The set of ratios of derangements to permutations in digraphs is dense in $[0, 1/2]$

Bethany Austhof
Department of Mathematics, Statistics, and Computer Science
University of Illinois Chicago
Chicago, IL, U.S.A.
baust2@uic.edu

Patrick Bennett *
Department of Mathematics
Western Michigan University
Kalamazoo, MI, U.S.A.
patrick.bennett@wmich.edu

Nick Christo
Department of Mathematics, Statistics, and Computer Science
University of Illinois Chicago
Chicago, IL, U.S.A.
nchrist5@uic.edu

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Abstract

A *permutation* in a digraph $G = (V, E)$ is a bijection $f : V \to V$ such that for all $v \in V$ we either have that f fixes v or $(v, f(v)) \in E$. A *derangement* in G is a permutation that does not fix any vertex. Bucic, Devlin, Hendon, Horne and Lund proved that in any digraph, the ratio of derangements to permutations is at most $1/2$. Answering a question posed by Bucic, Devlin, Hendon, Horne and Lund, we show that the set of possible ratios of derangements to permutations in digraphs is dense in the interval $[0, 1/2]$.

Mathematics Subject Classifications: 05A05, 05C80

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1 Introduction

A *permutation* in a digraph (with no loops) $G = (V, E)$ is a bijection $f : V \to V$ such that for all $v \in V$ we either have that f fixes v or $(v, f(v)) \in E$. A *derangement* in G is a permutation that does not fix any vertex. We define the parameter $(d/p)_G$ to be the ratio of derangements to permutations in G. As an aside, it is worth noting that if A is the adjacency matrix of the graph G, then the ratio we are studying can be written

$$(d/p)_G = \frac{\mathrm{per}(A)}{\mathrm{per}(A + I_n)},$$

where $\mathrm{per}(\cdot)$ refers to the permanent of a matrix and I_n is the $n \times n$ identity matrix.

Bucic, Devlin, Hendon, Horne and Lund [1] showed that $(d/p)_G \leqslant 1/2$ for all digraphs G, with equality if and only if G is a directed cycle. They also gave a construction (the blow-up of a directed cycle) that can achieve a ratio arbitrarily close to but not equal to $1/2$. Let $S = \{(d/p)_G \ : \ G \text{ is a digraph}\}$ be the set of values arising as a ratio (d/p). In [1] they analyzed the ratio $(d/p)_G$ for the random graph $G = G(n, m)$, and as a corollary of this analysis they showed that S is dense in $[0, 1/e]$. This corollary follows from two facts: $(d/p)_G$ is concentrated around its mean, and by choosing a suitable value of m one can make the expected ratio $(d/p)_G$ close to any given value in $[0, 1/e]$. At the end of the paper [1] they ask whether S is dense in $[0, 1/2]$. Our main theorem, below, answers this question in the positive.

Theorem 1. *The set of possible ratios of derangements to permutations in digraphs is dense in* $[0, 1/2]$.

The construction we use, described in more detail later, is a random subgraph of the blow-up of a directed cycle. The main part of the proof is an application of the second moment method (see [2], for example, for an introduction to the method) to show that the number of derangements and permutations are concentrated around their expectations.

2 Proof of Theorem 1

2.1 Outline

First we outline the proof. Suppose we are given a fixed real number $r \in [0, 1/2]$. We will show that there exists a sequence of digraphs G_k such that the ratio of derangements to permutations in G_k is $r + o(1)$ as $k \to \infty$ (which proves Theorem 1). If $r = 0$ or $1/2$ this is trivial. Indeed, for $r = 0$ observe that a digraph with one vertex and no edges has no derangements and one permutation, and for $r = 1/2$ observe that any directed cycle has one derangement and two permutations. So we assume $0 < r < 1/2$.

