Spectral radius, edge-disjoint cycles and cycles of the same length

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Abstract

In this paper, we provide spectral conditions for the existence of two edge-disjoint cycles and two cycles of the same length in a graph, which can be viewed as the spectral analogues of Erdős and Posa's condition and Erdős' classic problem about the maximum number of edges of a graph without two edge-disjoint cycles and two cycles of the same length, respectively. Furthermore, we give a spectral condition to guarantee the existence of k edge-disjoint triangles in a graph.

Mathematics Subject Classifications: 05C50, 05C35

1 Introduction

All graphs considered here are simple, finite and undirected. The study of cycles has a rich history in graph theory. A folklore result states that every graph with minimum degree at least 2 contains a cycle. On the other hand, a connected graph containing no cycles is a tree. Let Ω'_k denote the family of graphs with k edge-disjoint cycles, and $\overline{\Omega'_k}$ denote the family of graphs outside Ω'_k. In 1965, Erdős and Posa [11] proved the following theorem.

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Theorem 1 (Erdős and Posa [11]). *Let G be a graph on n vertices and m edges. If $G \in \overline{\Omega'_2}$, then $m \leqslant n+3$. Furthermore, if $G \in \overline{\Omega'_2}$ and $m = n+3$, then G is obtained from a subdivision G_0 of $K_{3,3}$ by adding a forest and exactly one edge, joining each tree of the forest to G_0.*

Bollobás in his classic book [2] raised a more general problem: What is the maximum number of edges of a graph $G \in \overline{\Omega'_k}$ of order n? Up to now, this problem is still widely open.

Another classic problem involves cycles of the same length. Let $f(n)$ be the maximum number of edges in a graph on n vertices without two cycles of the same length. Erdős posed the problem of determining $f(n)$, which was listed as one of 50 unsolved problems in the textbook of Bondy and Murty (see [1, p.247, Problem 11]). It has attracted the attentions of many scholars. In 1988, Shi [35] gave a lower bound $f(n) \geqslant n + \lfloor \frac{\sqrt{8n-15}-3}{2} \rfloor$ for $n \geqslant 3$, which was improved sequentially by Lai (see [23, 24]), including the current record [24] that $f(n) \geqslant n + 1.55\sqrt{n}$. Concerning the upper bound, Boros, Caro, Füredi and Yuster [4] obtained $f(n) < n + 1.98\sqrt{n}$, and further conjectured that every n-vertex 2-connected graph without two cycles of the same length satisfies $\lim\limits_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} = 1$. Very recently, Ma and Yang [27] confirmed their conjecture by showing that $f(n) < n + \sqrt{n} + o(n)$ for any n-vertex 2-connected graph. However, the exact value of $f(n)$ is still unknown. For more results related to this problem, we refer the reader to [10, 22, 24].

The main goal of this paper is to investigate the above problems from a spectral perspective. The eigenvalue conditions for cycles have been studied by a plenty of researchers (see [7, 25, 32, 41]). In 1995, Favaron, Mahéo and Saclé [12] proved that every graph on n vertices satisfying $\rho(G) > \rho(S_{n,1}) = \sqrt{n-1}$ contains a C_3 or a C_4. Generalizing this result, Nikiforov [31] conjectured that: (a) every graph of sufficiently large order n with $\rho(G) \geqslant \rho(S_{n,k})$ contains a C_{2k+1} or a C_{2k+2}, unless $G = S_{n,k}$. In the same paper, Nikiforov also conjectured that: (b) every graph of sufficiently large order n with $\rho(G) \geqslant \rho(S^+_{n,k})$ contains a C_{2k+2}, unless $G = S^+_{n,k}$. For $k = 2$, the conjectures (a) and (b) were confirmed in [37] and [38], respectively. For C_4, Nikiforov [30] and Zhai and Wang [40] characterized the extremal graphs for odd n and even n, respectively. For consecutive cycles, see Nikiforov [28], Ning and Peng [34], Zhai and Lin [39] and Li and Ning [19]. Very recently, confirming the starting case of a conjecture due to Bollobás and Nikiforov, Lin, Ning and Wu [20] obtained a new eigenvalue condition for triangles. Furthermore, they obtained a spectral analogue for a theorem of Erdős, which states that a non-bipartite graph G with $\rho(G) \geqslant \rho(S(K_{\lfloor\frac{n-1}{2}\rfloor,\lceil\frac{n-1}{2}\rceil}))$ contains a triangle unless $G \cong S(K_{\lfloor\frac{n-1}{2}\rfloor,\lceil\frac{n-1}{2}\rceil})$, where $S(K_{\lfloor\frac{n-1}{2}\rfloor,\lceil\frac{n-1}{2}\rceil})$ denotes a subdivision of $K_{\lfloor\frac{n-1}{2}\rfloor,\lceil\frac{n-1}{2}\rceil}$ on one edge. In the same paper, the authors [20] posed a more general problem: Which graphs attain the maximum spectral radius among all non-bipartite graphs with order n and odd girth at least $2k+3$? Recently, this problem has been independently solved by Li, Sun and Yu [25], and Lin and Guo [21]. There are also quite a lot of references on eigenvalues and long cycles, and we refer the reader to [13, 14, 19].

It seems that the investigation of cycles in terms of eigenvalues is closely related to extremal graph theory. For example, in order to solve an open problem raised by Caro and Yuster on degree powers of graphs with a forbidden even cycle, Nikiforov [29] proved an extension of the classical Erdős-Gallai theorem on cycles. Interestingly, Nikiforov also used it as a tool to study a spectral problem, showing that $\rho^2(G) - k\rho(G) \leqslant k(n-1)$ if G is C_{2k}-free. Only very recently, it is also shown that the spectral theorem on consecutive cycles [28] is a tool for studying some generalized Turán-type problems, such as estimating $\mathrm{ex}(n, P_l, C_{2k+1})$ (see [15]).

In this paper, we continue the project of studying cycles from spectral prospective. Compared with [20], we focus on eigenvalues conditions for edge-disjoint cycles and cycles of the same length.

A k-fan, denoted by F_k, is the graph obtained from k triangles by sharing a common vertex. As usual, we denote by K_r the complete graph on r vertices, C_r the cycle of length r, $S_{n,k}$ the graph obtained by joining $n-k$ isolated vertices with each vertex in K_k, and $K_{a,b}$ the complete bipartite graph with two parts of order a, b, respectively. Let $H \in \{C_r, K_r, K_{r,r}\}$. We denote by $H \bullet K_{a,b}$ the graph obtained by coalescing a vertex of $K_{a,b}$ belonging to the part of order a with a vertex in H and retaining the connection of edges in H and $K_{a,b}$. Let $K^+_{1,n-1}$ be the graph obtained by adding an edge within the independent set of $K_{1,n-1}$.

It is rather surprising that we find a spectral analogue of $f(n)$ and the extremal graph is also determined.

Theorem 2. *Let G be a graph of order $n \geqslant 26$. If $\rho(G) \geqslant \rho(K^+_{1,n-1})$, then G contains two cycles of the same length unless $G \cong K^+_{1,n-1}$.*

Moreover, we provide a spectral analogue for the result of Theorem 1.

Theorem 3. *Let G be a graph of order $n \geqslant 17$. If $\rho(G) \geqslant \rho(K_4 \bullet K_{1,n-4})$, then G contains two edge-disjoint cycles unless $G \cong K_4 \bullet K_{1,n-4}$.*

With regard to k edge-disjoint cycles, it is natural to propose the following problem.

Problem 4. What is the maximum spectral radius of a graph $G \in \overline{\Omega'_k}$ of order n?

However, it seems difficult to solve Problem 4, even for giving a conjecture on $k \geqslant 3$. Therefore, we turn to this problem in a special version, i.e., the Brualdi-Solheid-Turán type problem for k edge-disjoint triangles. Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-free if it does not contain any graph in $\mathcal{F}$ as a subgraph. The Turán number of $\mathcal{F}$, denoted by $\mathrm{ex}(n, \mathcal{F})$, is the maximum number of edges in an $\mathcal{F}$-free graph of order n. Let $\mathrm{EX}(n, \mathcal{F})$ be the family of $\mathcal{F}$-free graphs with $\mathrm{ex}(n, \mathcal{F})$ edges. In particular, if $\mathcal{F}$ contains exactly one element, say $\mathcal{F} = \{F\}$, then we denote $\mathrm{ex}(n, F) = \mathrm{ex}(n, \mathcal{F})$ and $\mathrm{EX}(n, F) = \mathrm{EX}(n, \mathcal{F})$. Let Γ_k be the family of graphs consisting of k-edge-disjoint triangles. Clearly, $F_k \in \Gamma_k$. Győri [17] determined the Turán number $\mathrm{ex}(n, \Gamma_k)$, and characterized the extremal graphs in $\mathrm{EX}(n, \Gamma_k)$.

Theorem 5. *([17]) Let G be a graph of sufficiently large order n that does not contain a subgraph belonging to Γ_k, $k \geqslant 1$. Then $e(G) \leqslant ex(n, \Gamma_k) = \lfloor \frac{n^2}{4} \rfloor + k - 1$, and the extremal graph is obtained from a complete bipartite graph with color classes of order $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ by embedding edges of size $k-1$.*

As usual, given a graph H, let $\mathrm{SPEX}(n, H)$ be the family of H-free graphs with the maximum spectral radius. Cioabă, Feng, Tait and Zhang [9] proved that $\mathrm{SPEX}(n, F_k) \subseteq \mathrm{EX}(n, F_k)$. To generalize their result, Li and Peng [26] showed that $\mathrm{SPEX}(n, H_{s,k}) \subseteq \mathrm{EX}(n, H_{s,k})$, where $H_{s,k}$ is the graph obtained from s triangles and k odd cycles of lengths at least 5 by sharing a common vertex. The odd wheel W_{2k+1} is the graph formed by joining a vertex to a cycle of length $2k$. Very recently, Cioabă, Desai and Tait [8] showed that $\mathrm{SPEX}(n, W_5) \subseteq \mathrm{EX}(n, W_5)$, and $\mathrm{SPEX}(n, W_{2k+1})$ ($k \geqslant 3$, $k \notin \{4, 5\}$) is obtained from a complete bipartite graph with parts L and R of order $\frac{n}{2} + s$ and $\frac{n}{2} - s$ with $|s| \leqslant 1$ by embedding a $(k-1)$-regular or nearly $(k-1)$-regular graph in $G[L]$ and exactly one edge in $G[R]$. Also, they posed the following conjecture for further research.

Conjecture 6. ([8]) Let F be any graph such that the graphs in $\mathrm{EX}(n, F)$ are Turán graphs plus $O(1)$ edges. Then $\mathrm{SPEX}(n, F) \subseteq \mathrm{EX}(n, F)$ for n large enough.

In this paper, we give a spectral version of Theorem 5, in which the extremal graph is completely characterized. This also provides a support for Conjecture 6.

Theorem 7. *Let $k \geqslant 2$, and let G be a Γ_k-free graph on n vertices with n sufficiently large. If G attains the maximum spectral radius, then*

$$G \in EX(n, \Gamma_k).$$

More precisely, G is obtained from $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ by embedding a graph H of order $k-1$ in the $\lfloor \frac{n}{2} \rfloor$-vertex partition set, where $H \cong C_3$ for $k = 4$ and $H \cong K_{1,k-1}$ otherwise.

2 Proof of Theorem 2

Before beginning our proof, we first give some notation not defined above. Let G be a graph, and let $d_G(v_i)$ be the degree of the vertex v_i in G. For $u, v \in V(G)$, denote by $d_G(u, v)$ the distance between u and v, i.e. the length of a shortest path between u and v. Set $N_G^d(u) = \{v \mid v \in V(G), d_G(v, u) = d\}$. Specially, we use $N_G(u)$ instead of $N_G^1(u)$ and $N_G[u] = \{v \mid v \in N_G(u)\} \cup \{u\}$. Denote by $E_G(V_1, V_2)$ the set of edges between V_1 and V_2, and denote $|E_G(V_1, V_2)|$ by $e_G(V_1, V_2)$. In particular, if $V_1 = V_2$, then we write $E_G(V_1, V_2)$ by $E_G(V_1)$ and $|E_G(V_1)|$ by $e_G(V_1)$. For the sake of simplicity, we shall omit all the subscripts if G is clear from the context.

An *internal path* of a graph is a path $v_1 v_2 \cdots v_k$ ($k \geqslant 2$), where $d_G(v_1), d_G(v_k) \geqslant 3$ and $d_G(v_i) = 2$ for $2 \leqslant i \leqslant k-1$ (unless $k = 2$). Denote by G_{uv} the graph obtained from G by subdividing the edge uv, that is, introducing a new vertex on the edge uv. Let Y_n be the graph obtained from a path $v_1 v_2 \cdots v_{n-4}$ by attaching two pendant vertices to v_1 and two pendant vertices to v_{n-4}. Hoffman and Smith [18] proved the following result, which is an important tool in spectral graph theory.

