point of l_0 or is a line that contains Q_0 and lies in the plane $\langle P, \pi_{BC} \rangle$. Hence, there are at most $2q + 1$ possibilities for the line r.

c) If $\mathcal{L}$ contains at least 4 elements and is contained in the union of two point-pencils through the points P and Q respectively such that $\mathcal{L}$ contains at least two lines through P and at least two lines through Q, then $|\mathcal{L}| \leqslant 2(q+1)$. In particular, if P and Q are on different lines, we note that $|\mathcal{L}| \leqslant 2q$, as the line PQ is not contained in $\mathcal{L}$. This follows since this line meets all other lines of $\mathcal{L}$, and so, this situation is discussed in Case 1. Therefore, in any case, a line r that meets all lines of $\mathcal{L}$ is the line π_{AC}, the line π_{BC} or the line PQ if it is distinct from π_{AC} and π_{BC}. Hence, there are at most 3 possibilities for the line r.

For every intersection plane D' in α, there are at most $\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = q^2$ ways to extend the plane to a solid $D \in \mathcal{S}'$, as this solid also has to meet several solids of $\mathcal{S}'$ in a point $Q \notin \langle A, B \rangle$. And since the number of planes D' equals the number of lines in $\mathcal{L}$, there are at most $(q+1) \cdot q^2$, $(q+1)^2 \cdot q^2$, $2 \cdot q^2$, $(q+2) \cdot q^2$, $2(q+1) \cdot q^2$ solids outside of $\langle A, B \rangle$, respectively, dependent on the five cases $ia), ib), iia), iib), iic)$ above.

For the solids inside $\langle A, B \rangle$, there are $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ solids in α and $(q+1) \cdot q^2, 2 \cdot q^2, (q+1)^2 \cdot q^2, (2q+1) \cdot q^2, 3 \cdot q^2$ solids of the second type of Proposition 21, respectively. We find these numbers by multiplying the number of possibilities for the line r and the number q^2 of 3-spaces through a plane in $\langle A, B \rangle$, not contained in α. So, in total, we have at most $\begin{bmatrix} 5 \\ 1 \end{bmatrix} + (q^2 + 2q + 3) \cdot q^2 = O(2q^4)$ solids, using case ib) or iia).

Remark 24. Note that the number of elements of $\mathcal{S}$ in this case is smaller than $f(3, q) = 3q^4 + 6q^3 + 5q^2 + q + 1$, and so we will not consider these maximal sets of solids in our classification result (Main Theorem 38).

2.3.2 General case $k > 3$ and α is a $(k+1)$-space

By Corollary 18, we can suppose w.l.o.g., that α is spanned by π_{AC}, π_{BC} and a point P_{AB} of π_{AB} outside of π_{ABC}, and that all $(k-1)$-spaces $D' = D \cap \langle A, B \rangle, D \in \mathcal{S}'$, contain $\langle P_{AB}, \pi_{ABC} \rangle$.

Proposition 25. [Using Notation 16] *If $\mathcal{S}$ contains three k-spaces that meet in a $(k-4)$-space and $\dim(\alpha) = k+1$, then a k-space of $\mathcal{S}$ in $\langle A, B \rangle$ is contained in α or contains π_{ABC}. More specifically, if $|\mathcal{S}| > f(k, q)$, then $\mathcal{S}$ is Example 9(vi).*

Proof. We suppose that E is a k-space of $\mathcal{S}$ in $\langle A, B \rangle$, not through π_{ABC}. As E contains at least a $(k-2)$-space of all the $(k-1)$-spaces D', with $D \in \mathcal{S}'$, we find that E contains a hyperplane τ_0 of π_{ABC}, a $(k-4)$-space τ_1 of $\alpha \cap \pi_{AB}$, a $(k-3)$-space τ_2 of π_{AC} and a $(k-3)$-space τ_3 of π_{BC}. As $\tau_1 \cap \tau_2 = \tau_1 \cap \tau_3 = \tau_2 \cap \tau_3 = \tau_0$, and by the Grassmann dimension property, we see that $E \subset \alpha$. For the k-spaces through π_{ABC}, we can investigate the solids $E/\pi_{ABC}, E \in \mathcal{S}$, in the quotient space $\mathrm{PG}(n, q)/\pi_{ABC}$, and use the results in Case 1 and Case 2 of Subsection 2.3.1. These results imply that a k-space in $\langle A, B \rangle$ through π_{ABC} is contained in α or contains $\langle P_{AB}, \pi_{ABC} \rangle$ and a line in $C \setminus \pi_{ABC}$ that meets all the $(k-2)$-spaces $D \cap C, D \in \mathcal{S}'$. Then there are two cases:

- CASE 1. If there is a line $l \in C \setminus \pi_{ABC}$ meeting the subspaces $D \cap C$ for all $D \in \mathcal{S}'$, then there are $\theta_{n-k} + q^4 + 5q^3 + q^2$ k-spaces of $\mathcal{S}$ that contain π_{ABC}.

- CASE 2. If there is no line $l \in C \setminus \pi_{ABC}$ meeting the subspaces $D \cap C$ for all $D \in \mathcal{S}'$, then there are at most $2q^4 + 3q^3 + 4q^2 + q + 1$ k-spaces of $\mathcal{S}$ that contain π_{ABC}.

It is clear that two elements of $\mathcal{S}$ in α meet in at least a $(k-1)$-space. From the investigation of the quotient space $\mathrm{PG}(n, q)/\pi_{ABC}$ it follows that two elements of $\mathcal{S}$ through π_{ABC}, not in α, meet in at least a $(k-2)$-space. A k-space E_1 of $\mathcal{S}$ in α and a k-space E_2 of $\mathcal{S}$ not in α, but through π_{ABC}, will also meet in a $(k-2)$-space. This follows since E_2 contains the $(k-3)$-space $\langle P_{AB}, \pi_{ABC} \rangle \subset \alpha$ and a line in $C \setminus \pi_{ABC} \subset \alpha$. Hence, E_2 meets α in a $(k-1)$-space. Since E_1 is contained in α, it follows that E_1 and E_2 meet in at least a $(k-2)$-space. Now, as every element of $\mathcal{S}$, not through π_{ABC}, is contained in α, there are $\theta_{k+1} - \theta_4$ elements of $\mathcal{S}$ not through π_{ABC}. Hence, in CASE 1, $\mathcal{S}$ is Example 9(vi) and $|\mathcal{S}| = \theta_{n-k} + \theta_{k+1} + 4q^3 - q - 1$. In CASE 2, $|\mathcal{S}| \leqslant \theta_{k+1} + q^4 + 2q^3 + 3q^2$, which proves the proposition. □

2.4 α is a $(k+2)$-space

Here again, we first consider the case $k = 3$.

2.4.1 α is a 5-space

We start with a lemma that will often be used in this subsection.

Lemma 26. [Using Notation 16] *If $\mathcal{S}$ contains three solids A, B, C, with $A \cap B \cap C = \emptyset$, then every two intersection planes D_1' and D_2', with $D_i' = D_i \cap \langle A, B \rangle, D_i \in \mathcal{S}', i = 1, 2$, share a point on π_{AB}, π_{AC} or π_{BC}.*

Proof. Consider two solids D_1 and D_2 in $\mathcal{S}'$, with corresponding intersection planes D_1' and D_2' in $\langle A, B \rangle$. Since D_1 and D_2 meet in at least a line, D_1' and D_2' have to meet in at least a point. If D_1' and D_2' do not meet in a point of π_{AB}, π_{AC} or π_{BC}, then these planes define 6 different intersection points $P_1, \ldots, P_6$ on the lines π_{AB}, π_{AC} and π_{BC}. As $\langle D_1', D_2' \rangle = \langle P_1, \ldots, P_6 \rangle = \langle \pi_{AB}, \pi_{AC}, \pi_{BC} \rangle$, we find that D_1' and D_2' span a 5-space, so these planes are disjoint, a contradiction. $\qquad\square$

If α is a 5-space, we distinguish two cases, depending on the planes $D' = D \cap \langle A, B \rangle$, $D \in \mathcal{S}'$.

