Chordal directed graphs are not directed χ-bounded

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Abstract

We show that digraphs with no transitive tournament on 3 vertices and in which every induced directed cycle has length 3 can have arbitrarily large dichromatic number. This answers in the negative a question of Carbonero, Hompe, Moore, and Spirkl (and strengthens one of their results).

Mathematics Subject Classifications: 05C15, 05C20

1 Introduction

Throughout this paper, we only consider simple graphs (resp. directed graphs) G, that is, for every two distinct vertices u and v, the graph G does not have multiple edges (resp. both arcs uv and vu).

Relations between the chromatic number $\chi(G)$ and the clique number $\omega(G)$ of a graph G have been studied for decades in structural graph theory. In particular, it is well known that there exist triangle-free graphs G with arbitrarily large chromatic number (see e.g. [4, 9]). A hereditary class of graphs is *χ-bounded* if there exists a function f such that for every $G \in \mathcal{G}$, $\chi(G) \leqslant f(\omega(G))$ (see e.g. a recent survey [7] on the topic). The following question received considerable attention in the last few years: Consider a hereditary class of graphs $\mathcal{G}$ in which every triangle-free graph has bounded chromatic number. Is it true that $\mathcal{G}$ is χ-bounded? Carbonero, Hompe, Moore and Spirkl [2] answered this in the negative in a recent breakthrough paper.

Their initial motivation was actually to prove a result on digraphs. Let D be a digraph. A *k-dicolouring* of D is a k-partition $(V_1, \dots, V_k)$ of $V(D)$ such that $D[V_i]$ is acyclic for every $1 \leqslant i \leqslant k$. Such a partition is also called an *acyclic colouring* of D. The *dichromatic number* of D, denoted by $\vec{\chi}(D)$ and introduced by Neumann-Lara in [6], is the smallest integer k such that D admits a k-dicolouring. We denote by $\omega(D)$ the size of a largest clique in the underlying graph of D. A *directed triangle* is a directed cycle of length 3. As for unoriented graphs, we say that a hereditary class of digraphs $\mathcal{G}$ is *$\vec{\chi}$-bounded* if for every $G \in \mathcal{G}$, $\vec{\chi}(G) \leqslant f(\omega(G))$.

Carbonero, Hompe, Moore and Spirkl [2] proved that the class of digraphs with no induced directed cycle of odd length at least 5 is not $\vec{\chi}$-bounded by giving a collection of digraphs with no induced directed cycle of odd length at least 5, no K_4 and with arbitrarily large dichromatic number. They ask (Question 3.2) if the class of digraphs in which every induced directed cycle has length 3 is $\vec{\chi}$-bounded. These digraphs can be seen as directed analogues of chordal graphs, where a *chordal directed graph* is a directed graph with no induced directed cycle of length at least 4.

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We give a negative answer to this question (and thus strengthen the construction of [2]). Let us denote by TT_3 the *transitive tournament* on 3 vertices (i.e. the triangle which is oriented acyclically). Let $\mathcal{C}_3$ be the class of digraphs with no TT_3 nor induced directed cycle of length at least 4. We prove the following.

Theorem 1. *For every k, there exists $G \in \mathcal{C}_3$ such that $\vec{\chi}(G) \geqslant k$.*

Since any orientation of a K_4 contains a TT_3, Theorem 1 answers Question 3.2 of [2].

2 Proof of Theorem 1

Our proof technique can be seen as a generalization of the construction of triangle-free graphs with arbitrarily large chromatic number due to Zykov [9]. Assume that we are given a triangle-free graph G_k with chromatic number at least k, and let us define G_{k+1} as follows (note that we can set G_1 as a single vertex graph). Let G be the graph made of k disjoint induced copies of G_k. Set $\mathcal{I}$ to be the set of all k-subsets of vertices of G containing exactly one vertex in each copy of G_k. Now, build the graph G_{k+1} from G as follows: for every set $I \in \mathcal{I}$, create a new vertex x_I adjacent to every vertex in I. The key observation is that, for any colouring of G_{k+1}, for each $I \in \mathcal{I}$, the vertex x_I forces I to miss at least one colour, namely the one received by x_I. This easily implies that G_{k+1} is not k-colourable. Indeed, if one tries to k-colour G_{k+1}, since G_k has chromatic number k, there must be a vertex x_i coloured i in the i^{th} copy of G_k for every $i \leqslant k$. A contradiction with the key observation above. Moreover, since each set of $\mathcal{I}$ is an independent set, G_{k+1} is triangle-free.

