

On the Subspace Choosability in Graphs*

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Abstract

A graph G is said to be k -subspace choosable over a field \mathbb{F} if for every assignment of k -dimensional subspaces of some finite-dimensional vector space over \mathbb{F} to the vertices of G , it is possible to choose for each vertex a nonzero vector from its subspace so that adjacent vertices receive orthogonal vectors over \mathbb{F} . The *subspace choice number* of G over \mathbb{F} is the smallest integer k for which G is k -subspace choosable over \mathbb{F} . This graph parameter, introduced by Haynes, Park, Schaeffer, Webster, and Mitchell (Electron. J. Comb., 2010), is inspired by well-studied variants of the chromatic number of graphs, such as the (color) choice number and the orthogonality dimension.

We study the subspace choice number of graphs over various fields. We first prove that the subspace choice number of every graph with average degree d is at least $\Omega(\sqrt{d/\ln d})$ over any field. We then focus on bipartite graphs and consider the problem of estimating, for a given integer k , the smallest integer m for which the subspace choice number of the complete bipartite graph $K_{k,m}$ over a field \mathbb{F} exceeds k . We prove upper and lower bounds on this quantity as well as for several extensions of this problem. Our results imply a substantial difference between the behavior of the choice number and that of the subspace choice number. We also consider the computational aspect of the subspace choice number, and show that for every $k \geq 3$ it is NP-hard to decide whether the subspace choice number of a given bipartite graph over \mathbb{F} is at most k , provided that \mathbb{F} is either the real field or any finite field.

Mathematics Subject Classifications: 05C15, 05C50, 05C62

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1 Introduction

Graph coloring is the problem of minimizing the number of colors in a vertex coloring of a graph G where adjacent vertices receive distinct colors. This minimum is known as the *chromatic number* of G and is denoted by $\chi(G)$. Being one of the most popular topics in graph theory, the graph coloring problem was extended and generalized over the years in various ways. One classical variant, initiated independently by Vizing in 1976 [19] and by Erdős, Rubin, and Taylor in 1979 [8], is that of *choosability*, also known as *list coloring*, which deals with vertex colorings with some restrictions on the colors available to each vertex. A graph $G = (V, E)$ is said to be *k-choosable* if for every assignment of a set S_v of k colors to each vertex $v \in V$, there exists a choice of colors $c_v \in S_v$ that form a proper coloring of G (that is, $c_v \neq c_{v'}$ whenever v and v' are adjacent in G). The *choice number* of a graph G , denoted $\text{ch}(G)$, is the smallest integer k for which G is k -choosable. It is well known that the choice number $\text{ch}(G)$ behaves quite differently from the standard chromatic number $\chi(G)$. In particular, it can be arbitrarily large even for bipartite graphs (see, e.g., [8]). The choice number of graphs enjoys an intensive study in graph theory involving combinatorial, algebraic, and probabilistic tools (see, e.g., [1]). The computational decision problem associated with the choice number is unlikely to be tractable, because it is known to be complete for the complexity class Π_2 of the second level of the polynomial-time hierarchy even for bipartite planar graphs [8, 10, 11].

Another interesting variant of graph coloring, introduced by Lovász [14] in the study of Shannon capacity of graphs, is that of *orthogonal representations*, where the vertices of the graph do not receive colors but vectors from some given vector space. A t -dimensional orthogonal representation of a graph $G = (V, E)$ over \mathbb{R} is an assignment of a nonzero vector $x_v \in \mathbb{R}^t$ to every vertex $v \in V$, such that $\langle x_v, x_{v'} \rangle = 0$ whenever v and v' are adjacent in G .¹ The *orthogonality dimension* of a graph G over \mathbb{R} is the smallest integer t for which there exists a t -dimensional orthogonal representation of G over \mathbb{R} . The orthogonality dimension parameter is closely related to several other well-studied graph parameters, and in particular, for every graph G it is bounded from above by the chromatic number $\chi(G)$. The orthogonality dimension of graphs and its extensions to fields other than the reals have found a variety of applications in combinatorics, information theory, and theoretical computer science (see, e.g., [15, Chapter 10] and [12]). As for the computational aspect, the decision problem associated with the orthogonality dimension of graphs is known to be NP-hard over every field [16] (see also [9]).

In 2010, Haynes, Park, Schaeffer, Webster, and Mitchell [13] introduced another variant of the chromatic number of graphs that captures both the choice number and the orthogonality dimension. In this setting, which we refer to as *subspace choosability*, each vertex of a graph G is assigned a k -dimensional subspace of some finite-dimensional vector space, and the goal is to choose for each vertex a nonzero vector from its subspace so that adjacent vertices receive orthogonal vectors. The smallest integer k for which such a

¹Orthogonal representations of graphs are sometimes defined in the literature as orthogonal representations of the complement, namely, the definition requires vectors associated with *non-adjacent* vertices to be orthogonal.

choice is guaranteed to exist for all possible subspace assignments is called the *subspace choice number* of the graph G , formally defined as follows.

Definition 1. For a graph $G = (V, E)$ and a function $f : V \rightarrow \mathbb{N}$, G is *f -subspace choosable* over a field \mathbb{F} if for every integer t and for every assignment of subspaces $W_v \subseteq \mathbb{F}^t$ with $\dim(W_v) = f(v)$ to the vertices $v \in V$ (which we refer to as an *f -subspace assignment*), there exists a choice of a nonzero vector $x_v \in W_v$ for each vertex $v \in V$, such that $\langle x_v, x_{v'} \rangle = 0$ whenever v and v' are adjacent in G . For an integer k , the graph G is *k -subspace choosable* over \mathbb{F} if it is f -subspace choosable over \mathbb{F} for the constant function f defined by $f(v) = k$. The *subspace choice number* of G over \mathbb{F} , denoted $\text{ch-s}(G, \mathbb{F})$, is the smallest k for which G is k -subspace choosable over \mathbb{F} .

Here and throughout the paper, we associate with the real field \mathbb{R} and with every finite field \mathbb{F} the inner product defined by $\langle x, y \rangle = \sum x_i y_i$, whereas for the complex field \mathbb{C} we consider, as usual, the one defined by $\langle x, y \rangle = \sum x_i \bar{y}_i$.

The work [13] has initiated the study of the subspace choice number of graphs over the real and complex fields. Among other things, it was shown there that a graph is 2-subspace choosable over \mathbb{R} if and only if it contains no cycles. We note that this is in contrast to the characterization given in [8] for the (chromatic) 2-choosable graphs, which include additional graphs such as even cycles. This implies that the choice number and the subspace choice number do not coincide even on the 4-cycle graph. Over the complex field \mathbb{C} , however, it was shown in [13] that a graph is 2-subspace choosable if and only if it either contains no cycles or contains only one cycle and that cycle is even. This demonstrates the possible effect of the field on the subspace choice number. It further follows from [13] that for every graph G and every field \mathbb{F} , it holds that $\text{ch-s}(G, \mathbb{F}) \leq \Delta(G) + 1$ where $\Delta(G)$ stands for the maximum degree in G . In fact, a similar argument shows that $\Delta(G)$ can be replaced in this bound by the degeneracy of G (i.e., the smallest integer k for which every subgraph of G contains a vertex of degree at most k).

1.1 Our Contribution

The current work studies the subspace choice number of graphs over various fields. Our first result provides a lower bound on the subspace choice number of a general graph over any field in terms of its average degree.

Theorem 2. *There exists a constant $c > 0$ such that for every graph G with average degree $d > 1$ and for every field \mathbb{F} ,*

$$\text{ch-s}(G, \mathbb{F}) > c \cdot \sqrt{\frac{d}{\ln d}}.$$

The proof of Theorem 2 is based on a probabilistic argument. For certain graphs, we provide an improved lower bound on the subspace choice number, avoiding the logarithmic term (see Theorem 16). This improvement relies on an explicit construction of finite projective planes.

We note that a result of Saxton and Thomason [17], improving on a result of Alon [2], asserts that for every graph G with average degree d , it holds that $\text{ch}(G) \geq (1+o(1)) \cdot \log_2 d$, where the $o(1)$ term tends to 0 when d tends to infinity. Erdős et al. [8] proved that the choice number of the complete bipartite graph $K_{n,n}$ satisfies $\text{ch}(K_{n,n}) = (1+o(1)) \cdot \log_2 n$, hence the lower bound of [17] is tight on these graphs. Theorem 2 thus shows a substantial difference between the behavior of the subspace choice number and that of the standard choice number in terms of the average degree.

For the complete graph K_n , it is easy to see that $\text{ch-s}(K_n, \mathbb{F}) = n$ whenever \mathbb{F} is a field over which no nonzero vector is self-orthogonal, such as \mathbb{R} and \mathbb{C} . For finite fields, however, we show that the subspace choice number of K_n is strictly smaller than n for every sufficiently large n . This in particular shows that the subspace choice number over finite fields can be smaller than the choice number.

Theorem 3. *There exists a constant $c > 0$ such that for every sufficiently large integer n and for every finite field \mathbb{F} ,*

$$\text{ch-s}(K_n, \mathbb{F}) \leq n - c \cdot \sqrt{n}.$$

We next put our focus on complete bipartite graphs. For the color choosability problem, it was observed in [8] that the graph $K_{k,m}$ is k -choosable for every $m < k^k$ whereas $\text{ch}(K_{k,m}) = k + 1$ for every $m \geq k^k$. Considering the subspace choice number of these graphs, for every field \mathbb{F} it holds that $\text{ch-s}(K_{k,m}, \mathbb{F}) \leq k + 1$, because $K_{k,m}$ is k -degenerate. We consider here the problem of identifying the values of m for which this $k + 1$ upper bound is tight. Namely, for an integer k and a field \mathbb{F} , let $m(k, \mathbb{F})$ denote the smallest integer m for which it holds that $\text{ch-s}(K_{k,m}, \mathbb{F}) = k + 1$. We provide the following lower bound.

Theorem 4. *For every integer k and for every field \mathbb{F} ,*

$$m(k, \mathbb{F}) > \sum_{i=1}^{k-1} \left\lfloor \frac{k-1}{i} \right\rfloor.$$

In particular, for every field \mathbb{F} it holds that $m(k, \mathbb{F}) = \Omega(k \cdot \log k)$.

We next provide a general approach for proving upper bounds on $m(k, \mathbb{F})$. The following theorem reduces this challenge to constructing families of vectors with certain linear independence constraints.

Theorem 5. *If there exists a collection of $m = k \cdot (t - 1) + 1$ nonzero vectors in \mathbb{F}^k satisfying that every t of them span the entire space \mathbb{F}^k , then $m(k, \mathbb{F}) \leq m$.*

The above theorem allows us to derive upper bounds on $m(k, \mathbb{F})$ for various fields \mathbb{F} .

Corollary 6. *Let k be an integer and let \mathbb{F} be a field.*

1. *If $|\mathbb{F}| \geq k^2 - k + 1$ then $m(k, \mathbb{F}) \leq k^2 - k + 1$.*

2. If \mathbb{F} is a finite field of size $q \geq k$ then $m(k, \mathbb{F}) \leq k \cdot \frac{q^{k-1}-1}{q-1} + 1$.

We remark that the first item of Corollary 6 is obtained by applying Theorem 5 with collections of vectors that form the columns of Vandermonde matrices. It implies that $m(k, \mathbb{F}) = O(k^2)$ whenever the field \mathbb{F} is infinite or sufficiently large as a function of k , leaving us with a nearly quadratic gap from the lower bound given in Theorem 4. This again demonstrates a significant difference between the behavior of the choice number and that of the subspace choice number.