Lemma 27. [Using Notation 16] *If $\mathcal{S}$ contains three solids A, B, C, with $A \cap B \cap C = \emptyset$, and if $\dim(\alpha) = 5$, then we have one of the following possibilities for the planes $D' = D \cap \langle A, B \rangle, D \in \mathcal{S}'$:*

    i) There are four possibilities for the planes D': $\langle P_1, P_3, P_6 \rangle, \langle P_1, P_4, P_5 \rangle, \langle P_2, P_4, P_6 \rangle$ and $\langle P_2, P_3, P_5 \rangle$, where $P_1 P_2 = \pi_{AB}, P_3 P_4 = \pi_{BC}$ and $P_5 P_6 = \pi_{AC}$.

    ii) There are three points $P \in \pi_{AB}, Q \in \pi_{BC}$ and $R \in \pi_{AC}$ so that every plane D' contains at least two of the three points of $\{P, Q, R\}$.

Proof. We prove the Lemma by construction and we start with a plane, we say D_1', intersecting π_{AB}, π_{BC} and π_{AC} in the points P, Q and R' respectively.
Case (a): There exists a plane D_2' such that $D_1' \cap D_2'$ is a point (w.l.o.g. P, see Lemma 26) *and let $D_2' \cap \pi_{BC}$ be Q' and $D_2' \cap \pi_{AC}$ be R.* In this case we know that there exists a third plane D_3' intersecting π_{AB} in a point P' different from P (as $\dim(\alpha) = 5$). Then D_3' needs at least a point of D_2' and D_1'. This implies that D_3' contains Q and R or Q' and R' (w.l.o.g. Q and R) by Lemma 26. Now there are two possibilities:

    i) There exists a plane $D_4' = \langle P', Q', R' \rangle$, and then, by construction, we cannot add another plane D_i'. (In the formulation of the lemma $P = P_1, P' = P_2, Q = P_3, Q' = P_4, R = P_5$ and $R' = P_6$.)

    ii) There exists no plane $D_4' = \langle P', Q', R' \rangle$, then, by construction, we see that all the planes need to contain at least two of the three points P, Q, R by Lemma 26.

Case (b): all the planes D'_i intersect pairwise in a line. Then all these planes have to lie in a solid (contradiction since they span a 5-space) or they go through a fixed line l. In this last case, l cannot be one of the lines $\pi_{AB}, \pi_{AC}, \pi_{BC}$ and also, l cannot intersect one of these lines, as otherwise all the planes D'_i would contain the intersection point of this line and l (which gives a contradiction since $\dim(\alpha) = 5$). Consider now the disjoint lines l and π_{AB}. Then all the planes D'_i would contain l and a point of π_{AB}, but this implies that $\dim(\alpha) = 3$ which also gives a contradiction. We conclude that this case cannot happen. $\qquad\square$

Case 1. There are four intersection planes D'

In this situation, using the notation from Lemma 27, there are four possibilities for the planes $D' = D \cap \langle A, B \rangle$, $D \in \mathcal{S}'$: $\langle P_1, P_3, P_6 \rangle, \langle P_1, P_4, P_5 \rangle, \langle P_2, P_4, P_6 \rangle$ and $\langle P_2, P_3, P_5 \rangle$, where $P_1, P_2 \in \pi_{AB}$, $P_3, P_4 \in \pi_{BC}$ and $P_5, P_6 \in \pi_{AC}$. We show that the only solids of $\mathcal{S}$ in $\langle A, B \rangle$ are A, B and C.
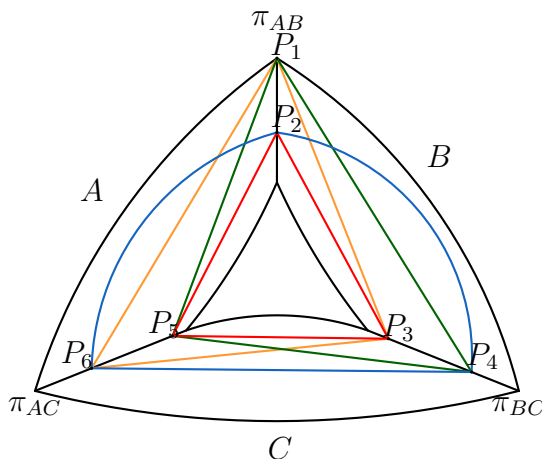

Figure 8: There are three elements A, B, C in $\mathcal{S}$ with $A \cap B \cap C = \emptyset$ and $\dim(\alpha) = 5$

Proposition 28. [Using Notation 16] *If $\mathcal{S}$ contains three solids A, B, C, $A \cap B \cap C = \emptyset$, $\dim(\alpha) = 5$, and so that there are exactly four intersection planes D', see Lemma 27(i), then the only solids of $\mathcal{S}$ in $\langle A, B \rangle$ are A, B and C.*

Proof. Let $P_1, \ldots, P_6$ be the intersection points of $D \cap \langle A, B \rangle$, $D \in \mathcal{S}'$, with the lines $\pi_{AB}, \pi_{AC}, \pi_{BC}$, and let E be a solid in $\langle A, B \rangle$ different from A, B, C. The solid E cannot contain all the points $P_1, \ldots, P_6$, by its dimension so we can suppose that $P_1 \notin E$. We will first show that E contains the point P_2. As E has a line in common with every plane intersection $D' = D \cap \langle A, B \rangle$, with $D \in \mathcal{S}'$, E has at least a point in common with every line of these planes D'. This implies that E has at least a point in common with P_1P_3, P_1P_4, P_1P_5, and P_1P_6 or equivalently, a line l_A in common with $\langle P_1, \pi_{AC} \rangle$ and a line

l_B in common with $\langle P_1, \pi_{BC} \rangle$. Hence, $E = \langle l_A, l_B \rangle$ and so $E \subset \langle P_1, C \rangle$. If $P_2 \notin E$ then we find by symmetry that $E \subset \langle P_2, C \rangle$, and so that $E \subseteq \langle P_1, C \rangle \cap \langle P_2, C \rangle$ and $E = C$, a contradiction. Then $P_2 \in E$; furthermore E cannot contain $P_2, \ldots, P_6$, by the dimension, and so we can suppose that $P_6 \notin E$. Then, by the previous arguments and symmetry, we know that P_5 lies in E. In A, the solid E needs an extra point P of $P_1 P_6$ since E shares a line with $\langle P_1, P_3, P_6 \rangle$. This gives that E contains the plane $\gamma = \langle P, P_2, P_5 \rangle$ of A. As E also needs at least a point of each line $P_1 P_3, P_1 P_4$, E needs at least one extra line, disjoint to γ. This gives the contradiction, again by the dimension, and so E cannot be different from A, B, C. $\qquad \square$

There are at most $4 \cdot \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$ solids in $\mathcal{S}'$. The first factor of this number follows since every solid in $\mathcal{S}'$ meets $\langle A, B \rangle$ in one of the four intersection planes. The second factor follows as each of these intersection planes is contained in at most $\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ solids of $\mathcal{S}'$: any two solids, intersecting $\langle A, B \rangle$ in different intersection planes, have to intersect in at least a point Q outside of $\langle A, B \rangle$. There are only 3 solids, A, B, C, in $\langle A, B \rangle$.

