Combinatorial Perspectives on the Crank and Mex Partition Statistics

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Abstract

Several authors have recently considered the smallest positive part missing from an integer partition, known as the minimum excludant or mex. In this work, we revisit and extend connections between Dyson's crank statistic, the mex, and Frobenius symbols, with a focus on combinatorial proof techniques. One highlight is a generating function expression for the number of partitions with a bounded crank that does not include an alternating sum. This leads to a combinatorial interpretation involving types of Durfee rectangles. A recurring combinatorial technique uses sign reversing involutions on certain triples of partitions to establish a result of Fine and other identities.

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1 Preliminaries

For a positive integer n, a partition of n is a finite sequence of integers $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_r$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 1$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Write $\ell(\lambda) = r$ for the length of λ or number of parts and $|\lambda| = n$ for its weight or sum. Let $p(n)$ denote the number of partitions of n and define $p(0) = 1$. Also, let $q(n)$ denote the number of partitions of n with distinct parts.

In 1944, Dyson [16] suggested the existence of an integer partition statistic, which he called the crank, to combinatorially prove a divisibility property of $p(n)$ proven 25 years earlier by Ramanujan via generating function manipulations. In 1988, Andrews and Garvan [4] provided a definition of the elusive crank, which we restate here.

For a partition λ, let $\omega(\lambda)$ be the number of parts 1 and $\mu(\lambda)$ the number of parts greater than $\omega(\lambda)$. Then the crank of λ is

$$\operatorname{crank}(\lambda) = \begin{cases} \lambda_1 & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0. \end{cases} \tag{1}$$

Example 1. $\operatorname{crank}(5, 4, 4, 2, 2) = 5$ and $\operatorname{crank}(5, 4, 2, 2, 1, 1) = 2 - 2 = 0$.

In a separate work, Garvan [19] provided very useful generating function results related to the crank statistic. Given integers m and $n > 1$, let $M(m, n)$ be the number of partitions of n with crank m. Also, define $M(0, 0) = M(1, 1) = -M(0, 1) = M(-1, 1) = 1$. His results incorporate the standard q-series notation

$$(a; q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n), \quad (a; q)_n = \frac{(aq; q)_\infty}{(aq^n; q)_\infty} \text{ for any integer } n.$$

Theorem 2 (Garvan)**.** *We have*

$$\sum_{m=-\infty}^{\infty} \sum_{n\geqslant 0} M(m, n)z^m q^n = \frac{(q; q)_\infty}{(zq; q)_\infty(q/z; q)_\infty}, \tag{2}$$

$$\sum_{n\geqslant 0} M(m, n)q^n = \frac{1}{(q; q)_\infty} \sum_{n\geqslant 1}(-1)^{n-1} q^{n(n-1)/2+n|m|}(1 - q^n). \tag{3}$$

Note that it is clear from (2) that, for all integers m,

$$M(m, n) = M(-m, n). \tag{4}$$

In recent years, another integer partition statistic has arisen which is much easier to define, commonly known as the mex of a partition, from minimal excludant. The mex of a set of integers is the smallest positive integer not in the set.

Example 3. $\operatorname{mex}(5, 4, 4, 2, 2) = 1$, $\operatorname{mex}(5, 4, 2, 1, 1) = 3$, and $\operatorname{mex}(4, 3, 2, 1) = 5$.

The application of the mex to partitions dates to at least 2006 with Grabner and Knopfmacher [21]; they called it the least gap. In 2011, Andrews [2] worked with "the smallest part that is *not* a summand." Recently, Andrews and Newman [5, 6] began using the term mex and introduced various related statistics.

The initial connection between crank and mex was found independently in [6, Theorem 2] and [22, Theorem 1]:

Theorem 4 (Andrews, Newman; Hopkins, Sellers)**.** *The number of partitions of n with nonnegative crank equals the number of partitions of n with odd mex.*

Further results of Hopkins and Sellers [22], followed by Hopkins, Sellers, and Stanton [23], establish additional connections between partitions satisfying certain crank conditions and partitions with certain mex properties. Those proofs primarily use generating function identities. Here we provide a combinatorial perspective on some of those results and establish new ones. Also, recent work of Huh and Kim [24] includes results involving some of these ideas; see Section 3.2.

Other results of Hopkins, Sellers, and Stanton [23] show unexpected connections between the crank and another representation of integer partitions: The Frobenius symbol of a partition consists of two rows of strictly decreasing nonnegative integers. Given the Ferrers diagram of a partition, the top row of the Frobenius symbol gives the number of boxes to the right of the diagonal entries and the bottom row gives the number of boxes below the diagonal entries.

Example 5. The partition $5, 4, 4, 2, 2$ has Frobenius symbol

$$\begin{pmatrix} 4 & 2 & 1 \\ 4 & 3 & 0 \end{pmatrix}.$$

In subsequent work, Andrews, Dastidar, and Morrill [3] have found additional connections between the crank and the Frobenius symbol and, at the end of their article, ask for combinatorial insight to these new relations. We believe the work in Section 4 helps answer that request.

Here are a few more concepts we will use.

In addition to the notation $(a; q)_n$, we need the q-binomial coefficients

$$\begin{bmatrix} n \\ d \end{bmatrix} = \frac{(q; q)_n}{(q; q)_d (q; q)_{n-d}}$$

which are also known as Gaussian polynomials. In particular, we will use the identity [20, (I.43)],

$$\frac{1}{(q; q)_d} = \frac{1}{(q^{n-d+1}; q)_d} \begin{bmatrix} n \\ d \end{bmatrix} \tag{5}$$

with the following combinatorial interpretation: Partitions into at most d parts are in bijection with pairs of partitions where the first is a partition into parts between $n-d+1$ and n and the second is a partition into at most d parts all at most $n - d$. For more details, see the material on the "kth excess" of a partition [1, p. 50].

For an integer j, we define the j-Durfee rectangle of a partition λ to be the largest rectangle of size $d \times (d + j)$ that fits inside the Ferrers diagram of λ. The well-known Durfee square, whose dimensions match the number of columns in the Frobenius symbol, corresponds to $j = 0$.

Example 6. The partition $5, 4, 4, 2, 2$ has the following nonempty j-Durfee rectangles.

j	-4	-3	-2	-1	0	1	2	3	4
j-Durfee rectangle	5×1	5×2	4×2	3×2	3×3	3×4	2×4	1×4	1×5

Many of our results concern the number of partitions having crank values bounded below by a given integer. For $j \geqslant 0$, the next theorem uses results of Garvan to express the number of partitions λ with $\mathrm{crank}(\lambda) \geqslant j$. The first identity follows from (4) and the second from (3).