Our construction is as follows. As defined in [1], let the digraph $D_{k,\ell}$ where $k \geqslant 1$ and $\ell \geqslant 2$ have vertices v_{ij} for $i \in [k]$ and $j \in [\ell]$ such that $(v_{ij}, v_{lm}) \in E(D_{k,\ell})$ if and only if $m = j + 1 \bmod l$. In other words, $D_{k,\ell}$ is the blow-up of a directed ℓ-cycle where each vertex is expanded to a set of k vertices. We let $V_i = \{v_{ij} : j \in [\ell]\}$. As was shown in [1], the number of derangements on $D_{k,\ell}$ is $(k!)^\ell$ and the number of permutations on $D_{k,\ell}$ is $\sum_{i=0}^{k}\left(\binom{k}{i}(k-i)!\right)^\ell$. Hence, $(d/p)_{k,\ell} = \left(\sum_{i=0}^{k}\left(\frac{1}{i!}\right)^\ell\right)^{-1}$ can be made arbitrarily close to 1/2 by choosing ℓ large enough (even for large k). This construction yields a graph for which the ratio of derangements to permutations is arbitrarily close to 1/2 but not exactly 1/2. We will also use this construction, but we will randomly remove some edges. By taking a random subgraph we can "interpolate" between $D_{k,\ell}$ (a dense digraph whose ratio of derangements to permutations is close to 1/2) and a sparse random digraph (whose ratio is 0).

In this paper all asymptotics are as $k \to \infty$. ℓ is treated as fixed. We use standard big-O, little-o and Ω notation. We write $x \sim y$ if $x = (1+o(1))y$. All logarithms are base e.

2.2 Proof details

Let the random graph $G_{k,\ell}(m)$ be chosen uniformly from among all subgraphs of $D_{k,\ell}$ with m edges. We will fix some p, ℓ and let $m = pk^2\ell$ (so p is the probability that any particular edge of $D_{k,\ell}$ becomes an edge of $G_{k,\ell}$). Let the random variables X, Y be the number of derangements and permutations in $G_{k,\ell}(m)$ respectively. Let $\mathcal{D}, \mathcal{P}$ be the collection of all possible derangements and permutations on $D_{k,\ell}(m)$.

2.2.1 First moments of X, Y

We have

$$\begin{aligned}\mathbb{E}[X] &= \sum_{D\in\mathcal{D}} \mathbb{P}[D \subseteq G_{k,\ell}] = (k!)^\ell \frac{\binom{k^2\ell-k\ell}{m-k\ell}}{\binom{k^2\ell}{m}} \\ &= (k!)^\ell \left(\frac{m}{k^2\ell}\right)^{k\ell} \exp\left\{\frac{k^2\ell^2}{2}\left(\frac{1}{k^2\ell}-\frac{1}{m}\right) + O\left(\frac{k^3}{m^2}+\frac{k}{m}\right)\right\} \\ &\sim (k!)^\ell p^{k\ell} \exp\left\{\frac{\ell}{2}\left(1-\frac{1}{p}\right)\right\}, \end{aligned} \tag{2.1}$$

where on the second line we have used the following fact:

Fact 2.

$$\frac{\binom{a-x}{b-x}}{\binom{a}{b}} = \frac{(b)_x}{(a)_x} = \left(\frac{b}{a}\right)^x \exp\left\{\frac{x^2}{2}\left(\frac{1}{a}-\frac{1}{b}\right) + O\left(\frac{x^3}{b^2}+\frac{x}{b}\right)\right\}.$$

For completeness we include the proof although it is well-known.

Proof.

$$\frac{(b)_x}{(a)_x} = \left(\frac{b}{a}\right)^x \cdot \frac{1\left(1-\frac{1}{b}\right)\left(1-\frac{2}{b}\right)\cdots\left(1-\frac{x-1}{b}\right)}{1\left(1-\frac{1}{a}\right)\left(1-\frac{2}{a}\right)\cdots\left(1-\frac{x-1}{a}\right)}$$