Case 2. Every intersection plane D' contains at least two of the points P, Q, R

Note that in this situation we have at least the red, green and blue plane (see Figure 9) as intersection planes $D' = D \cap \langle A, B \rangle, D \in \mathcal{S}'$. In the following proposition, we prove how the solids in $\langle A, B \rangle$ lie with respect to the points P, Q, R.
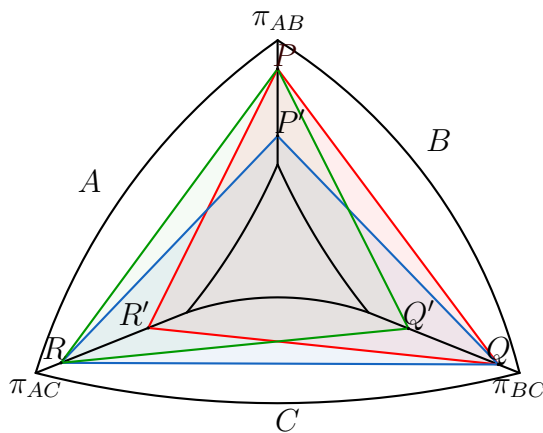

Figure 9: There are three elements A, B, C in $\mathcal{S}$ with $A \cap B \cap C = \emptyset$ and $\dim(\alpha) = 5$

Proposition 29. [Using Notation 16] *If $\mathcal{S}$ contains three solids A, B, C, $A \cap B \cap C = \emptyset$, $\dim(\alpha) = 5$, and so that every intersection plane D' contains at least two of the points P, Q, R, see Lemma 27(ii), then all the solids of $\mathcal{S}$ in $\langle A, B \rangle$, also contain at least two of the points P, Q, R.*

Proof. Let E be a solid of $\mathcal{S}$ in $\langle A, B \rangle$, different from A, B and C. Suppose $P \notin E$, then we have to prove that E contains the points R and Q. We find that $E \cap A$ and $E \cap B$ are subspaces that meet the lines $PR, PR', P'R$ and $PQ, PQ', P'Q$, respectively, as E meets every intersection plane D' in at least a line. Hence, E meets A in a line l_{AE} through R and a point of PR', or E has a plane γ_{AE} in common with A. By symmetry, E meets B in a line l_{BE} through Q and a point of PQ', or E has a plane γ_{BE} in common with B.

   a) If $\dim(A \cap E) = \dim(B \cap E) = 2$, then the planes γ_{AE} and γ_{BE} meet in a point of π_{AB} as they cannot contain the line π_{AB} since $P \notin E$. Hence, E contains two planes meeting in a point, which gives a contradiction since $\dim(E) = 3$.

   b) If $\dim(A \cap E) = 2$ and $\dim(B \cap E) = 1$, then $\gamma_{AE} \cap \pi_{AB} = l_{BE} \cap \pi_{AB}$. First note that $l_{BE} \cap \pi_{AB}$ is not empty by the dimension of E. Now, if $\gamma_{AE} \cap \pi_{AB} \neq l_{BE} \cap \pi_{AB}$, then $\pi_{AB} \subset E$, which gives a contradiction as $P \notin E$. Since l_{BE} can only meet π_{AB} in the point P, we find a contradiction, again as $P \notin E$. Clearly, by symmetry, an analogue argument holds also if $\dim(A \cap E)=1$ and $\dim(B \cap E) = 2$.

Hence we know that E contains a line $l_{AE} \subset A$ through R and a line $l_{BE} \subset B$ through Q, which proves the proposition. $\qquad\square$

    There are at most $\left(3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2\right)\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$ solids not in $\langle A, B \rangle$. This follows as two solids D_1, D_2, intersecting $\langle A, B \rangle$ in the intersection planes D_1' and D_2' meeting in a point, then D_1 and D_2 have to intersect in at least a point not in $\langle A, B \rangle$. And there are at most $3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2$ intersection planes D'. There are at most $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3q\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ solids in $\langle A, B \rangle$, namely all the solids through the plane $\langle P, Q, R \rangle$ and all solids through precisely two of the three points P, Q, R in $\langle A, B \rangle$.

Remark 30. Note that if $\mathcal{S}$ contains three elements A, B, C, with $A \cap B \cap C = \emptyset$, and if $\dim(\alpha) = 5$, then the number of elements of $\mathcal{S}$ is at most $f(3, q) = 3q^4 + 6q^3 + 5q^2 + q + 1$, and so we will not consider these maximal sets of solids in our classification.

2.4.2   General case $k > 3$ and α is a $(k+2)$-space

In this case we prove that all the k-spaces of $\mathcal{S}$ contain π_{ABC}. This implies that we can investigate this case by considering the quotient space of π_{ABC} and use the previous results for $k = 3$.

Proposition 31. [Using Notation 16] *If $\mathcal{S}$ contains three k-spaces A, B, C, with $\dim(A \cap B \cap C) = k - 4$, and if $\dim(\alpha) = k + 2$, then every k-space in $\mathcal{S}$ contains π_{ABC}.*

Proof. By Lemma 17, we know that all the k-spaces of $\mathcal{S}$ outside of $\langle A, B \rangle$ contain π_{ABC}. It is also clear that A, B and C contain π_{ABC}.
Suppose that there is a k-space E in $\langle A, B \rangle$, not through π_{ABC}. As E has to meet all the $(k-1)$-spaces D_i' in at least a $(k-2)$-space, E has to meet π_{ABC} in a $(k-5)$-space γ and $\pi_{AB}, \pi_{BC}, \pi_{AC}$ in three distinct $(k-3)$-spaces such that they meet pairwise in γ. This would imply that $\dim(E) = k + 1$, which gives a contradiction. $\qquad\square$

Clearly, the previous proposition implies that in order to have an estimate of the number of k-spaces in and outside of $\langle A, B \rangle$, we can use the results for $k = 3$ in Section 2.4.1: $|\mathcal{S}| \leqslant 4 \cdot \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) + 3$ or $|\mathcal{S}| \leqslant \left(3 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2 \right) \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \left(3q^2 + 1 \right)$. In both cases, $|\mathcal{S}| < \theta_{k+1} + q^4 + 2q^3 + 3q^2 = f(k, q)$.

To conclude this section we give a theorem which summarizes the cases studied in this section.