Lemma 8. ([18]) *Let G be a connected graph with $uv \in E(G)$. If uv belongs to an internal path of G and $G \ncong Y_n$, then $\rho(G_{uv}) < \rho(G)$.*

Now we are in a position to give the proof of Theorem 2.

Proof of Theorem 2. Let $G^\star$ be a graph with the maximum spectral radius among all graphs without two cycles with the same length. Then $G^\star$ is connected, since otherwise we can add some cut edges between the components of $G^\star$ such that the resulting graph is a connected graph containing no two cycles of the same length with larger spectral radius than $G^\star$. Let $Z = (z_1, z_2, \dots, z_n)^t$ be the Perron vector of $A(G^\star)$ and $z_{u^\star} = \max\{z_i : 1 \leqslant i \leqslant n\}$. Note that $K_{1,n-1}^+$ does not contain two cycles of the same length and $X = \left(\frac{\sqrt{2}}{2}, \frac{1}{\sqrt{2(n-1)}}, \dots, \frac{1}{\sqrt{2(n-1)}}\right)^t$ is the Perron vector of $K_{1,n-1}$. Obviously, $X \neq Z$. Then it follows that $\rho(G^\star) \geqslant \rho(K_{1,n-1}^+) > X^t A(K_{1,n-1}^+) X = \sqrt{n-1} + \frac{1}{n-1}$, which is equivalent to

$$\rho^2(G^\star) - \frac{2}{n-1}\rho(G^\star) + \frac{1}{(n-1)^2} - n + 1 > 0. \tag{1}$$

Now we begin to analyze the structure of $G^\star$. First we give the following two claims.

Claim 9. *For each vertex $u \in V(G^\star)\backslash\{u^\star\}$, u is not a cut vertex.*

Proof. By the way of contradiction, assume that u is a cut vertex. Then there exist two components of $G^\star\backslash\{u\}$, say G_1 and G_2. Without loss of generality, suppose that $u^\star \in V(G_1)$. Let G' be the graph obtained from $G^\star$ by deleting the edges between u and $N_{V(G_2)}(u)$ and adding the edges between $u^\star$ and $N_{V(G_2)}(u)$. Note that G' still does not contain two cycles of the same length, and

$$Z^t(\rho(G') - \rho(G^\star))Z \geqslant Z^t(A(G') - A(G^\star))Z = 2(z_{u^\star} - z_u) \sum_{v \in N_{V(G_2)}(u)} z_v \geqslant 0. \tag{2}$$

If $\rho(G') = \rho(G^\star)$, then $z_{u^\star} = z_u$, and Z is also the Perron vector of $A(G')$. On the other hand,

$$\rho(G')z_{u^\star} = \sum_{v \in N(u^\star)} z_v + \sum_{v \in N_{V(G_2)}(u)} z_v > \sum_{v \in N(u^\star)} z_v = \rho(G^\star)z_{u^\star}.$$

It follows that $\rho(G') > \rho(G^\star)$, a contradiction. □

Claim 10. *For any $u \in V(G^\star)$, we have $e(N(u)) \leqslant 1$ and $e(N(u), N^2(u)) \leqslant |N^2(u)| + 1$.*

Proof. By contradiction, assume that $e(N(u)) \geqslant 2$ or $e(N(u), N^2(u)) \geqslant |N^2(u)| + 2$. Then $G^\star$ contains two triangles or two C_4's, a contradiction. □

Let $N(u^\star) = \{v_{11}, v_{12}, \dots, v_{1q}\}$ and $N^2(u^\star) = \{v_{21}, v_{22}, \dots, v_{2t}\}$. For simplicity, we denote $d_{1i} = d_{N(u^\star)}(v_{1i})$ for $1 \leqslant i \leqslant q$ and $d_{2j} = d_{N(u^\star)}(v_{2j})$ for $1 \leqslant j \leqslant t$. Note that

$$\begin{aligned} &\rho^2(G^\star)z_{u^\star} - \frac{2}{n-1}\rho(G^\star)z_{u^\star} + (\frac{1}{(n-1)^2} - n + 1)z_{u^\star} \\ &= d(u^\star)z_{u^\star} + \sum_{i=1}^{q} d_{1i}z_{v_{1i}} + \sum_{j=1}^{t} d_{2j}z_{v_{2j}} - \frac{2}{n-1}\rho(G^\star)z_{u^\star} + (\frac{1}{(n-1)^2} - n + 1)z_{u^\star}, \end{aligned} \tag{3}$$

By (1), we have (3) > 0. On the other hand,

$$\begin{aligned}(3) &\leqslant z_{u^\star}(d(u^\star)+\sum_{i=1}^{q} d_{1i}+\sum_{j=1}^{t} d_{2j}-\frac{2}{n-1}\rho(G^\star)+\frac{1}{(n-1)^2}-n+1)\\ &= z_{u^\star}(d(u^\star)+2e(N(u^\star))+e(N(u^\star),N^2(u^\star))-\frac{2}{n-1}\rho(G^\star)+\frac{1}{(n-1)^2}-n+1).\end{aligned}$$

Thus we obtain

$$d(u^\star)+2e(N(u^\star))+e(N(u^\star),N^2(u^\star))-\frac{2}{n-1}\rho(G^\star)+\frac{1}{(n-1)^2}-n+1>0. \tag{4}$$

Let $C(u^\star)=V(G^\star)\backslash\{N[u^\star]\cup N^2(u^\star)\}$. By Claim 10,

$$e(N(u^\star),N^2(u^\star))\leqslant |N^2(u^\star)|+1=n-d(u^\star)-|C(u^\star)|.$$

Then (4) becomes

$$2e(N(u^\star))-|C(u^\star)|-\frac{2}{n-1}\rho(G^\star)+\frac{1}{(n-1)^2}+1>0. \tag{5}$$

Since $\rho(G^\star)>\sqrt{n-1}+\frac{1}{n-1}$, by (5), we see that $|C(u^\star)|=0$ if $e(N(u^\star))=0$ and $|C(u^\star)|\leqslant 2$ if $e(N(u^\star))=1$.

Note that $\rho(G^\star)z_{v_{2j}}=\sum\limits_{u\sim v_{2j}} z_u\leqslant d(v_{2j})z_{u^\star}$ and $\rho(G^\star)>\sqrt{n-1}\geqslant 5$ as $n\geqslant 26$. Then

$$\begin{aligned}(3) \leqslant& z_{u^\star}(d(u^\star)+\sum_{i=1}^{q} d_{1i}+\sum_{j=1}^{t} d_{2j}\frac{d(v_{2j})}{\rho(G^\star)}-\frac{2}{n-1}\rho(G^\star)+\frac{1}{(n-1)^2}-n+1)\\ =& z_{u^\star}(d(u^\star)+\sum_{i=1}^{q} d_{1i}+e(N(u^\star),N^2(u^\star))-\alpha-\frac{2}{n-1}\rho(G^\star)+\frac{1}{(n-1)^2}-n+1), \quad (6)\end{aligned}$$

where $\alpha=e(N(u^\star),N^2(u^\star))-\sum\limits_{j=1}^{t} d_{2j}\frac{d(v_{2j})}{\rho(G^\star)}$, that is, $\alpha=\sum\limits_{j=1}^{t} d_{2j}-\sum\limits_{j=1}^{t} d_{2j}\frac{d(v_{2j})}{\rho(G^\star)}$. Before proceeding, we need the following fact.

Fact 11. $5t-5\alpha<\sum_{j=1}^{t} d(v_{2j})\leqslant t+1+2e(N^2(u^\star))+e(N^2(u^\star),N^3(u^\star))$.

Proof. Recall that there is at most one vertex $v_{2j}\in N^2(u^\star)$ such that $d_{2j}=2$ and $d_{2j'}=1$ for any other vertex $v_{2j'}\in N^2(u^\star)$, that is, $t\leqslant\sum\limits_{j=1}^{t} d_{2j}\leqslant t+1$. This implies that $\sum\limits_{j=1}^{t} d(v_{2j})\leqslant t+1+2e(N^2(u^\star))+e(N^2(u^\star),N^3(u^\star))$. In the following, we only need to show that $\sum\limits_{j=1}^{t} d(v_{2j})>5t-5\alpha$. If $\sum\limits_{j=1}^{t} d_{2j}=t$, then $\alpha=t-\sum\limits_{j=1}^{t}\frac{d(v_{2j})}{\rho(G^\star)}$. If $\sum_{j=1}^{t} d_{2j}=t+1$, without loss of generality, suppose $d_{2t} = 2$. Then $d_{N^2(u^\star)}(v_{2t}) \leqslant 1$, since otherwise we can find two C_5's. Combining this with $|C(u^\star)| \leqslant 2$, we have $d(v_{2t}) \leqslant 5$. Then

$$\alpha = t - 1 - \sum_{j=1}^{t-1} \frac{d(v_{2j})}{\rho(G^\star)} + 2 - 2\frac{d(v_{2t})}{\rho(G^\star)} \geqslant t - 1 - \sum_{j=1}^{t-1} \frac{d(v_{2j})}{\rho(G^\star)} + 1 - \frac{d(v_{2t})}{\rho(G^\star)} = t - \sum_{j=1}^{t} \frac{d(v_{2j})}{\rho(G^\star)}.$$

Therefore, $\alpha \geqslant t - \sum_{j=1}^{t} \frac{d(v_{2j})}{\rho(G^\star)} > t - \frac{1}{5}\sum_{j=1}^{t} d(v_{2j})$, as required. □

Claim 12. $e(N(u^\star)) = 1$.

Proof. By the way of contradiction, assume that $e(N(u^\star)) = 0$. Recall that the only possible cut vertex is $u^\star$. Moreover, from (5), it follows that $|C(u^\star)| = 0$. Now we derive the proof by the following three cases.

Case 1. $e(N^2(u^\star)) \geqslant 3$. Without loss of generality, suppose that $e_1, e_2, e_3 \in E(G^\star[N^2(u^\star)])$. Clearly, each edge in $\{e_1, e_2, e_3\}$ is contained in some C_3 or C_5, which implies that $G^\star$ contains two C_3's or two C_5's, a contradiction.

Case 2. $1 \leqslant e(N^2(u^\star)) \leqslant 2$. Recall that $2e(N(u^\star)) = \sum_{i=1}^{q} d_{1i} = 0$ and $|C(u^\star)| = 0$. Then $e(N^2(u^\star), N^3(u^\star)) = 0$ and

$$(6) \leqslant z_{u^\star}(-\frac{2}{n-1}\rho(G^\star) + \frac{1}{(n-1)^2} + 1 - \alpha).$$

As $0 < (3) \leqslant (6)$, we have $-\frac{2}{n-1}\rho(G^\star) + \frac{1}{(n-1)^2} + 1 - \alpha > 0$, and hence $\alpha < 1$. Then by Fact 11,

$$5t - 5 < \sum_{j=1}^{t} d(v_{2j}) \leqslant t + 1 + 2e(N^2(u^\star)),$$

which is equivalent to $2|N^2(u^\star)| < e(N^2(u^\star)) + 3 \leqslant 5$, that is, $e(N^2(u^\star)) = 1$ and $|N^2(u^\star)| = 2$. Then $2|N^2(u^\star)| < e(N^2(u^\star)) + 3 = 4$, which contradicts the fact that $|N^2(u^\star)| = 2$.

Case 3. $e(N^2(u^\star)) = 0$. Combining $e(N(u^\star), N^2(u^\star)) \leqslant t + 1$ with the fact that $N(u^\star)$ contains no cut vertices, we have $t = |N^2(u^\star)| = 1$, and then $G^\star \cong C_4 \bullet K_{1,n-4}$. By Lemma 8, $\rho(C_4 \bullet K_{1,n-4}) < \rho(K^+_{1,n-2}) < \rho(K^+_{1,n-1})$, a contradiction. □

Since $e(N(u^\star)) = 1$, for every edge in $G^\star[V(G^\star)\backslash N[u^\star]]$ the two end-vertices of which have no common neighbors. Then each edge $e \in G^\star[N^2(u^\star)]$ is contained in some 5-cycle, and hence $e(N^2(u^\star)) \leqslant 1$. Now we characterize the structure of $G^\star$.

Claim 13. $G^\star \cong K^+_{1,n-1}$.