Theorem 7. *For an integer $j \geqslant 0$,*

$$\begin{aligned}\sum_{m \geqslant j} \sum_{n \geqslant 0} M(m,n)\, q^n &= \sum_{m \geqslant j} \sum_{n \geqslant 0} M(-m,n)\, q^n \\ &= \frac{1}{(q;q)_\infty} \sum_{n \geqslant 0} (-1)^n q^{n(n+1)/2 + j(n+1)}. \end{aligned} \tag{6}$$

There has been a surge of research on the mex and these related concepts. In addition to [3, 24] mentioned above and discussed in more detail below, the interested reader will want to look at [7, 8, 10, 11, 14, 15, 25]. There are even very recent results [9, 12, 26] that build on a preprint version of this current work.

In Section 2, we present a generating function result that is the foundation for much of the paper. Unlike (6), the generating function of Theorem 8 is not an alternating sum and leads more easily to combinatorial arguments. As with several of our results, we present both analytic and combinatorial proofs. In fact, Section 2 includes a second combinatorial proof of a lemma used in the analytic proof of Theorem 8. In Section 3 we turn to the mex statistic, in particular partitions with odd mex further refined by the parity of the length and the mex modulo 4 (deriving their generating functions uses Theorem 8). This leads to easier proofs of some earlier results and expanded connections between partitions with certain mex characteristics and certain ranges of crank values. Finally, in Section 4 we reconsider relations between the crank and the Frobenius symbol from a combinatorial perspective, including another application of Theorem 8.

2 Bounded crank generating function

Our first major result establishes another generating function for the number of partitions with crank bounded below by an arbitrary integer.

Theorem 8. *For any integer j,*

$$\sum_{m \geqslant j} \sum_{n \geqslant 0} M(m,n) q^n = \sum_{n \geqslant 0} \frac{q^{(n+1)(n+j)}}{(q;q)_n (q;q)_{n+j}}.$$

Note that, in contrast to (6) of Theorem 7, the right-hand side here does not involve an alternating sum. This makes this expression more amenable to combinatorial interpretation, as we will see in the proofs below. A key insight is to use the symmetry (4) and focus on partitions with nonpositive crank which arise only from the second part of the definition (1).

2.1 Analytic and combinatorial proofs of Theorem 8

For an analytic proof of Theorem 8, we need the following identity given by Fine [18, (20.51)].

Lemma 9 (Fine)**.** *We have*

$$(t;q)_\infty \sum_{n\geqslant 0} \frac{t^n}{(q;q)_n(bq;q)_n} = \frac{1}{(bq;q)_\infty} \sum_{n\geqslant 0} \frac{(t;q)_n}{(q;q)_n}(-b)^n q^{n(n+1)/2}$$
$$= \sum_{n\geqslant 0} \frac{(bt)^n q^{n^2}}{(q;q)_n(bq;q)_n}. \tag{7}$$

In particular, we use the following application of Lemma 9.

Lemma 10. *For $j \geqslant 0$,*

$$\frac{1}{(q;q)_\infty} \sum_{n\geqslant 0} (-1)^n q^{n(n+1)/2+j(n+1)} = \sum_{n\geqslant 0} \frac{q^{(n+1)(n+j)}}{(q;q)_n(q;q)_{n+j}}.$$

Analytic proof of Lemma 10. Setting $t \to q$ and $b \to q^j$ in (7) gives

$$\frac{1}{(q^{j+1};q)_\infty} \sum_{n\geqslant 0} (-1)^n q^{n(n+1)/2+nj} = \sum_{n\geqslant 0} \frac{q^{n^2+n(j+1)}}{(q,q^{j+1};q)_n}.$$

Now multiply both sides by $q^j/(q;q)_j$. □

Analytic proof of Theorem 8. For $j \geqslant 0$, the result follows from (6) and Lemma 10 directly. For $j < 0$, we start with the right-hand side in the theorem. For notational convenience, we replace j by $-j$ and assume $j > 0$. Then

$$\sum_{n\geqslant 0} \frac{q^{(n+1)(n-j)}}{(q;q)_n(q;q)_{n-j}} = \sum_{n\geqslant j} \frac{q^{(n+1)(n-j)}}{(q;q)_n(q;q)_{n-j}}$$
$$= \sum_{n\geqslant 0} \frac{q^{n(n+j+1)}}{(q;q)_n(q;q)_{n+j}}$$
$$= \frac{1}{(q;q)_\infty} \sum_{n\geqslant 0} (-1)^n q^{n(n+1)/2+jn}. \tag{8}$$

On the other hand, by summing over $m \geqslant -j$ in both sides of (3) we have

$$\sum_{m\geqslant -j} \sum_{n\geqslant 0} M(m,n)q^n$$
$$= \frac{1}{(q;q)_\infty} \sum_{m\geqslant -j} \sum_{n\geqslant 1} (-1)^{n-1} q^{n(n-1)/2+n|m|}(1-q^n)$$

$$
\begin{aligned}
&= \frac{1}{(q;q)_\infty}\left(\sum_{m\geqslant 0}\sum_{n\geqslant 1}(-1)^{n-1}q^{n(n-1)/2+nm}(1-q^n)+\sum_{m=1}^{j}\sum_{n\geqslant 1}(-1)^{n-1}q^{n(n-1)/2+nm}(1-q^n)\right)\\
&= \frac{1}{(q;q)_\infty}\left(\sum_{n\geqslant 1}(-1)^{n-1}q^{n(n-1)/2}+\sum_{n\geqslant 1}(-1)^{n-1}q^{n(n-1)/2+n}(1-q^{jn})\right)\\
&= \frac{1}{(q;q)_\infty}\left(1+\sum_{n\geqslant 1}(-1)^{n}q^{n(n+1)/2+jn}\right)\\
&= \frac{1}{(q;q)_\infty}\sum_{n\geqslant 0}(-1)^{n}q^{n(n+1)/2+jn}
\end{aligned}
$$

which is the last expression in (8). □

Our combinatorial proof uses the j-Durfee rectangles discussed in Section 1.

Combinatorial proof of Theorem 8. We begin with the case $j \geqslant 0$. By (4), it is sufficient to check the generating function for partitions λ with $\text{crank}(\lambda) \leqslant -j$.