In fact, Theorems 4 and 5 are proved in a more general form with respect to *asymmetric* subspace assignments, where the left and right vertices of the complete bipartite graphs might be assigned subspaces of different dimensions. For the precise generalized statements, see Theorems 26 and 29. We note that this is analogous to the asymmetric setting of color choosability that was recently studied by Alon, Cambie, and Kang [3].

We particularly consider the bipartite graph $K_{2,m}$ whose left side consists of only two vertices. For an integer n , we say that $K_{2,m}$ is $(n; 2, 2)$ -subspace choosable over a field \mathbb{F} if it is f -subspace choosable over \mathbb{F} for the function f that assigns the integer n to one vertex of the left side and the integer 2 to each of the other vertices. We consider the problem of determining, for a given integer n , the smallest m for which $K_{2,m}$ is $(n; 2, 2)$ -subspace choosable over a given field \mathbb{F} , and prove the following.

Theorem 7. *For every integer $n \geq 1$ the following holds.*

1. *The graph $K_{2,n-1}$ is $(n; 2, 2)$ -subspace choosable over every field \mathbb{F} .*
2. *The graph $K_{2,n}$ is $(n; 2, 2)$ -subspace choosable over \mathbb{C} .*
3. *The graph $K_{2,n}$ is $(n; 2, 2)$ -subspace choosable over \mathbb{R} if and only if n is odd.*

We finally consider the computational aspect of the subspace choice number and prove the following hardness result.

Theorem 8. *Let $k \geq 3$ be an integer and let \mathbb{F} be either \mathbb{R} or some finite field. Then, the problem of deciding whether a given bipartite graph G satisfies $\text{ch-s}(G, \mathbb{F}) \leq k$ is NP-hard.*

The proof of Theorem 8 is inspired by the approach taken in a proof due to Rubinfeld [8] for the Π_2 -hardness of the decision problem associated with the (color) choice number. His proof involves a delicate construction of several gadget graphs used to efficiently map an instance of the Π_2 -variant of the satisfiability problem to an instance of the color choosability problem. These gadgets, however, do not fit the setting of subspace choosability. In fact, the characterization of 2-subspace choosable graphs over the reals, given in [13], implies that the instances produced by the reduction of [8] are never subspace choosable over this field. To overcome this difficulty, we construct and analyze a different gadget graph that allows us, combined with ideas of Gutner and Tarsi [10, 11], to obtain the NP-hardness result stated in Theorem 8. Our analysis involves a characterization, stated below, of the 2-subspace choosable graphs over finite fields, extending the characterizations given in [13] for the real and complex fields.

Proposition 9. *For every finite field \mathbb{F} , a graph is 2-subspace choosable over \mathbb{F} if and only if it contains no cycles.*

While Theorem 8 indicates the hardness of efficiently determining the subspace choice number of bipartite graphs, it would be natural to expect the stronger notion of Π_2 -hardness to hold for this problem.

1.2 Outline

The rest of the paper is organized as follows. In Section 2, we prove Theorem 2, relating the subspace choice number of a graph over a general field to its average degree. We also prove there an improved bound for certain graphs and discuss a limitation of our approach. In section 3, we prove the upper bound on the subspace choice number of complete graphs over finite fields given in Theorem 3. In Section 4, we prove the characterization of 2-subspace choosable graphs over finite fields given in Proposition 9, which will be used in the following sections. In Section 5, we prove several upper and lower bounds on the subspace choosability of complete bipartite graphs in the asymmetric setting, and in particular confirm Theorems 4, 5, and 7. Finally, in Section 6, we prove our hardness result given in Theorem 8.

2 Subspace Choosability and Average Degree

In this section we relate the subspace choice number of a graph over a general field to its average degree and prove Theorem 2. We start with the following definition of k -partitioned graphs (for an example, see Lemma 15).

Definition 10. Let $G = (V, E)$ be a graph. For every vertex $v \in V$, let $E_v \subseteq E$ denote the set of edges of G that are incident with v . We say that the graph G is k -partitioned if it is possible to partition every set E_v , $v \in V$, into k sets $E_v^{(1)}, \dots, E_v^{(k)}$ (some of which may be empty), such that for every function $g : V \rightarrow [k]$ there exist two adjacent vertices $v_1, v_2 \in V$ such that $\{v_1, v_2\} \in E_{v_1}^{(g(v_1))} \cap E_{v_2}^{(g(v_2))}$.

The following theorem shows that the subspace choice number of a k -partitioned graph exceeds k over any field.

Theorem 11. *For every k -partitioned graph G and for every field \mathbb{F} , $\text{ch-s}(G, \mathbb{F}) > k$.*

Proof. Fix an arbitrary field \mathbb{F} . Let $G = (V, E)$ be a k -partitioned graph, and for every vertex $v \in V$, let $E_v = E_v^{(1)} \cup \dots \cup E_v^{(k)}$ be the corresponding partition of the edges incident with v , as in Definition 10. We use these partitions to define a k -subspace assignment over \mathbb{F} to the vertices of G involving vectors from the space $\mathbb{F}^{|E|}$, where each entry corresponds to an edge $e \in E$. To a vertex $v \in V$ we assign the subspace W_v spanned by the k vectors $w_v^{(1)}, \dots, w_v^{(k)}$, where $w_v^{(i)}$ is the 0, 1 indicator vector of the subset $E_v^{(i)}$ of E . In fact, some of the sets $E_v^{(i)}$ might be empty, and thus some of the vectors $w_v^{(i)}$ might be zeros, resulting in subspaces W_v of dimension smaller than k . To fix it, one can increase the length of the

vectors from $|E|$ to $|E| + k \cdot |V|$ and to add to each of the $k \cdot |V|$ vectors $w_v^{(i)}$ a nonzero entry in a coordinate on which all the others have zeros. These entries ensure that the dimension of every subspace W_v is precisely k . For simplicity of notation, we refer from now on to these modified vectors as $w_v^{(i)}$.

We show now that no choice of nonzero vectors from these subspaces satisfies that every two adjacent vertices receive orthogonal vectors over \mathbb{F} . To see this, consider some choice of a nonzero vector $x_v \in W_v$ for each vertex $v \in V$. We define a function $g : V \rightarrow [k]$ as follows. For every $v \in V$, x_v is a nonzero linear combination of the vectors $w_v^{(1)}, \dots, w_v^{(k)}$, hence there exists some $j_v \in [k]$ for which the coefficient of $w_v^{(j_v)}$ in this linear combination is nonzero. We define $g(v)$ to be such an index j_v . By assumption, there exist two adjacent vertices $v_1, v_2 \in V$ such that $\{v_1, v_2\} \in E_{v_1}^{(g(v_1))} \cap E_{v_2}^{(g(v_2))}$. This implies that the entry that corresponds to the edge $\{v_1, v_2\}$ of G is nonzero in both x_{v_1} and x_{v_2} . However, the supports of the subspaces W_{v_1} and W_{v_2} intersect at this single entry, implying that the vectors x_{v_1} and x_{v_2} are not orthogonal over \mathbb{F} . This implies that there exists a k -subspace assignment over \mathbb{F} to the vertices of G with no appropriate choice of nonzero vectors, yielding that $\text{ch-s}(G, \mathbb{F}) > k$, as required. \square

Theorem 11 motivates the problem of determining the largest integer k for which a given graph is k -partitioned. The following lemma uses a probabilistic argument to prove a lower bound on this quantity in terms of the average degree.

Lemma 12. *There exists a constant $c > 0$ such that every graph with average degree $d > 1$ is k -partitioned for some $k \geq c \cdot \sqrt{\frac{d}{\ln d}}$.*

Proof. Let $G = (V, E)$ be a graph with average degree $d > 1$. Note that $2 \cdot |E| = |V| \cdot d$. Let k be the largest integer satisfying

$$d > 2k^2 \cdot \ln k. \tag{1}$$

Observe that for an appropriate choice of the constant c , it holds that $k \geq c \cdot \sqrt{\frac{d}{\ln d}}$. We prove that G is k -partitioned by a probabilistic argument. For every vertex $v \in V$, we define a random partition of the set E_v of the edges incident with v , into k sets $E_v^{(1)}, \dots, E_v^{(k)}$ (some of which may be empty) as follows. For each edge $e \in E_v$, we pick at random, uniformly and independently, some $j \in [k]$, and put e in $E_v^{(j)}$. We claim that the obtained partitions satisfy with positive probability the condition given in Definition 10, namely, that for every function $g : V \rightarrow [k]$ there exist two adjacent vertices $v_1, v_2 \in V$ such that

$$\{v_1, v_2\} \in E_{v_1}^{(g(v_1))} \cap E_{v_2}^{(g(v_2))}. \tag{2}$$

Indeed, for every fixed function $g : V \rightarrow [k]$ and for every edge $\{v_1, v_2\} \in E$, the probability that the event (2) occurs is $1/k^2$. Hence, the probability that for all edges of E this event does not occur is $(1 - 1/k^2)^{|E|}$. By the union bound, the probability that there exists a

function $g : V \rightarrow [k]$ such that for all edges of E the event (2) does not occur is at most

$$k^{|V|} \cdot \left(1 - \frac{1}{k^2}\right)^{|E|} \leq k^{|V|} \cdot e^{-|E|/k^2} = (e^{\ln k - d/(2k^2)})^{|V|}.$$

By (1), the above is smaller than 1, hence with positive probability the random partition satisfies the required condition, and thus G is k -partitioned \square

Combining Theorem 11 and Lemma 12 completes the proof of Theorem 2.

It is natural to ask whether Theorem 11 can be used to obtain better lower bounds on the subspace choice number of graphs than the one achieved by Theorem 2. The following lemma shows that for graphs with similar average and maximum degrees, Lemma 12 is tight up to the logarithmic term. Hence, the approach suggested by Theorem 11 cannot yield significantly better bounds for such graphs.

Lemma 13. *There exists a constant $c > 0$ such that every graph with maximum degree D is not k -partitioned whenever $k \geq c \cdot \sqrt{D}$.*

The proof of Lemma 13 uses the Lovász local lemma stated below (see, e.g., [4, Chapter 5]).

Lemma 14 (Lovász Local Lemma). *Let \mathcal{E} be a collection of events such that for each $A \in \mathcal{E}$, it holds that $\Pr[A] \leq p < 1$ and that A is mutually independent of a set of all but at most d of the other events of \mathcal{E} . If $e \cdot p \cdot (d + 1) \leq 1$, then with positive probability none of the events of \mathcal{E} occurs.*

Proof of Lemma 13. Let $G = (V, E)$ be a graph with maximum degree D , and let $k \geq c \cdot \sqrt{D}$ be an integer for some constant c to be determined. We prove that G is not k -partitioned by a probabilistic argument. For every vertex $v \in V$, consider a partition $E_v = E_v^{(1)} \cup \dots \cup E_v^{(k)}$ of the set of the edges incident with v into k sets (some of the sets may be empty). We claim that there exists a function $g : V \rightarrow [k]$ such that no edge $\{v_1, v_2\}$ of G satisfies $\{v_1, v_2\} \in E_{v_1}^{(g(v_1))} \cap E_{v_2}^{(g(v_2))}$. To prove it, consider a random function $g : V \rightarrow [k]$ such that each value $g(v)$ for $v \in V$ is chosen uniformly and independently at random from $[k]$. For every edge $e = \{v_1, v_2\} \in E$, let A_e denote the event that $e \in E_{v_1}^{(g(v_1))} \cap E_{v_2}^{(g(v_2))}$. The probability of each event A_e is clearly $1/k^2$. In addition, every event A_e is mutually independent of the set of all the other events $A_{e'}$ but those satisfying $e \cap e' \neq \emptyset$, whose number is at most $2 \cdot (D - 1)$. By the Lovász local lemma (Lemma 14), it follows that if

$$e \cdot \frac{1}{k^2} \cdot (2D - 1) \leq 1$$

then with positive probability no event A_e occurs. This implies that for an appropriate choice of the constant c , there exists a function g with the required property. Since this holds for all possible partitions of the sets E_v into k sets, it follows that G is not k -partitioned, and we are done. \square

We end this section by proving that for certain graphs, the logarithmic term in Lemma 12 can be avoided. Here, the proof does not use a probabilistic construction of partitions, but an explicit one, based on finite projective planes.