$$
\begin{aligned}
&= \left(\frac{b}{a}\right)^x \cdot \exp\left\{\sum_{i=0}^{x-1}\left[\ln\left(1-\frac{i}{b}\right) - \ln\left(1-\frac{i}{a}\right)\right]\right\} \\
&= \left(\frac{b}{a}\right)^x \cdot \exp\left\{\sum_{i=0}^{x-1}\left[-\frac{i}{b}+\frac{i}{a}+O\left(\frac{i^2}{a^2}+\frac{i^2}{b^2}\right)\right]\right\} \\
&= \left(\frac{b}{a}\right)^x \cdot \exp\left\{\frac{x(x-1)}{2}\left(\frac{1}{a}-\frac{1}{b}\right)+O\left(\frac{x^3}{b^2}\right)\right\} \\
&= \left(\frac{b}{a}\right)^x \exp\left\{\frac{x^2}{2}\left(\frac{1}{a}-\frac{1}{b}\right)+O\left(\frac{x^3}{b^2}+\frac{x}{b}\right)\right\}. \qquad \square
\end{aligned}
$$

Before we calculate $\mathbb{E}[Y]$ we introduce a function $f_\ell(x)$. For any integer $\ell \geqslant 1$, let

$$
f_\ell(x) := \sum_{i=0}^{\infty} \frac{x^{i\ell}}{(i!)^\ell}. \tag{2.2}
$$

Note that the above power series for $f_\ell(x)$ converges for all x and therefore in particular each f_ℓ is continuous in x. We have

$$
\begin{aligned}
\mathbb{E}[Y] &= \sum_{P\in\mathcal{P}} \mathbb{P}[P \subseteq G_{k,\ell}] = \sum_{i=0}^{k}\left(\binom{k}{i}(k-i)!\right)^\ell \frac{\binom{k^2\ell-(k-i)\ell}{m-(k-i)\ell}}{\binom{k^2\ell}{m}} \\
&= \sum_{i=0}^{k}\left(\frac{k!}{i!}\right)^\ell \left(\frac{m}{k^2\ell}\right)^{(k-i)\ell} \exp\left\{\frac{(k-i)^2\ell^2}{2}\left(\frac{1}{k^2\ell}-\frac{1}{m}\right)+O\left(\frac{k^3}{m^2}+\frac{k}{m}\right)\right\} \\
&= (k!)^\ell p^{k\ell}\sum_{i=0}^{k}\left(\frac{1}{i!}\right)^\ell p^{-i\ell}\exp\left\{\frac{\ell}{2}\left(1-\frac{1}{p}\right)+O\left(\frac{i+1}{k}\right)\right\}. \qquad (2.3)
\end{aligned}
$$

We split the above sum into two ranges of i. Note that for $0 \leqslant i \leqslant \sqrt{k}$ we have $\exp\left\{O\left(\frac{i+1}{k}\right)\right\} = 1 + O\left(\frac{1}{\sqrt{k}}\right)$, while for $\sqrt{k} \leqslant i \leqslant k$ we have $\exp\left\{O\left(\frac{i+1}{k}\right)\right\} = O(1)$. Thus line (2.3) becomes

$$
\begin{aligned}
(k!)^\ell p^{k\ell}\Bigg[\left(1+O\left(\frac{1}{\sqrt{k}}\right)\right)\exp\left\{\frac{\ell}{2}\left(1-\frac{1}{p}\right)\right\}\sum_{0\leqslant i\leqslant\sqrt{k}}\left(\frac{1}{i!}\right)^\ell p^{-i\ell} \\
+O(1)\sum_{\sqrt{k}<i\leqslant k}\left(\frac{1}{i!}\right)^\ell p^{-i\ell}\Bigg]. \qquad (2.4)
\end{aligned}
$$

As $k \to \infty$ we have

$$
\sum_{0\leqslant i\leqslant\sqrt{k}}\left(\frac{1}{i!}\right)^\ell p^{-i\ell} \to f_\ell(1/p),
$$

and

$$\sum_{\sqrt{k}<i\leqslant k}\left(\frac{1}{i!}\right)^{\ell}p^{-i\ell}\leqslant\sum_{i=\sqrt{k}}^{\infty}\left(\frac{1}{i!}\right)^{\ell}p^{-i\ell}=o(1),$$

since the latter is the tail of a convergent series. Thus, returning to our estimate of $\mathbb{E}[Y]$ on line (2.4), we have

$$\mathbb{E}[Y]\sim(k!)^{\ell}p^{k\ell}\exp\left\{\frac{\ell}{2}\left(1-\frac{1}{p}\right)\right\}f_{\ell}(1/p). \tag{2.5}$$