Note that for a partition λ, if $\omega(\lambda) = 0$, then its crank is always positive. Thus, we suppose $\omega(\lambda) > 0$. Suppose the j-Durfee rectangle of λ is of size $d \times (d+j)$. Since $\lambda_d \geqslant d+j$, we see that if $\omega(\lambda) < d+j$, then $\mu(\lambda) \geqslant d$, thus

$$\text{crank}(\lambda) = \mu(\lambda) - \omega(\lambda) > d - (d+j) = -j,$$

so that if $\text{crank}(\lambda) \leqslant -j$, then $\omega(\lambda) \geqslant d+j$. Also, if $\omega(\lambda) \geqslant d+j$, then clearly $\mu(\lambda) < d$, hence

$$\text{crank}(\lambda) = \mu(\lambda) - \omega(\lambda) < d - (d+j) = -j.$$

Therefore, the generating function for such λ is

$$\sum_{d\geqslant 0}\frac{q^{d(d+j)+(d+j)}}{(q;q)_d(q;q)_{d+j}}$$

where the exponent $d(d+j)+(d+j)$ of q in the numerator accounts for the j-Durfee rectangle and the lower bound of $\omega(\lambda)$, while the factors $(q;q)_d$ and $(q;q)_{d+j}$ in the denominator account for the parts to the right of and below the j-Durfee rectangle, respectively.

We now consider the case $j < 0$. Rather than consider partitions λ with $\text{crank}(\lambda) \geqslant j$, we use complementarity and consider the smaller set of partitions with $\text{crank}(\lambda) < j$, where the previous argument applies. A more detailed argument follows.

For notational convenience, we replace j by $-j$ and assume $j > 0$. Then

$$
\begin{aligned}
\sum_{m\geqslant -j}\sum_{n\geqslant 0}M(m,n)q^n &= \sum_{m=-\infty}^{\infty}\sum_{n\geqslant 0}M(m,n)q^n - \sum_{m\leqslant -j-1}\sum_{n\geqslant 0}M(m,n)q^n\\
&= \sum_{n\geqslant 0}p(n)q^n - \sum_{m\leqslant -(j+1)}\sum_{n\geqslant 0}M(m,n)q^n.
\end{aligned}
$$

Thus, it suffices to prove that

$$\sum_{n\geqslant 0} p(n)q^n - \sum_{m\leqslant -(j+1)} \sum_{n\geqslant 0} M(m,n)q^n = \sum_{n\geqslant 0} \frac{q^{(n+1)(n-j)}}{(q;q)_n(q;q)_{n-j}} = \sum_{n\geqslant 0} \frac{q^{n(n+j+1)}}{(q;q)_{n+j}(q;q)_n}$$

which is equivalent to

$$\begin{aligned}
\sum_{m\leqslant -(j+1)} \sum_{n\geqslant 0} M(m,n)q^n &= \sum_{n\geqslant 0} p(n)q^n - \sum_{n\geqslant 0} \frac{q^{n(n+j+1)}}{(q;q)_{n+j}(q;q)_n} \\
&= \sum_{n\geqslant 0} \frac{q^{n(n+j+1)}}{(q;q)_{n+j+1}(q;q)_n} - \sum_{n\geqslant 0} \frac{q^{n(n+j+1)}}{(q;q)_{n+j}(q;q)_n} \\
&= \sum_{n\geqslant 0} \frac{q^{n(n+j+1)}}{(q;q)_{n+j+1}(q;q)_n} \left(1-(1-q^{n+j+1})\right) \\
&= \sum_{n\geqslant 0} \frac{q^{(n+1)(n+j+1)}}{(q;q)_{n+j+1}(q;q)_n} \\
&= \sum_{m\geqslant j+1} \sum_{n\geqslant 0} M(m,n)q^n
\end{aligned}$$

which must hold as seen in the proof of the j positive case. The first sum in the second line follows by taking $-(j+1)$-Durfee rectangles for partitions. $\square$

2.2 Combinatorial proof of Lemma 10

Keeping with the combinatorial theme of the paper, we provide a combinatorial proof of Lemma 10 that uses certain triples of partitions and two forms of cancellation.

Combinatorial proof of Lemma 10. Setting $t \to q$ and $b \to q^j$ in Lemma 9 and multiplying by $q^j/(q;q)_j$, we get

$$\begin{aligned}
(q;q)_\infty \sum_{n\geqslant 0} \frac{q^{n+j}}{(q;q)_n(q;q)_{n+j}} &= \frac{1}{(q;q)_\infty} \sum_{n\geqslant 0} (-1)^n q^{n(n+1)/2+j(n+1)} \\
&= \sum_{n\geqslant 0} \frac{q^{(n+1)(n+j)}}{(q;q)_n(q;q)_{n+j}}.
\end{aligned}$$

Thus, proving Lemma 10 combinatorially is equivalent to proving the following identities combinatorially:

$$(q;q)_\infty \sum_{n\geqslant 0} \frac{q^{n+j}}{(q;q)_n(q;q)_{n+j}} = \frac{1}{(q;q)_\infty} \sum_{n\geqslant 0} (-1)^n q^{n(n+1)/2+j(n+1)}, \tag{9}$$

$$(q;q)_\infty \sum_{n\geqslant 0} \frac{q^{n+j}}{(q;q)_n(q;q)_{n+j}} = \sum_{n\geqslant 0} \frac{q^{(n+1)(n+j)}}{(q;q)_n(q;q)_{n+j}}. \tag{10}$$

Let $\mathcal{T}_j$ be the set of triples of partitions $(\pi;\kappa;\nu)$ such that π is a partition into distinct parts, κ is a nonempty partition with largest part at least j, and ν is a partition into parts that are at least j less than the largest part of κ.

Example 11. The weight 5 elements of $\mathcal{T}_3$ are

$$(\emptyset;3,2;\emptyset),(\emptyset;3,1,1;\emptyset),(2;3;\emptyset),(1;3,1;\emptyset),(\emptyset;4,1;\emptyset),(1;4;\emptyset),(\emptyset;4;1),(\emptyset;5;\emptyset).$$

Since π is independent of κ and ν, whereas the largest part of ν is at least j less than the largest part of κ, we have

$$\sum_{(\pi;\kappa;\nu)\in\mathcal{T}_j}(-1)^{\ell(\pi)}q^{|\pi|+|\kappa|+|\nu|}=\sum_{\pi}(-1)^{\ell(\pi)}q^{|\pi|}\sum_{(\kappa;\nu)}q^{|\kappa|+|\nu|}$$

where the second sum on the right-hand side is over all pairs of κ and ν with the largest part of ν at least j less than the largest part of κ. Thus,

$$\sum_{(\pi;\kappa;\nu)\in\mathcal{T}_j}(-1)^{\ell(\pi)}q^{|\pi|+|\kappa|+|\nu|}=(q;q)_\infty\sum_{n\geqslant 0}\frac{q^{n+j}}{(q;q)_n(q;q)_{n+j}}. \tag{11}$$

We make cancellations in $\mathcal{T}_j$ in two ways; the first cancellation will lead to (9) and the second to (10). For a triple $(\pi;\kappa;\nu)$ in $\mathcal{T}_j$, call the first largest part of κ the peak and let the peak equal $n+j$ for some $n\geqslant 0$.