Lemma 15. *For a prime power q , let H be the $(q + 1)$ -partite graph with q vertices in every part. Let G be a graph obtained from H by removing at most $q - 1$ of its vertices. Then, G is q -partitioned.*

Proof. The proof is based on a well-known construction of projective planes, some of whose properties are described next (see, e.g., [6, Chapter 9]). For every prime power q , there exists a collection of $n = q^2 + q + 1$ elements called *points*, and n sets of points, called *lines*, satisfying that every two lines intersect at a single point, every two points belong together to a single line, every point belongs to precisely $q + 1$ of the lines, and every line includes precisely $q + 1$ of the points. Fix some point p , let L_1, \dots, L_{q+1} be the $q + 1$ lines that include p , and put $L'_i = L_i \setminus \{p\}$ for every $i \in [q + 1]$. Note that the sets L'_i are pairwise disjoint. We view the graph H as the graph on the vertex set $\cup_{i \in [q+1]} L'_i$ in which two vertices are adjacent if they belong to distinct sets L'_i . Observe that every two vertices of H are adjacent if and only if the line that includes their points does not include p .

Let $G = (V, E)$ be some subgraph of H obtained by removing at most $q - 1$ of its vertices, and observe that the number of its vertices satisfies

$$|V| \geq (q + 1) \cdot q - (q - 1) = q^2 + 1.$$

We show that G is q -partitioned. To do so, we assign to every edge of the graph G the line that includes the points represented by its vertices. This assignment induces for every vertex $v \in V$ a partition of the set E_v of the edges incident with v in G , where the sets of the partition correspond to the lines associated with the edges. Observe that this partition of E_v consists of at most q sets. Indeed, the vertex v represents a point that belongs to $q + 1$ lines, but no edge of E_v is assigned the line that includes the point p and the point of v .

In order to show that these partitions satisfy the condition of Definition 10, we shall verify that if one chooses for every point represented by a vertex in G a line that corresponds to an edge incident with it, then there exist two adjacent vertices in G for which the same line was chosen. This indeed follows from the fact that no edge of G corresponds to a line that includes p , hence the total number of lines associated with the edges of G is at most q^2 . Since the number of vertices in G exceeds q^2 , it follows that two vertices are assigned the same line. Since this line does not include p , the two vertices must be adjacent in G , completing the proof. \square

Note that the graph H from Lemma 15 is regular with degree q^2 , hence the minimum degree of its subgraph G is at least $q^2 - q + 1$, and yet G is q -partitioned. This shows that the logarithmic term from Lemma 12 is not needed for G . By combining Lemma 15 with Theorem 11, we derive the following.

Theorem 16. For a prime power q , let H be the $(q + 1)$ -partite graph with q vertices in every part. Let G be a graph obtained from H by removing at most $q - 1$ of its vertices. Then, for every field \mathbb{F} , $\text{ch-s}(G, \mathbb{F}) > q$.

3 Subspace Choosability in Complete Graphs over Finite Fields

In this section we prove Theorem 3, which provides an upper bound on the subspace choice number of complete graphs over finite fields. We start with two useful lemmas.

Lemma 17. For a finite field \mathbb{F} and an integer t , let w_1, w_2, w_3 and z_1, z_2, z_3 be two triples of vectors in \mathbb{F}^t . Then, there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$, not all zeros, such that

$$\left\langle \sum_{i \in [3]} \alpha_i \cdot w_i, \sum_{i \in [3]} \alpha_i \cdot z_i \right\rangle = 0.$$

Proof. Consider the function $f : \mathbb{F}^3 \rightarrow \mathbb{F}$ defined by

$$f(\alpha_1, \alpha_2, \alpha_3) = \left\langle \sum_{i \in [3]} \alpha_i \cdot w_i, \sum_{i \in [3]} \alpha_i \cdot z_i \right\rangle.$$

The function f is a degree 2 polynomial on 3 variables over \mathbb{F} , and $(0, 0, 0)$ forms a root of f . The Chevalley theorem (see, e.g., [18, Chapter IV, Theorem 1D]) implies that f has another root, as required. \square

Lemma 18. For a finite field \mathbb{F} and an integer t , let U_1, U_2, U_3 be three subspaces of \mathbb{F}^t whose dimensions satisfy

$$\dim(U_1) \geq 2, \quad \dim(U_2) \geq 2, \quad \text{and} \quad \dim(U_1) + \dim(U_2) - \dim(U_3) \geq 5.$$

Then, there exist nonzero vectors $x_1 \in U_1$ and $x_2 \in U_2$ such that $\langle x_1, x_2 \rangle = 0$ and

$$\dim(U_3 \cap (x_1)^\perp \cap (x_2)^\perp) \geq \dim(U_3) - 1.$$

Proof. Let $U_1, U_2, U_3 \subseteq \mathbb{F}^t$ be three subspaces as in the statement of the lemma.

Assume first that $\dim(U_1 \cap U_2) \geq 3$. In this case, there exist three linearly independent vectors in $U_1 \cap U_2$. By Lemma 17, there exists a nonzero self-orthogonal linear combination of them. By choosing x_1 and x_2 to be this vector, it obviously holds that $\langle x_1, x_2 \rangle = 0$ and that

$$\dim(U_3 \cap (x_1)^\perp \cap (x_2)^\perp) = \dim(U_3 \cap (x_1)^\perp) \geq \dim(U_3) - 1,$$

as required.

Assume next that $\dim(U_1 \cap U_3^\perp) \geq 1$. Here, x_1 can be chosen as an arbitrary nonzero vector of $U_1 \cap U_3^\perp$, and x_2 as an arbitrary nonzero vector of U_2 satisfying $\langle x_1, x_2 \rangle = 0$. Such a vector exists because $\dim(U_2) \geq 2$. By $x_1 \in U_3^\perp$, it follows that

$$\dim(U_3 \cap (x_1)^\perp \cap (x_2)^\perp) = \dim(U_3 \cap (x_2)^\perp) \geq \dim(U_3) - 1.$$

The case $\dim(U_2 \cap U_3^\perp) \geq 1$ is handled similarly.

Otherwise, we have $\dim(U_1 \cap U_2) \leq 2$ and $\dim(U_1 \cap U_3^\perp) = \dim(U_2 \cap U_3^\perp) = 0$. This implies that

$$\begin{aligned} \dim(U_1 + U_2) &= \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2) \\ &\geq \dim(U_1) + \dim(U_2) - 2 \geq \dim(U_3) + 3, \end{aligned}$$

where for the last inequality we have used the assumption

$$\dim(U_1) + \dim(U_2) - \dim(U_3) \geq 5.$$

It thus follows that

$$\begin{aligned} \dim((U_1 + U_2) \cap U_3^\perp) &= \dim(U_1 + U_2) + \dim(U_3^\perp) - \dim(U_1 + U_2 + U_3^\perp) \\ &\geq \dim(U_1 + U_2) + (t - \dim(U_3)) - t \geq 3. \end{aligned}$$

Hence, there exist vectors $w_1, w_2, w_3 \in U_1$ and $z_1, z_2, z_3 \in U_2$ for which the three sums

$$w_1 + z_1, w_2 + z_2, w_3 + z_3$$

are linearly independent vectors that belong to U_3^\perp . By Lemma 17, there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$, not all zeros, such that the vectors $x_1 = \sum_{i \in [3]} \alpha_i \cdot w_i$ and $x_2 = \sum_{i \in [3]} \alpha_i \cdot z_i$ satisfy $\langle x_1, x_2 \rangle = 0$. These vectors further satisfy that

$$x_1 + x_2 = \sum_{i \in [3]} \alpha_i \cdot (w_i + z_i) \in U_3^\perp.$$

It follows that $x_1 + x_2$ is nonzero, because the vectors $w_i + z_i$ are linearly independent. Observe that x_1 belongs to U_1 and that it is nonzero, because otherwise the vector x_2 would be a nonzero vector that belongs to $U_2 \cap U_3^\perp$, in contradiction to $\dim(U_2 \cap U_3^\perp) = 0$. By the same reasoning, x_2 is a nonzero vector of U_2 . Finally, notice that for every vector $u \in U_3$ such that $\langle u, x_1 \rangle = 0$, it also holds that $\langle u, x_2 \rangle = 0$, and thus

$$\dim(U_3 \cap (x_1)^\perp \cap (x_2)^\perp) = \dim(U_3 \cap (x_1)^\perp) \geq \dim(U_3) - 1,$$

so we are done. □

Remark 19. It can be shown that for the binary field \mathbb{F}_2 , the third condition of Lemma 18 can be slightly weakened to $\dim(U_1) + \dim(U_2) - \dim(U_3) \geq 4$.

We are ready to prove the following result, which implies Theorem 3.

Theorem 20. *For an integer $k \geq 1$, put $n = k^2 + 2k + 3$. Then, for every finite field \mathbb{F} ,*

$$\text{ch-s}(K_n, \mathbb{F}) \leq n - k.$$

Proof. For an integer $k \geq 1$ and a finite field \mathbb{F} , consider the complete graph K_n on $n = k^2 + 2k + 3$ vertices. Let $V = A \cup B \cup C$ be the vertex set of the graph, where $A = \{v_1, \dots, v_k\}$ is a set of k vertices, B is a set of $k^2 + k$ vertices, and C consists of the three remaining vertices. To prove that $\text{ch-s}(K_n, \mathbb{F}) \leq n - k$, suppose that for some integer t , we are given a subspace $U_v \subseteq \mathbb{F}^t$ with $\dim(U_v) = n - k$ for every $v \in V$. Our goal is to show that there exist pairwise orthogonal nonzero vectors $x_v \in U_v$ for $v \in V$. We describe now a process with several steps for choosing the vectors. Throughout the process we maintain for every vertex $v \in V$ a subspace U'_v defined as the subspace of the vectors currently available to the vertex v . Namely, for every partial choice of vectors, U'_v is the subspace of U_v that consists of all the vectors of U_v that are orthogonal to all the previously chosen vectors. Initially, we have $U'_v = U_v$ for every $v \in V$.

Consider some partition of the set B into k sets, $B = B_1 \cup \dots \cup B_k$, where $|B_i| = 2 \cdot (k - i + 1)$ for every $i \in [k]$. Note that this is possible, because $|B| = k^2 + k = 2 \cdot \sum_{i=1}^k (k - i + 1)$. Our process starts with k initial steps, where the role of the i th step ($i \in [k]$) is to choose vectors for the vertices of B_i in a way that poses only $k - i + 1$ linear constraints on the choice of the vector for v_i . Note that for the other vertices, the choice of the vectors for the vertices of B_i might pose twice this number of linear constraints.