For digraphs, such a naive construction fails since adjacent vertices are allowed to receive the same colour. A way to force a given independent set I of a digraph D to avoid a colour (without creating induced directed cycle of length at least 4 nor TT_3) is to connect each vertex of I to an arc uv (instead of a single vertex as in the directed case) in such a way that each vertex of I forms a directed triangle with uv and then hope that the two vertices u and v receive the same colour. Unfortunately we cannot force an arc to have both endpoints of the same colour. But we have the following slightly weaker property.

Remark 2. Let $G \in \mathcal{C}_3$ be a directed graph with at least one arc. Any $\vec{\chi}(G)$-dicolouring of G contains at least one monochromatic arc.

Proof. The result trivially holds if $\vec{\chi}(G) = 1$, so we may assume that $\vec{\chi}(G) \geqslant 2$. Let $V_1, \dots, V_{\vec{\chi}(G)}$ be a $\vec{\chi}(G)$-dicolouring of G. The set $V_1 \cup V_2$ must contain an induced directed cycle C since otherwise G would be $(\vec{\chi}(G) - 1)$-colourable. (Indeed, a colouring of the vertices of a digraph is acyclic if and only if none of its induced directed cycles is monochromatic). Hence, by definition of $\mathcal{C}_3$, $V_1 \cup V_2$ contains a directed triangle, and an arc of this directed triangle must have both endpoints in V_1 or both endpoints in V_2. $\square$

Let G be a k-chromatic digraph and I be an independent set of G. Using Remark 2, we prove that we can create a graph G' containing many induced copies of G such that, for every k-coloring of G', there is one copy of G in G' where the vertices of I (in that copy) miss at least one color (Lemma 3). We then extend this result for arbitrarily many independent sets (Lemma 4). We then prove Theorem 1 using Lemma 4 as in Zykov's construction.

Lemma 3. *Let k be an integer. Let $G \in \mathcal{C}_3$ with n vertices and m arcs, and such that $\vec{\chi}(G) = k$. Let I be an independent set of G. Then there exists a digraph $H \in \mathcal{C}_3$ such that H contains m pairwise disjoint induced copies $G_1, \dots, G_m$ of G and satisfy the following:*

- *For every $1 \leqslant i \neq j \leqslant m$, there is no arc between G_i and G_j;*
- *For every k-dicolouring of H, there exists an index $i \leqslant m$ and a colour $\alpha \in \{1, \dots, k\}$ such that no vertex of the copy of I in G_i is coloured with α.*

Moreover H has $n \cdot (m+1) \leqslant n^4$ vertices and at most $m(m+1) + mn^2 \leqslant n^4$ arcs.

Proof. Let us first describe the construction of H. We first create $m+1$ pairwise disjoint induced copies of G denoted by $G_1, \dots, G_m, G_{m+1}$. For every $i \leqslant m$, let I_i be the copy of I in G_i. Let us denote $u_1^{m+1}v_1^{m+1}, \dots, u_m^{m+1}v_m^{m+1}$ the arcs of G_{m+1}. We add in H some arcs between the G_i ($i \leqslant m$) and G_{m+1} as follows. For every $i \leqslant m$ and for every vertex $x \in I_i$, add the arcs $v_i^{m+1}x$ and xu_i^{m+1} in H.

Observe that H has $n \cdot (m+1)$ vertices and $m \cdot (m+1) + 2m \cdot |I| \leqslant n^4$ arcs as announced.

By construction, for every $1 \leqslant i \neq j \leqslant m$, there is no arc between G_i and G_j, so the first bullet holds.

Let c be a k-dicolouring of H. By Remark 2, G_{m+1} has a monochromatic arc, say $u_i^{m+1}v_i^{m+1}$. Let α be the colour of u_i^{m+1} and v_i^{m+1} in c. Then, for every vertex $x \in I_i$, x is not coloured with α since $H[\{u_i, v_i, x\}]$ is a directed triangle. This proves the second bullet.

