Covering Symmetric Sets of the Boolean Cube by Affine Hyperplanes

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Abstract

Alon and Füredi (European J. Combin., 1993) proved that any family of hyperplanes that covers every point of the Boolean cube $\{0,1\}^n$ except one must contain at least n hyperplanes. We obtain two extensions of this result, in characteristic zero, for hyperplane covers of symmetric sets of the Boolean cube (subsets that are closed under permutations of coordinates), as well as for *polynomial covers* of *weight-determined* sets of *strictly unimodal uniform* (SU^2) grids.

As a central tool for solving our problems, we give a combinatorial characterization of (finite-degree) Zariski (Z-) closures of symmetric sets of the Boolean cube. In fact, we obtain a characterization that concerns, more generally, weight-determined sets of SU^2 grids. However, in this generality, our characterization is not of the Z-closures but of a new variant of Z-closures defined exclusively for weight-determined sets, which coincides with the Z-closures in the Boolean cube setting, for symmetric sets. This characterization admits a linear time algorithm, and may also be of independent interest. Indeed, as further applications, we (i) give an alternate proof of a lemma by Alon et al. (IEEE Trans. Inform. Theory, 1988), and (ii) characterize the *certifying degrees* of weight-determined sets.

In the Boolean cube setting, our above characterization can also be derived using a result of Bernasconi and Egidi (Inf. Comput., 1999) that determines the affine Hilbert functions of symmetric sets. However, our proof is independent of this result, works for all SU^2 grids, and could be regarded as being more combinatorial.

We also introduce another new variant of Z-closures to better understand the difference between the hyperplane and polynomial covering problems over uniform grids. Finally, we conclude by introducing a third variant of our covering problems and conjecturing its solution in the Boolean cube setting.

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Contents

Notations. $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{Z}$ denotes the set of all integers, $\mathbb{N}$ denotes the set of all nonnegative integers, and $\mathbb{Z}^+$ denotes the set of all positive integers.

1 Introduction and overview

We will work over the field $\mathbb{R}$. For any two integers $a, b \in \mathbb{Z}$, $a \leqslant b$, by abuse of notation, we will denote the *interval* of all integers between a and b by $[a, b]$. Further, the interval of integers $[1, n]$ will also be denoted by $[n]$. By a uniform grid, we mean a finite grid of the form $[0, k_1 - 1] \times \cdots \times [0, k_n - 1]$, for some $k_1, \ldots, k_n \in \mathbb{Z}^+$. (This means for each $i \in [n]$, the values taken by the points of the grid in the i-th coordinate are equispaced.) Consider a uniform grid $G = [0, k_1 - 1] \times \cdots \times [0, k_n - 1]$. For any $x = (x_1, \ldots, x_n) \in G$, define the weight of x as $\mathrm{wt}(x) = \sum_{i=1}^{n} x_i$. We define a subset $S \subseteq G$ to be weight-determined if

$$x \in S,\ y \in G,\ \mathrm{wt}(y) = \mathrm{wt}(x) \quad \Longrightarrow \quad y \in S.$$

Let $N = \sum_{i=1}^{n}(k_i - 1)$. It follows that there is a one-to-one correspondence between weight-determined sets of G and subsets of $[0, N]$ – a subset $E \subseteq [0, N]$ corresponds to a weight-determined set $\underline{E} \subseteq G$, defined as the set of all elements $x \in G$ satisfying $\mathrm{wt}(x) \in E$. When $E = \{j\}$, a singleton set, we will denote $\underline{E}$ by $\underline{j}$. Further, we will freely use this one-to-one correspondence and identify the weight-determined set $\underline{E}$ with E without mention, whenever convenient. This identification will be clear from the context. In addition, for $E \subseteq [0, N]$, we will denote $|\underline{E}|$ by $\begin{bmatrix} G \\ E \end{bmatrix}$. It is immediate that $\begin{bmatrix} G \\ j \end{bmatrix} = \begin{bmatrix} G \\ N-j \end{bmatrix}$, for all $j \in [0, N]$.