For $i \in [k]$, the i th step is performed as follows. Consider an arbitrary partition of the set B_i into $k - i + 1$ pairs, denoted by $(a_1, b_1), \dots, (a_{k-i+1}, b_{k-i+1})$. For every $j \in [k - i + 1]$, we choose two nonzero vectors $u_{a_j} \in U'_{a_j}$ and $u_{b_j} \in U'_{b_j}$ such that $\langle u_{a_j}, u_{b_j} \rangle = 0$ and

$$\dim(U'_{v_i} \cap (u_{a_j})^\perp \cap (u_{b_j})^\perp) \geq \dim(U'_{v_i}) - 1.$$

Observe that such a choice, if it exists, satisfies that u_{a_j} and u_{b_j} are nonzero vectors that belong to the subspaces of the vertices a_j and b_j respectively, they are orthogonal to all the previously chosen vectors and to one another, and in addition, their choice reduces the dimension of U'_{v_i} by at most 1. To prove the existence of such a choice we apply Lemma 18. The number of vectors chosen before the (i, j) iteration is $\sum_{l=1}^{i-1} |B_l| + 2(j - 1)$, hence each of $\dim(U'_{a_j})$ and $\dim(U'_{b_j})$ is at least

$$(n - k) - \sum_{l=1}^{i-1} |B_l| - 2(j - 1).$$

Additionally, since the $2(j - 1)$ already chosen vectors of the i th step reduce the dimension of U'_{v_i} by at most $j - 1$, it can be assumed that

$$\dim(U'_{v_i}) = (n - k) - \sum_{l=1}^{i-1} |B_l| - (j - 1).$$

It thus follows that in the (i, j) iteration, it holds that

$$\dim(U'_{a_j}) + \dim(U'_{b_j}) - \dim(U'_{v_i}) \geq (n - k) - \sum_{l=1}^{i-1} |B_l| - 3(j - 1)$$

$$\begin{aligned}
&= \sum_{l=i}^k |B_l| + 3 - 3(j-1) \\
&= (k-i+1)(k-i+2) - 3j + 6 \\
&\geq (k-i+1)(k-i+2) - 3(k-i+1) + 6 \\
&\geq (k-i+1)(k-i-1) + 6 = (k-i)^2 + 5 \geq 5,
\end{aligned}$$

where for the first equality we use the fact that $n = |B| + k + 3$, and for the second inequality we use the fact $j \leq k - i + 1$. The above bound, which also implies that $\dim(U_{a_j}) \geq 2$ and that $\dim(U_{b_j}) \geq 2$, allows us to apply Lemma 18 and to obtain the required vectors u_{a_j} and u_{b_j} .

We next show that given the above choice for the vertices of B , one can choose vectors for the vertices of $A \cup C$ to obtain the required pairwise orthogonal vectors. First, for the three vertices of C , choose arbitrary pairwise orthogonal nonzero vectors from the currently available subspaces. This is indeed possible, because so far we chose $n - (k + 3)$ vectors, so the dimension of the subspace available to each of them is at least 3. The choice for the first one leaves the available subspaces of the other two with dimension at least 2, and the choice of the second one leaves the available subspace of the third with dimension at least 1, allowing us to choose its nonzero vector.

Finally, we choose the vectors for the vertices of A . For each $i \in [k]$, among the $n - k$ vectors chosen so far, there are $k - i + 1$ pairs of vectors whose choice reduced the dimension of U'_{v_i} by at most 1. This implies that we currently have

$$\dim(U'_{v_i}) \geq (n - k) - ((n - k) - (k - i + 1)) = k - i + 1.$$

This allows us to go over the vertices v_k, v_{k-1}, \dots, v_1 , in this order, and to choose a nonzero vector from the subspace currently available to each of them, completing the proof. \square

Remark 21. For the binary field \mathbb{F}_2 , it can be shown that $\text{ch-s}(K_n, \mathbb{F}_2) \leq n - k$ for $n = k^2 + 2k + 2$. This follows by applying the above proof with the version of Lemma 18 mentioned in Remark 19.

4 Characterization of 2-Subspace Choosable Graphs

In this section we prove Proposition 9, which asserts that for every finite field \mathbb{F} and for every graph G , $\text{ch-s}(G, \mathbb{F}) \leq 2$ if and only if G contains no cycles.

Proof of Proposition 9. If G contains no cycles then it is 1-degenerate, implying that it is 2-subspace choosable over every field \mathbb{F} . To complete the proof, we fix some finite field \mathbb{F} and turn to show that for every $\ell \geq 3$, the ℓ -cycle C_ℓ satisfies $\text{ch-s}(C_\ell, \mathbb{F}) > 2$. We first prove it for $\ell = 3$ and for $\ell = 4$.

- For $\ell = 3$, assign to the vertices of the cycle C_3 the subspaces of \mathbb{F}^3 defined by

$$U_1 = \text{span}(e_1, e_2), \quad U_2 = \text{span}(e_1, e_2 + e_3), \quad \text{and} \quad U_3 = \text{span}(e_1 + \alpha \cdot e_3, e_2),$$

where $\alpha \in \mathbb{F}$ is some field element to be determined. We claim that for some $\alpha \in \mathbb{F}$ it is impossible to choose three pairwise orthogonal nonzero vectors $x_i \in U_i$ ($i \in [3]$). Indeed, it is easy to verify that x_1 cannot be chosen as a scalar multiple of e_1 nor of e_2 . So assume without loss of generality that x_1 is proportional to $e_1 + z \cdot e_2$ for some $z \neq 0$. If x_1 is orthogonal to x_2 and to x_3 , then x_2 is proportional to $z \cdot e_1 - e_2 - e_3$ and x_3 is proportional to $z \cdot e_1 + \alpha z \cdot e_3 - e_2$. However, the inner product of the latter two is $z^2 - \alpha \cdot z + 1$, so it suffices to show that there exists $\alpha \in \mathbb{F}$ for which this quadratic polynomial has no root. Notice that in case that $z^2 - \alpha \cdot z + 1$ has a root, it can be written as $(z - \gamma) \cdot (z - \gamma^{-1})$ for some $\gamma \neq 0$. Since the number of possible values of α is larger than the number of possible invertible values of γ , it follows that the required α exists.

- For $\ell = 4$, suppose first that the field \mathbb{F} is of characteristic larger than 2, and assign to the vertices along the cycle C_4 the subspaces of \mathbb{F}^4 defined by

$$U_1 = U_2 = \text{span}(e_1, e_2), \quad U_3 = \text{span}(e_1 + e_4, e_2 + e_3), \quad \text{and} \quad U_4 = \text{span}(e_1 + \alpha \cdot e_3, e_2 + e_4),$$

where $\alpha \in \mathbb{F}$ is some nonzero field element to be determined. We claim that for some $\alpha \neq 0$ it is impossible to choose four nonzero vectors $x_i \in U_i$ ($i \in [4]$) that form a valid choice for C_4 . By $\alpha \neq 0$, it is easy to verify, as before, that x_1 cannot be chosen as a scalar multiple of e_1 nor of e_2 , so it can be assumed that it is proportional to $e_1 + z \cdot e_2$ for some $z \neq 0$. If the vectors x_i form a valid choice for C_4 , then x_2 is proportional to $z \cdot e_1 - e_2$, thus x_3 is proportional to $e_1 + e_4 + z \cdot e_2 + z \cdot e_3$, and x_4 is proportional to $z \cdot e_1 + \alpha z \cdot e_3 - e_2 - e_4$. However, the inner product of the latter two is $\alpha \cdot z^2 - 1$, so it suffices to show that there exists $\alpha \neq 0$ for which $\alpha \cdot z^2 \neq 1$ for all values of z . Since \mathbb{F} is of characteristic larger than 2, it has a non-square element, whose choice for α completes the argument.

If, however, \mathbb{F} is of characteristic 2, one can consider the subspaces of \mathbb{F}^5 defined by $U_1 = U_2 = \text{span}(e_1, e_2)$, $U_3 = \text{span}(e_1 + e_4, e_2 + e_3 + e_5)$, $U_4 = \text{span}(e_1 + e_3, e_2 + \alpha \cdot e_4 + e_5)$, where $\alpha \in \mathbb{F}$ is some nonzero field element for which $z^2 + z \neq \alpha$ for all values of z . Notice that such an α exists because the function $z \mapsto z^2 + z$ maps both 0 and 1 to 0, so some nonzero element does not belong to its image. It can be verified that for the above subspace assignment, no valid choice of vectors for C_4 exists.

Finally, observe that for every odd $\ell > 3$, one can extend the above subspace assignment for C_3 by adding $\ell - 3$ copies of the subspace $\text{span}(e_1, e_2)$ between U_1 and U_2 to get a subspace assignment showing that $\text{ch-s}(C_\ell, \mathbb{F}) > 2$. Similarly, for every even $\ell > 4$, one can extend the above subspace assignment for C_4 by adding $\ell - 4$ copies of the subspace $\text{span}(e_1, e_2)$ between U_1 and U_2 . \square

Remark 22. As shown in [13], the characterization given in Proposition 9 for finite fields holds for the real field \mathbb{R} too. In particular, for every integer $\ell \geq 3$, it holds that $\text{ch-s}(C_\ell, \mathbb{R}) > 2$. For an odd ℓ , this simply follows by assigning \mathbb{R}^2 to every vertex. For an even ℓ , this follows from the construction given above in the proof for fields of characteristic larger than 2, taking α to be some non-square over \mathbb{R} .

5 Subspace Choosability in Complete Bipartite Graphs

In this section we prove our results on subspace choosability in complete bipartite graphs.

5.1 Complete Balanced Bipartite Graphs

Erdős et al. [8] proved that the choice number of the complete balanced bipartite graph $K_{m,m}$ exceeds k for $m = \binom{2k-1}{k}$. We provide here a quick proof for an analogue result for subspace choosability. Note, however, that when the number of vertices is sufficiently large, the lower bound given by Theorem 2 is significantly better.

Proposition 23. *For an integer k and a field \mathbb{F} , $\text{ch-s}(K_{m,m}, \mathbb{F}) > k$ for $m = \binom{2k-1}{k}$.*

Proof. Let k be an integer and let \mathbb{F} be a field. Consider the graph $K_{m,m}$ for $m = \binom{2k-1}{k}$, and associate with the vertices of every side of the graph all the k -subsets of $[2k-1]$. For a vertex associated with a k -subset A of $[2k-1]$ we assign the k -subspace of \mathbb{F}^{2k-1} spanned by the vectors e_i with $i \in A$, where e_i stands for the vector of \mathbb{F}^{2k-1} with 1 on the i th entry and 0 everywhere else. We claim that there is no choice of nonzero vectors from these subspaces such that the vectors of the left side are orthogonal to those of the right side. To see this, suppose in contradiction that such a choice exists, and denote by x_1, \dots, x_m and y_1, \dots, y_m the vectors chosen for the vertices of the left and right sides respectively. Letting $U = \text{span}(x_1, \dots, x_m)$ and $V = \text{span}(y_1, \dots, y_m)$, it follows that $V \subseteq U^\perp$, and thus

$$\dim(U) + \dim(V) \leq \dim(U) + \dim(U^\perp) = 2k - 1,$$

implying that at least one of U and V has dimension at most $k-1$. Without loss of generality, assume that $\dim(U) \leq k-1$. Put $\ell = \dim(U)$, fix some ℓ vectors from x_1, \dots, x_m that span U , and consider the $(2k-1) \times \ell$ matrix whose columns are these vectors. Since the dimension of the subspace spanned by the rows of U is also ℓ , it follows that there exists a set $B \subseteq [2k-1]$ of ℓ indices whose rows are linearly independent. It follows that the only vector in U with zeros in all entries of B is the zero vector. However, by $|B| = \ell \leq k-1$, there exists a k -subset A of $[2k-1]$ disjoint from B , so the vertex associated with this A in the left side of the graph cannot receive any nonzero vector of U . This gives us the required contradiction and completes the proof. \square

5.2 Asymmetric Subspace Choosability in Complete Bipartite Graphs

We consider now complete bipartite graphs in the asymmetric setting, where the dimensions of the subspaces assigned to the vertices of the right and left sides might be different.

