Diagonal Hooks and a Schmidt-Type Partition Identity

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Abstract

In a recent paper of Andrews and Paule, several Schmidt-type partition identities are considered within the framework of MacMahon's Partition Analysis. Following their work, we derive a new Schmidt-type identity concerning diagonal hooks of partitions. We provide an analytic proof based on MacMahon's Partition Analysis and a combinatorial proof through an involution on the set of partitions. We also establish connections between Schmidt-type distinct partitions and partitions with nonpositive and negative cranks.

Mathematics Subject Classifications: 11P84, 05A17

1 Introduction

A *partition* of a natural number n is a nonincreasing sequence of positive integers whose sum equals n. For any partition λ, we define its *size* $|\lambda|$ as the sum of all parts in λ and define its *length* $\ell(\lambda)$ as the number of parts in λ. Throughout, $\mathscr{P}$ denotes the set of partitions, and $\mathscr{D}$ denotes the set of partitions into distinct parts.

In their most recent work on MacMahon's Partition Analysis, Andrews and Paule [3] revisited a *Monthly* problem proposed by Frank Schmidt [7].

Theorem S. *Let $f(n)$ denote the number of partitions μ into distinct parts $\mu_1 > \mu_2 > \mu_3 > \cdots$ such that $\mu_1 + \mu_3 + \mu_5 + \cdots = n$. Then $f(n) = p(n)$, the number of partitions of n.*

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The published solution of Theorem S is due to Peter Mork [6], and it is combinatorial, relying on a bijection concerning diagonal hooks of partitions. Graham Lord also found the same bijection without hooks involved; see the *Editorial comment* in [6].

Andrews and Paule observed that Theorem S, and other identities sharing the same nature, can be well fit into the framework of MacMahon's Partition Analysis. One example is as follows.

Theorem A–P. *Let $g(n)$ denote the number of partitions μ into parts $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \cdots$ such that $\mu_1 + \mu_3 + \mu_5 + \cdots = n$. Then $g(n) = p_2(n)$, the number of partitions of n into two colors.*

In Theorem S and Theorem A–P, the main ingredients are partitions with the sum of odd-indexed parts equal to n. A natural variant is to replace the odd-indexed parts with other odd-indexed statistics of partitions.

The object of our paper follows along these lines, focusing on odd-indexed diagonal hooks.

Definition 1 (Durfee square)**.** The *Durfee square* of a partition λ is the largest square that fits inside the Ferrers diagram of λ. We denote by $D(\lambda)$ the length of the Durfee square of λ.

Definition 2 (Diagonal hook lengths)**.** Let λ be an integer partition with Durfee square of length $D(\lambda)$. For $1 \leqslant i \leqslant D(\lambda)$, we denote by $\Gamma_i(\lambda)$ the hook length of the i-th diagonal entry of the Durfee square of λ, ordering from top-left to bottom-right. These $\Gamma_i(\lambda)$'s are called the *diagonal hook lengths* of λ. For example, the partition $\lambda = 4+4+3+3+2+1$ with $D(\lambda) = 3$ has diagonal hook lengths $\Gamma_1(\lambda) = 9$, $\Gamma_2(\lambda) = 6$ and $\Gamma_3(\lambda) = 2$; see Figure 1.

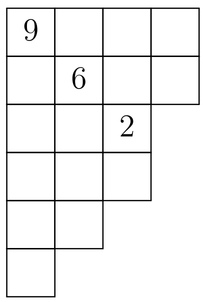

Figure 1: Diagonal hook lengths of $4+4+3+3+2+1$.

Our main result is stated as follows.

Theorem 3. *Let $\gamma_e(n)$ (resp. $\gamma_o(n)$) be the number of partitions λ such that its length $\ell(\lambda)$ and the length $D(\lambda)$ of its Durfee square has the same parity (resp. different parities) and such that the diagonal hook lengths satisfy $(\Gamma_1(\lambda)+1)+(\Gamma_3(\lambda)+1)+(\Gamma_5(\lambda)+1)+\cdots = n$. Then, $\gamma_e(n) - \gamma_o(n)$ equals the number of partitions of n into even parts. In particular, $\gamma_e(2n+1) = \gamma_o(2n+1)$.*

Example 4. The partitions counted by $\gamma_e(4)$ are 3, $1+1+1$ and $2+2$, and the partition counted by $\gamma_o(4)$ is $2+1$; see Figure 2. Then $\gamma_e(4) - \gamma_o(4) = 3 - 1 = 2$. On the other hand, 4 has two partitions into even parts: 4 and $2+2$.