• First cancellation: If all the parts of π are greater than $n+j$ and κ does not have any parts other than the peak, then we do nothing.

If the smallest part of π is at most the smallest part of κ, then we move the smallest part of π to κ. If the smallest part of κ that is not a peak is less than the smallest part of π, then we move the smallest part of κ to π. This process is clearly an involution. Also, it increases or decreases the number of parts of π by 1, so this is a sign reversing involution. Thus, after cancellation, the remaining triples in $\mathcal{T}_j$ are $(\pi;\kappa;\nu)$ where κ has only the peak $n+j$, the parts of π are greater than $n+j$ and distinct, and the parts of ν are at most n. Therefore,

$$\sum_{(\pi;\kappa;\nu)\in\mathcal{T}_j}(-1)^{\ell(\pi)}q^{|\pi|+|\kappa|+|\nu|}=\sum_{n\geqslant 0}\frac{q^{n+j}(q^{n+j+1};q)_\infty}{(q;q)_n}.$$

We now make further adjustment on the remaining $(\pi;\kappa;\nu)$: Suppose π has m distinct parts for some $m\geqslant 0$. Subtract j from the peak and successively subtract $j+1$ from the smallest part of π, $j+2$ from the second smallest part of π, and so on. The resulting parts of π are at least n. Thus, all the parts of the resulting π, κ, and ν form an ordinary partition and the subtracted sequence, i.e., $j,j+1,\dots,j+m$, forms a partition into $m+1$ distinct parts differing by exactly one with smallest part j. Conversely, for an ordinary partition λ and the sequence of $m+1$ consecutive numbers starting from j, i.e., $j,j+1,\dots,j+m$, we add $j+m$ to the largest part of λ, $j+m-1$ to the second largest part of λ, and so on. If λ has fewer than $m+1$ parts, then we temporarily add zeros as parts to make λ have length $m+1$. The first m parts of the resulting partition are clearly distinct and they form π, the $(m+1)$st part becomes the peak of κ, and any remaining parts form ν. Therefore,

$$\sum_{(\pi;\kappa;\nu)\in\mathcal{T}_j}(-1)^{\ell(\pi)}q^{|\pi|+|\kappa|+|\nu|}=\frac{1}{(q;q)_\infty}\sum_{m\geqslant 0}(-1)^m q^{m(m+1)/2+j(m+1)}.$$

Combining this with (11) completes the proof of (9).

• Second cancellation: We first make some adjustments on $(\pi;\kappa;\nu)\in\mathcal{T}_j$. Put the peak aside momentarily and take the j-Durfee rectangle of the non-peak parts of κ, which has size $d\times(d+j)$. Since the parts of κ are at most $n+j$, we see from the definition of the j-Durfee rectangle that to the right of the rectangle, there are at most d parts that are at most $n-d$. By applying the bijection associated with (5) to those parts to the right of the Durfee rectangle with parts in ν that are between $n-d+1$ and n (if they exist), we obtain a partition with at most d parts. Putting this back to the right of the j-Durfee rectangle yields a partition with a j-Durfee rectangle and no restriction on part sizes. By abuse of notation, we denote by κ the resulting partition with the j-Durfee rectangle and by ν the remaining parts at most $n-d$. After dividing the peak into $d+j$ and $n-d$, add $d+j$ back to the new κ and $n-d$ to the new ν. Therefore, we have

$$\sum_{(\pi;\kappa;\nu)\in\mathcal{T}_j}(-1)^{\ell(\pi)}q^{|\pi|+|\kappa|+|\nu|}=(q;q)_\infty\sum_{d\geqslant 0}\frac{q^{d(d+j)+d+j}}{(q;q)_{d+j}(q)_d}\sum_{n\geqslant d}\frac{q^{n-d}}{(q;q)_{n-d}}.$$

Let x be the smallest part of π and y the smallest part of ν. If $x\leqslant y$, then move x to ν. If $x>y$, then move y to π. This is clearly a sign reversing involution. After this cancellation, the remaining triples are $(\emptyset;\kappa;\emptyset)$. Therefore, we have

$$\sum_{(\pi;\kappa;\nu)\in\mathcal{T}_j}(-1)^{\ell(\pi)}q^{|\pi|+|\kappa|+|\nu|}=\sum_{d\geqslant 0}\frac{q^{d(d+j)+d+j}}{(q;q)_{d+j}(q)_d}.$$

Combining this with (11) completes the proof of (10). □

3 Connecting mex and crank

We now shift our attention to various functions related to the mex of an integer partition, ultimately with the goal of connecting such mex results with partitions satisfying certain crank conditions. Toward this end, define $m_{a,b}(n)$ to be the number of partitions of weight n with mex congruent to a modulo b. For example, $m_{1,2}(n)$ is the number of partitions of n with odd mex. In [23], it was productive to split the odd mex partitions modulo 4; in our notation, these are $m_{1,4}(n)$ and $m_{3,4}(n)$. (Note that the $m_{1,2}(n)$, $m_{1,4}(n)$, $m_{3,4}(n)$ are called $o(n)$, $o_1(n)$, $o_3(n)$, respectively, in [23].)

Next, we show that breaking these counts down further, by parity of partition length, shows new relations and allows for cleaner proofs of previous results. We conclude the section with more refined connections between partitions with odd mex and those with nonpositive crank, answering a question posed in [23].

3.1 Refinements on odd mex by parity of length

We will use the following lemma, which can be adapted from Ewell [17, (6)]:

$$\sum_{n\geqslant 0}(-q)^{n(n+1)/2} = \prod_{n=1}^{\infty}\frac{1-q^{2n}}{1+q^{2n-1}}.$$

Lemma 12. *We have*

$$\frac{1}{(q;q)_\infty}\sum_{n\geqslant 0}(-q)^{n(n+1)/2} = (-q^2;q^2)_\infty.$$

With this and Theorem 8, generating functions for the initial refinements of the odd mex statistics can be found rather easily.