Definition 24. The complete bipartite graph K_{ℓ_1, ℓ_2} with the vertex set A of size ℓ_1 on the left side and the vertex set B of size ℓ_2 on the right side is said to be (k_1, k_2) -subspace choosable over a field \mathbb{F} if it is f -subspace choosable over \mathbb{F} for the function $f : A \cup B \rightarrow \{k_1, k_2\}$ defined by $f(u) = k_1$ for every $u \in A$ and $f(u) = k_2$ for every $u \in B$.

In what follows, we provide several conditions that imply subspace choosability and subspace non-choosability in complete bipartite graphs, and in particular prove Theorems 4, 5, and 7.

5.2.1 Upper Bounds

We start with the following simple statement.

Proposition 25. *For every field \mathbb{F} , the graph K_{ℓ_1, ℓ_2} is (k_1, k_2) -subspace choosable over \mathbb{F} whenever $\ell_1 < k_2$ or $\ell_2 < k_1$.*

Proof. Suppose that $\ell_1 < k_2$, and let U_1, \dots, U_{ℓ_1} and V_1, \dots, V_{ℓ_2} be k_1 -subspaces and k_2 -subspaces, respectively, of \mathbb{F}^t for some integer t . Choose an arbitrary nonzero vector from each U_i for $i \in [\ell_1]$. Such a choice poses at most ℓ_1 linear constraints on the choice of a vector from each V_j , and since the dimension of those subspaces is $k_2 > \ell_1$, a nonzero choice exists, resulting in a valid choice for the whole graph. By symmetry, the result holds for the case $\ell_2 < k_1$ as well. \square

We next prove the following result, which confirms Theorem 4.

Theorem 26. *For every two integers k and n and for every field \mathbb{F} , the graph $K_{k,m}$ is (n, k) -subspace choosable over \mathbb{F} for $m = \sum_{i=0}^{k-1} \lfloor \frac{n-1}{k-i} \rfloor$.*

We need the following lemma.

Lemma 27. *Let W be a k -subspace of some finite-dimensional vector space over a field \mathbb{F} , let W_1, \dots, W_t be r -subspaces of W , and suppose that $t \leq \frac{k-1}{k-r}$. Then, there exists a nonzero vector in the intersection $\bigcap_{i \in [t]} W_i$.*

Proof. Using the standard equality $\dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 + V_2)$, observe that

$$\begin{aligned} \dim \left(\bigcap_{i \in [t]} W_i \right) &= \dim(W_1) + \dim \left(\bigcap_{i \geq 2} W_i \right) - \dim \left(W_1 + \bigcap_{i \geq 2} W_i \right) \\ &\geq \dim(W_1) + \dim \left(\bigcap_{i \geq 2} W_i \right) - \dim(W). \end{aligned}$$

By repeatedly applying this inequality we obtain that

$$\dim \left(\bigcap_{i \in [t]} W_i \right) \geq \sum_{i \in [t]} \dim(W_i) - (t-1) \cdot \dim(W) = t \cdot r - (t-1) \cdot k \geq 1,$$

hence there exists a nonzero vector in $\bigcap_{i \in [t]} W_i$, as required. \square

We are ready to prove Theorem 26.

Proof of Theorem 26. For integers k and n , put $m = \sum_{i=0}^{k-1} \lfloor \frac{n-1}{k-i} \rfloor$. We show that the graph $K_{k,m}$ is (n, k) -subspace choosable over every field \mathbb{F} . Denote the left and right vertices of the graph by u_1, \dots, u_k and v_1, \dots, v_m respectively, and consider an arbitrary assignment of n -subspaces and k -subspaces of \mathbb{F}^t to the left and right vertices, respectively, for some integer t . For every $i \in [k]$ let U_i be the subspace assigned to u_i , and for every $j \in [m]$ let V_j be the subspace assigned to v_j . We will show that it is possible to choose nonzero vectors from these subspaces such that the vectors of the left side are orthogonal over \mathbb{F} to those of the right side.

We first describe how the vectors x_1, \dots, x_k of the left vertices u_1, \dots, u_k are chosen. We choose them one by one, and to do so we maintain a set $J \subseteq [m]$ and some subspaces L_1, \dots, L_m of \mathbb{F}^t . Initially, we define $J = [m]$ and $L_j = V_j^\perp$ for all $j \in [m]$. Note that $\dim(L_j) = t - k$. Then, for every $i \in [k]$ we act as follows.

- Pick some set $J' \subseteq J$ of size $|J'| = \lfloor \frac{n-1}{k-(i-1)} \rfloor$.
- Let $J'' \subseteq J'$ be the set of indices $j \in J'$ satisfying $\dim(L_j) = t - k + (i - 1)$.
- Choose x_i to be some nonzero vector of U_i that belongs to the intersection $\bigcap_{j \in J''} L_j$.
- Add the vector x_i to every subspace L_j , that is, update every subspace L_j to be the subspace $L_j + \text{span}(x_i)$.
- Remove the elements of J' from J .

Observe that the number of elements removed from J during the above k iterations is $\sum_{i=1}^k \lfloor \frac{n-1}{k-(i-1)} \rfloor$. Since the latter coincides with our definition of m , it follows that after the k th iteration the set J is empty.

We show now that the vectors x_i are well defined, in the sense that in the i th iteration there exists a nonzero vector that belongs to U_i and to the intersection $\bigcap_{j \in J''} L_j$. To see this, put $W = U_i$ and consider its subspaces $W_j = L_j \cap W$ for $j \in J''$. By the definition of J'' , for every $j \in J''$ it holds that $\dim(L_j) = t - k + (i - 1)$, hence, using $\dim(W) = n$, it follows that

$$\begin{aligned} \dim(W_j) &= \dim(L_j) + \dim(W) - \dim(L_j + W) \\ &\geq t - k + (i - 1) + n - t = n - k + i - 1. \end{aligned}$$

By Lemma 27 applied to W and to its subspaces W_j , using the fact that $|J''| \leq \lfloor \frac{n-1}{k-(i-1)} \rfloor$, the required vector x_i is guaranteed to exist.

We finally show that the above choice of vectors for the left vertices can be extended to a valid choice of vectors for the whole graph. Fix some $j \in [m]$ and observe that if the subspace L_j obtained at the end of the k th iteration has dimension strictly smaller than t then it is possible to choose an appropriate vector y_j for the vertex v_j . Indeed, y_j can be chosen as any nonzero vector orthogonal to this L_j , because such a vector is orthogonal to V_j^\perp , hence belongs to V_j , and is orthogonal to all the vectors x_1, \dots, x_k that were chosen for the left vertices and were added to L_j during the k iterations. Since the

initial dimension of L_j is $t - k$ it suffices to show that in at least one of the k iterations, the chosen vector x_i was already inside L_j . So suppose that the set J' includes j in the i th iteration. If $j \in J''$ then the vector x_i chosen in this iteration belongs to the current L_j . Otherwise, the dimension of L_j in this iteration is smaller than $t - k + (i - 1)$, implying that in one of the previous $i - 1$ iterations a vector that already belongs to L_j was chosen, so we are done. \square

5.2.2 Lower Bounds

We start with the following simple statement.

Proposition 28. *For every two integers $n, k \geq 2$ and for every field \mathbb{F} , the graph K_{k,n^k} is not (n, k) -subspace choosable over \mathbb{F} .*

Proof. Denote the vertices of the left side of K_{k,n^k} by u_1, \dots, u_k . For every $i \in [k]$, assign to the vertex u_i the n -subspace of \mathbb{F}^{nk} spanned by the vectors e_i with $i \in [(i-1) \cdot n + 1, i \cdot n]$, where e_i stands for the vector in \mathbb{F}^{nk} with 1 on the i th entry and 0 everywhere else. Then, associate with each of the n^k vertices of the right side a distinct k -tuple $(a_1, \dots, a_k) \in [n]^k$, and assign to it the k -subspace of \mathbb{F}^{nk} spanned by the vectors $e_{(i-1)n+a_i}$ for $i \in [k]$.

We claim that there is no choice of nonzero vectors from these subspaces such that the vectors of the left side are orthogonal over \mathbb{F} to those of the right side. To see this, consider any choice of a nonzero vector x_i for each vertex u_i for $i \in [k]$. For every $i \in [k]$, consider the restriction $\tilde{x}_i \in \mathbb{F}^n$ of the vector x_i to the support of its subspace, that is, to the entries with indices in $[(i-1) \cdot n + 1, i \cdot n]$. Since x_i is nonzero, it follows that there exists some $a_i \in [n]$ such that the vector \tilde{x}_i is nonzero in its a_i th entry. However, the only vector in the subspace of the vertex (a_1, \dots, a_k) of the right side which is orthogonal to all the vectors x_i ($i \in [k]$) is the zero vector. This implies that no choice of nonzero vectors for the left side can be extended to a valid choice of vectors for the whole graph, so we are done. \square

We next prove the following result.

Theorem 29. *For every integers n, t, k and for every field \mathbb{F} , the following holds. If there exists a collection of $m = k \cdot (t - 1) + 1$ nonzero vectors in \mathbb{F}^n satisfying that every t of them span the entire space \mathbb{F}^n , then the graph $K_{k,m}$ is not (n, k) -subspace choosable over \mathbb{F} .*

Proof. Suppose that there exists a collection of $m = k \cdot (t - 1) + 1$ nonzero vectors b_1, \dots, b_m in \mathbb{F}^n satisfying that every t of them span the space \mathbb{F}^n . To prove that $K_{k,m}$ is not (n, k) -subspace choosable, we have to show that it is possible to assign n -subspaces and k -subspaces over \mathbb{F} to the left and right vertices of the graph $K_{k,m}$ respectively, so that no choice of a nonzero vector from each subspace satisfies that the vectors of the left vertices are orthogonal over \mathbb{F} to the vectors of the right vertices.

Let u_1, \dots, u_k be the vertices of the left side, and let v_1, \dots, v_m be the vertices of the right side. For every $i \in [k]$, we assign to the vertex u_i the subspace U_i of \mathbb{F}^{kn} that includes all the vectors whose support is contained in the entries indexed by $[(i-1) \cdot n + 1, i \cdot n]$.

In other words, viewing the vectors of \mathbb{F}^{kn} as a concatenation of k parts of length n , U_i is the n -subspace of all the vectors that have zeros in all the parts but the i th one. Then, for every $j \in [m]$, we assign to the vertex v_j the subspace V_j spanned by the k vectors $e_1 \otimes b_j, \dots, e_k \otimes b_j$ of \mathbb{F}^{kn} . Here, e_i stands for the vector in \mathbb{F}^k with 1 on the i th entry and 0 everywhere else, and \otimes stands for the tensor product operation of vectors. Hence, V_j is the k -subspace of all the vectors in \mathbb{F}^{kn} consisting of k parts, each of which is equal to the vector b_j multiplied by some element of \mathbb{F} .