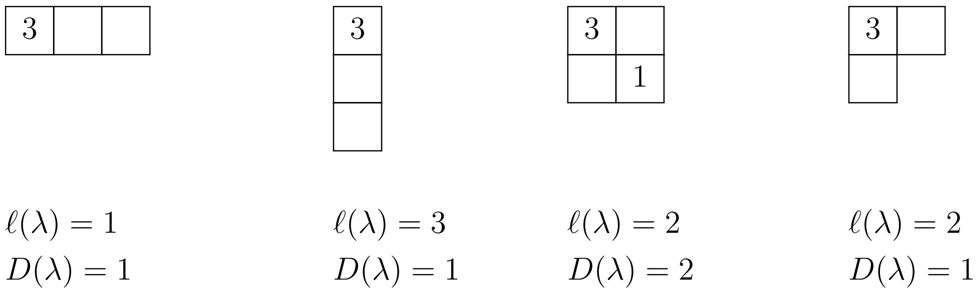

Figure 2: Partitions counted by $\gamma_e(4)$ and $\gamma_o(4)$.

This paper is organized as follows. We first prove Theorem 3 in Section 2 with MacMahon's Partition Analysis applied to compute related generating functions. Then in Section 3, we construct an involution on the set of partitions which leads to a combinatorial proof of Theorem 3. Finally, in Section 4, we give a variant of Mork's bijection that builds connections between Schmidt-type distinct partitions and partitions with nonpositive and negative cranks.

2 MacMahon's Partition Analysis

2.1 An identity from MacMahon's Partition Analysis

The main ingredient we require from MacMahon's Partition Analysis is the following result due to Andrews and Paule [3, Lemma 3.1]:

Lemma 5. *For any nonnegative integers a and b,*

$$\sum_{\substack{j_1,j_2,\ldots,j_m \geqslant a\\ j_1-j_2\geqslant b\\ j_2-j_3\geqslant b\\ \cdots\\ j_{m-1}-j_m\geqslant b}} x_1^{j_1}x_2^{j_2}\cdots x_m^{j_m} = \frac{x_1^b(x_1x_2)^b\cdots(x_1x_2\cdots x_{m-1})^b(x_1x_2\cdots x_m)^a}{(1-x_1)(1-x_1x_2)\cdots(1-x_1x_2\cdots x_m)}. \tag{1}$$

Recall that the MacMahon operator $\Omega_{\geqslant}$ is defined by

$$\underset{\geqslant}{\Omega}\sum_{s_1=-\infty}^{\infty}\cdots\sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r}\lambda_1^{s_1}\cdots\lambda_r^{s_r} := \sum_{s_1=0}^{\infty}\cdots\sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r},$$

where the functions $A_{s_1,\ldots,s_r}$ in several complex variables are rational over $\mathbb{C}$ and the λ_i are restricted to a neighborhood of the circle $|\lambda_i| = 1$. Furthermore, we require that the $A_{s_1,\ldots,s_r}$ are such that any of the series involved are absolutely convergent within the domain of definition of $A_{s_1,\ldots,s_r}$. Now, the left hand side of (1) can be written as

$$\underset{\geqslant}{\Omega} \sum_{j_1,\ldots,j_m \geqslant 0} x_1^{j_1} x_2^{j_2} \cdots x_m^{j_m} \lambda_1^{j_1-j_2-b} \lambda_2^{j_2-j_3-b} \cdots \lambda_{m-1}^{j_{m-1}-j_m-b} \lambda_m^{j_m-a}.$$

Then, as shown by Andrews and Paule, Lemma 5 follows by induction on m.

From a combinatorial perspective, Lemma 5 can be interpreted as follows. First, the left hand side of (1) can be treated as the generating function for colored partitions of the form $j_1 + j_2 + \cdots + j_m$ where $j_1, j_2, \ldots, j_m \geqslant a$ and $j_i - j_{i+1} \geqslant b$ for $1 \leqslant i \leqslant m-1$. Here, we color j_1 by x_1, j_2 by x_2, $\ldots$, and j_m by x_m.

Now, we subtract a from j_m, $a+b$ from j_{m-1}, $\ldots$, and $a+(m-1)b$ from j_1. Then the subtracted numbers are counted by

$$x_1^{a+(m-1)b} x_2^{a+(m-2)b} \cdots x_{m-1}^{a+b} x_m^a,$$

which gives the numerator on the right hand side of (1). Also, after subtracting these numbers, we are left with a partition μ with $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m \geqslant 0$ and μ_i colored by x_i for each $1 \leqslant i \leqslant m$.

Recall that in the Ferrers diagram of a partition, each part of size s is represented as a row of s nodes. Now, we color the first node of each row by x_1, the second node by x_2, and so on. Then the part of size s is counted by

$$x_1 x_2 \cdots x_s.$$

As an example, the partition $5+3+3+3+2+2+1$ is represented by Figure 3 with this coloring. Given any partition ν with parts at most m (so at most m columns in the Ferrers diagram), if we color it as above, then for each $1 \leqslant i \leqslant m$, the i-th column in the Ferrers diagram of ν is colored by x_i. Taking the conjugate of ν, we arrive at a partition with at most m parts and the i-th part colored by x_i. The gives a one-to-one correspondence with the partitions μ in the above. In other words, the partitions μ can be generated by

$$\frac{1}{(1-x_1)(1-x_1x_2)\cdots(1-x_1x_2\cdots x_m)}.$$

This gives the denominator on the right hand side of (1), and therefore, Lemma 5 holds true.