2.2.2 Choosing p,ℓ

Now that we know $\mathbb{E}[X],\mathbb{E}[Y]$ we will choose p,ℓ to make sure that the ratio of $\mathbb{E}[X]$ to $\mathbb{E}[Y]$ is close to r. Using lines (2.1) and (2.5) we have

$$\frac{\mathbb{E}[X]}{\mathbb{E}[Y]}\sim\frac{1}{f_{\ell}\left(\frac{1}{p}\right)},$$

so we would like to choose ℓ and $0<p<1$ so that $f_{\ell}(1/p)=1/r$. We have

$$\lim_{x\to\infty}f_{\ell}(x)=\infty,\qquad f_{\ell}(1)=\sum_{i=0}^{k}\left(\frac{1}{i!}\right)^{\ell}=1+1+\frac{1}{2^{\ell}}+\frac{1}{6^{\ell}}+\frac{1}{24^{\ell}}+\dots.$$

Note that we can make $f_{\ell}(1)$ arbitrarily close to 2 by taking ℓ large. Indeed, we have $f_{\ell}(1)\geqslant 2$ and

$$f_{\ell}(1)=2+\sum_{i\geqslant 2}\left(\frac{1}{i!}\right)^{\ell}\leqslant 2+\sum_{i\geqslant 2}\left(\frac{1}{2^{i-1}}\right)^{\ell}=2+\frac{1}{2^{\ell}-1}.$$

Since $r<1/2$, we can choose ℓ so that $f_{\ell}(1)<1/r$. Then by the intermediate value theorem there is some $x\in(1,\infty)$ such that $f_{\ell}(x)=1/r$. We choose p to be the value $1/x$, so $0<p<1$ and $f_{\ell}(1/p)=1/r$. So we view ℓ and p as constants determined entirely by r.

2.2.3 Second moments of X,Y

In this section we show that $\mathbb{E}[X^2]\sim\mathbb{E}[X]^2$ and $\mathbb{E}[Y^2]\sim\mathbb{E}[Y]^2$. This will complete the proof, since then by the second moment method we have that

$$\frac{X}{Y}\sim\frac{\mathbb{E}[X]}{\mathbb{E}[Y]}\sim\frac{1}{f_{\ell}(1/p)}=r.$$

with probability approaching 1 as k goes to infinity.

To help us estimate $\mathbb{E}[X^2],\mathbb{E}[Y^2]$ we will find the function $h(a,b)$ (defined below) useful. Suppose we have some fixed matching B of b many edges in the graph $K_{a,a}$. Then by inclusion-exclusion the number of perfect matchings that do not have any edges from B is

$$h(a,b) := \sum_{w=0}^{b} (-1)^w \binom{b}{w} (a-w)!.$$

Note that we always have $h(a,b) \leqslant a!$. We will now observe that, roughly speaking, $h(a,b) \approx \dfrac{a!}{e}$ whenever $b \approx a \to \infty$. More formally we have the following

Fact 3. *Suppose $a - a^{1/10} \leqslant b \leqslant a$. Then we have*

$$h(a,b) = \left(1 + O(a^{-4/5})\right) \frac{a!}{e}$$

as $a \to \infty$

Proof. We have

$$h(a,b) = \sum_{0 \leqslant w \leqslant b} (-1)^w \binom{b}{w} (a-w)! = a! \sum_{0 \leqslant w \leqslant b} \frac{(-1)^w}{w!} \frac{(b)_w}{(a)_w}. \tag{2.6}$$

Now, for $0 \leqslant w \leqslant a^{1/10}$ we have by Fact 2 that

$$\begin{aligned} \frac{(b)_w}{(a)_w} &= \left(\frac{b}{a}\right)^w \exp\left\{ \frac{w^2}{2}\left(\frac{1}{a} - \frac{1}{b}\right) + O\left(\frac{w^3}{b^2} + \frac{w}{b}\right)\right\} \\ &= \left(1 + O\left(a^{-9/10}\right)\right)^{O\left(a^{1/10}\right)} \exp\left\{O(a^{-4/5}\right\} = 1 + O(a^{-4/5}). \end{aligned}$$