Proposition 32. [Using Notation 16] *In the projective space* $\mathrm{PG}(n, q)$, *with* $n \geqslant k+2$ *and* $k \geqslant 3$, *let* $\mathcal{S}$ *be a maximal set of* k-*spaces pairwise intersecting in at least a* $(k-2)$-*space such that* $\mathcal{S}$ *contains three* k-*spaces* A, B, C, *with* $\dim(A \cap B \cap C) = k - 4$, *and such that* $|\mathcal{S}| \geqslant f(k, q)$. *Then we have one of the following possibilities:*

   i) there are no k-*spaces of* $\mathcal{S}$ *outside of* $\langle A, B \rangle$ *and* $\mathcal{S}$ *is Example 9(x),*

   ii) $\dim(\alpha) = k - 1$ *and* $\mathcal{S}$ *is Example 9(v),*

   iii) $\dim(\alpha) = k$ *and* $\mathcal{S}$ *is Example 9(iv),*

   iv) $\dim(\alpha) = k + 1$ *and* $\mathcal{S}$ *is Example 9(vi).*

3   Every three elements of $\mathcal{S}$ meet in at least a $(k-3)$-space

Throughout this section we suppose that every three elements of $\mathcal{S}$ meet in at least a $(k-3)$-space. Moreover, to avoid trivial cases, we can suppose that there exist two k-spaces in $\mathcal{S}$ intersecting in precisely a $(k-2)$-space. We can find those two k-spaces as otherwise all subspaces would pairwise intersect in a $(k-1)$-space and the classification in this case is known: all the k-spaces go through a fixed $(k-1)$-space or all the k-spaces lie in a $(k+1)$-dimensional space, see Theorem 6. We also suppose that $\mathcal{S}$ is not a $(k-2)$- or a $(k-3)$-pencil as in this case either $\mathcal{S}$ is Example 9(i) or we can investigate the quotient space and use the known Erdős-Ko-Rado results [8]. We begin this section with a useful lemma.

Lemma 33. *Let* $\mathcal{S}$ *be a maximal set of* k-*spaces in* $\mathrm{PG}(n, q)$ *pairwise intersecting in at least a* $(k-2)$-*space such that for every* $X, Y, Z \in \mathcal{S}$, $\dim(X \cap Y \cap Z) \geqslant k - 3$, *and such that there is no point contained in all elements of* $\mathcal{S}$. *Then there exist three elements* A, B, C *of* $\mathcal{S}$ *such that*

   a) $\pi = A \cap B \cap C$ *is a* $(k-3)$-*space,*

   b) at least two of the three subspaces $\pi_{AB} = A \cap B, \pi_{BC} = B \cap C, \pi_{AC} = A \cap C$ *have dimension* $k - 2$, *and at most one of them has dimension* $k - 1$.

   c) $\zeta = \langle \pi_{AB}, \pi_{BC}, \pi_{AC} \rangle$ *has dimension* k *or* $k + 1$.

Every k-*space in* $\mathcal{S}$ *not through* π *meets the space* $\zeta = \langle \pi_{AB}, \pi_{BC}, \pi_{AC} \rangle$ *in at least a* $(k-1)$-*space.*

Proof. If every three k-spaces in $\mathcal{S}$ meet (at least) in a $(k-2)$-space, then $\mathcal{S}$ is a $(k-2)$-pencil, and so there is a point contained in all the k-spaces of $\mathcal{S}$. Therefore, there exist three elements $A, B, C \in \mathcal{S}$ such that $\pi = A \cap B \cap C$ is a $(k-3)$-space. Let $\pi_{AB} = A \cap B$, $\pi_{BC} = B \cap C$ and $\pi_{AC} = A \cap C$, and let $\zeta = \langle \pi_{AB}, \pi_{BC}, \pi_{AC} \rangle$. Note that at least two of the three subspaces $\pi_{AB}, \pi_{BC}, \pi_{AC}$ have dimension $k-2$. Otherwise, if, for example, $\dim(\pi_{AB}) = \dim(\pi_{AC}) = k-1$, then the k-space A contains two $(k-1)$-spaces, π_{AB} and π_{AC}, meeting in at most $(k-3)$-space, a contradiction. W.l.o.g. we can suppose that $\dim(\pi_{AB}) = \dim(\pi_{AC}) = k-2$ and $\dim(\pi_{BC}) \in \{k-1, k-2\}$. This also implies that the dimension of ζ is at most $k+1$. On the other hand, note that ζ has at least dimension k. Otherwise, if $\zeta = \langle \pi_{AB}, \pi_{BC}, \pi_{AC} \rangle$ is a $(k-1)$-space, then $\zeta = \langle \pi_{AB}, \pi_{AC} \rangle$ and so $\zeta \subset A$. By the same argument, $\zeta \subset B$, and $\zeta \subset C$. Hence, $\zeta \subset A \cap B \cap C = \pi$, a contradiction.

CASE 1. Suppose that π_{AB}, π_{AC} and π_{BC} are $(k-2)$-spaces. Then, ζ is a k-space and consider a k-space G in $\mathcal{S}$ not through π. This k-space exists since there is no point contained in all elements of $\mathcal{S}$, and hence not all elements of $\mathcal{S}$ contain π. Then G meets π in a $(k-4)$-space π_G and it contains at least a $(k-3)$-space of π_{AB}, π_{BC} and π_{AC}. This follows since any three elements of $\mathcal{S}$ meet in at least a $(k-3)$-space and $\pi \nsubseteq G$. Since the three subspaces $G \cap \pi_{AB}, G \cap \pi_{BC}$ and $G \cap \pi_{AC}$ have dimension at least $k-3$, since they pairwise meet in the $(k-4)$-space π_G, and since π_{AB}, π_{AC} and π_{BC} span at least a k-space, G contains the subspace $\langle G \cap \pi_{AB}, G \cap \pi_{BC}, G \cap \pi_{AC} \rangle$, with at least dimension $k-1$, in ζ.

CASE 2. Suppose that $\dim(\pi_{AB}) = \dim(\pi_{AC}) = k-2$ and $\dim(\pi_{BC}) = k-1$. They meet in the $(k-3)$-space π. Now, ζ is a $(k+1)$-space and consider a k-space G not through π. As before G meets π in a $(k-4)$-space; the spaces $G \cap \pi_{AB}$ and $G \cap \pi_{AC}$ are $(k-3)$-spaces otherwise G goes through π and finally $\dim(G \cap \pi_{BC}) \in \{k-3, k-2\}$.

Case 2a. $\dim(G \cap \pi_{BC}) = k-3$. Then $G \cap \pi_{AC}$ and $G \cap \pi_{BC}$ cannot be contained in π_{AB} otherwise $\dim(G \cap \pi) = k-3$. Hence, $G \cap \pi_{AC}$, $G \cap \pi_{BC}$ and $G \cap \pi_{AB}$ are linearly independent spaces (i.e. the span of two of them does not meet the other space) $(k-3)$-spaces pairwise intersecting in $G \cap \pi$. Therefore

$$\dim\langle \pi_{AB} \cap G, \pi_{AC} \cap G, \pi_{BC} \cap G \rangle = k-1.$$

Case 2b. $\dim(G \cap \pi_{BC}) = k-2$. Note that $G \cap \pi_{BC}$ cannot meet π_{AB} in a $(k-3)$-space, otherwise G goes through π. Then, again $G \cap \pi_{XY}$ with $\{X, Y\} \subset \{A, B, C\}$ are linearly independent spaces $(k-3)$-spaces pairwise intersecting in $G \cap \pi$ and

$$\dim\langle \pi_{AB} \cap G, \pi_{AC} \cap G, \pi_{BC} \cap G \rangle = k.$$

Hence, the k-space G is inside of ζ.