Assume for the sake of contradiction that there exist nonzero vectors $x_i \in U_i$ ($i \in [k]$) and $y_j \in V_j$ ($j \in [m]$) such that $\langle x_i, y_j \rangle = 0$ for all i and j . For any $i \in [k]$, let $\tilde{x}_i \in \mathbb{F}^n$ be the (nonzero) restriction of the vector x_i to the i th part. For any $j \in [m]$, write $y_j = \sum_{i \in [k]} \alpha_{i,j} \cdot e_i \otimes b_j$ for some coefficients $\alpha_{i,j} \in \mathbb{F}$. Since all the vectors y_j are nonzero, it clearly follows that at least m of the coefficients $\alpha_{i,j}$ are nonzero. Now, observe that for all $i \in [k]$ and $j \in [m]$, $\langle x_i, y_j \rangle = 0$ implies that $\langle \tilde{x}_i, \alpha_{i,j} \cdot b_j \rangle = 0$. However, combining the facts that \tilde{x}_i is nonzero and that every t vectors among b_1, \dots, b_m span \mathbb{F}^n , it follows that for every $i \in [k]$, at most $t - 1$ of the coefficients $\alpha_{i,j}$ with $j \in [m]$ are nonzero. This yields that the total number of nonzero coefficients $\alpha_{i,j}$ is at most $k \cdot (t - 1) < m$, providing the desired contradiction. \square

We derive the following.

Corollary 30. *Let k be an integer, and let \mathbb{F} be a field.*

1. *For an integer n , set $m = k \cdot (n - 1) + 1$. If $|\mathbb{F}| \geq m$ then the graph $K_{k,m}$ is not (n, k) -subspace choosable over \mathbb{F} .*
2. *For integers n and q , set $m = k \cdot \frac{q^{n-1}-1}{q-1} + 1$. If \mathbb{F} is a finite field of size $q \geq k$ then the graph $K_{k,m}$ is not (n, k) -subspace choosable over \mathbb{F} .*

Proof. For Item 1, set $m = k \cdot (n - 1) + 1$, and let $\gamma_1, \dots, \gamma_m$ be some distinct elements of the field \mathbb{F} . For each $i \in [m]$, let b_i be the vector in \mathbb{F}^n defined by $b_i = (1, \gamma_i, \gamma_i^2, \dots, \gamma_i^{n-1})$. As follows from standard properties of the Vandermonde matrix, every n of the vectors b_1, \dots, b_m are linearly independent and thus span the space \mathbb{F}^n . By Theorem 29 applied to these vectors with $t = n$, it follows that $K_{k,m}$ is not (n, k) -subspace choosable over \mathbb{F} , as required.

For Item 2, set $t = \frac{q^{n-1}-1}{q-1} + 1$ and $m = k \cdot (t - 1) + 1$. Consider the equivalence relation on the nonzero vectors of \mathbb{F}^n defined by calling two vectors equivalent if one is a multiple of the other by an element of \mathbb{F} . Let B be a collection of vectors in \mathbb{F}^n that consists of one vector from every equivalence class, and note that $|B| = \frac{q^n-1}{q-1}$. We observe that every t vectors of B span the space \mathbb{F}^n . Indeed, every strict subspace of \mathbb{F}^n has dimension at most $n - 1$, so it includes at most $q^{n-1} - 1$ nonzero vectors, and thus at most $t - 1$ vectors that represent different equivalence classes. The assumption $q \geq k$ implies that

$$m = k \cdot \frac{q^{n-1} - 1}{q - 1} + 1 \leq \frac{q^n - 1}{q - 1} = |B|,$$

so by applying Theorem 29 to m of the vectors of B , we get that $K_{k,m}$ is not (n, k) -subspace choosable over \mathbb{F} , and we are done. \square

Theorem 5 and Corollary 6 follow, respectively, from Theorem 29 and Corollary 30.

We note that the approach proposed by Theorem 29 for proving subspace non-choosability results seems to be more beneficial for large fields. This is justified by the following lemma that relates the size of the collection needed in Theorem 29 to the size of the field. Its proof is inspired by an argument given in [5].

Lemma 31. *Let \mathbb{F} be a finite field of size q , and let $m \geq t \geq n$ be integers. If there exists a collection of m nonzero vectors in \mathbb{F}^n satisfying that every t of them span the space \mathbb{F}^n , then*

$$m \leq n - 2 + (q + 1) \cdot (t - n + 1).$$

Proof. Let $S \subseteq \mathbb{F}^n$ be a set of m nonzero vectors in \mathbb{F}^n satisfying that every t of them span the space \mathbb{F}^n . Let x_1, \dots, x_{n-2} be $n - 2$ linearly independent vectors of S , and consider all the $(n - 1)$ -subspaces of \mathbb{F}^n that include all of these vectors. Observe that the number of such subspaces is $q + 1$, and that these subspaces cover together the entire space \mathbb{F}^n . Since every t vectors of S span \mathbb{F}^n , it follows that each of these $q + 1$ subspaces includes less than $t - (n - 2)$ of the vectors of $S \setminus \{x_1, \dots, x_{n-2}\}$. We thus conclude that $m = |S| \leq (n - 2) + (q + 1) \cdot (t - (n - 2) - 1)$, and we are done. \square

5.2.3 Two Vertices on the Left Side

We next consider the particular case of the complete bipartite graph $K_{2,m}$ with two vertices on the left side and m vertices on the right side. It will be convenient to use the following definition.

Definition 32. The complete bipartite graph $K_{2,m}$ with the vertex set $A = \{u_1, u_2\}$ on the left side and the vertex set B of size m on the right side is said to be $(k_1; k_2, k_3)$ -subspace choosable over a field \mathbb{F} if it is f -subspace choosable over \mathbb{F} for the function $f : A \cup B \rightarrow \{k_1, k_2, k_3\}$ defined by $f(u_1) = k_1$, $f(u_2) = k_2$ and $f(u) = k_3$ for every $u \in B$.

In what follows, we prove Theorem 7. We start with its first item, restated and proved below.

Proposition 33. *For every integer n and for every field \mathbb{F} , the graph $K_{2,n-1}$ is $(n; 2, 2)$ -subspace choosable over \mathbb{F} .*

Proof. For an integer n , consider the graph $K_{2,n-1}$. To prove that it is $(n; 2, 2)$ -subspace choosable over a field \mathbb{F} , consider for some integer t arbitrary subspaces U_1, U_2 and V_1, \dots, V_{n-1} of \mathbb{F}^t whose dimensions satisfy $\dim(U_1) = n$, $\dim(U_2) = 2$, and $\dim(V_j) = 2$ for $j \in [n - 1]$. Choose an arbitrary nonzero vector $x_2 \in U_2$, and for every $j \in [n - 1]$ choose a nonzero vector $y_j \in V_j$ such that $\langle x_2, y_j \rangle = 0$. Note that this is possible since $\dim(V_j) = 2$. Finally, choose a vector $x_1 \in U_1$ satisfying $\langle x_1, y_j \rangle = 0$ for all $j \in [n - 1]$, which is possible by $\dim(U_1) = n$. This gives us the required choice of vectors. \square

By the above proposition, $K_{2,n-1}$ is $(n; 2, 2)$ -subspace choosable over every field. We consider the question of whether this holds even after adding another vertex to the right side of the graph. Under certain conditions the answer is positive, as shown by the following result, confirming Item 2 and the “if” part of Item 3 in Theorem 7.

Proposition 34. *The graph $K_{2,n}$ is $(n; 2, 2)$ -subspace choosable for every integer n over \mathbb{C} and for every odd integer n over \mathbb{R} .*

We need the following lemma, which is essentially given in [13].

Lemma 35 ([13, Lemma 2.9]). *Let $t \geq 2$ be an integer, and let \mathbb{F} be either \mathbb{R} or \mathbb{C} . Let U, V be two 2-subspaces of \mathbb{F}^t such that for every nonzero vector $x \in U$ there exists a nonzero vector $y \in V$ such that $\langle x, y \rangle \neq 0$. Then, for every basis $u^{(1)}, u^{(2)}$ of U satisfying $\langle u^{(i)}, u^{(j)} \rangle \neq 0$ if and only if $i = j$, there exists a basis $v^{(1)}, v^{(2)}$ of V satisfying $\langle v^{(i)}, v^{(j)} \rangle \neq 0$ if and only if $i = j$ and, in addition, $\langle u^{(i)}, v^{(j)} \rangle = 0$ if and only if $i = j$.*

Proof of Proposition 34. Let n be an integer, and let \mathbb{F} be either \mathbb{R} or \mathbb{C} . Consider the graph $K_{2,n}$ with the vertex set $A = \{u_1, u_2\}$ on the left side and the vertex set $B = \{v_1, \dots, v_n\}$ on the right side. To prove that the graph is $(n; 2, 2)$ -subspace choosable over \mathbb{F} , consider some subspaces $U_1, U_2, V_1, \dots, V_n$ of \mathbb{F}^t for some integer t , where $\dim(U_1) = 2$, $\dim(U_2) = n$, and $\dim(V_j) = 2$ for all $j \in [n]$. We will show now that there exist nonzero vectors $x_i \in U_i$ ($i \in [2]$) and $y_j \in V_j$ ($j \in [n]$) such that $\langle x_i, y_j \rangle = 0$ over \mathbb{F} for all i and j .

Suppose first that there exists a nonzero vector $x_1 \in U_1$ such that x_1 is orthogonal to the subspace $V_{j'}$ for some $j' \in [n]$. In this case, choose x_1 for the vertex u_1 , and for every $j \in [n] \setminus \{j'\}$ let $y_j \in V_j$ be a nonzero choice for the vertex v_j satisfying $\langle x_1, y_j \rangle = 0$. Note that such a choice exists because $\dim(V_j) = 2$. These choices pose at most $n - 1$ linear constraints on the choice for u_2 , so by $\dim(U_2) = n$, there exists a nonzero vector $x_2 \in U_2$ that is orthogonal to all the vectors y_j with $j \in [n] \setminus \{j'\}$. Finally, choose $y_{j'} \in V_{j'}$ as a nonzero vector orthogonal to x_2 , whose existence is guaranteed by $\dim(V_{j'}) = 2$. The assumption on x_1 implies that $\langle x_1, y_{j'} \rangle = 0$, so we obtain the required choice of vectors.

Otherwise, let $u_1^{(1)}, u_1^{(2)}$ be a basis of U_1 satisfying $\langle u_1^{(i)}, u_1^{(j)} \rangle \neq 0$ if and only if $i = j$. Since no nonzero vector of U_1 is orthogonal to some V_j , we can apply Lemma 35 to obtain for every $j \in [n]$ a basis $v_j^{(1)}, v_j^{(2)}$ of V_j that satisfies the assertion of the lemma. Note that it can be assumed that $\langle u_1^{(1)}, v_j^{(2)} \rangle = \langle u_1^{(2)}, v_j^{(1)} \rangle = 1$ for all $j \in [n]$. Let M_1 and M_2 be the $n \times t$ matrices over \mathbb{F} whose j th rows are $v_j^{(1)}$ and $v_j^{(2)}$ respectively.