Meanwhile for $w \geqslant a^{1/10}$ we have that the corresponding term in line (2.6) has absolute value

$$\frac{1}{w!}\frac{(b)_w}{(a)_w} \leqslant \frac{1}{(a^{1/10})!} = \exp\left\{-\Omega\left(a^{1/10}\log a\right)\right\}$$

by Stirling's approximation. Thus, the sum of all such terms in line (2.6) is at most

$$b \exp\left\{-\Omega\left(a^{1/10}\log a\right)\right\} = O(a^{-4/5})$$

(this bound is quite comfortable). By the Alternating Series Test we have that

$$\sum_{0 \leqslant w \leqslant a^{1/10}} \frac{(-1)^w}{w!} = \frac{1}{e} + O\left(\frac{1}{(a^{1/10})!}\right) = \frac{1}{e} + O(a^{-4/5}).$$

Breaking up the sum for $h(a,b)$ we have

$$\begin{aligned} h(a,b) &= a!\left[\sum_{0 \leqslant w \leqslant a^{1/10}} \frac{(-1)^w}{w!}\frac{(b)_w}{(a)_w} + \sum_{a^{1/10} < w \leqslant b} \frac{(-1)^w}{w!}\frac{(b)_w}{(a)_w}\right] \\ &= a!\left[\left(1 + O(a^{-4/5})\right) \sum_{0 \leqslant w \leqslant a^{1/10}} \frac{(-1)^w}{w!} + O(a^{-4/5})\right] \\ &= \left(1 + O(a^{-4/5})\right)\frac{a!}{e}. \end{aligned}$$

□

We find that

$$\begin{aligned}\mathbb{E}[X^2] &= \sum_{D,D' \in \mathcal{D}} \mathbb{P}[D, D' \subseteq G_{k,\ell}] = (k!)^\ell \sum_{D' \in \mathcal{D}} \mathbb{P}[D_0, D' \subseteq G_{k,\ell}] \\ &= (k!)^\ell \sum_{b=0}^{k\ell} \left[\frac{\binom{k^2\ell-(2k\ell-b)}{m-(2k\ell-b)}}{\binom{k^2\ell}{m}} \sum_{\vec{b} \in S_b} \prod_{c=1}^{\ell} \binom{k}{b_c} h(k-b_c, k-b_c) \right], \qquad (2.7)\end{aligned}$$

where D_0 is a fixed derangement and in the inner sum, S_b is the set of ℓ-dimensional vectors $\vec{b} = (b_1, \dots b_\ell)$ whose components are nonnegative integers summing to b.

By 2, if $b \leqslant k^{1/10}$ then we have

$$\begin{aligned}\frac{\binom{k^2\ell-(2k\ell-b)}{m-(2k\ell-b)}}{\binom{k^2\ell}{m}} &= p^{2k\ell-b} \exp\left\{ \frac{(2k\ell-b)^2}{2} \left(\frac{1}{k^2\ell} - \frac{1}{m} \right) + O\left(\frac{k^3}{m^2} + \frac{k}{m} \right) \right\} \\ &= \left(1 + O(k^{-9/10})\right) p^{2k\ell-b} \exp\left\{ 2\ell \left(1 - \frac{1}{p} \right) \right\},\end{aligned}$$

and by Fact 3 we have $h(k-b_c, k-b_c) = \left(1 + O(k^{-4/5})\right) \dfrac{(k-b_c)!}{e}$. Therefore the term corresponding to b in (2.7) is