2.2 Proof of Theorem 3

To prove Theorem 3, we also need the *Frobenius symbol* of a partition λ, which is a two-rowed array

$$\begin{pmatrix} s_1 & s_2 & \cdots & s_m \\ t_1 & t_2 & \cdots & t_m \end{pmatrix}$$

$$
\begin{array}{|c|c|c|c|c|}
\hline
x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
x_1 & x_2 & x_3 \\
\cline{1-3}
x_1 & x_2 & x_3 \\
\cline{1-3}
x_1 & x_2 & x_3 \\
\cline{1-3}
x_1 & x_2 \\
\cline{1-2}
x_1 & x_2 \\
\cline{1-2}
x_1 \\
\cline{1-1}
\end{array}
$$

Figure 3: Partition $5+3+3+3+2+2+1$.

with $s_1 > s_2 > \cdots > s_m \geqslant 0$ and $t_1 > t_2 > \cdots > t_m \geqslant 0$, where s_i (resp. t_i) counts the number of nodes to the right of (resp. below) the i-th diagonal entry of the Durfee square of λ in its Ferrers diagram.

Thus, the partition λ with Frobenius symbol above has

$$|\lambda| = \sum_{1\leqslant i\leqslant m} s_i + \sum_{1\leqslant i\leqslant m} t_i + m.$$

We also have

$$\ell(\lambda) = t_1 + 1, \qquad D(\lambda) = m$$

and for $1 \leqslant i \leqslant m$,

$$\Gamma_i(\lambda) = s_i + t_i + 1.$$

Notice that our desired result is equivalent to

$$\sum_{\lambda\in\mathscr{P}} (-1)^{\ell(\lambda)+D(\lambda)} q^{(\Gamma_1(\lambda)+1)+(\Gamma_3(\lambda)+1)+(\Gamma_5(\lambda)+1)+\cdots} = \frac{1}{(q^2;q^2)_\infty}, \tag{2}$$

where the q-Pochhammer symbol is defined for $n \in \mathbb{N}\cup\{\infty\}$ by

$$(A;q)_n := \prod_{k=0}^{n-1}(1-Aq^k).$$

In terms of the Frobenius symbol, it is also equivalent to show

$$1 + \sum_{\substack{m\geqslant 1\\ s_1>s_2>\cdots>s_m\geqslant 0\\ t_1>t_2>\cdots>t_m\geqslant 0}} (-1)^{t_1+m+1} q^{(s_1+t_1+2)+(s_3+t_3+2)+(s_5+t_5+2)+\cdots} = \frac{1}{(q^2;q^2)_\infty}. \tag{3}$$

Now, in (1), we set $a=0$, $b=1$, $x_1 = x_3 = \cdots = q$ and $x_2 = x_4 = \cdots = 1$. Then

$$\sum_{s_1>s_2>\cdots>s_m\geqslant 0} q^{s_1+s_3+s_5+\cdots} = \begin{cases} \dfrac{q^{k^2}}{(q;q)_k(q;q)_k} & \text{if } m=2k, \\[2ex] \dfrac{q^{k^2-k}}{(q;q)_k(q;q)_{k-1}} & \text{if } m=2k-1. \end{cases}$$

On the other hand, if we set $a = 0$, $b = 1$, $x_1 = -q$, $x_3 = x_5 = \cdots = q$ and $x_2 = x_4 = \cdots = 1$ in (1), then

$$\sum_{t_1>t_2>\cdots>t_m\geqslant 0} (-1)^{t_1} q^{t_1+t_3+t_5+\cdots} = \begin{cases} -\dfrac{q^{k^2}}{(-q;q)_k(-q;q)_k} & \text{if } m = 2k, \\ \dfrac{q^{k^2-k}}{(-q;q)_k(-q;q)_{k-1}} & \text{if } m = 2k-1. \end{cases}$$

Therefore,

$$\begin{aligned} &\sum_{\substack{s_1>s_2>\cdots>s_m\geqslant 0\\ t_1>t_2>\cdots>t_m\geqslant 0}} (-1)^{t_1+m+1} q^{(s_1+t_1+2)+(s_3+t_3+2)+(s_5+t_5+2)+\cdots} \\ &\quad = q^{2\lfloor\frac{m+1}{2}\rfloor} \sum_{s_1>s_2>\cdots>s_m\geqslant 0} q^{s_1+s_3+s_5+\cdots} \sum_{t_1>t_2>\cdots>t_m\geqslant 0} (-1)^{t_1+m+1} q^{t_1+t_3+t_5+\cdots} \\ &\quad = \begin{cases} \dfrac{q^{2k^2+2k}}{(q^2;q^2)_k(q^2;q^2)_k} & \text{if } m = 2k, \\ \dfrac{q^{2k^2}}{(q^2;q^2)_k(q^2;q^2)_{k-1}} & \text{if } m = 2k-1. \end{cases} \end{aligned} \tag{4}$$