We say the uniform grid G is strictly unimodal if

$$\begin{bmatrix} G \\ 0 \end{bmatrix} < \cdots < \begin{bmatrix} G \\ \lfloor N/2 \rfloor \end{bmatrix} = \begin{bmatrix} G \\ \lceil N/2 \rceil \end{bmatrix} > \cdots > \begin{bmatrix} G \\ N \end{bmatrix}.$$

We will abbreviate the term 'strictly unimodal uniform grid' by 'SU^2 grid'. The SU^2 condition on uniform grids is not very restrictive; there are enough interesting uniform grids which are SU^2. For instance, the uniform grid $[0, k-1]^n$ is SU^2. There is a simple characterization of the SU^2 grids in terms of their dimensions, given by Dhand [Dha14] (stated in our work as Theorem 19).

We will stick with the above notations whenever we consider uniform grids. Further, we will assume throughout that $k_i \geqslant 2$, for all $i \in [n]$.

The Boolean cube setting. Consider the Boolean cube $\{0, 1\}^n$. In this case, for any $x \in \{0, 1\}^n$, the weight $\mathrm{wt}(x)$ is equal to $|x|$, the Hamming weight of x, and $\begin{bmatrix} \{0,1\}^n \\ j \end{bmatrix} = \binom{n}{j}$, $j \in [0, n]$. It is easy to check that $\{0, 1\}^n$ is strictly unimodal. We define $S \subseteq \{0, 1\}^n$ to be symmetric if S is closed under permutations of coordinates. It follows quite trivially that a subset is weight-determined if and only if it is symmetric. This is not true though for any other uniform grid. The Boolean cube $\{0, 1\}^n \subseteq [0, k_1 - 1] \times \cdots \times [0, k_n - 1]$ is clearly symmetric but not weight-determined, if any of the k_i-s is at least 3. Also without loss of generality, if $k_i < k_j$ for some $i, j \in [n]$, $i \neq j$, then trivially the grid $[0, k_1 - 1] \times \cdots \times [0, k_n - 1]$ is weight-determined but not symmetric. In the case all the k_i-s are equal, every weight-determined set is indeed symmetric.

1.1 Our hyperplane and polynomial covering problems

The term *hyperplane covering problem* is commonly used in the literature (see Subsection 1.2) to refer to any problem of finding the minimum number of hyperplanes covering a finite set in a finite-dimensional vector space (over a field) while satisfying some conditions. Borrowing this terminology, we use the term *polynomial covering problem* to refer to any problem of finding the minimum degree of a polynomial covering[1] a finite set in a finite-dimensional vector space (over a field) while satisfying some conditions.

[1] We say a polynomial $P(X_1, \ldots, X_n)$ covers a point $(a_1, \ldots, a_n)$ if $P(a_1, \ldots, a_n) = 0$.

We will denote the polynomial ring $\mathbb{R}[X_1, \ldots, X_n]$ by $\mathbb{R}[\mathbb{X}]$. Further, for any $\alpha \in \mathbb{N}^n$, we denote the monomial $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ by $\mathbb{X}^\alpha$.

Alon and Füredi [AF93] considered the following hyperplane covering problem: what is the minimum number of hyperplanes required to cover every point of the Boolean cube $\{0,1\}^n$, except the origin $0^n \coloneqq (0, \ldots, 0)$? They proved a lower bound of n, which was clearly tight – the family of n hyperplanes defined by the polynomials $\sum_{j=1}^n X_j - i$, for $i \in [n]$, satisfies the required conditions. In fact, they proved the following stronger result.

Theorem 1 ([AF93, Theorem 1]). [2] *If $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ is a polynomial and $a \in [0, k_1 - 1] \times \cdots \times [0, k_n - 1]$ such that $P(x) = 0$, for all $x \in [0, k_1 - 1] \times \cdots \times [0, k_n - 1]$, $x \neq a$ and $P(a) \neq 0$, then $\deg P \geqslant \sum_{i \in [n]} (k_i - 1)$.*

In this work, we will consider extensions of this result to weight-determined sets of a uniform grid. Let G be a uniform grid. For a weight-determined set $\underline{E}$, where $E \subsetneq [0, N]$, we say a family of hyperplanes $\mathcal{H}$ in $\mathbb{R}^n$ is