To conclude, we simply have to prove that $H \in \mathcal{C}_3$. First assume for contradiction that H contains a copy X of a TT_3 as a subgraph. Since there is no arc between G_i and G_j for $1 \leqslant i \neq j \leqslant m$, X intersects at most one of the graphs G_i for $i \leqslant m$. Moreover, since G is in $\mathcal{C}_3$, X is not included in G_i for $i \leqslant m+1$. So X must intersect G_{m+1} and G_i for some $i \leqslant m$. Assume first that X contains two vertices of G_{m+1}. By construction, the only vertices of G_{m+1} connected to G_i are u_i^{m+1} and v_i^{m+1}. So both vertices are in X. Moreover, the only vertices of G_i connected to G_{m+1} are the vertices of I_i so the third vertex must be a vertex x of I_i. But by construction, $G[\{x, u_i^{m+1}, v_i^{m+1}\}]$ is a directed triangle, a contradiction. So we can assume that X contains two vertices of G_i. Since X is a TT_3, they must be adjacent and both be adjacent to a vertex of G_{m+1}. But, by construction, the only vertices of G_i connected to G_{m+1} are the vertices of I_i which is an independent set, a contradiction. So H contains no TT_3.

Finally, assume for contradiction that H contains a directed cycle C of length at least 4 as an induced subgraph. Since $G \in \mathcal{C}_3$, C is not contained in G_i for $i = 1, \dots, m+1$. Since there is no arc between G_i and G_j for $1 \leqslant i \neq j \leqslant m$, the cycle C intersects G_{m+1} and we may assume without loss of generality that C also intersects G_1. So C contains u_1^{m+1} or v_1^{m+1}. Since, by construction, u_1^{m+1} has no out-neighbour in G_1 and v_1^{m+1} has no in-neighbour in G_1, C must contain both u_1^{m+1} and v_1^{m+1} (since the deletion of u_1^{m+1} and v_1^{m+1} disconnects G_1 from the rest of the graph). But now all the vertices of G_1 incident to u_1 or v_1 are the vertices x of I. And by construction, for every $x \in I$, $H[\{u_1^{m+1}, v_1^{m+1}, x\}]$ is a directed triangle, a contradiction. ☐

Lemma 4. *Let k, r be two integers. Let $G \in \mathcal{C}_3$ such that $\vec{\chi}(G) = k$ and let $I_1, \dots, I_r$ be r independent sets of G. There exist an integer ℓ_r and a digraph $H \in \mathcal{C}_3$ such that H contains ℓ_r pairwise disjoint induced copies $G_1, \dots, G_{\ell_r}$ of G such that:*

- *For every $1 \leqslant i \neq j \leqslant \ell_r$, there is no arc between G_i and G_j;*
- *For every k-dicolouring of H, there exists an index $j \leqslant \ell_r$ such that, for every $s \leqslant r$, there exists a colour $\alpha_s \in \{1, \dots, k\}$ such that no vertex of the copy of I_s in G_j is coloured with α_s.*

Moreover H contains at most n^{4r} vertices and arcs.

Proof. We now have all the ingredients to prove Lemma 4 by induction on r. By Lemma 3, the case $r = 1$ holds.

Assume that the conclusion holds for $r \geqslant 1$ and let us prove the result for $r+1$. Let $G \in \mathcal{C}_3$ with $\vec{\chi}(G) = k$ and let $I_1, \dots, I_{r+1}$ be $r+1$ independent sets of G. By induction applied to G and independent sets $I_1, \dots, I_r$, there exists an integer ℓ_r and a digraph $H_r \in \mathcal{C}_3$ such that H_r contains ℓ_r pairwise disjoint induced copies $G_1, \dots, G_{\ell_r}$ of G such that:

- For every $1 \leqslant i \neq j \leqslant \ell_r$, there is no arc between G_i and G_j;
- For every k-dicolouring of H_r, there exists an index $j \leqslant \ell_r$ such that, for $s = 1, \dots, r$, there exists a colour α_s such that no vertex of the copy of I_s in G_j is coloured with α_s.