Proposition 13. *We have*

$$\begin{aligned}
\sum_{n\geqslant 0} m_{1,4}(n)q^n &= \frac{1}{(q;q)_\infty}\sum_{k\geqslant 0} q^{2k(4k+1)}(1-q^{4k+1}) && (12)\\
&= \frac{1}{2}\left(\sum_{n\geqslant 0}\frac{q^{n(n+1)}}{(q;q)_n^2} + \sum_{n\geqslant 0}\frac{q^{n(n+1)}}{(q^2;q^2)_n}\right), && (13)\\
\sum_{n\geqslant 0} m_{3,4}(n)q^n &= \frac{1}{(q;q)_\infty}\sum_{k\geqslant 0} q^{(2k+1)(4k+3)}(1-q^{4k+3}) && (14)\\
&= \frac{1}{2}\left(\sum_{n\geqslant 0}\frac{q^{n(n+1)}}{(q;q)_n^2} - \sum_{n\geqslant 0}\frac{q^{n(n+1)}}{(q^2;q^2)_n}\right). && (15)
\end{aligned}$$

Proof. The generating functions (12) and (14) follow directly from the definitions of $m_{1,4}(n)$ and $m_{3,4}(n)$.

The verification of the other generating functions uses Theorem 8 and Lemma 12:

$$\begin{aligned}
\sum_{n\geqslant 0}(m_{1,4}(n)+m_{3,4}(n))q^n &= \sum_{n\geqslant 0} m_{1,2}(n)q^n\\
&= \frac{1}{(q;q)_\infty}\sum_{k\geqslant 0} q^{k(2k+1)}(1-q^{2k+1})\\
&= \sum_{n\geqslant 0}\frac{q^{n(n+1)}}{(q;q)_n^2},\\
\sum_{n\geqslant 0}(m_{1,4}(n)-m_{3,4}(n))q^n &= \frac{1}{(q;q)_\infty}\sum_{k\geqslant 0}(-1)^k q^{k(2k+1)}(1-q^{2k+1})\\
&= \frac{1}{(q;q)_\infty}\sum_{k\geqslant 0}(-q)^{k(k+1)/2}\\
&= (-q^2;q^2)_\infty
\end{aligned}$$

$$= \sum_{n \geqslant 0} \frac{q^{n^2+n}}{(q^2;q^2)_n}.$$

The expressions (13) and (15) follow. □

The following result [23, Proposition 9] connects $m_{1,4}(n)$ and $m_{3,4}(n)$.

Proposition 14 (Hopkins, Sellers, Stanton)**.** *For any $n \geqslant 1$,*

$$m_{1,4}(n) = \begin{cases} m_{3,4}(n) & \text{if } n \text{ is odd,} \\ m_{3,4}(n) + q(n/2) & \text{if } n \text{ is even.} \end{cases}$$

A different proof than the one given in [23] will follow from Theorem 15 as detailed below.

Here, we consider further refinements of these statistics incorporating the parity of partition length. A superscript o denotes the number of designated partitions with odd length, similarly a superscript e for even length. For instance, $m^o_{3,4}(n)$ is the number of partitions of n with an odd number of parts and mex congruent to 3 modulo 4, while $q^e(n)$ is the number of partitions of n into an even number of distinct parts. Values of some of these statistics for small values of n are given in Table 1. The $m_{1,2}(n)$ sequence matches [27, A064428]; at the time of writing, no other rows are currently in that encyclopedia.

Several relations between these statistics follow by definition:

$$\begin{gathered} m_{1,2}(n) = m_{1,4}(n) + m_{3,4}(n) = m^o_{1,2}(n) + m^e_{1,2}(n), \\ m_{1,4}(n) = m^o_{1,4}(n) + m^e_{1,4}(n), \end{gathered}$$

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$m_{1,2}(n)$	1	2	3	4	6	8	12	16	23	30	42	54	73	94
$m_{1,4}(n)$	1	1	2	2	4	4	7	8	13	15	23	27	39	47
$m_{3,4}(n)$	0	1	1	2	2	4	5	8	10	15	19	27	34	47
$m^o_{1,2}(n)$	1	1	2	2	3	4	6	8	11	15	21	27	36	47
$m^e_{1,2}(n)$	0	1	1	2	3	4	6	8	12	15	21	27	37	47
$m^o_{1,4}(n)$	1	1	1	1	2	2	3	4	6	8	11	14	19	24
$m^e_{1,4}(n)$	0	0	1	1	2	2	4	4	7	7	12	13	20	23
$m^o_{3,4}(n)$	0	0	1	1	1	2	3	4	5	7	10	13	17	23
$m^e_{3,4}(n)$	0	1	0	1	1	2	2	4	5	8	9	14	17	24

Table 1: Values of various refined odd mex statistics for small n.

$$
\begin{aligned}
m_{3,4}(n) &= m^o_{3,4}(n) + m^e_{3,4}(n),\\
m^o_{1,2}(n) &= m^o_{1,4}(n) + m^o_{3,4}(n),\\
m^e_{1,2}(n) &= m^e_{1,4}(n) + m^e_{3,4}(n).
\end{aligned}
$$

We prove another relation between these statistics that will simplify several previous results and serves as a natural refinement of Proposition 14.

Theorem 15. *For any $n \geqslant 1$,*

$$
m^o_{1,4}(n) = \begin{cases} m^e_{3,4}(n) & \text{if } n \text{ is odd,}\\ m^e_{3,4}(n) + q^o(n/2) & \text{if } n \text{ is even;}\end{cases}
$$

$$
m^e_{1,4}(n) = \begin{cases} m^o_{3,4}(n) & \text{if } n \text{ is odd,}\\ m^o_{3,4}(n) + q^e(n/2) & \text{if } n \text{ is even.}\end{cases}
$$