Proof. Recall that $|C(u^\star)| \leqslant 2$, and $V(G^\star)\backslash\{u^\star\}$ contains no cut vertices. It follows that $|N^4(u^\star)| = 0$ and $|N^3(u^\star)| \leqslant 2$. Note that $\rho(G) \cdot z_u = \sum_{v\sim u} z_v \leqslant d(u)z_{u^\star}$ and $\rho(G) > \sqrt{n-1} \geqslant 5$ as $n \geqslant 26$. Then $z_u < \frac{d(u)}{5} z_{u^\star}$ for each $u \in V(G^\star)\backslash\{u^\star\}$. We derive the proof by the following three cases.

Case 1. $|N^3(u^\star)| = 2$. Let $N^3(u^\star) = \{u_1, u_2\}$. Recall that $e(N(u^\star), N^2(u^\star)) \leqslant n - d(u^\star) - |C(u^\star)| = t + 1$. We assert that $e(N(u^\star), N^2(u^\star)) = t + 1$. If not, then $e(N(u^\star), N^2(u^\star)) = t = n - d(u^\star) - |C(u^\star)| - 1$ and (4) becomes $-\frac{2}{n-1}\rho(G^\star) + \frac{1}{(n-1)^2} < 0$, a contradiction. It implies that $u^\star$ is contained in a 4-cycle. Therefore, for each $u \in N^3(u^\star)$, if $w, v \in N^2(u^\star)$ are two neighbors of u, then $N_{N(u^\star)}(w) \neq N_{N(u^\star)}(v)$. It follows that $e(N^2(u^\star), N^3(u^\star)) \leqslant 3$ (otherwise, there are two C_6's) and $u_1u_2 \in E(G^\star)$ since $N^2(u^\star)$ contains no cut vertices. By the above discussion, we have $\sum_{i=1}^{q} d_{1i} = 2e(N(u^\star)) = 2$, $e(N^2(u^\star), N(u^\star)) = t + 1$, $e(N^2(u^\star)) \leqslant 1$ and $e(N^2(u^\star), N^3(u^\star) \leqslant 3$. Then

$$(6) = z_{u^\star}(-\frac{2}{n-1}\rho(G^\star) + \frac{1}{(n-1)^2} + 1 - \alpha).$$

Combining the fact with $0 < (3) \leqslant (6)$, we have $\alpha < 1$. By Fact 11, we have $5t - 5 < t + 6$, that is, $t \leqslant 2$. Then $t = 2$ since $N^2(u^\star)$contains no cut vertices, and $e(N^2(u^\star)) \neq 1$ since $u^\star$ belongs to a 4-cycles. Since u_1, u_2 have no common neighbors in $N^2(u^\star)$, we obtain $e(N^2(u^\star), N^3(u^\star)) = 2$. Again by Fact 11, we have $5t - 5 < t + 3$, that is, $t \leqslant 1$, contrary to $t = 2$.

Case 2. $|N^3(u^\star)| = 1$. Let $N^3(u^\star) = \{u\}$. If $d(u) \geqslant 3$, then $G^\star$ contains two C_4's or two C_6's, which is impossible. Note that $u^\star$ is the unique possible cut vertex. Thus $d(u) = 2$. Without loss of generality, set $N(u) = \{v_{21}, v_{22}\}$. Then $v_{21}v_{22} \notin E(G^\star)$ and $d_{21} = d_{22} = 1$ (otherwise, $G^\star$ contains two C_4's or two C_6's). Thus $d(v_{21}) + d(v_{22}) \leqslant 5$ since $e(N^2(u^\star)) \leqslant 1$. Let $G_1 = G^\star - \{uv_{21}, uv_{22}\} + \{u^\star u\}$. Recall that $z_{v_{2i}} < \frac{d(u)}{5} z_{u^\star}$ for $i = 1, 2$. Then

$$Z^t(\rho(G_1) - \rho(G^\star))Z = 2(z_{u^\star} z_u - z_u(z_{v_{21}} + z_{v_{22}})) > 2z_{u^\star} z_u(1 - \frac{d(v_{21}) + d(v_{22})}{5}) \geqslant 0,$$

which implies that $\rho(G_1) > \rho(G^\star)$, a contradiction.

Case 3. $|N^3(u^\star)| = 0$. Note that $e(N^2(u^\star)) \leqslant 1$ and $e(N(u^\star), N^2(u^\star)) \leqslant t + 1$. Then $t \leqslant 3$ since $N(u^\star)$ contains no cut vertics. If $t = 1$ or $t = 3$, then there exists a vertex $\{v_{21}\} \in N^2(u^\star)$ such that $d(v_{21}) = 2$ and let $N(v_{21}) = \{v_{11}, v_{12}\} \subseteq N(u^\star)$. Besides, $v_{11}v_{12} \notin E(G^\star)$. It is clear that $d(v_{11}) + d(v_{12}) \leqslant 5$ for $t = 1$. Moreover, $d(v_{11}) + d(v_{12}) \leqslant 5$ also holds for $t = 3$ since otherwise there are two C_5's. Let $G_2 = G^\star - \{v_{21}v_{11}, v_{21}v_{12}\} + \{u^\star v_{21}\}$. Similarly as in Case 2, we have $\rho(G_2) > \rho(G^\star)$, a contradiction. If $t = 2$, say $\{v_{21}, v_{22}\} \subseteq N^2(u^\star)$. Since no vertex in $N(u^\star)$ is a cut vertex and v_{21}, v_{22} have no common neighbor, we have $v_{21}v_{22} \in E(G^\star)$. It follows that $d_{21} = d_{22} = 1$ (otherwise, there are two C_5's), that is, $d(v_{21}) = d(v_{22}) = 2$. Let $N_{N(u^\star)}(v_{21}) = v_{11}$. Then $d(v_{11}) \leqslant 3$ and $d(v_{11}) + d(v_{22}) \leqslant 5$. Let $G_3 = G^\star - \{v_{21}v_{11}, v_{21}v_{22}\} + \{u^\star v_{21}\}$. Similarly as in Case 2, we have $\rho(G_3) > \rho(G^\star)$, a contradiction. Therefore, $t = 0$, i.e., $G^\star \cong K_{1,n-1}^+$. □

To sum up, we complete the proof. □

3 Proof of Theorems 3 and 7

In this section, we shall prove Theorems 3 and 7. The following is a well-known result in spectral graph theory.

Lemma 14. *(Brouwer and Haemers [5, p. 30]; Godsil and Royle [16, pp. 196–198].) Let A be a real symmetric matrix, and let B be an equitable quotient matrix of A. Then the eigenvalues of B are also eigenvalues of A. Furthermore, if A is nonnegative and irreducible, then*

$$\rho(A) = \rho(B).$$

3.1 Proof of Theorem 3

Now, we give the proof of Theorem 3.

Proof of Theorem 3. Suppose that $G^\star$ is a graph attaining the maximum spectral radius among all graphs of order n without two edge-disjoint cycles. Clearly, $G^\star$ is connected. Since the graph $K_4 \bullet K_{1,n-4}$ does not contain two edge-disjoint cycles, we have $\rho(G^\star) \geqslant \rho(K_4 \bullet K_{1,n-4})$.

Let $X = (\frac{\sqrt{2}}{2}, \frac{1}{\sqrt{2(n-1)}}, \ldots, \frac{1}{\sqrt{2(n-1)}})^t$ be the Perron vector of $K_{1,n-1}$. Note that X cannot be the Perron vector of $K_4 \bullet K_{1,n-4}$. Then, by the Rayleigh quotient, we obtain

$$\rho(G^\star) \geqslant \rho(K_4 \bullet K_{1,n-4}) > X^t A(K_4 \bullet K_{1,n-4})X = \sqrt{n-1} + \frac{3}{n-1},$$

which implies that

$$\rho^2(G^\star) > n - 1 + \frac{6}{\sqrt{n-1}} + \frac{9}{(n-1)^2}. \tag{7}$$

Let $Y = (y_{v_1}, y_{v_2}, \ldots, y_{v_n})^t$ be the unique positive eigenvector of $\rho(G^\star)$ with $\max\{y_{v_i} : 1 \leqslant i \leqslant n\} = 1$, and let $u^\star \in V(G^\star)$ be such that $y_{u^\star} = 1$. Similarly as in the proof of Theorem 2, $u^\star$ is the only possible cut vertex.

Claim 15. *Aside from some triangles, there are no internal paths in $G^\star$.*

Proof. By Lemma 8, contracting an edge in an internal path will increase the spectral radius but not increase the number of edge-disjoint cycles. The claim follows. □

Claim 16. *Each vertex of $V(G^\star)\backslash\{u^\star\}$ has degree at most* 3.

Proof. By contradiction, assume that there exists some $v \in V(G^\star)\backslash\{u^\star\}$ such that $d(v) \geqslant 4$. Let u_1, u_2, u_3, u_4 be four neighbors of v. By Claim 9, v is not a cut vertex. Thus, there is a path P_1 from u_1 to u_4 with $v \notin V(P_1)$. We call it a (u_1, u_4)-path. Similarly, we can find a (u_2, u_3)-path P_2 with $v \notin V(P_2)$. Let $|V(P_1) \cap V(P_2)| = k$. Whether $k \geqslant 2$, $k = 1$ or $k = 0$ (see Figure. 1), we can always find two edge-disjoint cycles, a contradiction. □

Suppose that $N(u^\star) = \{v_{11}, v_{12}, \ldots, v_{1q}\}$ and $N^2(u^\star) = \{v_{21}, v_{22}, \ldots, v_{2t}\}$. For simplicity, let $d_{1i} = d_{G^\star[N(u^\star)]}(v_{1i})$ $(1 \leqslant i \leqslant q)$ and $d_{2j} = d_{G^\star[N(u^\star)]}(v_{2j})$ $(1 \leqslant j \leqslant t)$. In order to characterize the structure of $G^\star$, we need the following claim.

Claim 17. $e(N(u^\star)) \geqslant 1$

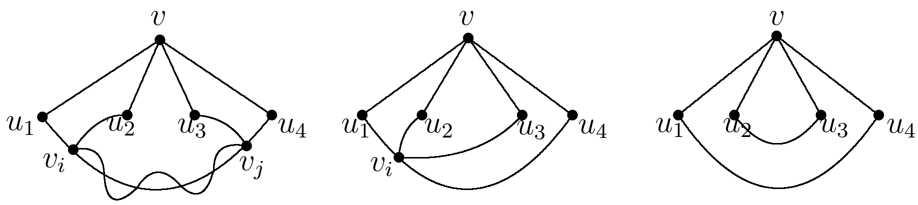

Figure 1: All cases of (u_1, u_4)-path and (u_2, u_3)-path.

Proof. Suppose to the contrary that $e(G^\star[N(u^\star)]) = 0$. Let $B_1, B_2, \dots, B_l$ be the components of $V(G^\star)\backslash N[u^\star]$. Clearly, $l \geqslant 1$ because $G^\star \ncong K_{1,n-1}$ due to $\rho(G^\star) > \rho(K_{1,n-1})$. By Claim 9, each vertex of $V(G^\star)\backslash\{u^\star\}$ is not a cut vertex. Then $|N_{N(u^\star)}(B_i)| \geqslant 2$, and $G^\star[N[u^\star] \cup B_i]$ contains a cycle for all $1 \leqslant i \leqslant l$. We derive the proof by the following two cases.

Case 1. $l \geqslant 2$. We claim that $G^\star[B_i]$ is a tree for $1 \leqslant i \leqslant l$. If not, assume that $G^\star[B_j]$ contains a cycle, say C_1 for some $j \in [1, l]$. Then C_1 together with a cycle in $G^\star[N[u^\star] \cup B_i]$ $(i \neq j)$ are two edge-disjoint cycles in $G^\star$, a contradiction. Similarly, we have $e(v, B_i) \leqslant 1$ for each vertex $v \in N(u^\star)$. Then we claim that $|B_i| = 1$ for $1 \leqslant i \leqslant l$. In fact, if $|B_j| \geqslant 2$ for some $j \in [1, l]$, then there are at least two leaves in B_j, say v_{21} and v_{22}. Note that $N_{N(u^\star)}(v_{21}) \cap N_{N(u^\star)}(v_{22}) = \varnothing$. Moreover, by Claims 9 and 15, we have $|N_{N(u^\star)}(v_{21})| = |N_{N(u^\star)}(v_{22})| = 2$. Without loss of generality, assume that $N_{N(u^\star)}(v_{21}) = \{v_{11}, v_{12}\}$ and $N_{N(u^\star)}(v_{22}) = \{v_{13}, v_{14}\}$. Then $u^\star v_{11} v_{21} v_{12} u^\star$ and $u^\star v_{13} v_{22} v_{14} u^\star$ are two edge-disjoint cycles, a contradiction. Hence, by Claim 2, $e(N(u^\star), B_i) = 3$ for $1 \leqslant i \leqslant l$. It follows that $l = 2$, since otherwise one can easily find two edge-disjoint cycles. Then $G^\star \cong K_{3,3} \bullet K_{1,n-6}$. Note that

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ n-6 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 2 & 0 \end{bmatrix},$$

is an equitable quotient matrix of $A(K_{3,3} \bullet K_{1,n-6})$, and the characteristic polynomial of B is $f(x) = x^4 - (n+3)x^2 + 6n - 36$. By Lemma 14,

$$\rho^2(G^\star) = \rho^2(B) = \frac{n + 3 + \sqrt{(n+3)^2 - 4(6n-36)}}{2} < n - 1$$

as $n \geqslant 17$, a contradiction.