Now, to obtain the required choice of nonzero vectors, let $x_1 = \alpha \cdot u_1^{(1)} + \beta \cdot u_1^{(2)}$ be our nonzero choice for the vertex u_1 for some $\alpha, \beta \in \mathbb{F}$ to be determined. Observe that this choice forces us to choose, up to a multiplicative constant, the vector $y_j = \alpha \cdot v_j^{(1)} - \beta \cdot v_j^{(2)}$ for the vertex v_j for each $j \in [n]$. For the vertex u_2 , let $U \in \mathbb{F}^{t \times n}$ denote a matrix whose columns form a basis of the subspace U_2 , and denote its choice by $x_2 = U \cdot \gamma$ for $\gamma \in \mathbb{F}^n$. We consider the question of whether there exist α, β as above and a nonzero γ such that $\langle x_2, y_j \rangle = 0$ for all $j \in [n]$. Observe that this condition is equivalent to

$$(\alpha \cdot M_1 - \beta \cdot M_2) \cdot (U \cdot \gamma) = 0.$$

Letting M'_1 and M'_2 be the $n \times n$ matrices defined by $M'_1 = M_1 \cdot U$ and $M'_2 = M_2 \cdot U$, we ask whether there exist $\alpha, \beta \in \mathbb{F}$, that are not both zeros, and a nonzero vector $\gamma \in \mathbb{F}^n$ satisfying

$$(\alpha \cdot M'_1 - \beta \cdot M'_2) \cdot \gamma = 0.$$

If $\det(M'_1) = 0$ then we can take $\alpha = 1$ and $\beta = 0$, for which a nonzero γ is guaranteed to exist. Otherwise, if $\det(M'_1) \neq 0$, we take, say, $\beta = -1$, and show that for some $\alpha \in \mathbb{F}$, the matrix $\alpha \cdot M'_1 + M'_2$ is singular, implying the existence of the required vector γ . To see this, observe that $\alpha \cdot M'_1 + M'_2$ is singular if and only if $\alpha \cdot I_n + N$ is singular as well, where $N = M'_2 \cdot (M'_1)^{-1}$. This reduces our question to whether for some $\alpha \in \mathbb{F}$ it holds that $\det(\alpha \cdot I_n + N) = 0$. This determinant is a degree n polynomial in α . Over $\mathbb{F} = \mathbb{C}$, this polynomial clearly has a root, and over $\mathbb{F} = \mathbb{R}$, assuming that n is odd, it has a root as well. This completes the proof. \square

We end this section by proving that adding a vertex to the right side of $K_{2,n-1}$ for an even integer n results in a graph which is no longer $(n; 2, 2)$ -subspace choosable over the real field \mathbb{R} and over every finite field. This, in particular, gives us the “only if” part of Item 3 of Theorem 7.

Proposition 36. *Let \mathbb{F} be either \mathbb{R} or any finite field. Then, for every even integer n , the graph $K_{2,n}$ is not $(n; 2, 2)$ -subspace choosable over \mathbb{F} .*

Proof. For a field \mathbb{F} as above, Proposition 9 and Remark 22 imply that $\text{ch-s}(K_{2,2}, \mathbb{F}) > 2$. Hence, for some integer t , there exist 2-subspaces $L_1, L_2, R_1, R_2 \subseteq \mathbb{F}^t$ such that no choice of nonzero vectors $\tilde{x}_i \in L_i$ and $\tilde{y}_j \in R_j$ for $i, j \in [2]$ satisfies $\langle \tilde{x}_i, \tilde{y}_j \rangle = 0$ for all i, j .

For an even integer $n = 2k$, we define a subspace assignment to the vertices of $K_{2,n}$ that lies in the vector space $\mathbb{F}^{t \cdot k}$ as follows. To the left vertices we assign the subspaces $U_1, U_2 \subseteq \mathbb{F}^{t \cdot k}$ defined by

$$U_1 = \text{span}(e_1 \otimes L_1, \dots, e_k \otimes L_1) \quad \text{and} \quad U_2 = \left(\sum_{i=1}^k e_i \right) \otimes L_2,$$

and to the right vertices we assign the subspaces $V_1, \dots, V_n \subseteq \mathbb{F}^{t \cdot k}$, defined by

$$V_{2j-1} = e_j \otimes R_1 \quad \text{and} \quad V_{2j} = e_j \otimes R_2$$

for each $j \in [k]$. Note that $\dim(U_1) = n$, $\dim(U_2) = 2$, and $\dim(V_j) = 2$ for all $j \in [n]$. Intuitively, the assignment is designed so that the t -dimensional restriction of $U_1, U_2, V_{2j-1}, V_{2j}$ to the j th block is the assignment L_1, L_2, R_1, R_2 .

To complete the proof, we show that there is no choice of nonzero vectors $x_i \in U_i$ and $y_j \in V_j$ for $i \in [2]$ and $j \in [n]$ that satisfies $\langle x_i, y_j \rangle = 0$ for all i, j . So suppose for contradiction that such a choice exists, and let $j \in [k]$ be an integer for which the restriction of x_1 to the j th block is nonzero. Denote by $\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2$ the restrictions of the vectors $x_1, x_2, y_{2j-1}, y_{2j}$ to the j th block. Observe that these are nonzero vectors that satisfy $\tilde{x}_i \in L_i, \tilde{y}_j \in R_j$, and $\langle \tilde{x}_i, \tilde{y}_j \rangle = 0$ for all $i, j \in [2]$, in contradiction. \square

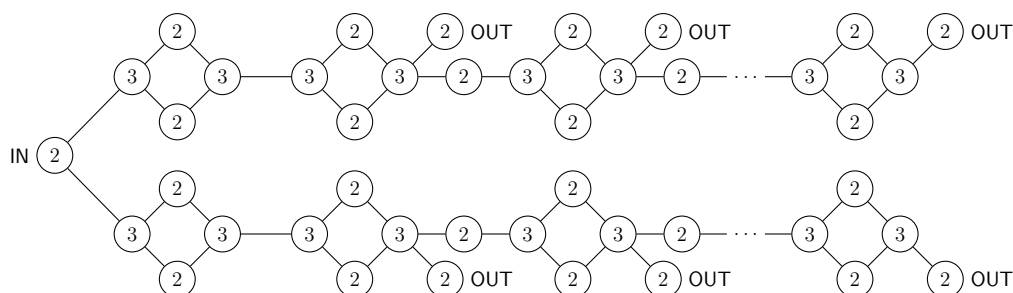
6 Hardness Result

In this section we prove our hardness result, given in Theorem 8. We start by presenting a gadget graph that will be used in the proof.

6.1 Gadget Graph

The main component of our hardness proof is the \exists -graph defined as follows.

Definition 37 (\exists -graph). For any integers n_1, n_2 , define the \exists -graph $H = H_{n_1, n_2}$ and the function $f_H : V(H) \rightarrow \{2, 3\}$ as follows. The graph consists of a vertex labelled **IN** with degree 2, whose two neighbors serve as the starting points of two subgraphs to which we will refer as the top and bottom branches. Each branch is composed of a sequence of 4-cycles connected by edges, as described in the figure below. In each branch, the vertex of largest distance from **IN** in every 4-cycle but the first has a neighbor labelled **OUT** and another neighbor separating it from the next 4-cycle (except for the last 4-cycle). The numbers of **OUT** vertices in the top and bottom branches are n_1 and n_2 respectively. The function f_H is defined on the vertices of H as indicated in the figure.



We need the following two claims.

Claim 38. *Let \mathbb{F} be any field. Let A denote a neighbor of **IN** in the \exists -graph, and let B denote another vertex adjacent to A . Then, for every f_H -subspace assignment for H over \mathbb{F} , there exists a choice of nonzero vectors for **IN** and B which poses a single linear constraint on the choice for A .*

Proof. Let W_{IN}, W_A, W_B denote the subspaces assigned to the vertices **IN**, A , B respectively, and recall that $\dim(W_{\text{IN}}) = 2$, $\dim(W_A) = 3$, and $\dim(W_B) = 2$. If there exists some nonzero vector in $W_{\text{IN}} \cap W_B$, then choosing it for both **IN** and B completes the proof. Otherwise, it must hold that $\dim(W_{\text{IN}} + W_B) = 4 > \dim(W_A)$, hence there exists some nonzero vector $x \in (W_{\text{IN}} + W_B) \cap W_A^\perp$. Write $x = x_1 + x_2$ for $x_1 \in W_{\text{IN}}$ and $x_2 \in W_B$. If both of x_1 and x_2 are nonzero, choose them for **IN** and B . Since every vector $y \in W_A$ satisfies $\langle y, x \rangle = 0$, it follows that if $\langle y, x_1 \rangle = 0$ then $\langle y, x_2 \rangle = 0$. This implies that the only linear constraint that this choice poses on the vector of A is the orthogonality to x_1 . If, however, x_1 is zero, then we have that $x_2 \in W_A^\perp$, so one can choose an arbitrary nonzero vector from W_{IN} for **IN** and x_2 for B . Similarly, if x_2 is zero, we have that $x_1 \in W_A^\perp$, so one can choose x_1 for **IN** and an arbitrary nonzero vector from W_B for B , completing the proof. \square

Claim 39. *Let \mathbb{F} be either \mathbb{R} or any finite field, and let x be either e_6 or e_7 in \mathbb{F}^7 . Then, there exists a subspace assignment $W_1, \dots, W_4 \subseteq \mathbb{F}^7$ to the vertices u_1, \dots, u_4 of C_4 , with $\dim(W_1) = 3$ and $\dim(W_i) = 2$ for $i \in \{2, 3, 4\}$, for which any valid choice of vectors assigns to u_1 a vector proportional to x .*

Proof. The proof of Proposition 9 (see also Remark 22) describes for every field \mathbb{F} as above, a 2-subspace assignment for C_4 in \mathbb{F}^5 that admits no valid choice of vectors. Let $W_1, \dots, W_4 \subseteq \mathbb{F}^7$ be the subspaces obtained from the subspaces of this assignment by adding two additional entries with values zero to their vectors. Define $W'_1 = W_1 + \text{span}(x)$, and observe that any valid choice of vectors from the subspace assignment W'_1, W_2, W_3, W_4 assigns to u_1 a vector proportional to x , as otherwise, the restriction of such a choice to the first five entries would provide a valid choice for the given 2-subspace assignment for C_4 . \square

The following lemma summarizes some properties of the \exists -graph.

Lemma 40. *The \exists -graph H and the function f_H given in Definition 37 satisfy the following.*

1. *The graph H is bipartite, and every bipartition of H puts all OUT vertices in the same part.*
2. *For every f_H -subspace assignment for H over any field \mathbb{F} , any choice of a nonzero vector for IN can be extended to all vertices of each of the branches.*
3. *For every f_H -subspace assignment for H over any field \mathbb{F} and for each of the branches of H , there exists a choice of a nonzero vector for IN which is compatible with any choice of vectors for the OUT vertices of that branch.*
4. *Let \mathbb{F} be either \mathbb{R} or any finite field, and let $t \geq 8$ and $j \in [t]$ be some integers. Then, there exists an f_H -subspace assignment for H in \mathbb{F}^t such that for every valid choice of vectors for H there exists a branch all of whose OUT vertices are assigned vectors proportional to e_j .*

Proof. For Item 1, it can be easily seen that the graph defined in Definition 37 is bipartite. Since the distance between every two OUT vertices is even, it follows that every bipartition puts all of them in the same part.

For Item 2, consider some f_H -subspace assignment for H over a field \mathbb{F} , and notice that any choice of a vector for IN reduces the dimension of the subspaces available to its neighbors by at most 1. So given any choice for IN, one can choose, in each branch, an available nonzero vector for IN's neighbor, reducing the dimension of the subspaces available to its other neighbors to not less than 1, allowing us to choose for them nonzero vectors as well. Their common neighbor has a subspace of dimension 3, so the two chosen vectors of its neighbors reduce the dimension of the subspace available to it to not less than 1, again allowing us to choose a nonzero vector. Proceeding this way for vertices with increasing distances from IN allows us to choose vectors for all vertices of each of the branches of H .