So, in any case, we get that a k-space not through π meets ζ in at least a $(k-1)$-space. $\square$

Theorem 34. *Let $\mathcal{S}$ be a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space in $\mathrm{PG}(n, q)$. If for every three elements X, Y, Z of $\mathcal{S}$: $\dim(X \cap Y \cap Z) \geqslant k-3$, and if there is no point contained in all elements of $\mathcal{S}$, then $\mathcal{S}$ is one of the following examples:*

(i) *Example 9(ii): Star.*

(ii) *Example 9(iii): Generalized Hilton-Milner example.*

Proof. From Lemma 33, it follows that we can suppose that there are three k-spaces A, B, C with $\dim(A \cap B \cap C) = k - 3$, $\dim(\pi_{AB}) = \dim(\pi_{AC}) = k - 2$ and $\dim(\pi_{BC}) \in \{k - 1, k - 2\}$

CASE 1. $\dim(\pi_{BC}) = k - 2$. In this case we know, again from Lemma 33, that $\zeta = \langle \pi_{AB}, \pi_{AC}, \pi_{BC} \rangle$ has dimension k and that any element of $\mathcal{S}$, not through $\pi = A \cap B \cap C$, meets ζ in at least a $(k - 1)$-space.

Case 1.1. Suppose that there exists a k-space D, not containing π, with $\dim(D \cap A) = \dim(D \cap B) = \dim(D \cap C) = k - 2$. Let π_{AD}, π_{BD} and π_{CD} be these $(k-2)$-spaces. Note that each of them contains the $(k-4)$-space $\pi_D = D \cap \pi$ and that they are contained in ζ. We prove that all elements of $\mathcal{S}$ meet ζ in at least a $(k-1)$-space. From Lemma 33, it follows that we only have to check that all elements of $\mathcal{S}$ through π have this property. Let E be a k-space in $\mathcal{S}$ through π. Then E contains a $(k-3)$-space of π_{AD}, π_{BD} and π_{CD}. At least two of these $(k-3)$-spaces are different, since π is not contained in D, and span together with π at least a $(k-1)$-space contained in the k-space ζ. Hence, every k-space of $\mathcal{S}$ meets ζ in at least a $(k-1)$-space. Then $\mathcal{S}$ is Example 9(ii).

Case 1.2. There exists no k-space D in $\mathcal{S}$, not containing π, with $\dim(D \cap A) = \dim(D \cap B) = \dim(D \cap C) = k - 2$.

In this case we will prove that if not every k-space of $\mathcal{S}$ meets ζ in a $(k-1)$-space, that then $\mathcal{S}$ is the second example described in the theorem. Let D be a k-space of $\mathcal{S}$ not containing π and meeting A, B or C in a $(k-1)$-space. W.l.o.g. we can suppose that $C \cap D$ is the $(k-1)$-space π_{CD} and that $A \cap D$ and $B \cap D$ are $(k-2)$-spaces (π_{AD} and π_{BD} respectively). Note that these subspaces $\pi_{AD}, \pi_{BD}, \pi_{CD}$ contain the $(k-4)$-space $\pi_D = D \cap \pi$ and that $\pi_{AD}, \pi_{BD} \subset \zeta$. This follows since D meets $\pi_{AB}, \pi_{AC}, \pi_{BC}$ in a $(k-3)$-space, and $D \cap \pi_{AB}$ and $D \cap \pi_{AC}$ span π_{AD}. The same argument holds for the space B. Suppose that $\mathcal{S}$ is not a Star, then there exists no k-space γ such that each of $\mathcal{S}$ meets γ in at least a $(k-1)$-space. In particular there exists a k-space $F \in \mathcal{S}$ that meets ζ in (at most) a $(k-2)$-space. As every k-space in $\mathcal{S}$, not containing π, meets ζ in a $(k-1)$-space (Lemma 33), we see that F contains π. Now, since every three elements of $\mathcal{S}$ meet in a $(k-3)$-space, F also contains a $(k-3)$-space of the two $(k-2)$-spaces π_{AD} and π_{BD} in ζ (π_{ADF}, π_{BDF} respectively). As F has no $(k-1)$-space in common with ζ, and since $\pi_{AD}, \pi_{BD} \subset \zeta$, $\pi_{CD} \not\subseteq \zeta$, we find that $\pi_{ADF} = \pi_{BDF} = \pi_{AB} \cap D$ and that $\pi_{CDF} \not\subseteq \zeta$. Hence, $F \cap \zeta = \pi_{AB}$ and $C \cap F = \langle \pi_{CDF}, \pi \rangle$. Let $\nu = \langle \zeta, C \rangle$. Then we prove that every k-space in $\mathcal{S}$ is contained in ν or contains π_{AB} and meets ν in a $(k-1)$-space. Every k-space in $\mathcal{S}$ containing π_{AB} must contain at least a $(k-2)$-space of C. Hence, this k-space meets ν in at least a $(k-1)$-space. Consider now a k-space $E \in \mathcal{S}$ not through π_{AB}. From the arguments above it follows that, if $\pi \subset E$, then $E \subset \nu$. Indeed, if $\pi \not\subseteq E$, then, by Lemma 33, E contains a $(k-1)$-space in ζ and a point in $C \setminus \zeta$ as otherwise we have *Case* 1.1, and so $\mathcal{S}$ would be a Star, a contradiction. Hence, $E \subset \nu$.

CASE 2. *For every three k-spaces $X, Y, Z \in \mathcal{S}$, we have that $\dim(X \cap Y \cap Z) \geqslant k - 2$ or two of these spaces meet in a $(k - 1)$-space.* Since we suppose that there is no point

contained in all elements of $\mathcal{S}$, we see that not every three elements meet in a fixed $(k-2)$-space. Recall that $A \cap B = \pi_{AB}$ is a $(k-2)$-space. Hence, every other element of $\mathcal{S}$ contains π_{AB} or meets A or B in a $(k-1)$-space. Note that the elements of $\mathcal{S}$, not through π_{AB}, are contained in $\langle A, B \rangle$. By Example 9(x), we may suppose that not all elements of $\mathcal{S}$ are contained in $\langle A, B \rangle$. Hence, let $D \in \mathcal{S}$ be a k-space not contained in $\langle A, B \rangle$.

If $D \cap A = D \cap B = \pi_{AB}$ then, by symmetry, it follows that every element of $\mathcal{S}$, not through π_{AB}, meets two of the three elements A, B, D in a $(k-1)$-space. This is a contradiction since a k-space cannot contain two $(k-1)$-spaces, meeting in a $(k-3)$-space. Hence, every k-space in $\mathcal{S}$, not in $\langle A, B \rangle$, meets A or B in a $(k-1)$-space through π_{AB}. W.l.o.g. we suppose that $B \cap D = \pi_{BD}$ is a $(k-1)$-space, and so $A \cap D = \pi_{AD} = \pi_{AB}$. Consider now an element $E \in \mathcal{S}$ not through π_{AB}. Then, $E \subset \langle A, B \rangle$, and since both A, B and A, D meet in a $(k-2)$-space, E contains a $(k-1)$-space in A or E contains a $(k-1)$-space in both D and B. Note that E cannot contain a $(k-1)$-space of D, since $E \subset \langle A, B \rangle$, but $D \cap \langle A, B \rangle$ is a $(k-1)$-space through $\pi_{AB} \not\supseteq E$. Hence, E must contain a $(k-1)$-space of A and a $(k-2)$-space of $B \cap D$ and so every element of $\mathcal{S}$, not through π_{AB}, is contained in $\nu = \langle A, \pi_{BD} \rangle$.

To conclude this proof, we show that every element of $\mathcal{S}$, through π_{AB}, meets $\nu = \langle A, \pi_{BD} \rangle$ in at least a $(k-1)$-space, which proves that $\mathcal{S}$ is the Generalized Hilton-Milner example. So, consider a k-space $F \in \mathcal{S}$, $\pi_{AB} \subset F$. Then F must contain a $(k-2)$-space π_{EF} of E. Hence, F contains the $(k-1)$-space $\langle \pi_{EF}, \pi_{AB} \rangle \subset \langle A, \pi_{BD} \rangle$. $\qquad \square$

4 There is at least a point contained in all k-spaces of $\mathcal{S}$

To classify all maximal sets of k-spaces pairwise intersecting in at least a $(k-2)$-space, we also have to investigate the families of k-spaces such that there is a subspace contained in all its elements.