Case 2. $l = 1$. We claim that $|B_1| \geqslant 2$. Otherwise, suppose that $B_1 = \{v\}$, then the vertex $w \in N(v)$ lies in an internal path, a contradiction. If $|B_1| = 2$, then $G^\star \cong G_1$ (as shown in Figure. 2). Let $G_2 = G_1 - \{v_{11}v_{21}, v_{11}v_{22}\} + \{v_{21}u^\star, v_{22}u^\star\}$ (see Figure. 2). Note that G_2 does not contain two edge-disjoint cycles. However,

$$Y^t(\rho(G_2) - \rho(G^\star))Y \geqslant 2(y_{v_{21}} + y_{v_{22}})(y_{u^\star} - y_{v_{11}}) > 0$$

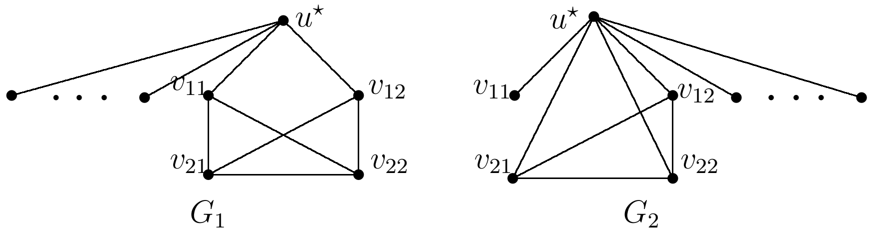

Figure 2: The graphs G_1 and G_2.

since $y_{v_{11}} \leqslant \frac{3y_{u^\star}}{\rho(G^\star)} < y_{u^\star}$ as $n \geqslant 17$. It follows that $\rho(G^\star) < \rho(G_2)$, a contradiction. Thus we may assume that $|B_1| \geqslant 3$. Note that

$$\rho^2(G^\star) = \rho^2(G^\star)y_{u^\star} = d(u^\star) + \sum_{i=1}^{q} d_{1i}y_{v_{1i}} + \sum_{j=1}^{t} d_{2j}y_{v_{2j}}.$$

According to inequality (7) and the assumption that $e(N(u^\star)) = 0$, we have

$$n - 1 + \frac{6}{\sqrt{n-1}} + \frac{9}{(n-1)^2} < d(u^\star) + \sum_{j=1}^{t} d_{2j}y_{2j} \leqslant d(u^\star) + e(N(u^\star), B_1),$$

which gives that $e(N(u^\star), B_1) > n - 1 - d(u^\star) = |B_1|$. Then there exists some $v_{21} \in B_1$ such that $d_{21} \geqslant 2$. Hence, $G^\star[B_1]$ is a tree. Otherwise, a cycle in $G^\star[B_1]$ together with a cycle in $G^\star[v_{21} \cup N[u^\star]]$ are two edge-disjoint cycles, a contradiction. By Claims 9 and 16, we have $2 \leqslant d(v) \leqslant 3$ for each $v \in V(B_1)$. Set $S = \{v \in V(B_1) | d(v) = 2\}$. By Claim 15, each vertex of S belongs to some triangle of $G^\star$. We aim to show that $|S| = 0$. By contradiction, suppose that $v_{22} \in S$, then $v_{21}v_{22} \notin E(G^\star)$ since $|B_1| \geqslant 3$ and $d(v_{21}) = 3$. Thus the 4-cycle from $G^\star[N[u^\star] \cup \{v_{21}\}]$ together with the triangle containing v_{22} are two edge-disjoint cycles, a contradiction. Therefore, $e(N(u^\star), B_1) = 3|B_1| - 2(|B_1| - 1) = |B_1| + 2 \geqslant 5$. By Claim 16, $|N_{N(u^\star)}(B_1)| \geqslant 3$. Suppose that $\{v_{11}, v_{12}, v_{13}\} \subseteq N(B_1)$ and $N_{N(u^\star)}(v_{21}) = \{v_{11}, v_{12}\}$. Since $e(v_{13}, B_1) = 2$ by Claims 15 and 16, the cycle $v_{21}v_{11}u^\star v_{12}v_{21}$ together with a cycle in $G^\star[B_1 \cup \{v_{13}\}]$ are two edge-disjoint cycles, a contradiction. □

According to Claim 17, $e(G^\star[N(u^\star)]) \geqslant 1$. Now we shall prove that $V(G^\star) = N[u^\star]$. If not, suppose that $B_1, B_2, \ldots, B_l$ are components of $G^\star - N[u^\star]$. Then each $G^\star[B_i]$ is a tree and $e(v, B_i) \leqslant 1$ for each $v \in N(u^\star)$. Furthermore, we claim that $|B_i| = 1$ for $i = 1, \ldots, l$, since otherwise there are two leaves in $G^\star[B_i]$, and we can find two edge-disjoint C_4's in $G^\star$, a contradiction. If $l \geqslant 2$, we set $B_1 = \{u\}$ and $B_2 = \{v\}$. If $|N(u) \cap N(v)| \geqslant 2$, then a triangle in $G^\star[N[u^\star]]$ together with a C_4 in $G^\star[N(u^\star) \cup \{u, v\}]$ are two edge-disjoint cycles, a contradiction. If $|N(u) \cap N(v)| \leqslant 1$, then we also can find two edge-disjoint cycles since $d(u), d(v) \geqslant 2$, a contradiction. Therefore, $l = 1$. Set $B_1 = \{u\}$. Then $2 \leqslant d(u) \leqslant 3$. If $d(u) = 3$ and $N(u) = \{v_{11}, v_{12}, v_{13}\}$, again by Claims 15 and 16, we have $d(v_{1i}) = 3$ for $i = 1, 2, 3$, and hence $G^\star$ contains two edge-disjoint triangles, a contradiction. If

$d(u) = 2$ and $N(u) = \{v_{11}, v_{12}\}$, by Claim 15, we have $v_{11}v_{12} \in E(G^\star)$, then $G^\star + u^\star u$ does not contain two edge-disjoint cycles but $\rho(G^\star + u^\star u) > \rho(G^\star)$, a contradiction. Note that $e(N(u^\star)) \leqslant 3$. Then by the maximality of $\rho(G^\star)$, we have $G^\star \cong K_4 \bullet K_{1,n-4}$. This completes the proof of Theorem 3. □

3.2 Proof of Theorem 7

First we list some lemmas, which are useful in the proof of Theorem 7. Let $k_r(G)$ denote the number of r-cliques in G. In 2007, Bollobás and Nikiforov [3] gave the following result.

Lemma 18. *([3]) Let G be a graph and $r \geqslant 2$. Then*

$$\rho^{r+1}(G) \leqslant (r+1)k_{r+1}(G) + \sum_{s=2}^{r}(s-1)k_s(G)\rho(G)^{r+1-s}.$$

The special case for $r = 2$ was also proved in [9]. Since we will frequently use the result, we cite it as our lemma.

Lemma 19. *Let G be a graph, and let t denote the number of triangles in G. Then*

$$e(G) \geqslant \rho^2(G) - \frac{3t}{\rho(G)}.$$

It is worth mentioning that Nikiforov [33] gave a strengthened result for the number of triangles.

Lemma 20. *(Cioabă, Feng, Tait and Zhang, [9]) Let G be an F_k-free graph of order n. For sufficiently large n, if G has the maximal spectral radius, then*

$$G \in EX(n, F_k).$$

Denote by $\beta(G)$ and $\Delta(G)$ the matching number and maximum degree of G, respectively. For any two positive integers β and Δ, we define $f(\beta,\Delta) = \max\{|E(G)| : \beta(G) \leqslant \beta, \Delta(G) \leqslant \Delta\}$. Chvátal and Hanson [6] proved the following result.

Lemma 21 (Chvátal and Hanson [6])**.** *For positive integers β, $\Delta \geqslant 1$,*

$$f(\beta,\Delta) = \Delta\beta + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\beta}{\lceil \Delta/2 \rceil} \right\rfloor \leqslant \Delta\beta + \beta.$$

Lemma 22. *([36]) Let G be a connected graph and let X be the eigenvector corresponding to $\rho(G)$. If $x_u \geqslant x_v$, and let $G' = G - \{vw | w \in N(v)\backslash N(u)\} + \{uw | w \in N(v)\backslash N(u)\}$, then $\rho(G) < \rho(G')$.*

Denote by $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor} \diamond K_{1,k-1}$ the graph obtained by embedding a copy of $K_{1,k-1}$ in the part of order $\lfloor \frac{n}{2} \rfloor$ in $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. Now we are in a position to give the proof of Theorem 7.

Proof of Theorem 7. Assume that $G^\star$ is a graph attaining the maximum spectral radius among all graphs of order n containing no Γ_k. Clearly, $G^\star$ is connected. Let us first prove that $G^\star \in \mathrm{EX}(n, \Gamma_k)$. Since the proof method is almost the same as the one in Lemma 20 (cf. [9]), we omit some details.

Claim 23. *$G^\star \in EX(n, \Gamma_k)$.*

Proof. Note that $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor} \diamond K_{1,k-1}$ is Γ_k-free. By the maximality of $\rho(G^\star)$, we have $\rho(G^\star) \geqslant \rho(K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor} \diamond K_{1,k-1}) \geqslant \frac{2}{n} e(K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor} \diamond K_{1,k-1}) = \frac{2}{n}(\lfloor \frac{n^2}{4} \rfloor + k - 1) > \frac{n}{2}$. Also, since $G^\star$ is Γ_k-free, F_k cannot be a subgraph of $G^\star$, i.e., for any $v \in V(G^\star)$, $G^\star[N(v)]$ contains no kK_2. Let t be the number of triangles in $G^\star$. By Lemma 21,

$$3t = \sum_{v \in V(G^\star)} e(G^\star[N(v)]) \leqslant \sum_{v \in V(G^\star)} \mathrm{ex}(d(v), kK_2) \leqslant \sum_{v \in V(G^\star)} \mathrm{ex}(n, kK_2) \leqslant \sum_{v \in V(G^\star)} kn = kn^2,$$

which gives that $t \leqslant \frac{kn^2}{3}$. Therefore, by Lemma 19, we have $e(G^\star) \geqslant \rho^2(G^\star) - \frac{6t}{n} \geqslant \rho^2(G^\star) - 2kn > \frac{n^2}{4} - 2kn$. Let ε and δ be fixed positive constants with $\delta < \frac{1}{10(k+1)^2}$, $\varepsilon < \frac{\delta^2}{16}$. Note that $\mathrm{ex}(n, \Gamma_k) = \lfloor \frac{n^2}{4} \rfloor + k - 1 \leqslant \mathrm{ex}(n, F_k) \leqslant \lfloor \frac{n^2}{4} \rfloor + k^2 - k$. Then, according to the proof in [9, Lemma 10], we have the following fact.