For Item 3, consider some f_H -subspace assignment for H over a field \mathbb{F} and an arbitrary branch of H . Let A denote the neighbor of IN in this branch, let B and C denote the other neighbors of A , and let D denote the remaining vertex of their 4-cycle. By Claim 38, there exists a choice of nonzero vectors for IN and B which poses a single linear constraint

on the choice for A . We claim that this choice for IN and B is compatible with any choice of vectors for the OUT vertices of that branch. To see this, consider an arbitrary choice of nonzero vectors for these OUT vertices. The single neighbor of each OUT vertex is assigned a 3-subspace, so having made our choice for the OUT vertices, each of these must still have a 2-subspace from which its vector can be chosen. Starting from the neighbor of the OUT vertex of largest distance from IN , we choose an arbitrary nonzero vector from its available 2-subspace, allowing us to choose a nonzero vector for each of its two neighbors. Their other common neighbor has a 3-subspace, so it includes a nonzero vector orthogonal to the vectors chosen for its neighbors. We proceed this way along the branch until we arrive to the 4-cycle closest to IN . Given the vectors chosen for the previous 4-cycle and the choice for B , it is possible to choose from the 3-subspace of D some nonzero vector orthogonal to the vectors already chosen for its neighbors. Given this choice, we choose a nonzero vector orthogonal to it from the 2-subspace of C , and since the choice for IN and B poses a single linear constraint on A , it is possible to choose a nonzero vector for A from its 3-subspace. By Item 2 of the lemma, our choice can be extended to the other branch, and we are done.

For Item 4, let \mathbb{F} be either \mathbb{R} or any finite field, and let $t \geq 8$ and $j \in [t]$ be some integers. Assume without loss of generality that $j \geq 8$. We define an f_H -subspace assignment for H in \mathbb{F}^t as follows. The vertex IN is assigned the subspace $\text{span}(e_6, e_7)$. By Claim 39, for x being either e_6 or e_7 in \mathbb{F}^7 , there exists a subspace assignment $W_1, \dots, W_4 \subseteq \mathbb{F}^7$ to the vertices u_1, \dots, u_4 of C_4 , with $\dim(W_1) = 3$ and $\dim(W_i) = 2$ for $i \in \{2, 3, 4\}$, for which any valid choice of vectors assigns to u_1 a vector proportional to x . By extending these subspaces to \mathbb{F}^t with zeros in the last $t - 7$ entries, one can get such a subspace assignment in \mathbb{F}^t . We put this subspace assignment with $x = e_6$ on the 4-cycle closest to IN in each branch, where the 3-subspace is assigned to the vertex with largest distance from IN . To the subspace of the top neighbor of IN , we add the vector e_6 , and to the one of the bottom, we add the vector e_7 . For all remaining 4-cycles in the graph, we assign the subspaces of \mathbb{F}^t given by Claim 39 with $x = e_7$, again with the 3-subspace assigned to the vertex of largest distance from IN , and add the vector e_6 to the subspace of the vertex closest to IN . Finally, to all OUT vertices we assign the subspace $\text{span}(e_7, e_j)$, and to the remaining vertices separating the 4-cycles, we assign the subspace $\text{span}(e_6, e_7)$.

We claim that this f_H -subspace assignment for H satisfies that for every valid choice of vectors there exists a branch all of whose OUT vertices are assigned vectors proportional to e_j . To see this, consider such a valid choice of vectors, and recall that it assigns to IN a nonzero vector from $\text{span}(e_6, e_7)$. In such a vector, at least one of the sixth and seventh entries is nonzero. We show that in the former case all the vectors of the OUT vertices of the top branch are proportional to e_j . A similar argument shows that in the latter case, the same holds for the bottom branch. Our assumption on the vector of IN implies that its neighbor in the top branch is orthogonal to e_6 . This essentially restricts its 4-cycle to the subspace assignment given by Claim 39, thus ensuring that the vertex of largest distance from IN in this 4-cycle is assigned a vector proportional to e_6 . Applying this argument again to the next 4-cycle yields that its vertex of largest distance from IN is assigned a vector proportional to e_7 . This ensures that the vector of its OUT neighbor

is proportional to e_j and that the vector of its neighbor that separates its cycle from the next one is proportional to e_6 . By repeating this argument for all the following 4-cycles, the proof is completed. \square

6.2 Proof of Theorem 8

To prove Theorem 8, we first prove the following.

Theorem 41. *Let \mathbb{F} be either \mathbb{R} or any finite field. It is NP-hard to decide given a bipartite graph $G = (V, E)$ and a function $f : V \rightarrow \{2, 3\}$ whether G is f -subspace choosable over \mathbb{F} .*

Proof. Let \mathbb{F} be a field as in the statement of the theorem. Given a 3SAT formula ϕ with clauses C_1, \dots, C_m over the variables x_1, \dots, x_n , we efficiently construct a graph $G_\phi = (V, E)$ and a function $f : V \rightarrow \{2, 3\}$ such that ϕ is satisfiable if and only if G_ϕ is f -subspace choosable over \mathbb{F} . Note that it can be assumed that each clause of ϕ contains three literals involving three distinct variables.

First, for each variable x_j , construct an \exists -graph H_{n_1, n_2} (see Definition 37), where n_1 and n_2 are, respectively, the numbers of occurrences of the literals x_j and \bar{x}_j in ϕ . Label the OUT vertices of the top branch of H_{n_1, n_2} by x_j , and the OUT vertices of its bottom branch by \bar{x}_j . Define the function f on the vertices of this graph as in Definition 37. Next, for each clause C_i of ϕ , add a vertex representing C_i and define its f value to be 3. For each literal x_j occurring in a clause C_i , add an edge between the vertex representing C_i and a previously unchosen vertex labelled x_j , and likewise for the literals of the form \bar{x}_j . Observe that G_ϕ is bipartite, as Item 1 of Lemma 40 implies that there exists a bipartition placing all OUT vertices of all \exists -graphs in the same part, thus the clause vertices may all belong to the opposite part. Note that G_ϕ can be constructed in polynomial running time.

We prove now the correctness of the reduction. Suppose first that there exists a satisfying assignment for ϕ , and consider an arbitrary f -subspace assignment for G_ϕ over \mathbb{F} . Then, for each variable x_j with value **True**, choose for the IN vertex of its \exists -graph a vector, promised by Item 3 of Lemma 40, which is compatible with any choice of vectors for the OUT vertices labelled x_j . If, however, x_j has value **False**, choose instead a vector for IN which is compatible with any choice of vectors for the OUT vertices labelled \bar{x}_j . By Item 2 of the lemma, such a choice can be extended to all the vertices in the opposite branch. Now, since every clause has at most two literals which evaluate to **False** under the given satisfying assignment, we find that, so far, vectors have been chosen for at most two of the neighbors of each clause vertex. Since each clause vertex has a subspace of dimension 3, we can make a choice for it which is compatible with all of its neighbors whose vectors have already been chosen. Observe that this choice can be extended to all the OUT vertices for which no vectors have been chosen so far, because their subspaces have dimension 2 whereas a vector has been chosen only for one of their neighbors. Finally, by our choice of the vectors of the IN vertices, using Item 3 of Lemma 40, one can properly choose vectors for the rest of the graph. This implies that G_ϕ is f -subspace choosable over \mathbb{F} .

For the other direction, suppose that G_ϕ is f -subspace choosable over \mathbb{F} . Put $t = n + 7$, and apply Item 4 of Lemma 40 to obtain an f_H -subspace assignment in \mathbb{F}^t for each \exists -gadget, such that, for each $j \in [n]$, every valid choice of vectors assigns vectors proportional to e_j either to all vertices labelled x_j or to all vertices labelled $\overline{x_j}$. Finally, to the vertex of a clause C_i that involves the three variables $x_{j_1}, x_{j_2}, x_{j_3}$, assign the subspace spanned by $e_{j_1}, e_{j_2}, e_{j_3}$. Since G_ϕ is f -subspace choosable over \mathbb{F} , there exists a valid choice for G_ϕ from these subspaces. By our definition of the subspace assignment, for every $j \in [n]$, this choice assigns vectors proportional to e_j to all vertices labelled x_j or to all vertices labelled $\overline{x_j}$. In the former case assign x_j to **False**, and in the latter to **True**. We claim that this assignment satisfies ϕ . To see this, observe that each vertex representing a clause C_i must have for some $j \in [n]$ a neighbor labelled x_j or $\overline{x_j}$ whose chosen vector is not proportional to e_j . This neighbor corresponds to a literal whose value is **True** according to our assignment, as desired. \square

We also need the following simple lemma, whose proof employs ideas from [11].

Lemma 42. *For every field \mathbb{F} and for every integer $k \geq 3$, the following holds. There exists a polynomial-time reduction from the problem of deciding for a given input of a bipartite graph $G = (V, E)$ and a function $f : V \rightarrow \{2, 3\}$ whether G is f -subspace choosable over \mathbb{F} , to the problem of deciding whether a given bipartite graph is k -subspace choosable over \mathbb{F} .*

Proof. We start by proving the statement of the lemma for $k = 3$. Given a bipartite graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$ and given a function $f : V \rightarrow \{2, 3\}$, consider the graph G' that consists of nine copies of G , labelled $G_{i,j}$ for $i, j \in [3]$, and two additional vertices v_1, v_2 such that, for each $\ell \in \{1, 2\}$, the vertex v_ℓ is adjacent to all vertices u with $f(u) = 2$ in the copies of V_ℓ . It is easy to see that G' is bipartite and that it can be constructed in polynomial running time.

For correctness, suppose first that G is f -subspace choosable over \mathbb{F} , and consider an arbitrary assignment of 3-subspaces over \mathbb{F} to the vertices of G' . Any choice of nonzero vectors for v_1 and v_2 will reduce the dimensions of the subspaces of the vertices of the graphs $G_{i,j}$ to not less than their original values under f . Since each $G_{i,j}$ is f -subspace choosable over \mathbb{F} , it follows that there exists a valid choice of vectors for the vertices of G' , as required. For the other direction, suppose that for some integer t , there exists an f -subspace assignment for G such that no choice of nonzero vectors from the subspaces is valid. To the vertices of each subgraph $G_{i,j}$ in G' we assign the subspaces of \mathbb{F}^{t+3} obtained by adding three zeros to the head of all vectors of those subspaces. To the subspaces of dimension 2 in $G_{i,j}$, we add the vector e_i for the vertices adjacent to v_1 and the vector e_j for the vertices adjacent to v_2 . To each of the vertices v_1 and v_2 we assign the subspace of \mathbb{F}^{t+3} spanned by e_1, e_2, e_3 . Now, for any choice of nonzero vectors for v_1, v_2 , the subspaces of at least one of the graphs $G_{i,j}$ will be restricted to their initial f -subspace assignment, and will thus admit no valid choice of vectors for its vertices.

It remains to consider the case of $k > 3$. It suffices to show a polynomial-time reduction from the problem of deciding whether a given bipartite graph is $(k-1)$ -subspace choosable

over \mathbb{F} to that of deciding whether a given bipartite graph is k -subspace choosable over \mathbb{F} . Here, given a bipartite graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$, consider the bipartite graph that consists of k^2 copies of G and two additional vertices v_1, v_2 such that, for each $\ell \in \{1, 2\}$, the vertex v_ℓ is adjacent to all the vertices in the copies of V_ℓ . The correctness proof is similar to the one given above, so we omit the details. \square

By combining Theorem 41 with Lemma 42, the proof of Theorem 8 is completed.

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