$$\begin{aligned}&\frac{\binom{k^2\ell-(2k\ell-b)}{m-(2k\ell-b)}}{\binom{k^2\ell}{m}} \sum_{\vec{b} \in S_b} \prod_{c=1}^{\ell} \binom{k}{b_c} h(k-b_c, k-b_c) \\ &= \left(1 + O(k^{-4/5})\right) p^{2k\ell-b} \exp\left\{ 2\ell \left(1 - \frac{1}{p} \right) \right\} \sum_{\vec{b} \in S_b} \prod_{c=1}^{\ell} \binom{k}{b_c} \frac{(k-b_c)!}{e} \\ &= \left(1 + O(k^{-4/5})\right) (k!)^\ell p^{2k\ell-b} \exp\left\{ 2\ell \left(1 - \frac{1}{p} \right) - \ell \right\} \sum_{\vec{b} \in S_b} \prod_{c=1}^{\ell} \frac{1}{b_c!} \\ &= \left(1 + O(k^{-4/5})\right) (k!)^\ell p^{2k\ell-b} \exp\left\{ \ell \left(1 - \frac{2}{p} \right) \right\} \frac{\ell^b}{b!},\end{aligned}$$

where on the last line we used the multinomial formula. Meanwhile if $b \geqslant k^{1/10}$ then the term corresponding to b in (2.7) is

$$\begin{aligned}&\frac{\binom{k^2\ell-(2k\ell-b)}{m-(2k\ell-b)}}{\binom{k^2\ell}{m}} \sum_{\vec{b} \in S_b} \prod_{c=1}^{\ell} \binom{k}{b_c} h(k-b_c, k-b_c) \\ &\leqslant \sum_{\vec{b} \in S_b} \prod_{c=1}^{\ell} \binom{k}{b_c} (k-b_c)! \\ &= (k!)^\ell \frac{\ell^b}{b!}\end{aligned}$$

$$= (k!)^\ell \cdot \exp\left\{-\Omega\left(k^{1/10}\log k\right)\right\},$$

and so the sum of all terms in (2.7) with $b \geqslant k^{1/10}$ is at most

$$k\ell \cdot (k!)^\ell \cdot \exp\left\{-\Omega\left(k^{1/10}\log k\right)\right\} = (k!)^\ell \cdot O\left(k^{-4/5}\right).$$

Therefore,

$$\begin{aligned}
\mathbb{E}[X^2] &= (k!)^\ell \sum_{b=0}^{k\ell}\left[\frac{\binom{k^2\ell-(2k\ell-b)}{m-(2k\ell-b)}}{\binom{k^2\ell}{m}}\sum_{\vec{b}\in S_b}\prod_{c=1}^{\ell}\binom{k}{b_c}h(k-b_c,k-b_c)\right]\\
&= (k!)^\ell\left[\sum_{0\leqslant b\leqslant k^{1/10}}\left(1+O(k^{-4/5})\right)(k!)^\ell p^{2k\ell-b}\exp\left\{\ell\left(1-\frac{2}{p}\right)\right\}\frac{\ell^b}{b!}\right.\\
&\qquad\qquad\left. + (k!)^\ell \cdot O\left(k^{-4/5}\right)\right]\\
&= \left(1+O(k^{-4/5})\right)(k!)^{2\ell}p^{2k\ell}\exp\left\{\ell\left(1-\frac{2}{p}\right)\right\}\sum_{0\leqslant b\leqslant k^{1/10}}p^{-b}\frac{\ell^b}{b!}\\
&= \left(1+O(k^{-4/5})\right)(k!)^{2\ell}p^{2k\ell}\exp\left\{\ell\left(1-\frac{2}{p}\right)\right\}\cdot\left(\exp\left\{\frac{\ell}{p}\right\}+O(k^{-4/5})\right)\\
&= \left(1+O(k^{-4/5})\right)(k!)^{2\ell}p^{2k\ell}\exp\left\{\ell\left(1-\frac{1}{p}\right)\right\}\\
&\sim \mathbb{E}[X]^2.
\end{aligned}$$

For $\mathbb{E}[Y^2]$ we find an exact expression to be cumbersome, but the following upper bound will suffice. We claim (with justification below) that $\mathbb{E}[Y^2]$ is at most

$$\sum_{\substack{0\leqslant i,j\leqslant k\\0\leqslant b\leqslant k\ell\\ \vec{b}\in S_b}}\frac{\binom{k^2\ell-(2k\ell-(i+j)\ell-b)}{m-(2k\ell-(i+j)\ell-b)}}{\binom{k^2\ell}{m}}\left(\frac{k!}{i!}\right)^\ell\prod_{c=1}^{\ell}\binom{k-i}{b_c}\binom{k-b_c}{j}h(k-j-b_c,k-i-b_c-2j). \tag{2.8}$$