Fact 24. *The graph $G^\star$ has a partition $V(G^\star) = S \cup T$ which gives a maximum cut such that $e(S, T) \geqslant (\frac{1}{4} - \varepsilon) n^2$. Furthermore, $\left(\frac{1}{2} - \sqrt{\varepsilon}\right) n \leqslant |S|, |T| \leqslant \left(\frac{1}{2} + \sqrt{\varepsilon}\right) n$.*

Let $L = \left\{ v \in V(G^\star) : d(v) \leqslant \left(\frac{1}{2} - \frac{1}{4(k+1)}\right) n \right\}$ and $W = \{v \in S : d_S(v) \geqslant \delta n\} \cup \{v \in T : d_T(v) \geqslant \delta n\}$. We assert that $|L| < 16k^2$. If not, there exists some $L' \subseteq L$ with $|L'| = 16k^2$. Then

$$\begin{aligned} e(G^\star - L') \geqslant e(G^\star) - \sum_{v \in L'} d(v) &\geqslant \frac{n^2}{4} - 2kn - 16k^2 \left(\frac{1}{2} - \frac{1}{4(k+1)} \right) n \\ &> \frac{(n - 16k^2)^2}{4} + k - 1 \end{aligned}$$

for sufficiently large n. By Theorem 5, $G^\star - L'$ contains k edge-disjoint triangles, and so is $G^\star$, a contradiction. Combining this with Fact 24, as in [9, Lemmas 13–17], we can prove the following results successively:

- $|W| < \frac{2\varepsilon}{\delta} n + \frac{2k^2}{\delta n}$ and $W \subseteq L$;
- $L = \emptyset$, and both $G^\star[S]$ and $G^\star[T]$ are $K_{1,k}$- and kK_2-free;
- $e(G^\star) \geqslant \frac{n^2}{4} - 12k^2$, $\frac{n}{2} - 4k \leqslant |S|, |T| \leqslant \frac{n}{2} + 4k$, and $\frac{n}{2} - 14k^2 \leqslant \delta(G^\star) \leqslant \rho(G^\star) \leqslant \Delta(G^\star) \leqslant \frac{n}{2} + 5k$;
- For any $u \in V(G^\star)$, $\mathbf{x}_u \geqslant 1 - \frac{120k^2}{n}$;
- $\big||S| - |T|\big| \leqslant 1$.

Now we prove that $G^\star \in \mathrm{EX}(n,\Gamma_k)$. By contradiction, assume that $e(G^\star) \leqslant \mathrm{ex}(n,\Gamma_k)-1$. By Theorem 5, every graph in $\mathrm{EX}(n,\Gamma_k)$ has a maximum cut (S,T) of size $\lfloor n^2/4 \rfloor$. Since $\big||S|-|T|\big| \leqslant 1$, there exists some $H \in \mathrm{EX}(n,\Gamma_k)$ with $V(H)=V(G^\star)$ such that the edges between S and T in H form a complete bipartite graph. Let $E_+ = E(H)\setminus E(G^\star)$ and $E_- = E(G^\star)\setminus E(H)$. Then $(E(G^\star)\cup E_+)\setminus E_- = E(H)$, and $|E_+| \geqslant |E_-|+1$ because $|E(G^\star)\cap E(H)| + |E_-| = e(G^\star) < e(H) = |E(G^\star)\cap E(H)| + |E_+|$. Also note that $|E_-| \leqslant e(S)+e(T) < 2k^2$. Thus,

$$\begin{aligned}
\rho(H) &\geqslant \frac{\mathbf{x}^t A(H)\mathbf{x}}{\mathbf{x}^t\mathbf{x}} \\
&= \rho(G^\star) + \frac{2}{\mathbf{x}^t\mathbf{x}} \sum_{ij\in E_+} \mathbf{x}_i\mathbf{x}_j - \frac{2}{\mathbf{x}^t\mathbf{x}} \sum_{ij\in E_-} \mathbf{x}_i\mathbf{x}_j \\
&\geqslant \rho(G^\star) + \frac{2}{\mathbf{x}^t\mathbf{x}} \left(|E_+| \left(1-\frac{120k^2}{n}\right)^2 - |E_-| \right) \\
&= \rho(G^\star) + \frac{2}{\mathbf{x}^t\mathbf{x}} \left(|E_+| - |E_-| - \frac{240k^2}{n}|E_+| + \frac{(120k^2)^2}{n^2}|E_+| \right) \\
&\geqslant \rho(G^\star) + \frac{2}{\mathbf{x}^t\mathbf{x}} \left(1 - \frac{240k^2}{n}|E_+| + \frac{(120k^2)^2}{n^2}|E_+| \right) \\
&> \rho(G^\star),
\end{aligned}$$

contrary to the maximality of $\rho(G^\star)$. Therefore, we conclude that $e(G^\star) = \mathrm{ex}(n,\Gamma_k)$. $\square$

By Claim 23 and Theorem 5, $G^\star$ has a partition $V(G^\star) = S\cup T$ with $|S| = s = \lfloor \frac{n}{2} \rfloor$ and $|T| = t = \lceil \frac{n}{2} \rceil$ such that the edges between S and T form a complete bipartite graph and $e(S)+e(T) = k-1$. Now we focus on characterizing the exact structure of $G^\star$.

Let $X = (x_v : v \in V(G^\star))^t$ be the Perron vector of $G^\star$. Set $S = \{u_0, u_1, \ldots, u_{s-1}\}$ and $T = \{v_0, v_1, \ldots, v_{t-1}\}$. Without loss of generality, suppose that $x_{u_0} \geqslant x_{u_1} \geqslant \cdots \geqslant x_{u_{s-1}}$ and $x_{v_0} \geqslant x_{v_1} \geqslant \cdots \geqslant x_{v_{t-1}}$. We first assert that

$$N_S(u_j) \subseteq N_S[u_i] \ \text{ for } \ 0 \leqslant i < j \leqslant s-1, \quad N_T(v_j) \subseteq N_T[v_i] \ \text{ for } \ 0 \leqslant i < j \leqslant t-1. \tag{8}$$

Without loss of generality, we only prove the former by the way of contradiction. Suppose that there exist i,j with $i<j$ such that $N_S(u_j) \nsubseteq N_S[u_i]$. Let $u \in N_S(u_j)\setminus N_S[u_i]$ and $G' = G^\star - uu_j + uu_i$. By Lemma 22, $\rho(G') > \rho(G^\star)$. This is a contradiction, and hence (8) follows. According to (8), we have $N_S(u_j) \subseteq N_S[u_0]$ for all $j \in [1, s-1]$ (resp. $N_T(v_j) \subseteq N_T[v_0]$ for all $j \in [1, t-1]$), and it follows that $E(S) = E(N_S[u_0])$ (resp. $E(T) = E(N_T[v_0])$). Similarly, we can deduce that $E(N_S(u_0)) = E(N_S[u_1])$ and $E(N_T(v_0)) = E(N_T[v_1])$.

In the remaining part of the proof, we will frequently construct a Γ_k-free graph G' from $G^\star$ by modifying some edges. For convenience, we always use $Y = (y_v : v \in V(G'))^t$ to denote the Perron vector of G'. Also, we write $\rho' = \rho(G')$ and $\rho = \rho(G^\star)$ for short.

Let $s_0 = d_S(u_0)$ and $t_0 = d_T(v_0)$. Clearly, $e(S) \geqslant s_0$ and $e(T) \geqslant t_0$. In the following, we shall analyze the structure of $G^\star$ according to the structure of $G^\star[S]$ and $G^\star[T]$. Before proceeding, we need the following two claims.

Claim 25. *$e(S) \in \{s_0, s_0+1\}$ and $e(T) \in \{t_0, t_0+1\}$.*

Proof. We first prove that $e(S) \in \{s_0, s_0+1\}$. By contradiction, suppose that $e(S) \geqslant s_0+2$. Let $E^\star = E(N_S(u_0))$. Since $e(S) = e(N_S[u_0]) \geqslant d_S(u_0)+2$, we have $|E^\star| \geqslant 2$. Let $E' = \{u_0u_i | s_0+1 \leqslant i \leqslant e(S)\}$ and $G' = G^\star - E^\star + E'$. Then $|E'| = e(S) - s_0 = |E^\star|$, $G'[S] \cong K_{1,e(S)} \cup (s-e(S)-1)K_1$, and G' is Γ_k-free. Recall that X and Y are the Perron vectors of $G^\star$ and G', respectively. Then

$$Y^t(\rho' - \rho)X = Y^t(A(G') - A(G^\star))X = \sum_{i=s_0+1}^{e(S)} (x_{u_0}y_{u_i} + y_{u_0}x_{u_i}) - \sum_{u_iu_j \in E^\star} (x_{u_i}y_{u_j} + x_{u_j}y_{u_i}).$$

Observe that $x_{u_i} = x_{u_j}$ for $i, j \in [s_0+1, e(S)]$ and $y_{u_i} = y_{u_j}$ for $i, j \in [1, e(S)]$. Then

$$\sum_{i=s_0+1}^{e(S)} (x_{u_0}y_{u_i} + y_{u_0}x_{u_i}) = |E^\star|(x_{u_0}y_{u_1} + y_{u_0}x_{u_{s_0+1}})$$

and

$$\sum_{u_iu_j \in E^\star} (x_{u_i}y_{u_j} + x_{u_j}y_{u_i}) \leqslant |E^\star|y_{u_1}(x_{u_1} + x_{u_2}) \leqslant 2|E^\star|x_{u_0}y_{u_1}.$$

It follows that $Y^t(\rho'-\rho)X \geqslant |E^\star|(y_{u_0}x_{u_{s_0+1}} - x_{u_0}y_{u_1})$. Note that $\rho x_{u_0} - \rho x_{u_{s_0+1}} = \sum_{i=1}^{s_0} x_{u_i} \leqslant s_0 x_{u_0}$. Then $x_{u_{s_0+1}} \geqslant \frac{\rho - s_0}{\rho} x_{u_0}$. Also, recall that $G'[S] \cong K_{1,e(S)} \cup (s-e(S)-1)K_1$. Then $y_{u_i} = y_{u_j}$ for $i, j \in [1, e(S)]$, and by considering the eigen-equation of $A(G')$ with respect to ρ', we obtain

$$\rho' y_{u_0} = \sum_{i=1}^{e(S)} y_{u_i} + \sum_{i=0}^{t-1} y_{v_i} = e(S) \cdot y_{u_1} + \sum_{i=0}^{t-1} y_{v_i} \quad \text{and} \quad \rho' y_{u_1} = y_{u_0} + \sum_{i=0}^{t-1} y_{v_i}.$$

This implies that $y_{u_1} = \frac{\rho'+1}{\rho'+e(S)} y_{u_0} \leqslant \frac{\rho+1}{\rho+e(S)} y_{u_0}$ because $\rho' \leqslant \rho$ and $e(S) > 0$. Thus,

$$\begin{aligned} y_{u_0}x_{u_{s_0+1}} - x_{u_0}y_{u_1} &\geqslant x_{u_0}y_{u_0}\left(\frac{\rho - s_0}{\rho} - \frac{\rho+1}{\rho+e(S)}\right) \\ &= x_{u_0}y_{u_0}\left(\frac{\rho(e(S)-(s_0+1)) - s_0e(S)}{\rho(\rho+e(S))}\right) \\ &> 0 \end{aligned}$$

for sufficiently large n. Then $Y^t(\rho'-\rho)X \geqslant |E^\star|(y_{u_0}x_{u_{s_0+1}} - x_{u_0}y_{u_1}) > 0$, and so $\rho' > \rho$, a contradiction. Similarly, $e(T) \in \{t_0, t_0+1\}$. This completes the proof of the claim. ☐

Claim 26. *If $e(S) \neq 3$ (resp. $e(T) \neq 3$), then $e(S) = s_0$ and $G^\star[S] \cong K_{1,e(S)} \cup (s - e(S) - 1)K_1$ (resp. $e(T) = t_0$ and $G^\star[T] \cong K_{1,e(T)} \cup (t - e(T) - 1)K_1$), and if $e(S) = 3$ (resp. $e(T) = 3$), then $e(S) = s_0 + 1$ and $G^\star[S] \cong K_3 \cup (s-3)K_1$ (resp. $e(T) = t_0 + 1$ and $G^\star[T] \cong K_3 \cup (t-3)K_1$).*