More precisely, in this section we will consider a set $\mathcal{S}$ of k-spaces of $\mathrm{PG}(n, q)$ such that there is at least a point contained in all elements of $\mathcal{S}$. So, let g, with $0 \leqslant g \leqslant k - 3$, be the dimension of the maximal subspace γ contained in all elements of $\mathcal{S}$. In the quotient space of $\mathrm{PG}(n, q)$ with respect to γ, the set $\mathcal{S}$ of k-spaces corresponds to a set $\mathcal{T}$ of $(k-g-1)$-spaces in $\mathrm{PG}(n-g-1, q)$ that pairwise intersect in at least a $(k-g-3)$-space, and so that there is no point contained in all elements of $\mathcal{T}$. Since we are interested in sets $\mathcal{S}$ of k-spaces with $|\mathcal{S}| > f(k, q)$, this corresponds with sets $\mathcal{T}$ of $(k-g-1)$-spaces with $|\mathcal{T}| > f(k, q)$.

Since $f(k, q) \geqslant f(k-g-1, q)$, if $k-g-1 > 2$, we can use Theorem 32 and Theorem 34 for the sets $\mathcal{T}$ in $\mathrm{PG}(n-g-1, q)$. For each example we show that it can be extended to one of the examples discussed in the previous sections.

1. $\mathcal{T}$ is the set of k'-spaces of Theorem 32(i), so that $\mathcal{T}$ is Example 9(x) : There exists a $(k'+2)$-space ρ' such that $\mathcal{T}$ is the set of all k'-spaces in ρ. Then $\mathcal{S}$ can be extended to Example 9(x) in $\mathrm{PG}(n, q)$, with $\rho = \langle \rho', \gamma \rangle$.

2. $\mathcal{T}$ is the set of k'-spaces of Theorem 32(ii), so that $\mathcal{T}$ is Example 9(v) : There are a $(k'+2)$-space ρ', and a $(k'-1)$-space $\alpha' \subset \rho'$ so that $\mathcal{T}$ contains all k'-spaces in ρ' that meets α' in at least a $(k'-2)$-space, and all k'-spaces in $\mathrm{PG}(n-g-1,q)$ through α'. Then $\mathcal{S}$ can be extended to Example 9(v) in $\mathrm{PG}(n,q)$, with $\rho = \langle \rho', \gamma \rangle$ and $\alpha = \langle \alpha', \gamma \rangle$.

3. $\mathcal{T}$ is the set of k'-spaces of Theorem 32(iii), so that $\mathcal{T}$ is Example 9(iv) : There are a $(k'+2)$-space ρ', a k'-space $\alpha' \subset \rho'$ and a $(k'-2)$-space $\pi' \subset \alpha'$ so that $\mathcal{T}$ contains all k'-spaces in ρ' that meets α' in at least a $(k'-1)$-space, all k'-spaces in ρ' through π', and all k'-spaces in $\mathrm{PG}(n-g-1,q)$ that contain a $(k'-1)$-space of α' through π'. Then $\mathcal{S}$ can be extended to Example 9(iv) in $\mathrm{PG}(n,q)$, with $\pi = \langle \pi', \gamma \rangle$, $\rho = \langle \rho', \gamma \rangle$ and $\alpha = \langle \alpha', \gamma \rangle$.

4. $\mathcal{T}$ is the set of k'-spaces of Theorem 32(iv). Since we suppose that $|\mathcal{S}| = |\mathcal{T}| > f(k,q)$, we know that $\mathcal{T}$ is Example 9(vi): There are two $(k'+2)$-spaces ρ_1', ρ_2' intersecting in a $(k'+1)$-space $\alpha' = \rho_1' \cap \rho_2'$. There are two $(k'-1)$-spaces $\pi_A', \pi_B' \subset \alpha'$, with $\pi_A' \cap \pi_B'$ the $(k'-2)$-space l', there is a point $P' \in \alpha' \setminus \langle \pi_A', \pi_B' \rangle$, and let $P_A', P_B' \subset l'$ be two different $(k'-3)$-spaces. Then $\mathcal{T}$ contains

   ○ all k'-spaces in α',
   ○ all k'-spaces through $\langle P', l' \rangle$,
   ○ all k'-spaces in ρ_1' through P' and a $(k'-2)$-space in π_A' through P_A',
   ○ all k'-spaces in ρ_1' through P' and a $(k'-2)$-space in π_B' through P_B',
   ○ all k'-spaces in ρ_2' through P' and a $(k'-2)$-space in π_A' through P_B',
   ○ all k'-spaces in ρ_2' through P' and a $(k'-2)$-space in π_B' through P_A'.

   Then $\mathcal{S}$ can be extended to Example 9(vi) in $\mathrm{PG}(n,q)$, with $P_A = \langle P_A', \gamma \rangle$, $P_B = \langle P_B', \gamma \rangle$, $\pi_A = \langle \pi_A', \gamma \rangle$, $\pi_B = \langle \pi_B', \gamma \rangle$, $l = \langle l', \gamma \rangle$, $\alpha = \langle \alpha', \gamma \rangle$, $\rho_1 = \langle \rho_1', \gamma \rangle$ and $\rho_2 = \langle \rho_2', \gamma \rangle$.

5. $\mathcal{T}$ is the set of k'-spaces of Theorem 34(i): There exists a k'-space ζ' such that $\mathcal{T}$ is the set of all k'-spaces that have a $(k'-1)$-space in common with ζ'. Then $\mathcal{S}$ can be extended to example (i) in Theorem 34 with $\zeta = \langle \zeta', \gamma \rangle$.

6. $\mathcal{T}$ is the set of k'-spaces of Theorem 34(ii): There exists a $(k'+1)$-space ν' and a $(k'-2)$-space $\pi' \subset \nu$ such that $\mathcal{T}$ consists of all k'-spaces in ν', together with all k'-spaces through π' that intersect ν' in at least a $(k'-1)$-space. Then $\mathcal{S}$ can be extended to example (ii) in Theorem 34 with $\nu = \langle \nu', \gamma \rangle$, $\pi = \langle \pi', \gamma \rangle$.

We note that if $\mathcal{T}$ is one of the set of k'-spaces described in Section 2.4 then $\mathcal{S}$ can be extended to a set $\mathcal{S}'$ of k-spaces pairwise intersecting in a $(k-2)$-space such that $\mathcal{S}'$ contains three k-spaces that meet in a $(k-4)$-space with $\dim(\alpha) = k+2$. Hence, $|\mathcal{S}'| < f(k,q)$ and so these sets $\mathcal{T}$ do not lead to large examples of $\mathcal{S}$.