- a nontrivial hyperplane cover of $\underline{E}$ if
$$\underline{E} \subseteq G \cap \left(\bigcup_{H \in \mathcal{H}} H \right) \neq G.$$
- a proper hyperplane cover of $\underline{E}$ if
$$\underline{E} \subseteq \bigcup_{H \in \mathcal{H}} H \quad \text{and} \quad \underline{j} \nsubseteq \bigcup_{H \in \mathcal{H}} H, \text{ for every } j \in [0, N] \setminus E.$$

Let $\mathsf{HC}_G(E)$ and $\mathsf{PHC}_G(E)$ denote the minimum sizes of a nontrivial hyperplane cover and a proper hyperplane cover respectively, for a weight-determined set $\underline{E}$, $E \subsetneq [0, N]$. In the case $G = \{0,1\}^n$, we will instead use the notations $\mathsf{HC}_n(E)$ and $\mathsf{PHC}_n(E)$ respectively.

For any $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$, we denote by $\mathcal{Z}_G(P)$, the set of all $a \in G$ such that $P(a) = 0$. For a weight-determined set $\underline{E}$, where $E \subsetneq [0, N]$, we say a polynomial $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ is

- a nontrivial polynomial cover of $\underline{E}$ if
$$\underline{E} \subseteq \mathcal{Z}_G(P) \neq G.$$
- a proper polynomial cover of $\underline{E}$ if
$$\underline{E} \subseteq \mathcal{Z}_G(P) \quad \text{and} \quad \underline{j} \nsubseteq \mathcal{Z}_G(P), \text{ for every } j \in [0, N] \setminus E.$$

Let $\mathsf{PC}_G(E)$ and $\mathsf{PPC}_G(E)$ denote the minimum degree of a nontrivial polynomial cover and a proper polynomial cover respectively, for a weight-determined set $\underline{E}$, $E \subsetneq [0, N]$. In the case $G = \{0,1\}^n$, we will instead use the notations $\mathsf{PC}_n(E)$ and $\mathsf{PPC}_n(E)$.

We are interested in the following problems.

[2] In [AF93], this result was proven true over any field.

Problem 2. Let G be a uniform grid. For all $E \subsetneq [0, N]$,

(a) find $\mathsf{HC}_G(E)$ and $\mathsf{PHC}_G(E)$.

(b) find $\mathsf{PC}_G(E)$ and $\mathsf{PPC}_G(E)$.

All the above variants of hyperplane and polynomial covers coincide for the symmetric set $\underline{[N]}$, and hence in our notation, Alon and Füredi [AF93] proved the following.

Theorem 3 ([AF93]). $\mathsf{HC}_G([N]) = \mathsf{PHC}_G([N]) = \mathsf{PC}_G([N]) = \mathsf{PPC}_G([N]) = N$.

It is immediate from the definitions that $\mathsf{PHC}_G(E) \geqslant \mathsf{HC}_G(E)$, $\mathsf{PPC}_G(E) \geqslant \mathsf{PC}_G(E)$, $\mathsf{HC}_G(E) \geqslant \mathsf{PC}_G(E)$, and $\mathsf{PPC}_G(E) \geqslant \mathsf{PHC}_G(E)$, for all $E \subsetneq [0, N]$. In this work, we will give combinatorial characterizations of $\mathsf{PC}_G(E)$ and $\mathsf{PPC}_G(E)$, for all $E \subsetneq [0, N]$, for an SU^2 grid G. The characterization of $\mathsf{HC}_G(E)$ and $\mathsf{PHC}_G(E)$, for all $E \subsetneq [0, N]$, for an SU^2 grid G is left open. However, in the Boolean cube setting, we will obtain $\mathsf{HC}_n(E) = \mathsf{PC}_n(E)$ and $\mathsf{PHC}_n(E) = \mathsf{PPC}_n(E)$, for all $E \subsetneq [0, n]$. In short, we will solve Problem 2 (b) for SU^2 grids, and further solve Problem 2 (a) for the Boolean cube. Problem 2 (a) for SU^2 grids is open.

For every $i \in [0, N]$, let $T_{N,i} = [0, i-1] \cup [N-i+1, N]$. (This means $T_{N,0} = \emptyset$.) For any $d, i \in [0, N]$, $i \leqslant d$ and $E \subseteq [0, N]$, we define E to be (d, i)-admitting if $E \cup T_{N,i} \neq [0, N]$ and $|E \setminus T_{N,i}| \leqslant d - i$. Further, we define E to be d-admitting if E is (d, i)-admitting, for some $i \in [0, d]$. Our combinatorial characterizations that answer Problem 2 (b) for SU^2 grids are as follows.