It follows that

$$\begin{aligned} 1 + &\sum_{\substack{m\geqslant 1\\ s_1>s_2>\cdots>s_m\geqslant 0\\ t_1>t_2>\cdots>t_m\geqslant 0}} (-1)^{t_1+m+1} q^{(s_1+t_1+2)+(s_3+t_3+2)+(s_5+t_5+2)+\cdots} \\ &= 1 + \sum_{k\geqslant 1} \left(\frac{q^{2k^2+2k}}{(q^2;q^2)_k(q^2;q^2)_k} + \frac{q^{2k^2}}{(q^2;q^2)_k(q^2;q^2)_{k-1}} \right) \\ &= 1 + \sum_{k\geqslant 1} \frac{q^{2k^2}}{(q^2;q^2)_k(q^2;q^2)_k} \\ &= \frac{1}{(q^2;q^2)_\infty}, \end{aligned}$$

where we use an identity due to Euler [1, p. 21, (2.2.9)] in the last equality. Now, (3) is proved, and thus, we arrive at Theorem 3. $\square$

3 An involution

Recall Mork's bijection [6] between partitions of n and partitions μ into distinct parts $\mu_1 > \mu_2 > \mu_3 > \cdots$ such that $\mu_1 + \mu_3 + \mu_5 + \cdots = n$. The same correspondence gives a bijection between partitions of n into even parts and partitions μ into distinct even parts $\mu_1 > \mu_2 > \mu_3 > \cdots$ such that $\mu_1 + \mu_3 + \mu_5 + \cdots = n$.

Therefore, to derive a combinatorial proof of Theorem 3, it suffices to show the following result.

Theorem 6. *Let $\gamma(m,n)$ denote the weighted count of partitions λ with weight $w(\lambda) = (-1)^{\ell(\lambda)+D(\lambda)}$ such that the Durfee square has length $D(\lambda) = m$ and $(\Gamma_1(\lambda)+1)+(\Gamma_3(\lambda)+1)+(\Gamma_5(\lambda)+1)+\cdots = n$. Let $d(m,n)$ denote the number of partitions μ into m distinct even parts $\mu_1 > \mu_2 > \cdots > \mu_m$ such that $\mu_1+\mu_3+\mu_5+\cdots = n$. Then for any positive integers m and n, $\gamma(m,n) = d(m,n)$.*

Notice that by (4), we have, for any positive integer m,

$$\sum_{n\geqslant 0} \gamma(m,n)q^n = \begin{cases} \dfrac{q^{2k^2+2k}}{(q^2;q^2)_k(q^2;q^2)_k} & \text{if } m = 2k, \\ \dfrac{q^{2k^2}}{(q^2;q^2)_k(q^2;q^2)_{k-1}} & \text{if } m = 2k-1. \end{cases}$$

On the other hand, in (1), we may set $a = 1$, $b = 1$, $x_1 = x_3 = \cdots = q$ and $x_2 = x_4 = \cdots = 1$ to obtain

$$\sum_{j_1>j_2\cdots>j_m\geqslant 1} q^{j_1+j_3+j_5+\cdots} = \begin{cases} \dfrac{q^{k^2+k}}{(q;q)_k(q;q)_k} & \text{if } m = 2k, \\ \dfrac{q^{k^2}}{(q;q)_k(q;q)_{k-1}} & \text{if } m = 2k-1. \end{cases} \tag{5}$$

Replacing q by q^2 in the above yields

$$\sum_{n\geqslant 0} d(m,n)q^n = \begin{cases} \dfrac{q^{2k^2+2k}}{(q^2;q^2)_k(q^2;q^2)_k} & \text{if } m = 2k, \\ \dfrac{q^{2k^2}}{(q^2;q^2)_k(q^2;q^2)_{k-1}} & \text{if } m = 2k-1. \end{cases}$$

Thus, $\gamma(m,n) = d(m,n)$.

Below, we also give a combinatorial proof of this relation. Then, combining with Mork's bijection, we arrive at a combinatorial proof of Theorem 3.

Our starting point is an involution on the set $\mathscr{P}$ of partitions. We construct a map $\phi : \mathscr{P} \to \mathscr{P}$ as follows.

- Given any partition λ with Durfee square of length m, we decompose it as in Figure 4. Here, the block below the Durfee square gives a partition π and the block to the right of the Durfee square gives a partition $\overline{\mu}$ whose conjugate is μ.

- Let x be the smallest part in π that appears an odd number of times. Let y be the smallest part in μ. If x or y does not exist, we assume that it has size ∞.

- If $x \leqslant y$, then we delete the part x from π and add a part of size x to μ (and thus one column is added to $\overline{\mu}$). If $x > y$, we delete the part y from μ (and thus one column is deleted from $\overline{\mu}$) and add a part of size y to π. We call the new partition $\phi(\lambda)$.