Note that by induction, H_r has at most n^{4r} vertices and edges. Let us denote by J the union of the vertices of the copies of I_{r+1} in the subgraphs $G_1, \dots, G_{\ell_r}$ and observe that J is an independent set. By Lemma 3 applied to H_r and J, there exists a digraph $H_{r+1} \in \mathcal{C}_3$ that contains $m = |E(H_r)|$ pairwise disjoint induced copies $H_r^1, \dots, H_r^m$ of H_r such that:

- For $1 \leqslant i \neq j \leqslant m$, there is no arc between H_r^i and H_r^j;
- For every k-dicolouring of H_{r+1}, there exists an index $j \leqslant m$ and a colour α_{r+1} such that no vertex of the copy of J in H_r^j is coloured with α_{r+1}.

Moreover, H has at most $|V(H_r)|^4 = n^{4(r+1)}$ vertices and arcs.

Let us prove that H_{r+1} satisfies the conclusion of Lemma 4. For every $i \leqslant m$, H_r^i being an induced copy of H_r, it contains ℓ_r pairwise disjoint induced copies of G, denoted by $G_1^i, \dots, G_{\ell_r}^i$. Thus, by construction of H_{r+1}, the graph H_{r+1} contains $\ell_{r+1} := m \cdot \ell_r$ induced copies of G and by construction there is no arc linking any of these copies.

Fix a k-dicolouring of H_{r+1}. There exists an index $j \leqslant m$ and a colour α_{r+1} such that no vertex of the induced copy of J in H_r^j is coloured α_{r+1}. Since H_r^j is an induced copy of H_r there exists an index $k \leqslant \ell_r$ such that, for $s = 1, \dots, r$, there exists a colour α_s such that no vertex of the induced copy of I_s in G_k^j is coloured with α_s. Hence, the second bullet holds, which completes the proof. $\square$

Proof of Theorem 1. Let us construct a sequence $(G_k)_{k \in \mathbb{N}}$ such that for every k, $G_k \in \mathcal{C}_3$ and $\vec{\chi}(G_k) \geqslant k$. Let G_1 be the graph consisting of a single vertex and let G_2 be the directed triangle. Let $k \geqslant 2$ and assume that we have obtained a k-dichromatic digraph G_k which is in $\mathcal{C}_3$, let us define G_{k+1} as follows. Let G be the digraph consisting of k pairwise disjoint induced copies of G_k, denoted by $G_k^1, \dots, G_k^k$. Let $\mathcal{I}$ be the set of independent sets that intersect each G_k^i in a single vertex. Since $\vec{\chi}(G_k) \geqslant k$, in any k-dicolouring of G, there exists a vertex x_i coloured i in G_k^i for every $i = 1, \dots, k$. By definition of $\mathcal{I}$, $\{x_1, \dots, x_k\} \in \mathcal{I}$. Hence, for every k-dicolouring of G, a set of $\mathcal{I}$ receives all the colours.

By Lemma 4 applied to G and $\mathcal{I}$, there exists a digraph $G_{k+1} \in \mathcal{C}_3$ such that, for every k-dicolouring of G_{k+1} (if such a colouring exists), there exists an induced copy of G in G_{k+1} such that each set $\mathcal{I}$ in that copy of G avoids a colour, a contradiction. So $\vec{\chi}(G_{k+1}) \geqslant k+1$. $\square$

3 Further works

Our $(k+1)$-dichromatic graph G_{k+1} has size $|G_k|^{2^{poly(|G_k|)}}$, which is larger than the graphs obtained using Zykov's construction which have size of order $2^{poly(|G_k|)}$. It would be interesting to know if the size of our example can be reduced.

One can wonder if directed triangles play a particular role in Theorem 1. More formally, one can wonder (as also asked in [2], Question 3.3) for which integer k, the class of digraphs which only contain induced directed cycles of length exactly k are $\vec{\chi}$-bounded? Our main result is that it is not the case for $k = 3$. Based on the idea of our construction, Carborero et al. [3] very recently answered in the negative this question for any possible value of k.

On the same flavour, we recall here the following conjecture of Aboulker, Charbit and Naserasr which can be seen as a directed analogue of the well-known Gyárfás-Sumner conjecture [5, 8]. An *oriented tree* is an orientation of a tree.

Conjecture 5. [1] For every oriented tree T, the class of digraphs with no induced T is $\vec{\chi}$-bounded.

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