The term corresponding to a tuple $(i,j,b,\vec{b})$ above is an upper bound on the contribution to $\mathbb{E}[Y^2]$ due to pairs of permutations (P,P') such that P fixes i vertices per part, P' fixes j vertices per part, and P and P' share a total of b edges where b_c of the shared edges are between V_c and part V_{c+1}. The first factor is the edge probability, and the next factor is the number of choices for P. The next factor is an upper bound on the number of choices for P'. Indeed, we choose the edges of P' from V_c to V_{c+1} by first choosing b_c edges of P to be shared, then we choose j vertices in V_c to be fixed by P', and finally we choose a matching between the remaining vertices (the vertices of $V_c \cup V_{c+1}$ that are not fixed by P' and are not endpoints of the b_c shared edges already chosen). This matching must avoid any edges of P, and the vertices to be matched induce at least $k-i-b_c-2j$ edges of P, explaining the last factor above.

We will now estimate the significant terms in (2.8). Assume $i,j,b \leqslant k^{1/10}$. Then by Fact 2

$$\frac{\binom{k^2\ell-(2k\ell-(i+j)\ell-b)}{m-(2k\ell-(i+j)\ell-b)}}{\binom{k^2\ell}{m}} = p^{2k\ell-(i+j)\ell-b}\exp\left\{\frac{(2k\ell-(i+j)\ell-b)^2}{2k^2\ell}\left(1-\frac{1}{p}\right)+O\left(\frac{1}{k}\right)\right\}$$
$$= p^{2k\ell-(i+j)\ell-b}\exp\left\{2\ell\left(1-\frac{1}{p}\right)+O\left(k^{-9/10}\right)\right\}.$$

Next we estimate

$$h(k-j-b_c,k-i-b_c-2j) = \left(1+O\left(k^{-4/5}\right)\right)\frac{(k-j-b_c)!}{e}$$

by Fact 3. So the product in (2.8) is

$$\prod_{c=1}^{\ell}\binom{k-i}{b_c}\binom{k-b_c}{j}h(k-j-b_c,k-i-b_c-2j)$$
$$= \left(1+O\left(k^{-4/5}\right)\right)\prod_{c=1}^{\ell}\frac{(k-i)_{b_c}}{b_c!}\frac{(k-b_c)_j}{j!}\frac{(k-j-b_c)!}{e}$$
$$\leqslant \left(1+O\left(k^{-4/5}\right)\right)\left(\frac{k!}{ej!}\right)^{\ell}\prod_{c=1}^{\ell}\frac{1}{b_c!}. \tag{2.9}$$

The sum of terms in (2.8) corresponding to small i,j,b is at most

$$\left(1+O\left(k^{-4/5}\right)\right)\sum_{\substack{0\leqslant i,j,b\leqslant k^{1/10}\\ \vec{b}\in S_b}} p^{2k\ell-(i+j)\ell-b}\exp\left\{2\ell\left(1-\frac{1}{p}\right)\right\}\cdot\left(\frac{k!}{i!}\right)^{\ell}\left(\frac{k!}{ej!}\right)^{\ell}\prod_{c=1}^{\ell}\frac{1}{b_c!}$$
$$= \left(1+O\left(k^{-4/5}\right)\right)\exp\left\{\ell\left(1-\frac{2}{p}\right)\right\}\sum_{0\leqslant i,j,b\leqslant k^{1/10}} p^{2k\ell-(i+j)\ell-b}\left(\frac{k!}{i!}\right)^{\ell}\left(\frac{k!}{j!}\right)^{\ell}\frac{\ell^b}{b!}$$
$$\leqslant \left(1+O\left(k^{-4/5}\right)\right)(k!)^{2\ell}p^{2k\ell}\exp\left\{\ell\left(1-\frac{1}{p}\right)\right\}\sum_{0\leqslant i,j,b}\frac{p^{-i\ell}}{(i!)^{\ell}}\cdot\frac{p^{-j\ell}}{(j!)^{\ell}}\cdot\frac{(\ell/p)^b}{b!}$$
$$= \left(1+O\left(k^{-4/5}\right)\right)(k!)^{2\ell}p^{2k\ell}\exp\left\{\ell\left(1-\frac{1}{p}\right)\right\}f_\ell\left(\frac{1}{p}\right)^2$$
$$\sim \mathbb{E}[Y]^2,$$