Proof. By the definitions, we see that

$$
\begin{aligned}
\sum_{n\geqslant 0} \left(m^e_{1,4}(n) + m^o_{1,4}(n)\right) q^n &= \frac{1}{(q;q)_\infty} \sum_{k\geqslant 0} q^{2k(4k+1)}(1-q^{4k+1}),\\
\sum_{n\geqslant 0} \left(m^e_{1,4}(n) - m^o_{1,4}(n)\right) q^n &= \frac{1}{(-q;q)_\infty} \sum_{k\geqslant 0} q^{2k(4k+1)}(1+q^{4k+1}),
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{n\geqslant 0} \left(m^o_{3,4}(n) + m^e_{3,4}(n)\right) q^n &= \frac{1}{(q;q)_\infty} \sum_{k\geqslant 0} q^{(2k+1)(4k+3)}(1-q^{4k+3}),\\
\sum_{n\geqslant 0} \left(m^o_{3,4}(n) - m^e_{3,4}(n)\right) q^n &= \frac{1}{(-q;q)_\infty} \sum_{k\geqslant 0} q^{(2k+1)(4k+3)}(1+q^{4k+3}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sum_{n\geqslant 0} 2\left(m^e_{1,4}(n) - m^o_{3,4}(n)\right) q^n &= \frac{1}{(q;q)_\infty} \sum_{k\geqslant 0} (-q)^{k(k+1)/2} + \frac{1}{(-q;q)_\infty} \sum_{k\geqslant 0} q^{k(k+1)/2}\\
&= (-q^2;q^2)_\infty + (q^2;q^2)_\infty\\
&= \sum_{n\geqslant 0} 2q^e(n/2) q^n
\end{aligned}
$$

where we apply Lemma 12 for the second equality, gives the result relating $m^e_{1,4}(n)$ and $m^e_{3,4}(n)$. The analogous computation

$$
\begin{aligned}
\sum_{n\geqslant 0} 2(m^o_{1,4}(n) - m^e_{3,4}(n)) q^n &= \frac{1}{(q;q)_\infty} \sum_{k\geqslant 0} (-q)^{k(k+1)/2} - \frac{1}{(-q;q)_\infty} \sum_{k\geqslant 0} q^{k(k+1)/2}\\
&= (-q^2;q^2)_\infty - (q^2;q^2)_\infty\\
&= \sum_{n\geqslant 0} 2q^o(n/2) q^n
\end{aligned}
$$

establishes the relation between $m^o_{1,4}(n)$ and $m^e_{3,4}(n)$. $\square$

Next, we give the more direct proof of Proposition 14.

Proof of Proposition 14. We have

$$\begin{aligned} m_{1,4}(n) &= m^o_{1,4}(n) + m^e_{1,4}(n) \\ &= \begin{cases} m^e_{3,4}(n) + m^o_{3,4}(n) & \text{if } n \text{ is odd,} \\ m^e_{3,4}(n) + q^o(n/2) + m^o_{3,4}(n) + q^e(n/2) & \text{if } n \text{ is even} \end{cases} \\ &= \begin{cases} m_{3,4}(n) & \text{if } n \text{ is odd,} \\ m_{3,4}(n) + q(n/2) & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

by rearranging the results of Theorem 15. □

Theorem 15 also leads to the following relation between $m^o_{1,2}(n)$ and $m^e_{1,2}(n)$; we provide both analytic and combinatorial proofs.

Corollary 16.

$$m^o_{1,2}(n) = \begin{cases} m^e_{1,2}(n) + (-1)^{k+1} & \textit{when } n = k(3k \pm 1), \\ m^e_{1,2}(n) & \textit{otherwise.} \end{cases}$$

Analytic proof of Corollary 16. Rearranging the second equation of Theorem 15 as

$$m^o_{3,4}(n) = \begin{cases} m^e_{1,4}(n) & \text{if } n \text{ is odd,} \\ m^e_{1,4}(n) - q^e(n/2) & \text{if } n \text{ is even} \end{cases}$$

allows us to write

$$\begin{aligned} m^o_{1,2}(n) &= m^o_{1,4}(n) + m^o_{3,4}(n) \\ &= \begin{cases} m^e_{3,4}(n) + m^e_{1,4}(n) & \text{if } n \text{ is odd,} \\ m^e_{3,4}(n) + q^o(n/2) + m^e_{1,4}(n) - q^e(n/2) & \text{if } n \text{ is even} \end{cases} \\ &= \begin{cases} m^e_{1,2}(n) & \text{if } n \text{ is odd,} \\ m^e_{1,2}(n) + q^o(n/2) - q^e(n/2) & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

and the result follows from Euler's pentagonal number theorem. □

Our combinatorial proof of Corollary 16 uses an equivalent generating function formulation. One step uses the following result of Carlitz [13].

Lemma 17 (Carlitz)**.** *We have*

$$\prod_{n=1}^{\infty}(1 - x^n y^n)(1 + x^n y^{n-1})(1 + x^{n-1} y^n) = \sum_{n=-\infty}^{\infty} x^{n(n+1)/2} y^{n(n-1)/2}.$$

The right-hand side of this identity is now known as Ramanujan's theta series. In terms of producing a purely combinatorial argument for Corollary 16, note that Wright [28] gave a combinatorial verification of Lemma 17.

Combinatorial proof of Corollary 16. From the definitions of $m^o_{1,2}(n)$ and $m^e_{1,2}(n)$, a generating function statement of Corollary 16 is

$$\sum_{k\geqslant 0} \frac{q^{k(2k+1)}}{(-q;q)_{2k}(-q^{2k+2};q)_\infty} = (q^2;q^2)_\infty$$

which is equivalent to

$$\frac{1}{(-q;q)_\infty} \sum_{k\geqslant 0} q^{k(2k+1)}(1+q^{2k+1}) = (q^2;q^2)_\infty.$$

Note that the left-hand side can be rewritten

$$\frac{1}{(-q;q)_\infty} \sum_{k\geqslant 0} q^{k(2k+1)}(1+q^{2k+1}) = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \frac{1}{(q^4;q^4)_\infty} \sum_{k=-\infty}^{\infty} q^{k(2k+1)}. \tag{16}$$

Now Lemma 17 with $x = q^3$ and $y = q$ gives

$$\frac{1}{(q^4;q^4)_\infty} \sum_{k=-\infty}^{\infty} q^{k(2k+1)} = \prod_{n=1}^{\infty}(1+q^{4n-1})(1+q^{4n-3}) = \sum_{\nu} q^{|\nu|}$$

where the last sum is over all partitions into distinct odd parts. Thus, the right-hand side of (16) is the weighted generating function of triples $(\pi;\mu;\nu)$ where π is a partition into distinct even parts, μ is a partition into odd parts, and ν is a partition into distinct odd parts. That is,

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \frac{1}{(q^4;q^4)_\infty} \sum_{k=-\infty}^{\infty} q^{k(2k+1)} = \sum_{(\pi;\mu;\nu)} (-1)^{\ell(\pi)+\ell(\mu)} q^{|\pi|+|\mu|+|\nu|}.$$

To complete the proof, we cancel out many of the triples $(\pi;\mu;\nu)$.