If $k - g - 1 = 2$, the set $\mathcal{T}$ is a set of planes in $\mathrm{PG}(n - k + 2, q)$ pairwise intersecting in at least a point, i.e. an Erdős-Ko-Rado set of planes. In [3, Section 6], Blokhuis *et al.* classified the maximal Erdős-Ko-Rado sets $\mathcal{T}$ of planes in $\mathrm{PG}(5, q)$ with $|\mathcal{T}| \geqslant 3q^4 + 3q^3 + 2q^2 + q + 1$. In [8], M. De Boeck generalized these results and classified the largest examples of sets of planes pairwise intersecting in at least a point in $\mathrm{PG}(n, q), n \geqslant 5$. Below we retrace the examples in [3] and [8] with size at least $f(k, q)$ and such that there is no point contained in all their elements. For each example, we show that it can be extended to one of the examples discussed in the previous sections, or that it gives rise to a new maximal example.

a) $\mathcal{T}$ is the set of planes of Example II in [8]: Consider a 3-space σ and a point $P_0 \in \sigma$. Let $\mathcal{T}$ be the set of all planes that either are contained in σ or else intersect σ in a line through P_0. Then $\mathcal{S}$ can be extended to example (ii) in Theorem 34, with ζ the $(k + 1)$-space spanned by σ and γ, and $\pi_{AB} = \langle \gamma, P_0 \rangle$.

b) $\mathcal{T}$ is the set of planes of Example III in [8]: Consider a plane π, then $\mathcal{T}$ is the set of planes meeting π in at least a line. Then $\mathcal{S}$ can be extended to example (i) in Theorem 34 with ζ the k-space spanned by π and γ.

c) $\mathcal{T}$ is the set of planes of Example IV in [8]: Consider a 4-space τ, a plane $\delta \subset \tau$ and a point $P_0 \in \delta$. Then $\mathcal{T}$ is the set containing the planes in τ intersecting δ in a line, the planes intersecting δ in a line through P_0 and the planes in τ through P_0. Then we can refer to Subsection 2.2 and so $\mathcal{S}$ can be extended to Example $9(iv)$, with $\rho = \langle \gamma, \tau \rangle$, $\alpha = \langle \gamma, \delta \rangle$ and $\pi = \langle \gamma, P_0 \rangle$.

d) $\mathcal{T}$ is the set of planes of Example V in [8]: Consider a 4-space τ, and a line $l \subset \tau$. Then $\mathcal{T}$ is the set containing the planes through l and all planes in τ containing a point of l. Then we can refer to Subsection 2.1 and $\mathcal{S}$ can be extended to Example $9(v)$, with $\rho = \langle \gamma, \tau \rangle$ and $\alpha = \langle \gamma, l \rangle$.

e) $\mathcal{T}$ is the set of planes of Example VI in [8]: Let τ_1 and τ_2 be two 4-spaces such that $\sigma = \tau_1 \cap \tau_2$ is a 3-space. Let π_1 and π_2 be two planes in σ with intersection line l_0 and let P_1 and P_2 be two different points on l_0. Then $\mathcal{T}$ is the set of planes through l_0, the planes in σ, the planes in τ_1 containing a line through P_1 in π_1 or a line through P_2 in π_2, and the planes in τ_2 containing a line through P_1 in π_2 or a line through P_2 in π_1. Then by using Section 2.3.1, Case 1, $\mathcal{S}$ can be extended to Example $9(vi)$ with $\rho_i = \langle \gamma, \tau_i \rangle$, $\alpha = \langle \gamma, \sigma \rangle$, $\pi_A = \langle \gamma, \pi_1 \rangle$, $\pi_B = \langle \gamma, \pi_2 \rangle$, $\lambda = \langle \gamma, l_0 \rangle$, $\lambda_A = \langle \gamma, P_1 \rangle$, $\lambda_B = \langle \gamma, P_2 \rangle$ and P_{AB} a point in γ.

f) $\mathcal{T}$ is the set of planes of Example VII in [8]: Let ρ be a 5-space. Consider a line $l \subset \rho$ and a 3-space $\sigma \subset \rho$ disjoint to l. Choose three points P_1, P_2, P_3 on l and choose four non-coplanar points Q_1, Q_2, Q_3, Q_4 in σ. Denote $l_1 = Q_1 Q_2$, $\bar{l}_1 = Q_3 Q_4$, $l_2 = Q_1 Q_3$, $\bar{l}_2 = Q_2 Q_4$, $l_3 = Q_1 Q_4$, and $\bar{l}_3 = Q_2 Q_3$. Then $\mathcal{T}$ is the set containing all planes through l and all planes through P_i in $\langle l, l_i \rangle$ or in $\langle l, \bar{l}_i \rangle$, $i = 1, 2, 3$. Note that this set $\mathcal{S}$ is the set described in Example $9(ix)$. We can prove the following lemma.

Lemma 35. *The set $\mathcal{S}$ of k-spaces described in Example 9(ix) is a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space.*

Proof. We have to prove that there exists no k-space E in $\mathrm{PG}(n,q)$, with $\gamma \nsubseteq E$ and so that E meets all elements of $\mathcal{S}$ in at least a $(k-2)$-space. Suppose there exists such a k-space E. As $\mathcal{S}$ contains all k-spaces through the $(k-1)$-space $\langle \gamma, l \rangle$, E contains a $(k-2)$-space π_0 of $\langle \gamma, l \rangle$, not through γ. Hence, $\dim(E \cap \gamma) = g - 1 = k - 4$. As $\mathcal{S}$ contains all k-spaces through $\langle \gamma, P_i \rangle$ in the $(k+1)$-space $\langle \gamma, l, l_i \rangle$ (or $\langle \gamma, l, \bar{l}_i \rangle$), E contains a $(k-1)$-space of each of those $(k+1)$-spaces. Consider now the quotient space $\mathrm{PG}(n,q)/\gamma$, and let $E' = \langle \gamma, E \rangle / \gamma$, $Q_i' = \langle Q_i, \gamma \rangle / \gamma$, $P_i' = \langle P_i, \gamma \rangle / \gamma$, and $l' = \langle l, \gamma \rangle / \gamma$. Then E' is a solid in $\mathrm{PG}(n,q)/\gamma$ through l' that contains a point of each of the lines $Q_i' Q_j'$, $1 \leqslant i < j \leqslant 4$, but this gives a contradiction as $\dim(E') = 3$. $\qquad\square$

g) $\mathcal{T}$ is the set of planes of Example $VIII$ in $\mathrm{PG}(n-k+2,q)$ in [8]: Consider two solids σ_1 and σ_2, intersecting in a line l. Take the points P_1 and P_2 on l. Then $\mathcal{T}$ is the set containing all planes through l, all planes through P_1 that contain a line in σ_1 and a line in σ_2, and all planes through P_2 in σ_1 of σ_2. Note that this set $\mathcal{S}$ is the set described in Example9($viii$). We can prove that the set $\mathcal{S}$ of k-spaces is not extendable.