where on the second-to-last line we have used

$$\sum_{0\leqslant i}\frac{p^{-i\ell}}{(i!)^{\ell}} = f_\ell\left(\frac{1}{p}\right), \qquad \sum_{0\leqslant b}\frac{(\ell/p)^b}{b!} = \exp\left\{\frac{\ell}{p}\right\}.$$

It remains to show that the sum of all other terms (i.e. terms where i, j, or b is at least $k^{1/10}$) is negligible compared to $\mathbb{E}[Y]^2$, which is of order $(k!)^{2\ell} p^{2k\ell}$. Note that by Fact 2

$$\frac{\binom{k^2\ell-(2k\ell-(i+j)\ell-b)}{m-(2k\ell-(i+j)\ell-b)}}{\binom{k^2\ell}{m}} = p^{2k\ell-(i+j)\ell-b} \exp\left\{\frac{(2k\ell-(i+j)\ell-b)^2}{2k^2\ell}\left(1-\frac{1}{p}\right)+O\left(\frac{1}{k}\right)\right\}$$
$$= O\left(p^{2k\ell-(i+j)\ell-b}\right).$$

Thus, the sum (over $\vec{b}$) of terms corresponding to a fixed triple (i, j, b) in line (2.8) big-O of

$$p^{2k\ell-(i+j)\ell-b} \sum_{\vec{b}\in S_b} \left(\frac{k!}{i!}\right)^{\ell} \prod_{c=1}^{\ell} \binom{k-i}{b_c}\binom{k-b_c}{j}(k-j-b_c)!$$
$$\leqslant p^{2k\ell-(i+j)\ell-b} \left(\frac{k!}{i!}\right)^{\ell}\left(\frac{k!}{j!}\right)^{\ell} \sum_{\vec{b}\in S_b}\prod_{c=1}^{\ell}\frac{1}{b_c!}$$
$$= \left((k!)^{2\ell}p^{2k\ell}\right)\cdot\left(\frac{\ell^b}{p^{(i+j)\ell+b}(i!)^{\ell}(j!)^{\ell}b!}\right).$$

It is easy to see that if i, j or b is at least $k^{1/10}$ then the second factor above is $\exp\left\{-\Omega\left(k^{1/10}\log k\right)\right\}$. Since the number of triples (i, j, b) is polynomial in k, the sum of all such terms (i.e. where i, j or b is at least $k^{1/10}$) is $o\left((k!)^{2\ell}p^{2k\ell}\right)$ which is a negligible contribution to $\mathbb{E}[Y^2]$. Therefore we have $\mathbb{E}[Y^2] \sim \mathbb{E}[Y]^2$.

3 Remarks and Open Problems

The reader should note that we did not use a "binomial" random construction (e.g. keep each edge of $D_{k,\ell}$ with probability p independently) because such a model lacks the concentration we need here. Indeed, for example Janson ([3]) showed that the number of perfect matchings in $G(n, p)$ is not concentrated even when it is quite large, while the number of perfect matchings of $G(n, m)$ is concentrated. We tried to use a binomial random construction and found that the second moments were too large, which in light of Janson's result makes sense (for example derangements in our graph are just a union of several perfect matchings on bipartite graphs).

There are still interesting open problems in [1]. In particular it is still open whether S, the set of possible ratios $(d/p)_G$, is equal to $\mathbb{Q} \cap [0, 1/2]$. Here we would like to pose another open problem that is mostly unrelated to our result. In particular, we ask about stability for digraphs whose ratio $(d/p)_G$ is close to $1/2$: is it possible that, if a graph has (d/p) in the interval $[1/2 - c, 1/2]$ for sufficiently small c, then it must be "nearly" a blowup of a t-cycle with $t > t(c)$?

References

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