Let x be the smallest part of μ and y the smallest part of ν. If $x < y$, then move x to ν. Otherwise, move y to μ. This is clearly a sign reversing involution with no fixed points. Thus, after cancellations, we are left with only π.

Now apply Franklin's bijection [1, Theorem 1.6] to $\pi_1/2, \ldots, \pi_r/2$ for each π. This matches partitions with an even number of parts and those with an odd number of parts except for exactly one partition π of n when n is twice a generalized pentagonal number. □

Corollary 16 gives the most succinct proof yet of [5, Theorem 1.2], that $m_{1,2}(n)$ is almost always even (see also [23, Theorem 10]).

Theorem 18 (Andrews, Newman). *$m_{1,2}(n)$ is almost always even and is odd exactly when $n = k(3k \pm 1)$ for some integer k.*

Proof. We have

$$\begin{aligned} m_{1,2}(n) &= m^o_{1,2}(n) + m^e_{1,2}(n) \\ &= \begin{cases} 2m^e_{1,2}(n) + (-1)^{k+1} & \text{when } n = k(3k \pm 1), \\ 2m^e_{1,2}(n) & \text{otherwise} \end{cases} \end{aligned}$$

by Corollary 16. □

Note that, for odd n, many of these statistics introduced in this section are equal. Specifically, for all integers $k \geqslant 0$,

$$m_{1,4}(2k+1) = m_{3,4}(2k+1) = m^o_{1,2}(2k+1) = m^e_{1,2}(2k+1). \tag{17}$$

It would be nice to have combinatorial proofs of the identities in (17).

3.2 Connecting odd mex and nonpositive crank

Given Theorem 4 which connects partitions with odd mex and partitions with nonnegative crank, the split of the odd mex partitions leads to a natural question, posed in [23]: Which partitions of n with nonnegative crank correspond to the partitions counted by $m_{1,4}(n)$, and which to those counted by $m_{3,4}(n)$? This was answered recently by Huh and Kim [24], whose Proposition 3.4 is the even case of the following theorem.

Let $M_{\leqslant 0}(n)$ be the number of partitions λ of n with $\text{crank}(\lambda) \leqslant 0$. Using this notation, we know from Theorem 4 and (4) that

$$M_{\leqslant 0}(n) = m_{1,2}(n).$$

Let $M^e_{\leqslant 0}(n)$ be the number of partitions λ of n with $\text{crank}(\lambda) \leqslant 0$ having even length, similarly $M^o_{\leqslant 0}(n)$ for odd length.

Theorem 19 (Huh, Kim). *We have $M^e_{\leqslant 0}(n) = m_{1,4}(n)$ and $M^o_{\leqslant 0}(n) = m_{3,4}(n)$.*

Proof. Note that the generating function for the number $M_{\leqslant 0}(k, n)$ of partitions λ of n into k parts with $\text{crank}(\lambda) \leqslant 0$ is

$$\sum_{k,n \geqslant 0} M_{\leqslant 0}(k, n) z^k q^n = \sum_{n \geqslant 0} \frac{z^{2n} q^{n(n+1)}}{(zq; q)_n (q; q)_n}.$$

Substituting $z = -1$ gives

$$\sum_{n \geqslant 0} \left(M^e_{\leqslant 0}(n) - M^o_{\leqslant 0}(n) \right) q^n = \sum_{n \geqslant 0} \frac{q^{n(n+1)}}{(-q; q)_n (q; q)_n} = \sum_{n \geqslant 0} \frac{q^{n(n+1)}}{(q^2; q^2)_n}.$$

Thus, the generating function for the number of partitions λ of n into an even number of parts with $\operatorname{crank}(\lambda) \leqslant 0$ is

$$\frac{1}{2}\left(\sum_{n\geqslant 0}\frac{q^{n(n+1)}}{(q;q)_n^2}+\sum_{n\geqslant 0}\frac{q^{n(n+1)}}{(q^2;q^2)_n}\right).$$

The theorem then follows easily with (13) and (15). $\square$

The interested reader will want to compare our proof with that of [24].

Theorem 19 and Proposition 14 immediately give the following corollary. We provide a combinatorial verification, similar to the second cancellation of the combinatorial proof of Lemma 10.

Corollary 20. *For any $n \geqslant 1$,*

$$M^e_{\leqslant 0}(n) = \begin{cases} M^o_{\leqslant 0}(n) & \text{if } n \text{ is odd,}\\ M^o_{\leqslant 0}(n) + q(n/2) & \text{if } n \text{ is even.}\end{cases}$$

Proof. We construct a sign reversing involution on $M_{\leqslant 0}(n)$.

Suppose $\lambda \in M_{\leqslant 0}(n)$ has Durfee square size $d \times d$. Since $\operatorname{crank}(\lambda) \leqslant 0$, there are at least d parts 1. Let π be the partition consisting of parts below the Durfee square excluding d parts 1. Let ν be the conjugate of the partition consisting of parts to the right of the Durfee square.

Let x be the smallest part of π appearing an odd number of times and y be the smallest part of ν. If $x \leqslant y$, then move a part of size x to ν. If $x > y$, then we move a part of size y to π. This decreases or increases $\ell(\pi)$ by one, so it indeed decreases or increases $\ell(\lambda)$ by one. Hence, it is a sign reversing involution.

Partitions λ with each part in π appearing an even number of times and ν being the empty partition remain unchanged by the involution. Note that this occurs only when $n = |\lambda|$ is even. Thus

$$\begin{aligned}\sum_{n\geqslant 0}\left(M^e_{\leqslant 0}(n) - M^o_{\leqslant 0}(n)\right)q^n &= \sum_{\lambda}(-1)^{\ell(\lambda)}q^{|\lambda|}\\ &= \sum_{d\geqslant 0}\sum_{\pi,\nu}(-1)^{\ell(\pi)}q^{d^2+d+|\pi|+|\nu|}\\ &= \sum_{d\geqslant 0}\frac{q^{d^2+d}}{(q^2;q^2)_d}\end{aligned}$$

where the second equality follows from the decomposition of λ with Durfee square size $d \times d$, d parts 1, π and ν, and the last equality follows from the sign involution.