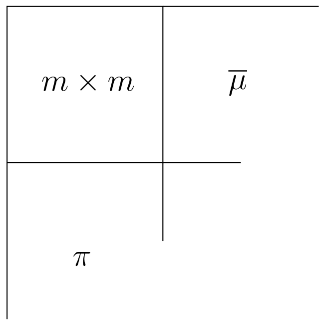

Figure 4: Decomposition of λ with Durfee square of length m.

Lemma 7. *The map ϕ is an involution on $\mathscr{P}$, that is, $\phi(\phi(\lambda)) = \lambda$ for any $\lambda \in \mathscr{P}$. Also, ϕ preserves the size of the Durfee square and each diagonal hook length.*

Proof. First, it is obvious from the construction of ϕ that it preserves the size of the Durfee square and each diagonal hook length.

For any partition λ, we decompose it as in Figure 4 and get π and $\overline{\mu}$ (and thus μ). For the image $\phi(\lambda)$, we decompose in the same way and obtain π^* and $\overline{\mu^*}$ (and thus μ^*). Also, $x(\pi)$ (resp. $x(\pi^*)$) denotes the smallest part in π (resp. π^*) that appears an odd number of times and $y(\mu)$ (resp. $y(\mu^*)$) denotes the smallest part in μ (resp. μ^*).

If both $x(\pi)$ and $y(\mu)$ are ∞, then $\pi^* = \pi$ and $\mu^* = \mu$, and thus, $\phi(\lambda) = \lambda$. Below we assume that the smaller one of $x(\pi)$ and $y(\mu)$ is not ∞.

If $x(\pi) \leqslant y(\mu)$, then π^* is obtained by deleting $x(\pi)$ from π. Thus, $x(\pi)$ appears an even number (including zero) of times in π^* and thus $x(\pi^*) > x(\pi)$. Also, $x(\pi)$ is added to μ to get μ^*. Since $x(\pi) \leqslant y(\mu)$, we have $y(\mu^*) = x(\pi)$. Hence, $x(\pi^*) > y(\mu^*)$. By the arguments in the next paragraph, we have $\phi(\phi(\lambda)) = \lambda$.

If $x(\pi) > y(\mu)$, then μ^* is obtained by deleting $y(\mu)$ from μ. Thus, $y(\mu^*) \geqslant y(\mu)$. Also, π^* is obtained by adding $y(\mu)$ to π. Since $x(\pi) > y(\mu)$, we know that $y(\mu)$ appears an even number (including zero) of times in π. Thus, $y(\mu)$ appears an odd number of times in π^*, and therefore, $x(\pi^*) = y(\mu)$. Hence, $x(\pi^*) \leqslant y(\mu^*)$. By the arguments in the previous paragraph, we also have $\phi(\phi(\lambda)) = \lambda$. $\square$

Lemma 8. *The only partitions that stay invariant under ϕ are those with μ the empty partition and π a partition with even multiplicities.*

Furthermore, for partitions λ not staying invariant under ϕ (that is, $\phi(\lambda) \neq \lambda$), we have $w(\phi(\lambda)) = -w(\lambda)$ where $w(\lambda)$ is as in Theorem 6.

Proof. The first part has already been shown in the proof of Lemma 7. For the second part, we also notice from Lemma 7 that $D(\phi(\lambda)) = D(\lambda)$. Furthermore, if λ does not stay invariant under ϕ, then the block below the Durfee square of $\phi(\lambda)$ is obtained by adding one part to or deleting one part from the block below the Durfee square of λ. Therefore, $\ell(\phi(\lambda))$ differs by ± 1 to $\ell(\lambda)$. We conclude that

$$w(\phi(\lambda)) = (-1)^{D(\phi(\lambda))+\ell(\phi(\lambda))} = (-1)^{D(\lambda)+\ell(\lambda)\pm 1} = -w(\lambda).$$

This gives the second part of the lemma. □

Now, we are ready to show Theorem 6.

Combinatorial proof of Theorem 6. For convenience, we denote by $\mathscr{P}_m$ the subset of $\mathscr{P}$ including partitions with Durfee square of length m. By Lemmas 7 and 8,

$$\sum_{\lambda\in\mathscr{P}_m} w(\lambda)q^{(\Gamma_1(\lambda)+1)+(\Gamma_3(\lambda)+1)+\cdots} = \sum_{\substack{\lambda\in\mathscr{P}_m\\ \lambda \text{ invariant under } \phi}} w(\lambda)q^{(\Gamma_1(\lambda)+1)+(\Gamma_3(\lambda)+1)+\cdots}.$$

Therefore, by Lemma 8, we are left with partitions λ in $\mathscr{P}_m$ such that in its decomposition, μ is the empty partition and π is a partition with even multiplicities. Notice that each column of π contains an even number of nodes. That is, if $\overline{\pi}$ is the conjugate of π, then all $\overline{\pi}_i$ are nonnegative even numbers for $1 \leqslant i \leqslant m$. Thus, $\Gamma_i(\lambda)+1 = \overline{\pi}_i+2(m+1-i)$. So we arrive at a partition into m distinct even parts.