Proof. By symmetry, we only need to prove the claim for S. First assume that $e(S) \neq 3$. By Claim 25, we have $e(S) \in \{s_0, s_0+1\}$. If $e(S) = e(N_S[u_0]) = s_0$, then the result follows. Thus we may assume that $e(S) = e(N_S[u_0]) = s_0+1 = d_S(u_0)+1 = |N_s[u_0]|$. Since $e(S) \neq 3$, we claim that $s_0 \geqslant 3$ and $G^\star[N_S[u_0]] \cong K_{1,s_0}+u_1u_2$. Let $G' = G-u_1u_2+u_0u_{s_0+1}$. It is easy to see that $G'[S] \cong K_{1,s_0+1} \cup (s-s_0-2)K_1$, and G' is Γ_k-free. Thus

$$\begin{aligned} Y^t(\rho'-\rho)X = Y^t(A(G')-A(G^\star))X &= x_{u_0}y_{u_{s_0+1}} + y_{u_0}x_{u_{s_0+1}} - x_{u_1}y_{u_2} - x_{u_2}y_{u_1} \\ &= x_{u_0}y_{u_1} + y_{u_0}x_{u_{s_0+1}} - 2x_{u_1}y_{u_1}, \end{aligned}$$

where the last equality follows from the fact that $x_{u_1} = x_{u_2}$ and $y_{u_i} = y_{u_j}$ for $i,j \in [1, s_0+1]$. Note that $G^\star[S] \cong (K_{1,s_0}+u_1u_2) \cup (s-s_0-1)K_1$, $x_{u_1} = x_{u_2}$ and $x_{u_i} = x_{u_j}$ for $i,j \in [3, s_0]$. By considering the eigen-equation of $A(G^\star)$ with respect to ρ, we obtain

$$\begin{aligned} \rho x_{u_0} &= x_{u_1} + x_{u_2} + \sum_{i=3}^{s_0} x_{u_i} + \sum_{i=0}^{t-1} x_{v_i} = 2x_{u_1} + (s_0-2)x_{u_3} + \sum_{i=0}^{t-1} x_{v_i}, \\ \rho x_{u_1} &= x_{u_0} + x_{u_2} + \sum_{i=0}^{t-1} x_{v_i} = x_{u_0} + x_{u_1} + \sum_{i=0}^{t-1} x_{v_i}, \\ \rho x_{u_3} &= x_{u_0} + \sum_{i=0}^{t-1} x_{v_i}, \quad \rho x_{u_{s_0+1}} = \sum_{i=0}^{t-1} x_{v_i}, \end{aligned}$$

which leads to $x_{u_1} = x_{u_2} = \frac{\rho(\rho+1)}{\rho^2+(s_0-1)\rho-(s_0-2)}x_{u_0} \leqslant \frac{\rho(\rho+1)}{\rho^2+2\rho-1}x_{u_0}$ and $x_{u_{s_0+1}} = \frac{\rho-1}{\rho}x_{u_1} - \frac{1}{\rho}x_{u_0}$. Therefore, $x_{u_0}y_{u_1} + y_{u_0}x_{u_{s_0+1}} - 2x_{u_1}y_{u_1}$ is equal to

$$x_{u_0}y_{u_1} + y_{u_0}\left(\frac{\rho-1}{\rho}x_{u_1} - \frac{1}{\rho}x_{u_0}\right) - 2x_{u_1}y_{u_1} = x_{u_0}\left(y_{u_1} - \frac{y_{u_0}}{\rho}\right) - x_{u_1}\left(2y_{u_1} - \frac{\rho-1}{\rho}y_{u_0}\right).$$

Recall that $G'[S] \cong K_{1,s_0+1} \cup (s-s_0-2)K_1$. According to $\rho' y_{u_0} = (s_0+1)\cdot y_{u_1} + \sum_{i=0}^{t-1} y_{v_i}$, $\rho' y_{u_1} = y_{u_0} + \sum_{i=0}^{t-1} y_{v_i}$ and the fact that $s_0 \geqslant 3$, we get

$$y_{u_1} = \frac{\rho'+1}{\rho'+s_0+1}y_{u_0} \leqslant \frac{\rho+1}{\rho+4}y_{u_0},$$

and therefore,

$$2y_{u_1} - \frac{\rho-1}{\rho}y_{u_0} > \left(\frac{2(\rho'+1)}{\rho'+s_0+1} - 1\right)y_{u_0} > 0.$$

Note that $x_{u_1} \leqslant \frac{\rho(\rho+1)}{\rho^2+2\rho-1}x_{u_0}$. Then

$$\begin{aligned} & x_{u_0}\left(y_{u_1} - \frac{y_{u_0}}{\rho}\right) - x_{u_1}\left(2y_{u_1} - \frac{\rho-1}{\rho}y_{u_0}\right) \\ &\geqslant x_{u_0}\left(y_{u_1} - \frac{y_{u_0}}{\rho}\right) - \frac{\rho(\rho+1)}{\rho^2+2\rho-1}x_{u_0}\left(2y_{u_1} - \frac{\rho-1}{\rho}y_{u_0}\right) \\ &= \left(\frac{\rho^2-1}{\rho^2+2\rho-1} - \frac{1}{\rho}\right)x_{u_0}y_{u_0} - \frac{\rho^2+1}{\rho^2+2\rho-1}x_{u_0}y_{u_1} \\ &\geqslant \left(\frac{\rho^2-1}{\rho^2+2\rho-1} - \frac{1}{\rho} - \frac{(\rho^2+1)(\rho+1)}{(\rho^2+2\rho-1)(\rho+4)}\right)x_{u_0}y_{u_0} > 0. \end{aligned}$$

Thus $Y^t(\rho'-\rho)X = x_{u_0}y_{u_1}+y_{u_0}x_{u_{s_0+1}}-2x_{u_1}y_{u_1} = x_{u_0}\left(y_{u_1}-\frac{y_{u_0}}{\rho}\right)-x_{u_1}\left(2y_{u_1}-\frac{\rho-1}{\rho}y_{u_0}\right) > 0$, implying that $\rho' > \rho$, a contradiction. Therefore, if $e(S) \neq 3$, then $e(S) = s_0$ and $G^\star[S] \cong K_{1,e(S)} \cup (s-e(S)-1)K_1$.

Now suppose that $e(S) = 3$. If $e(S) = e(N_S[u_0]) = s_0+1$, then $s_0 = 2$ and $G^\star[S] \cong K_3 \cup (s-3)K_1$, as required. So the remaining case is $e(S) = e(N_S[u_0]) = s_0$. In this situation, we have $s_0 = 3$ and $G^\star[S] \cong K_{1,3} \cup (s-4)K_1$. Let $G' = G^\star - u_0u_3 + u_1u_2$. Then $G'[S] \cong K_3 \cup (s-3)K_1$, and

$$\begin{aligned} Y^t(\rho'-\rho)X = Y^t(A(G')-A(G^\star))X &= x_{u_1}y_{u_2} + x_{u_2}y_{u_1} - x_{u_0}y_{u_3} - x_{u_3}y_{u_0} \\ &= x_{u_1}y_{u_0} - x_{u_0}y_{u_3}, \end{aligned}$$

where the last equality follows from the fact that $x_{u_1} = x_{u_2} = x_{u_3}$ and $y_{u_0} = y_{u_1} = y_{u_2}$. As above, from $\rho x_{u_0} = 3x_{u_1} + \sum_{i=0}^{t-1} x_{v_i}$, $\rho x_{u_1} = x_{u_0} + \sum_{i=0}^{t-1} x_{v_i}$, $\rho' y_{u_0} = 2y_{u_0} + \sum_{i=0}^{t-1} y_{v_i}$ and $\rho' y_{u_3} = \sum_{i=0}^{t-1} y_{v_i}$, we obtain

$$x_{u_1} = \frac{\rho+1}{\rho+3}x_{u_0} \quad \text{and} \quad y_{u_3} = \frac{\rho'-2}{\rho'}y_{u_0} \leqslant \frac{\rho-2}{\rho}y_{u_0}.$$

Hence,

$$x_{u_1}y_{u_0} - x_{u_0}y_{u_3} \geqslant x_{u_0}y_{u_0}\left(\frac{\rho+1}{\rho+3} - \frac{\rho-2}{\rho}\right) = \frac{6}{\rho(\rho+3)}x_{u_0}y_{u_0} > 0.$$

Then $Y^t(\rho'-\rho)X = x_{u_1}y_{u_0} - x_{u_0}y_{u_3} > 0$, and so $\rho' > \rho$, contrary to the maximality of ρ. Hence, if $e(S) = 3$, then $e(S) = s_0+1$ and $G^\star[S] \cong K_3 \cup (s-3)K_1$. $\square$

According to Claim 26, we only need to consider the following four cases. For simplicity, we denote by $s^\star = e(S)$ and $t^\star = e(T)$.

Case 1. $G^\star[S] \cong K_3 \cup (s-3)K_1$ and $G^\star[T] \cong K_3 \cup (t-3)K_1$.

In this situation, we have $x_{u_0} = x_{u_1} = x_{u_2}$, $x_{u_i} = x_{u_j}$ for $i,j \in [3, s-1]$ and $x_{v_0} = x_{v_1} = x_{v_2}$, $x_{v_i} = x_{v_j}$ for $i,j \in [3,t-1]$. Combining $\rho x_{u_0} = x_{u_1} + x_{u_2} + \sum_{i=0}^{t-1} x_{v_i} = 2x_{u_0} + \rho x_{u_3}$ with $\rho x_{v_0} = x_{v_1} + x_{v_2} + \sum_{i=0}^{s-1} x_{u_i} = 2x_{v_0} + \rho x_{v_3}$ yields that

$$x_{v_3} = \frac{\rho-2}{\rho}x_{v_0}, \; x_{u_3} = \frac{\rho-2}{\rho}x_{u_0} \quad \text{and} \quad (\rho-2)(x_{v_0}-x_{u_0}) = \sum_{i=0}^{s-1} x_{u_i} - \sum_{i=0}^{t-1} x_{v_i}. \qquad (9)$$

Furthermore, we assert that $x_{v_0} \leqslant x_{u_0}$. Suppose to the contrary that $x_{v_0} > x_{u_0}$, then we obtain $\sum_{i=0}^{s-1} x_{u_i} > \sum_{i=0}^{t-1} x_{v_i}$. On the other hand,

$$\begin{aligned} \sum_{i=0}^{t-1} x_{v_i} = \sum_{i=0}^{2} x_{v_i} + \sum_{i=3}^{t-1} x_{v_i} = 3x_{v_0} + (t-3)x_{v_3} &= 3x_{v_0} + (t-3)\frac{\rho-2}{\rho}x_{v_0} \\ &> 3x_{u_0} + (s-3)\frac{\rho-2}{\rho}x_{u_0} = \sum_{i=0}^{s-1} x_{u_i}, \end{aligned}$$

a contradiction.

Let $E^\star = \{v_0v_1, v_0v_2, v_1v_2,\}$, $E' = \{u_0u_3, u_0u_4, u_3u_4\}$ and $G' = G^\star - E^\star + E'$. Then $G'[S] \cong (K_1\nabla 2K_2) \cup (s-5)K_1$ and G' is Γ_k-free, where $K_1\nabla 2K_2$ is a graph obtained from the disjoint union $K_1 \cup 2K_2$ by adding all edges between K_1 and $2K_2$. Then $y_{u_1} = y_{u_2} = y_{u_3} = y_{u_4}$, $y_{u_i} = y_{u_j}$ for $i, j \in [5, s-1]$ and $y_{v_i} = y_{v_j}$ for $i, j \in [0, t-1]$. By considering the eigen-equation of $A(G')$ with respect to ρ', we obtain

$$\rho' y_{u_0} = 4y_{u_1} + \sum_{i=0}^{t-1} y_{v_i},\ \ \rho' y_{u_1} = y_{u_0} + y_{u_1} + \sum_{i=0}^{t-1} y_{v_i},$$
$$\rho' y_{u_5} = \sum_{i=0}^{t-1} y_{v_i},\quad \rho' y_{v_0} = y_{u_0} + 4y_{u_1} + (s-5)y_{u_5},$$

which gives that

$$y_{u_5} = \left(1 - \frac{4(\rho'+1)}{\rho'(\rho'+3)}\right) y_{u_0} \text{ and } y_{u_1} = \frac{\rho'+1}{\rho'+3} y_{u_0}. \tag{10}$$

Hence, $y_{v_0} = \frac{y_{u_0}}{\rho'}\left(s - 4 + \frac{4(\rho'+1)(\rho'-s+5)}{\rho'(\rho'+3)}\right) = \frac{y_{u_0}}{\rho'}\left(s - 4 + \frac{4(\rho'+3)^2 - 4s(\rho'+3) + 8(s-2)}{\rho'(\rho'+3)}\right)$. Combining this with $\rho' > s = \lfloor \frac{n}{2} \rfloor$ (since G' contains $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ as a proper subgraph), we deduce that

$$\begin{aligned} y_{v_0} &\leqslant \frac{y_{u_0}}{\rho'}\left(\rho' - 4 + \frac{4(\rho'+3)^2 - 4\rho'(\rho'+3) + 8(\rho'-2)}{\rho'(\rho'+3)}\right) \\ &= \frac{y_{u_0}}{\rho'}\left((\rho'-4) + O\left(\frac{1}{\rho'^2}\right)\right). \end{aligned} \tag{11}$$

Recall that $x_{v_0} \leqslant x_{u_0}$, and X and Y are the Perron vectors of $G^\star$ and G', respectively. Combining (9), (10) and (11), we have

$$\begin{aligned} Y^t(\rho' - \rho)X &= Y^t(A(G') - A(G))X = \sum_{ij\in E'} (x_i y_j + y_i x_j) - \sum_{ij \in E^\star} (x_i y_j + y_i x_j) \\ &= 2((y_{u_1} x_{u_0} + (y_{u_0} + y_{u_1}) x_{u_3}) - 3x_{v_0} y_{v_0}) \\ &\geqslant 2((y_{u_1} x_{u_0} + (y_{u_0} + y_{u_1}) x_{u_3}) - 3x_{u_0} y_{v_0}) \\ &= 2x_{u_0} y_{u_0}\left(\frac{\rho'+1}{\rho'+3} + \left(1 + \frac{\rho'+1}{\rho'+3}\right)\frac{\rho-2}{\rho}\right) - 6x_{u_0} y_{v_0} \\ &\geqslant 2x_{u_0} y_{u_0}\left(\frac{\rho'+1}{\rho'+3} + \left(1 + \frac{\rho'+1}{\rho'+3}\right)\frac{\rho'-2}{\rho'}\right) - 6x_{u_0} y_{v_0} \\ &\geqslant 2x_{u_0} y_{u_0}\left(3 - \frac{4}{\rho'} - \frac{4}{\rho'+3} - \frac{3}{\rho'}(\rho'-4) + O\left(\frac{1}{\rho'^2}\right)\right) \\ &= 2x_{u_0} y_{u_0}\left(\frac{12}{\rho'} - \frac{4}{\rho'} - \frac{4}{\rho'+3} + O\left(\frac{1}{\rho'^2}\right)\right) > 0. \end{aligned}$$

It follows that $\rho' > \rho$, a contradiction.