Finally,

$$\sum_{d\geqslant 0}\frac{q^{d^2+d}}{(q^2;q^2)_d} = (-q^2;q^2)_\infty$$

which has the following combinatorial proof: Rearrange the d^2 boxes of the Durfee square and d parts 1 in rows of length $2d, 2d-2, \ldots, 2$, then add the conjugate of π to these d consecutive even parts. This produces a partition of n into distinct even parts, and there are $q(n/2)$ such partitions. $\square$

We hope that this proof contributes to combinatorial verifications of Theorem 4 and its refinement, Theorem 19, which have eluded us so far.

4 Frobenius symbols and crank

In this final section, we revisit a theme begun by Andrews in 2011 [2], the relationships between partitions whose Frobenius symbols satisfy certain restrictions and partitions with certain crank or mex characteristics. In keeping with the theme of this paper, we provide combinatorial proofs for two results of Hopkins, Sellers, and Stanton. The first is [23, Proposition 7].

Proposition 21 (Hopkins, Sellers, Stanton)**.** *The number of partitions of n with crank 0 equals the number of partitions of n whose Frobenius symbol has no 0 minus the number of partitions of $n-1$ whose Frobenius symbol has no 0.*

Proof. Let λ be a partition with $\operatorname{crank}(\lambda) = 0$. If $\omega(\lambda) = 0$, then $\operatorname{crank}(\lambda) > 0$, which is a contradiction. So, assume $\omega(\lambda) > 0$ and let the size of the Durfee square of λ be $d \times d$. If $d < \omega(\lambda)$, then $\mu(\lambda) \leqslant d$ since $\lambda_{d+1} \leqslant d < \omega(\lambda)$, so $\operatorname{crank}(\lambda) < 0$. Similarly, we can check that if $d > \omega(\lambda)$, then $\operatorname{crank}(\lambda) > 0$. Hence, if $\operatorname{crank}(\lambda) = 0$, then its Durefee square must be of size $d \times d$ and $d = \omega(\lambda)$. Also, the first d parts of λ must be greater than d since, if $\lambda_d \leqslant d$, then $\mu(\lambda) \leqslant d-1$, so

$$\operatorname{crank}(\lambda) = \mu(\lambda) - \omega(\lambda) \leqslant d-1-d \leqslant -1.$$

Thus, the generating function for partitions with crank 0 is

$$1 + \sum_{d=1}^{\infty} \frac{q^{d^2+2d}}{(q;q)_d (q^2;q)_{d-1}}.$$

A Ferrers diagram of this type is shown in the left-hand side of Figure 1. To produce the partition on the right-hand side, delete the d parts 1 and create a row of size d just below the Durfee square.

Now consider partitions whose Ferrers diagrams are of the type shown in the right-hand side of Figure 1 with parts 1 allowed and let $a(n)$ be the number of such partitions of n. Such partitions can be divided into two groups: partitions with and without parts 1. If such a partition of n has at least one part 1, then deleting one part 1 gives a partition of $n-1$. Thus, the number of partitions of n of the type shown in the right-hand side with no parts 1 equals $a(n) - a(n-1)$.

Also, because of the length d row below the Durfee square and the height d column to the right of the Durfee square, the Frobenius symbol for this right-hand partition has

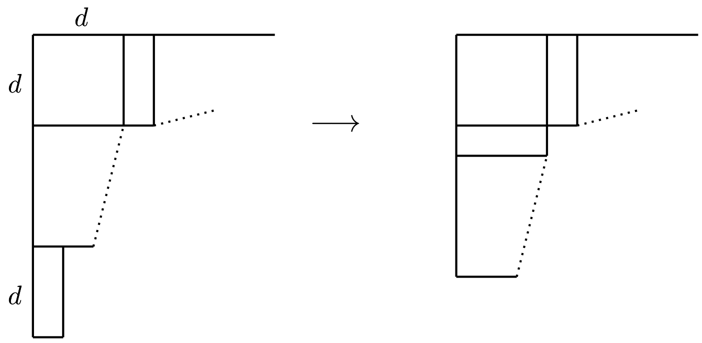

Figure 1: Mapping a crank 0 partition to a partition with no 0 in its Frobenius symbol.

no 0 entries. Thus, the number of partitions of n in question is the number of partitions of n whose Frobenius symbol has no 0 minus the number of partitions of $n-1$ whose Frobenius symbol has no 0. $\square$

Examining the Ferrers diagram on the right-hand side of Figure 1 shows that Proposition 21 is equivalent to the following corollary.

Corollary 22. *The number of partitions of n with crank 0 equals the number of partitions of n whose Frobenius symbol has no 0 and the first two entries of the bottom row differ by 1.*

Our last proof gives a combinatorial argument for [23, Theorem 8]. We return to j-Durfee rectangles and the foundational Theorem 8.

Theorem 23 (Hopkins, Sellers, Stanton)**.** *The number of partitions of n with crank at least j equals the number of partitions of $n-j$ whose Frobenius symbol has no j in its top row.*

Proof. By Theorem 8, we know that partitions λ of n with $\mathrm{crank}(\lambda) \geqslant j$ are in bijection with partitions of n with at least $d+j$ parts 1 where d is the parameter of its j-Durfee rectangle. Modify λ by deleting $d+j$ parts 1 and increasing each of the d largest parts of λ by 1. The resulting λ' is therefore a partition of $n-j$ which satisfies

$$\lambda'_d \geqslant d+j+1, \quad \lambda'_{d+1} \leqslant d+j.$$

Thus the top entries in columns d and $d+1$ of the Frobenius symbol of λ' are at least $j+1$ and at most $j-1$, respectively. This shows that there are no values of j in the top row of the Frobenius symbol. See the Figure 2. $\square$

Andrews, Dastidar, and Morrill give an alternative analytic proof of [23, Theorem 8] in their [3, Theorem 2], stated in an equivalent form. They conclude their paper,

> Prior to the discoveries of this paper and [Hopkins, Sellers, Stanton], there was no reason to suspect that there would be any connection between Cranks and Frobenius symbols. So one would hope that there is something combinatorial underlying Theorem 2 that would shed light on this mystery. [3, p. 6]

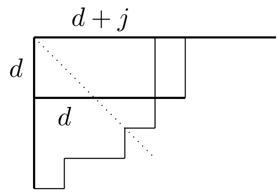

Figure 2: The result of deleting $d+j$ parts 1 and adding 1 to each of the d largest parts.

We hope that the combinatorial proof of our Theorem 23 contributes to better understanding this interesting and previously unexpected connection.

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