Conversely, if we are given a partition ν into m distinct even parts, we subtract $2(m+1-i)$ from each part ν_i for $1 \leqslant i \leqslant m$. Then we append the conjugate of the resulting partition below an $m \times m$ box. The partitions that are invariant under ϕ are uniquely determined.

We therefore conclude that $\gamma(m,n) = d(m,n)$. □

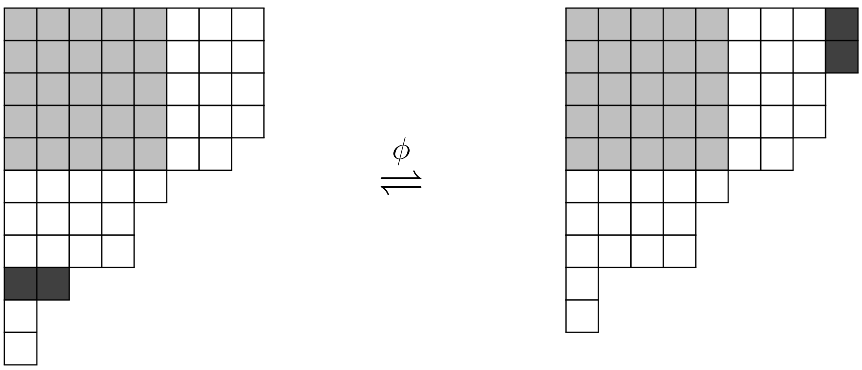

Figure 5: Partitions λ and λ^* in Example 9 under ϕ.

Example 9. Let $\lambda = 8+8+8+8+7+5+4+4+2+1+1$ and $\lambda^* = 9+9+8+8+7+5+4+4+1+1$. Then $\lambda^* = \phi(\lambda)$ and $\lambda = \phi(\lambda^*)$. See Figure 5.

Example 10. The partition $\lambda = 3+3+3+2+2+2+2+1+1$ with Durfee square of length 3 stays invariant under ϕ. Its diagonal hook lengths are given in Figure 6. Also, it corresponds to the partition $12+8+2$ into three distinct even parts.

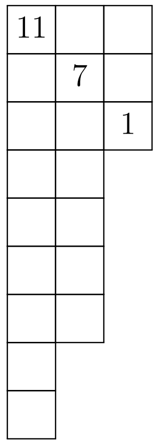

Figure 6: Partition λ in Example 10 that stays invariant under ϕ.

4 A variant of Mork's bijection and cranks of partitions

In [5], Hopkins, Sellers and Yee considered partitions with bounded cranks: for any non-negative integer j,

$$\sum_{\substack{\lambda\in\mathscr{P}\\ \operatorname{crank}(\lambda)\leqslant -j}} q^{|\lambda|} = \sum_{n\geqslant 0} \frac{q^{(n+1)(n+j)}}{(q;q)_n(q;q)_{n+j}}. \tag{6}$$

Here the *crank* of a partition λ is defined by Andrews and Garvan [2]:

$$\operatorname{crank}(\lambda) := \begin{cases} \ell(\lambda) & \text{if } \omega(\lambda)=0,\\ \mu(\lambda)-\omega(\lambda) & \text{if } \omega(\lambda)>0, \end{cases}$$

where $\omega(\lambda)$ denotes the number of ones in λ, and $\mu(\lambda)$ denotes the number of parts in λ that are larger than $\omega(\lambda)$. The existence of the crank statistic was first predicted by Dyson [4] to give a unified combinatorial interpretation of Ramanujan's congruences for the partition function.

Comparing (6) for $j=0$ and 1 with (5), it is natural to expect connections between Schmidt-type distinct partitions and partitions with nonpositive and negative cranks.

To start our investigation of such connections, let us review Mork's bijection given in [6].

Theorem 11 (Mork)**.** *For any positive integer k, there exists a bijection between partitions μ into $2k$ or $2k-1$ distinct parts such that $\mu_1+\mu_3+\mu_5+\cdots=n$ and partitions of n with Durfee square of length k.*

Let μ be as in Theorem 11. If μ has $2k-1$ parts, we append an empty part $\mu_{2k}=0$. Now, Mork's bijection ψ can be illustrated by Figure 7. Here, the value below (resp. to