Case 2. $G^\star[S] \cong K_{1,s^\star} \cup (s-s^\star-1)K_1$ and $G^\star[T] \cong K_{1,t^\star} \cup (t-t^\star-1)K_1$, where $s^\star \neq 3$ and $t^\star \neq 3$.

Note that $s_0 = e(N_S[u_0])$, $t_0 = e(N_T[v_0])$ and $s_0 + t_0 = k-1$. If $s_0 = k-1$, then $G^\star \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor} \diamond K_{1,k-1}$, as desired. Thus we may assume that $s_0 \leqslant k-2$. Observe that $x_{u_i} = x_{u_j}$ for $i,j \in [1,s_0]$ and $x_{v_i} = x_{v_j}$ for $i,j \in [1,t_0]$. Let $E^\star = \{v_0v_i : 1 \leqslant i \leqslant t_0\}$, $E' = \{u_0u_i : s_0+1 \leqslant i \leqslant k-1\}$, and $G' = G^\star - E^\star + E'$. Then $G'[S] \cong K_{1,k-1} \cup (s-k)K_1$ and G' is Γ_k-free. By symmetry, $y_{u_i} = y_{u_j}$ for $i,j \in [1,k-1]$. Hence, we have

$$\rho' y_{u_0} = \sum_{i=1}^{k-1} y_{u_i} + \sum_{i=0}^{t-1} y_{v_i} = (k-1)y_{u_1} + \sum_{i=0}^{t-1} y_{v_i},\ \rho' y_{u_1} = y_{u_0} + \sum_{i=0}^{t-1} y_{v_i},\ \rho' y_{u_k} = \sum_{i=0}^{t-1} y_{v_i},$$

which leads to

$$y_{u_1} = \frac{\rho'+1}{\rho'+k-1} y_{u_0},\ y_{u_k} = \frac{\rho' y_{u_1} - y_{u_0}}{\rho'},$$

and

$$\sum_{i=0}^{s-1} y_{u_i} = \frac{\rho'^2 s + 2\rho'(k-1) - (k-1)s + k(k-1)}{\rho'(\rho'+k-1)} y_{u_0} \leqslant \frac{\rho'^3 + \rho'(k-1) + k(k-1)}{\rho'(\rho'+k-1)} y_{u_0}.$$

Note that $\rho' y_{v_0} = \sum\limits_{i=0}^{s-1} y_{u_i}$. Then

$$y_{v_0} \leqslant \frac{\rho'^3 + \rho'(k-1) + k(k-1)}{\rho'^2(\rho'+k-1)} y_{u_0}. \tag{12}$$

Similarly, we have $x_{u_1} = \frac{\rho+1}{\rho+s_0} x_{u_0}$, $x_{u_{s_0+1}} = \frac{\rho^2 - s_0}{\rho(\rho+s_0)} x_{u_0}$. Since $s = \lfloor \frac{n}{2} \rfloor < \rho$, it follows that

$$\sum_{i=0}^{s-1} x_{u_i} = \frac{\rho^2 s + 2\rho s_0 - s_0 s + s_0^2 + s_0}{\rho(\rho+s_0)} x_{u_0} \leqslant \frac{\rho^3 + \rho s_0 + s_0^2 + s_0}{\rho(\rho+s_0)} x_{u_0}.$$

Hence, $x_{v_1} = \frac{\rho+1}{\rho+t_0} x_{v_0}$. According to $\rho x_{v_0} = \sum\limits_{i=1}^{t_0} x_{v_i} + \sum\limits_{i=0}^{s-1} x_{u_i} = t_0 x_{v_1} + \sum\limits_{i=0}^{s-1} x_{u_i}$, we obtain

$$x_{v_0} \leqslant \frac{(\rho+k-1-s_0)(\rho^3 + \rho s_0 + s_0^2 + s_0)}{\rho(\rho+s_0)(\rho^2 - (k-1-s_0))} x_{u_0}.$$

It follows that

$$\begin{aligned}
&\frac{2\rho+k-s_0}{\rho+k-1-s_0} x_{v_0} y_{v_0} \\
&\leqslant \frac{2\rho+k-s_0}{\rho+k-1-s_0} \cdot \frac{(\rho+k-1-s_0)(\rho^3 + \rho s_0 + s_0^2 + s_0)}{\rho(\rho+s_0)(\rho^2 - (k-1-s_0))} \cdot \\
&\quad \frac{\rho'^3 + \rho'(k-1) + k(k-1)}{\rho'^2(\rho'+k-1)} x_{u_0} y_{u_0} \\
&= \frac{(2\rho+k-s_0)(\rho^3 + \rho s_0 + s_0(s_0+1))(\rho'^3 + \rho'(k-1) + k(k-1))}{\rho\rho'^2(\rho+s_0)(\rho^2 - (k-1-s_0))(\rho'+k-1)} x_{u_0} y_{u_0} \\
&= \left(\frac{2\rho^4\rho'^3 + \rho^3\rho'^3(k-s_0)}{\rho\rho'^2(\rho+s_0)(\rho^2-(k-1-s_0))(\rho'+k-1)} + O\left(\frac{1}{\rho^2}\right) \right) x_{u_0} y_{u_0}.
\end{aligned} \tag{13}$$

Note that X and Y are the Perron vectors of $G^\star$ and G', respectively. Then

$$\begin{aligned}
&Y^t(\rho'-\rho)X = Y^t(A(G')-A(G^\star))X \\
&= \sum_{u_0u_i\in E'}(x_{u_0}y_{u_i}+y_{u_0}x_{u_i}) - \sum_{v_0v_i\in E^\star}(x_{v_0}y_{v_i}+y_{v_0}x_{v_i}) \\
&= (k-1-s_0)(x_{u_0}y_{u_1}+y_{u_0}x_{u_{s_0+1}}-x_{v_0}y_{v_0}-x_{v_1}y_{v_0}) \\
&= (k-1-s_0)\left(\left(\frac{\rho'+1}{\rho'+k-1}+\frac{\rho^2-s_0}{\rho(\rho+s_0)}\right)x_{u_0}y_{u_0} - \frac{2\rho+k-s_0}{\rho+k-1-s_0}x_{v_0}y_{v_0}\right).
\end{aligned} \tag{14}$$

Recall that $s_0 \leqslant k-2$. We shall prove (14) > 0 by showing

$$\left(\frac{\rho'+1}{\rho'+k-1}+\frac{\rho^2-s_0}{\rho(\rho+s_0)}\right)x_{u_0}y_{u_0} > \frac{2\rho+k-s_0}{\rho+k-1-s_0}x_{v_0}y_{v_0},$$

which leads to $\rho' > \rho$, and we derive a contradiction. According to (13), it suffices to show

$$\frac{\rho'+1}{\rho'+k-1}+\frac{\rho^2-s_0}{\rho(\rho+s_0)} > \frac{2\rho^4\rho'^3+\rho^3\rho'^3(k-s_0)}{\rho\rho'^2(\rho+s_0)(\rho^2-(k-1-s_0))(\rho'+k-1)} + O\left(\frac{1}{\rho^2}\right).$$

Indeed, the above inequality holds by $\frac{\rho'+1}{\rho'+k-1}+\frac{\rho^2-s_0}{\rho(\rho+s_0)} = \frac{2\rho^4\rho'^3+k\rho^4\rho'^2+s_0\rho^3\rho'^3}{\rho\rho'^2(\rho+s_0)(\rho^2-(k-1-s_0))(\rho'+k-1)} + O\left(\frac{1}{\rho^2}\right)$, as required.

<u>Case 3.</u> $G^\star[S]\cong K_{1,s^\star}\cup(s-s^\star-1)K_1$ and $G^\star[T]\cong K_3\cup(t-3)K_1$, where $s^\star\neq 3$.

Clearly, $k\geqslant 4$ and $s_0=k-4$. First we may assume that $k\geqslant 5$. Then $x_{u_i}=x_{u_j}$ for $i,j\in[1,k-4]$ and $x_{v_0}=x_{v_1}=x_{v_2}$. It follows that

$$x_{u_1} = \frac{\rho+1}{\rho+k-4}x_{u_0}, \tag{15}$$

and

$$\begin{aligned}
x_{u_{k-3}} = \frac{\rho^2-k+4}{\rho(\rho+k-4)}x_{u_0} &= \left(\frac{\rho}{\rho+k-4}+O\left(\frac{1}{\rho^2}\right)\right)x_{u_0} \\
&= \left(1-\frac{k-4}{\rho+k-4}+O\left(\frac{1}{\rho^2}\right)\right)x_{u_0}.
\end{aligned} \tag{16}$$

Combining with $s=\lfloor\frac{n}{2}\rfloor<\rho$, we have

$$\begin{aligned}
\sum_{i=0}^{s-1}x_{u_i} &= x_{u_0}+(k-4)x_{u_1}+(s-k+3)x_{u_{k-3}} \\
&= \frac{\rho^2 s+2\rho(k-4)-s(k-4)+(k-4)(k-3)}{\rho(\rho+k-4)}x_{u_0} \\
&\leqslant \frac{\rho^3+2\rho(k-4)-\rho(k-4)+(k-4)(k-3)}{\rho(\rho+k-4)}x_{u_0} \\
&= \left(\frac{\rho^2+k-4}{\rho+k-4}+O\left(\frac{1}{\rho^2}\right)\right)x_{u_0}.
\end{aligned}$$

Note that $\rho x_{v_0} = x_{v_1} + x_{v_2} + \sum_{i=0}^{s-1} x_{u_i} = 2x_{v_0} + \sum_{i=0}^{s-1} x_{u_i}$. Then

$$x_{v_0} \leqslant \frac{1}{\rho-2}\left(\frac{\rho^2+k-4}{\rho+k-4} + O\left(\frac{1}{\rho^2}\right)\right) x_{u_0} = \left(\frac{\rho^2+k-4}{(\rho-2)(\rho+k-4)} + O\left(\frac{1}{\rho^3}\right)\right) x_{u_0}. \quad (17)$$

Let $E^\star = \{v_0v_1, v_0v_2, v_1v_2\}$, $E' = \{u_0u_{k-3}, u_0u_{k-2}, u_0u_{k-1}\}$, and $G' = G^\star - E^\star + E'$. Then $G' \cong K_{\lceil \frac{n}{2}\rceil, \lfloor \frac{n}{2}\rfloor} \diamond K_{1,k-1}$ and G' is Γ_k-free. Note that

$$y_{u_1} = \frac{\rho'+1}{\rho'+k-1} y_{u_0} = (1 - \frac{k-2}{\rho'+k-1}) y_{u_0}, \ y_{u_k} = \frac{\rho' y_{u_1} - y_{u_0}}{\rho'}, \quad (18)$$

and

$$\begin{aligned}\sum_{i=0}^{s-1} y_{u_i} &= \frac{\rho'^2 s + 2\rho'(k-1) - s(k-1) + k(k-1)}{\rho'(\rho'+k-1)} y_{u_0} \\ &\leqslant \frac{\rho'^3 + 2\rho'(k-1) - \rho'(k-1) + k(k-1)}{\rho'(\rho'+k-1)} y_{u_0} \\ &= \left(\frac{\rho'^2+k-1}{\rho'+k-1} + O\left(\frac{1}{\rho'^2}\right)\right) y_{u_0},\end{aligned}$$

where the inequality follows from $s = \lfloor \frac{n}{2} \rfloor < \rho'$. Then by using $\rho' y_{v_0} = \sum_{i=0}^{s-1} y_{u_i}$, we have

$$y_{v_0} \leqslant \frac{1}{\rho'}\left(\frac{\rho'^2+k-1}{\rho'+k-1} + O\left(\frac{1}{\rho'^2}\right)\right) y_{u_0} = \left(\frac{\rho'^2+k-1}{\rho'(\rho'+k-1)} + O\left(\frac{1}{\rho'^3}\right)\right) y_{u_0}.$$