Lemma 36. *The set $\mathcal{S}$ of k-spaces described in Example 9(vii) is a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space.*

Proof. We have to prove that there exists no k-space E in $\mathrm{PG}(n,q)$, with $\gamma \nsubseteq E$ and so that E meets all elements of $\mathcal{S}$ in at least a $(k-2)$-space. Suppose there exists such a k-space E. As $\mathcal{S}$ contains all k-spaces through the $(k-1)$-space $\langle \gamma, l \rangle$, E contains a $(k-2)$-space π_0 of $\langle \gamma, l \rangle$, not through γ. Hence, $\dim(\gamma \cap E) = k - 4$. As $\mathcal{S}$ contains all k-spaces through $\langle \gamma, P_2 \rangle$ in the $(k+1)$-space $\langle \gamma, \sigma_1 \rangle$ (or $\langle \gamma, \sigma_2 \rangle$), E contains a $(k-1)$-space of each of those $(k+1)$-spaces. These two $(k-1)$-spaces, α_1 and α_2 respectively, span E and meet in a $(k-2)$-space π_0. Then we show that there exists a k-space $A \in \mathcal{S}$, containing γ, that meets E in precisely a $(k-3)$-space. Consider the quotient space $\mathrm{PG}(n,q)/\gamma$, and let $E' = \langle \gamma, E \rangle / \gamma$, $\sigma_i' = \langle \sigma_i, \gamma \rangle / \gamma$, $P_i' = \langle P_i, \gamma \rangle / \gamma$, $A' = \langle A, \gamma \rangle / \gamma$ and $l' = \langle l, \gamma \rangle / \gamma = \langle \pi_0, \gamma \rangle / \gamma$. Then E' is a solid in $\mathrm{PG}(n,q)/\gamma$ through l' that contains planes α_1', α_2' in σ_1' and σ_2' respectively. Note that $\alpha_1' \cap \alpha_2' = l'$. Let $l_1 \in \sigma_1'$ and $l_2 \in \sigma_2'$ be two lines containing P_1' so that $l_1 \cap \alpha_1' = l_2 \cap \alpha_2' = P_1'$, and let A' be the plane spanned by l_1 and l_2. Then $E' \cap A'$ is a point in $\mathrm{PG}(n,q)/\gamma$. Since $\gamma \subseteq A$ and $\gamma \nsubseteq E$ we find that $E \cap A$ is a $(k-3)$-space of $\langle \gamma, P_1 \rangle$ in $\mathrm{PG}(n,q)$, and so these elements of $\mathcal{S}$ meet in a $(k-3)$-space, a contradiction. $\qquad\square$

h) $\mathcal{T}$ is the set of planes of Example IX in $\mathrm{PG}(n-k+2,q)$ in [8]: Let l be a line and σ a solid skew to l. Denote $\langle l, \sigma \rangle$ by ρ. Let P_1 and P_2 be two points on l and let $\mathcal{R}_1$ and $\mathcal{R}_2$ be a regulus and its opposite regulus in σ. Then $\mathcal{T}$ is the set containing

all planes through l, all planes through P_1 in the solid generated by l and a line of $\mathcal{R}_1$, and all planes through P_2 in the solid generated by l and a line of $\mathcal{R}_2$. Note that this set $\mathcal{S}$ is the set described in Example 9(*viii*). We can prove the following lemma.

Lemma 37. *The set $\mathcal{S}$ of k-spaces described in Example 9(viii) is a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space.*

Proof. We have to prove that there exists no k-space E in $\mathrm{PG}(n,q)$, with $\gamma \nsubseteq E$, and so that E meets all elements of $\mathcal{S}$ in at least a $(k-2)$-space. Suppose there exists such a k-space E. Let $\mathcal{R}_1 = \{l_1, l_2, \ldots, l_{q+1}\}$ and $\mathcal{R}_2 = \{\bar{l}_1, \bar{l}_2, \ldots, \bar{l}_{q+1}\}$. As $\mathcal{S}$ contains all k-spaces through the $(k-1)$-space $\langle \gamma, l \rangle$, E contains a $(k-2)$-space π_0 of $\langle \gamma, l \rangle$, not through γ. Hence, $\dim(\gamma \cap E) = k-4$. As $\mathcal{S}$ contains all k-spaces through $\langle \gamma, P_i \rangle$ in the $(k+1)$-spaces $\langle \gamma, l, l_i \rangle$ (or $\langle \gamma, l, \bar{l}_i \rangle$), E contains a $(k-1)$-space of each of those $(k+1)$-spaces. Consider now the quotient space $\mathrm{PG}(n,q)/\gamma$, and let $E' = \langle \gamma, E \rangle / \gamma$, $l_i' = \langle l_i, \gamma \rangle / \gamma$, $\bar{l}_i' = \langle \bar{l}_i, \gamma \rangle / \gamma$, $P_i' = \langle P_i, \gamma \rangle / \gamma$, and $l' = \langle l, \gamma \rangle / \gamma = \langle \pi_0, \gamma \rangle / \gamma$. Then E' is a solid in $\mathrm{PG}(n,q)/\gamma$ through l' that contains a point of each of the lines l_i' and $\bar{l}_i'$, $1 \leqslant i \leqslant q+1$, but this gives a contradiction as $\dim(E') = 3$. $\qquad\square$

We see that example $(f), (g)$ and (h) give rise to maximal examples of sets $\mathcal{S}$ of k-spaces pairwise intersecting in at least a $(k-2)$-space, described in Example 9(ix), (vii), ($viii$) respectively. From [8], it follows that the number of elements in $\mathcal{S}$ equals $\theta_{n-k} + 6q^2$, $\theta_{n-k} + q^4 + 2q^3 + 3q^2$ and $\theta_{n-k} + 2q^3 + 2q^2$ respectively.

Finally, if $k - g - 1 = 1$, then $g = k - 2$ and so, there is a $(k-2)$-space contained in all solids of $\mathcal{S}$. This case gives rise to Example 9(i).

5   Main Theorem

By collecting the results from Propositions 32, Theorem 34 and Section 4, we find the following result.

Main Theorem 38. Let $\mathcal{S}$ be a maximal set of k-spaces pairwise intersecting in at least a $(k-2)$-space in $\mathrm{PG}(n,q)$, $n \geqslant 2k$, $k \geqslant 3$. Let

$$f(k,q) = \begin{cases} 3q^4 + 6q^3 + 5q^2 + q + 1 & \text{if } k = 3, q \geqslant 2 \text{ or } k = 4, q = 2 \\ \theta_{k+1} + q^4 + 2q^3 + 3q^2 & \text{otherwise.} \end{cases}$$

If $|\mathcal{S}| > f(k,q)$, then $\mathcal{S}$ is one of the families described in Example 9. Note that for $n > 2k + 1$, the examples $(i) - (ix)$ are stated in decreasing order of the sizes.

Proof.   - If there is no point contained in all elements of $\mathcal{S}$ and $\mathcal{S}$ contains three k-spaces A, B, C with $\dim(A \cap B \cap C) = k - 4$, then we distinguished the possibilities

for $\mathcal{S}$ depending on the dimension of $\alpha = \langle D \cap \langle A, B \rangle \,|\, D \in \mathcal{S}' \rangle$, where $\mathcal{S}' = \{D \in \mathcal{S} \,|\, D \not\subset \langle A, B \rangle\}$, see Section 2. By Proposition 32, it follows that $\mathcal{S}$ is one of the examples $(iv), (v), (vi), (x)$ in Example 9.

- If there is no point contained in all elements of $\mathcal{S}$ and if for every three elements A, B, C in $\mathcal{S}$, we have that $\dim(A \cap B \cap C) \geqslant k - 3$, then the only possibilities for $\mathcal{S}$ are described in Example 9 (ii) and (iii), see Theorem 34.

- If there is at least a point contained in all k-spaces of $\mathcal{S}$, then we refer to Section 4. Let γ be the maximal subspace contained in all k-spaces of $\mathcal{S}$, with $\dim(\gamma) = g$. Then $\mathcal{T} = \{D/\gamma \,|\, D \in \mathcal{S}\}$ is a set of $(k-g-1)$-spaces of $\mathrm{PG}(n-g-1, q) \simeq \mathrm{PG}(n, q)/\gamma$ pairwise intersecting in at least a $(k - g - 3)$-space. The only examples of sets $\mathcal{T}$ that give rise to maximal examples of sets of k-spaces are described in Section 4 in the examples $(f), (g), (h)$ and when $g = k - 3$. They correspond to Example $9(i), (ix), (vii), (viii)$.

$\square$

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