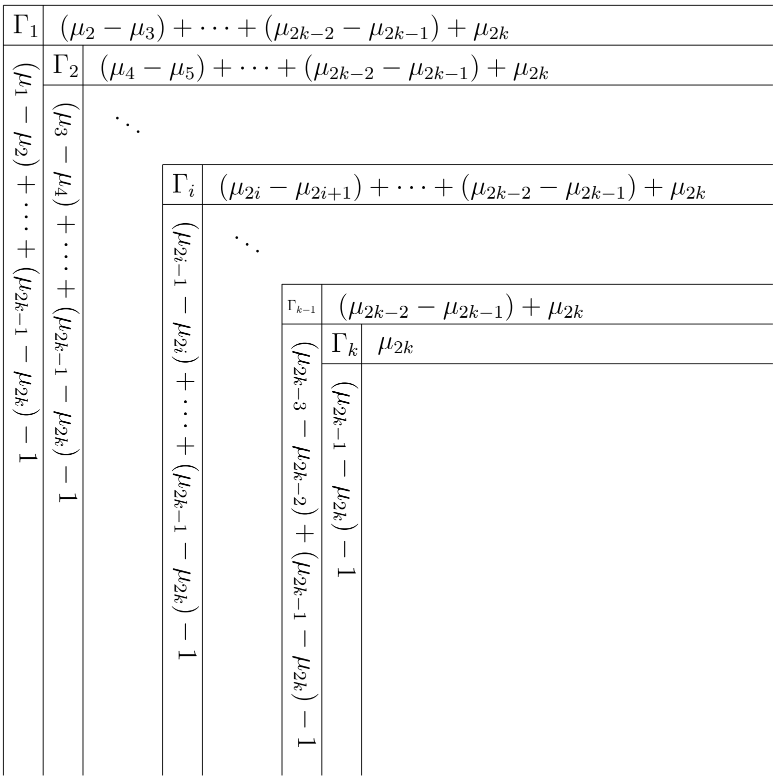

Figure 7: Mork's bijection $\psi(\mu)$.

the right of) the i-th diagonal node denotes the number of nodes in the Ferrers diagram of $\psi(\mu)$ that are below (resp. to the right of) the i-th diagonal node.

Next, we introduce the j-Durfee rectangle of a partition for j a nonnegative integer.

Definition 12 (j-Durfee rectangle)**.** The *j-Durfee rectangle* of a partition λ is the largest rectangle of size $d \times (d + j)$ that fits inside the Ferrers diagram of λ. We denote by $D_j(\lambda) = d$ the length of the j-Durfee rectangle. In particular, the 0-Durfee rectangle is the same as the Durfee square.

Now, we define a variant of Mork's bijection, denoted by ψ^*, as follows. Let $\mu \in \mathscr{D}$ be a partition into m distinct parts.

- If $m = 2k$, then $\psi^*(\mu)$ is illustrated by Figure 8.
- If $m = 1$, then $\psi^*(\mu)$ is $1 + 1 + \cdots + 1$ with 1 appearing μ_1 times.
- If $m = 2k - 1$ with $k \geqslant 2$, then $\psi^*(\mu)$ is illustrated by Figure 9.

Evidently, for any $\mu \in \mathscr{D}$, $\psi^*(\mu)$ is a partition.

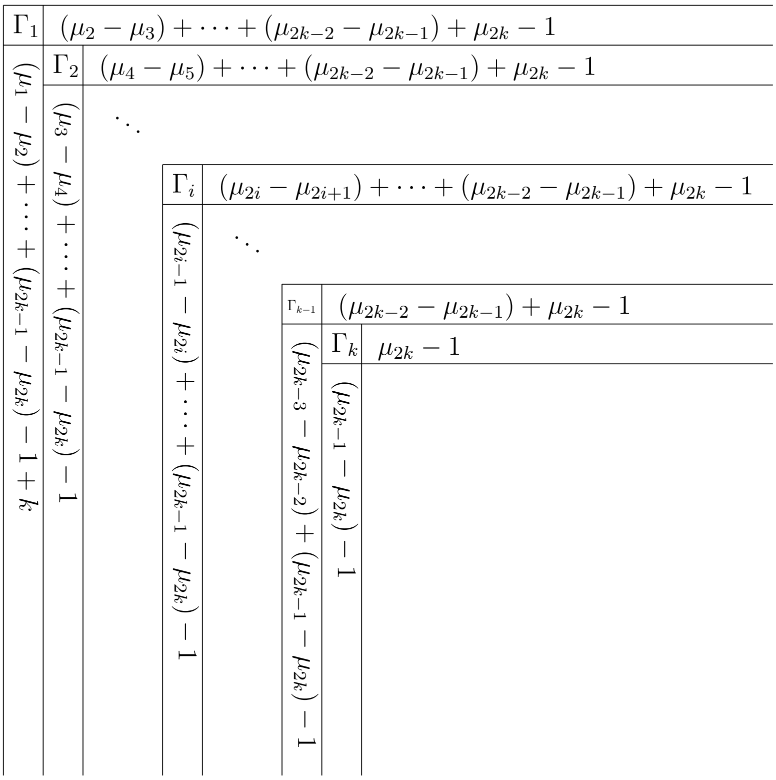

Figure 8: The map $\psi^*(\mu)$ with $\mu \in \mathscr{D}$ and $\ell(\mu) = 2k$.

Theorem 13. *Let k be any positive integer.*

The map ψ^* gives a bijection between partitions μ into $2k$ distinct parts such that $\mu_1 + \mu_3 + \mu_5 + \cdots = n$ and partitions λ of n with $D(\lambda) = k$ and crank nonpositive.