Thus, combining this with (17), we deduce

$$\begin{aligned}x_{v_0}y_{v_0} &\leqslant \left(\frac{\rho^2+k-4}{(\rho-2)(\rho+k-4)} + O\left(\frac{1}{\rho^3}\right)\right)\left(\frac{\rho'^2+k-1}{\rho'(\rho'+k-1)} + O\left(\frac{1}{\rho'^3}\right)\right) x_{u_0}y_{u_0} \\ &= \left(\frac{(\rho^2+k-4)(\rho'^2+k-1)}{\rho'(\rho'+k-1)(\rho-2)(\rho+k-4)} + O\left(\frac{1}{\rho^3}\right)\right) x_{u_0}y_{u_0}.\end{aligned} \quad (19)$$

By symmetry, we have

$$\begin{aligned}Y^t(\rho'-\rho)X = Y^t(A'-A)X &= \sum_{ij\in E'}(x_iy_j + x_jy_i) - \sum_{ij\in E^\star}(x_iy_j + x_jy_i) \\ &= 3x_{u_0}y_{u_1} + 3y_{u_0}x_{u_{k-3}} - 6x_{v_0}y_{v_0}.\end{aligned} \quad (20)$$

Now, we shall show (20) > 0. According to (16) and (18), we change (20) to $3x_{u_0}y_{u_0}(\frac{\rho'+1}{\rho'+k-1} + \frac{\rho^2-k+4}{\rho(\rho+k-4)}) - 6x_{v_0}y_{v_0}$. Thus, combining with (19), it suffices to prove

$$\frac{\rho'+1}{\rho'+k-1} + \frac{\rho^2-k+4}{\rho(\rho+k-4)} > 2\left(\frac{(\rho^2+k-4)(\rho'^2+k-1)}{\rho'(\rho'+k-1)(\rho-2)(\rho+k-4)} + O\left(\frac{1}{\rho^3}\right)\right).$$

Multiplying both sides by $\rho'\rho(\rho'+k-1)(\rho-2)(\rho+k-4)$. Then it suffices to show

$$\rho\rho'(\rho'+1)(\rho-2)(\rho+k-4)+\rho'(\rho'+k-1)(\rho-2)(\rho^2-k+4) > 2\rho(\rho^2+k-4)(\rho'^2+k-1)+O(\rho^2).$$

By calculation, we only need to prove $\rho^2\rho'^2(k-8)+\rho^3\rho'k > 0$ since $\rho = O(n), \rho' = O(n)$. Note that $\rho > \rho'$. Then

$$\rho^2\rho'^2(k-8)+\rho^3\rho'k > \rho^2\rho'^2(k-8)+\rho^2\rho'^2k = \rho^2\rho'^2(2k-8).$$

Thus, for $k \geqslant 5$, we have $\rho' > \rho$ by (20), a contradiction.

For $k = 4$, we see that $G^\star[N_T[v_0]] \cong K_3$, and $G^\star[N_S[u_0]]$ is an empty graph. If $s = t = \frac{n}{2}$, then $G^\star \cong K_{\frac{n}{2},\frac{n}{2}} \diamond K_3$, as desired. If $s = t-1 = \frac{n-1}{2}$, let $E^\star = \{v_0v_1, v_0v_2, v_1v_2\}$, $E' = \{u_0u_1, u_0u_2, u_1u_2\}$ and $G' = G^\star - E^\star + E'$, then $G' \cong K_{\lceil \frac{n}{2}\rceil,\lfloor \frac{n}{2}\rfloor} \diamond K_3$ and G' is Γ_4-free. Note that the quotient matrix of $A(G^\star)$ with respect to the partition $\Pi^\star : V(G^\star) = \{u_0, \ldots, u_{s-1}\} \cup \{v_0, v_1, v_2\} \cup \{v_3, \ldots, v_{t-1}\}$ is given by

$$B_1 = \begin{pmatrix} 0 & 3 & t-3 \\ s & 2 & 0 \\ s & 0 & 0 \end{pmatrix}.$$

Furthermore, the characteristic polynomial of B_1 is $\varphi(B_1, x) = x^3 - 2x^2 - stx + 2s(t-3)$, and its largest root coincides with ρ by Lemma 14 since the partition is equitable. By symmetry, the characteristic polynomial $\varphi(B_2, x)$ of the quotient matrix B_2 of $A(G')$ with respect to the partition $\Pi' : V(G') = \{v_0, \ldots, v_{t-1}\} \cup \{u_0, u_1, u_2\} \cup \{u_3, \ldots, u_{s-1}\}$ can be obtained from $\varphi(B_1, x)$ by switching s and t. The partition is also equitable, by Lemma 14, $\rho' = \rho(B_2)$. Note that $\rho > s = \frac{n-1}{2}$ and $s + t = n$. By a simple calculation, we have

$$\varphi(B_2, \rho) = \varphi(B_2, \rho) - \varphi(B_1, \rho) = -4\rho^2 - \left(\frac{n^2}{2} - \frac{1}{2}\right)\rho + n^2 - 6n - 1.$$

Thus $\varphi(B_2, \rho) < \varphi(B_2, \frac{n-1}{2}) = -\frac{n^3}{4} + \frac{n^2}{4} - \frac{15n}{4} - \frac{9}{4} < 0$, as required. Therefore, $\rho < \rho'$, which leads to a contradiction.

Case 4. $G^\star[S] \cong K_3 \cup (s-3)K_1$ and $G^\star[T] \cong K_{1,t^\star} \cup (t - t^\star - 1)K_1$, where $t^\star \neq 3$.

Clearly, $k \geqslant 4$ and $t_0 = k - 4$. For $k = 4$, we have $G^\star \cong K_{\lceil \frac{n}{2}\rceil,\lfloor \frac{n}{2}\rfloor} \diamond K_3$, as desired. Now, suppose $k \geqslant 5$. Note that $s = \lfloor \frac{n}{2} \rfloor$ and $t = \lceil \frac{n}{2} \rceil$. Thus, if n is even, i.e., $s = t = \frac{n}{2}$, then we have completed the proof by Case 3. In the following, we assume that n is odd. In this situation, $s = \frac{n-1}{2}$ and $t = \frac{n+1}{2}$. By symmetry, we obtain $x_{u_0} = x_{u_1} = x_{u_2}$ and $x_{v_i} = x_{v_j}$ for $i, j \in [1, k-4]$. It follows that

$$x_{v_1} = \frac{\rho+1}{\rho+k-4}x_{v_0},\ x_{u_3} = \frac{\rho-2}{\rho}x_{u_0}, \tag{21}$$

and

$$\sum_{i=0}^{s-1} x_{u_i} = 3x_{u_0} + (s-3)x_{u_3} = \frac{\rho s - 2s + 6}{\rho}x_{u_0} = \frac{(\rho-2)\cdot\frac{n-1}{2}+6}{\rho}x_{u_0} \leqslant \frac{\rho(\rho-\frac{5}{2})+7}{\rho}x_{u_0},$$

where the last inequality follows from $\frac{n}{2} < \rho$. Note that

$$\rho x_{v_0} = \sum_{i=1}^{k-4} x_{v_i} + \sum_{i=0}^{s-1} x_{u_i} = (k-4)x_{v_1} + \sum_{i=0}^{s-1} x_{u_i}.$$

Then

$$x_{v_0} \leqslant \left(1 + \frac{\rho(k-\frac{13}{2})}{\rho^2 - k + 4} + O\left(\frac{1}{\rho^2}\right)\right) x_{u_0} \tag{22}$$

Let $E^\star = \{v_0v_1, v_0v_2, \ldots, v_0v_{k-4}, u_1u_2\}$, $E' = \{u_0u_3, u_0u_4, \ldots, u_0u_{k-1}\}$ and $G' = G^\star - E^\star + E'$. Then $G' \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor} \diamond K_{1,k-1}$ and G' is Γ_k-free. Recall that $s = \frac{n-1}{2}$. Combining (18) with $\rho' y_{v_0} = \sum_{i=0}^{s-1} y_{u_i} = y_{u_0} + (k-1)y_{u_1} + (s-k)y_{u_k}$, we deduce $y_{v_0} \leqslant \left(\frac{\rho' - \frac{1}{2}}{\rho' + k - 1} + O\left(\frac{1}{\rho'^2}\right)\right) y_{u_0}$. According to (22), we have

$$\begin{aligned} x_{v_0} y_{v_0} &\leqslant \left(1 + \frac{\rho(k-\frac{13}{2})}{\rho^2 - k + 4} + O\left(\frac{1}{\rho^2}\right)\right)\left(\frac{\rho' - \frac{1}{2}}{\rho' + k - 1} + O\left(\frac{1}{\rho'^2}\right)\right) x_{u_0} y_{u_0} \\ &= \left(1 - \frac{k - \frac{1}{2}}{\rho' + k - 1} + \frac{\rho'\rho(k - \frac{13}{2})}{(\rho' + k - 1)(\rho^2 - k + 4)} + O\left(\frac{1}{\rho^2}\right)\right) x_{u_0} y_{u_0}. \end{aligned}$$

Combining this with (18) and (21), we have

$$\begin{aligned} & Y^t(\rho' - \rho)X = Y^t(A' - A)X = \sum_{ij \in E'} (x_i y_j + x_j y_i) - \sum_{ij \in E^\star} (x_i y_j + x_j y_i) \\ &= (k-3)(x_{u_0} y_{u_1} + x_{u_3} y_{u_0}) - (k-4)(x_{v_0} y_{v_0} + y_{v_0} x_{v_1}) - 2x_{u_0} y_{u_1} \\ &= \left((k-5)\left(1 - \frac{k-2}{\rho' + k - 1}\right) + (k-3)(1 - \frac{2}{\rho})\right) x_{u_0} y_{u_0} - (k-4)\left(2 - \frac{k-5}{\rho + k - 4}\right) x_{v_0} y_{v_0} \\ &\geqslant \left(2k - 8 - \frac{(k-5)(k-2)}{\rho' + k - 1} - \frac{2(k-3)}{\rho}\right) x_{u_0} y_{u_0} - \left(2k - 8 - \frac{(k-5)(k-4)}{\rho + k - 4}\right) \cdot \\ &\quad \left(1 - \frac{k - \frac{1}{2}}{\rho' + k - 1} + \frac{\rho'\rho(k - \frac{13}{2})}{(\rho' + k - 1)(\rho^2 - k + 4)} + O\left(\frac{1}{\rho^2}\right)\right) x_{u_0} y_{u_0} \\ &\geqslant \left(\frac{k^2 - 2k - 6}{\rho' + k - 1} + \frac{k^2 - 11k + 26}{\rho + k - 4} - \frac{\rho'\rho(2k^2 - 21k + 52)}{(\rho' + k - 1)(\rho^2 - k + 4)} + O\left(\frac{1}{\rho^2}\right)\right) x_{u_0} y_{u_0} \\ &= \left(\frac{(k^2 - 2k - 6)\rho^3 - (k^2 - 10k + 26)\rho^2\rho'}{(\rho' + k - 1)(\rho + k - 4)(\rho^2 - k + 4)} + O\left(\frac{1}{\rho^2}\right)\right) x_{u_0} y_{u_0} \\ &\geqslant \left(\frac{(8k - 32)\rho^3}{(\rho' + k - 1)(\rho + k - 4)(\rho^2 - k + 4)} + O\left(\frac{1}{\rho^2}\right)\right) x_{u_0} y_{u_0} > 0. \end{aligned}$$

Thus $\rho' > \rho$, a contradiction. This completes the proof of Case 4.

Considering Cases 1-4, we complete the proof of Theorem 7. □

4 Conclusion remark

In [2], Bollobás asked for the maximum size of an n-vertex graph $G \in \overline{\Omega'_k}$? Based on the result of Theorem 3, we pose a spectral analogue for Bollobás's problem as follows.

Problem 27. What is the maximum spectral radius of an n-vertex graph $G \in \overline{\Omega'_k}$ for $k \geqslant 3$?

Ma and Yang [27] proved that $f(n) < n + \sqrt{n} + o(n)$ for any n-vertex 2-connected graph. Theorem 2 shows that the graph with the maximum spectral radius among all graphs without two cycles of the same length has a cut vertex. So it is natural to ask the following problem.

Problem 28. What is the maximum spectral radius among all 2-connected n-vertex graphs without two cycles of the same length?

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