Also, the map ψ^* gives a bijection between partitions μ into $2k-1$ distinct parts such that $\mu_1 + \mu_3 + \mu_5 + \cdots = n$ and partitions λ of n with $D_1(\lambda) = k-1$ and crank negative.

For its proof, we require the following result due to Hopkins, Sellers and Yee [5].

Lemma 14. *Let j be a nonnegative integer. Then for any nonempty partition λ,* crank$(\lambda) \leqslant -j$ *if and only if $\omega(\lambda) - D_j(\lambda) \geqslant j$ where $\omega(\lambda)$ is the number of ones in λ and $D_j(\lambda)$ is the length of the j-Durfee rectangle of λ.*

Proof of Theorem 13. Let $\mu \in \mathscr{D}$. If $\ell(\mu) = 2k$, we have $\omega(\psi^*(\mu)) = k + (\mu_1 - \mu_2) - 1 \geqslant k$, and $D(\psi^*(\mu)) = k$. Thus, $\omega(\psi^*(\mu)) - D(\psi^*(\mu)) \geqslant 0$, and by Lemma 14, we have $\operatorname{crank}(\psi^*(\mu)) \leqslant 0$. Conversely, given any partition λ with $D(\lambda) = k$ and $\operatorname{crank}(\lambda) \leqslant 0$, we can compute each μ_1, μ_2, $\ldots$, μ_{2k} through the construction of ψ^*.

If $\ell(\mu) = 2k-1$, we have $\omega(\psi^*(\mu)) = \mu_1 \geqslant 1$ if $k = 1$ and $\omega(\psi^*(\mu)) = k+(\mu_1-\mu_2)-1 \geqslant k$ if $k \geqslant 2$, and $D_1(\psi^*(\mu)) = k-1$ (since $\mu_{2k-2} - \mu_{2k-1} \geqslant 1$ in the case $k \geqslant 2$). Thus, $\omega(\psi^*(\mu)) - D(\psi^*(\mu)) \geqslant 1$, and by Lemma 14, we have $\operatorname{crank}(\psi^*(\mu)) \leqslant -1$. Conversely, given any partition λ with $D_1(\lambda) = k-1$ and $\operatorname{crank}(\lambda) \leqslant -1$, we can recover each μ_1, μ_2, $\ldots$, μ_{2k-1} through the construction of ψ^*. $\square$

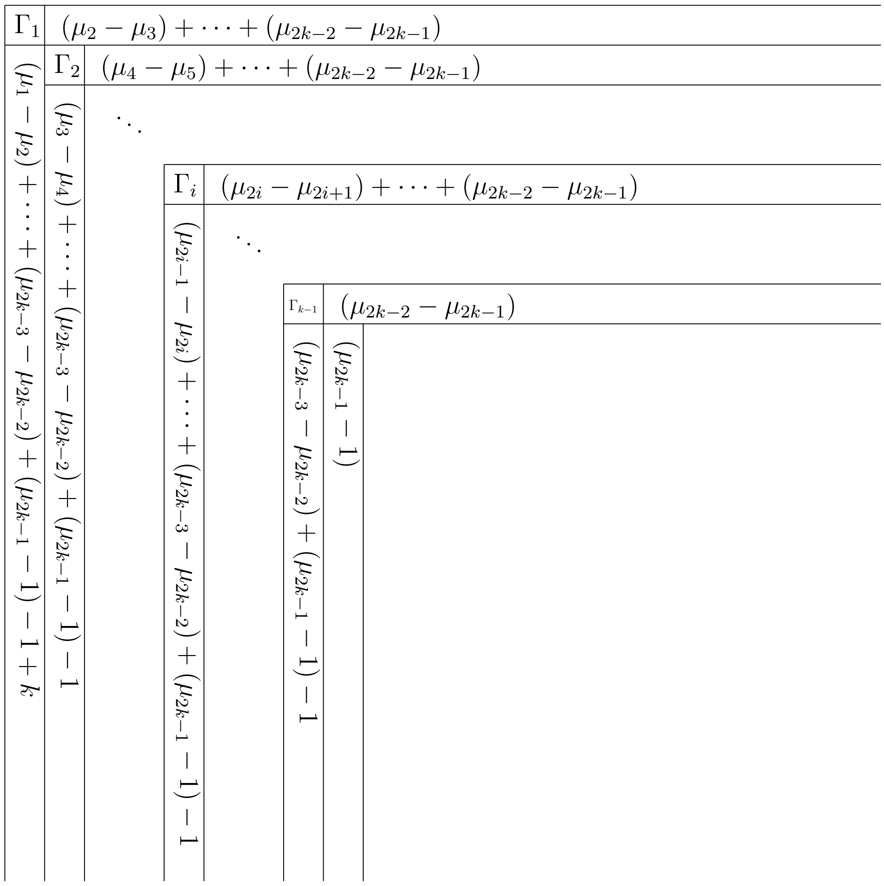

Figure 9: The map $\psi^*(\mu)$ with $\mu \in \mathscr{D}$ and $\ell(\mu) = 2k-1$ $(k \geqslant 2)$.

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