Theorem 4. *Let G be an SU^2 grid. For any $E \subsetneq [0, N]$,*

(a) $\mathsf{PC}_G(E) = \min\{d \in [0, N] : E \text{ is } d\text{-admitting}\}$.

(b) $\mathsf{PPC}_G(E) = |E| - \max\{i \in [0, \lfloor N/2 \rfloor] : T_{N,i} \subseteq E\}$.

Further, we answer Problem 2 (a), in the Boolean cube setting, as follows.

Theorem 5. *Consider the Boolean cube $\{0, 1\}^n$. For any $E \subsetneq [0, n]$,*

(a) $\mathsf{HC}_n(E) = \mathsf{PC}_n(E) = \min\{d \in [0, n] : E \text{ is } d\text{-admitting}\}$.

(b) $\mathsf{PHC}_n(E) = \mathsf{PPC}_n(E) = |E| - \max\{i \in [0, \lfloor n/2 \rfloor] : T_{n,i} \subseteq E\}$.

1.2 Related work

Alon and Füredi [AF93] mention that their hyperplane covering problem was extracted by Bárány from Komjáth [Kom94]. Some of the extensions and variants studied subsequently (over $\mathbb{R}$) are as follows.

- Linial and Radhakrishnan [LR05] gave an upper bound of $\lceil n/2 \rceil$ and a lower bound of $\Omega(\sqrt{n})$ on the minimum size of *essential hyperplane covers* of the Boolean cube – a family of hyperplanes $\mathcal{H}$ is an essential hyperplane cover if $\mathcal{H}$ is a minimal family covering $\{0,1\}^n$, and each coordinate is influential for a linear polynomial representing some hyperplane in $\mathcal{H}$. Saxton [Sax13] later gave a tight bound of $n+1$ for this problem, in the case where the linear polynomials representing the hyperplanes are restricted to be of the form $\sum_{i=1}^{n} a_i X_i - b$, where $a_i \geqslant 0$ for all $i \in [n]$, and $b \geqslant 0$. Recently, Yehuda and Yehudayoff [YY21] improved the lower bound in the unrestricted case to $\Omega(n^{0.52})$.

- Kós, Mészáros and Rónyai [KR12] introduced the following multiplicity extension: given a finite grid $S = S_1 \times \cdots \times S_n$ with each (S_i, m_i) being a multiset such that $0 \in S_i$, $m_i(0) = 1$, find the minimum number of hyperplanes such that each point $s \in S \setminus \{0\}$ is covered by at least $\sum_{i \in [n]} m_i(s_i) - n + 1$ hyperplanes and the point 0 is not covered by any hyperplane. They gave a tight lower bound of $\sum_{i \in [n]} m_i(S_i) - n$. This bound is in fact true over any field.

- Aaronson, Groenland, Grzesik, Johnston and Kielak [AGG$^+$20] considered *exact hyperplane covers* of subsets of the Boolean cube – a family of hyperplanes $\mathcal{H}$ is an exact hyperplane cover of a subset $S \subseteq \{0,1\}^n$ if $\left(\bigcap_{H \in \mathcal{H}} H\right) \cap \{0,1\}^n = S$. They obtained tight bounds on the minimum size of exact hyperplane covers for subsets S with $|\{0,1\}^n \setminus S| \leqslant 4$, and asymptotic bounds for general subsets.

- Clifton and Huang [CH20] considered another multiplicity version of the hyperplane covering problem: find the least number of hyperplanes required to cover all points of the Boolean cube except the origin k times and not cover the origin at all. They proved the tight bound of $n+1$ and $n+3$, for $k=2$ and $k=3$ respectively, and gave a lower bound of $n+k+1$ for $k \geqslant 4$. Sauermann and Wigderson [SW20] considered the polynomial version of this problem: find the least degree of a polynomial that vanishes at all points of the Boolean cube, except the origin, k times and vanishes at the origin j times, for some $j < k$. They gave the tight bounds $n + 2k - 3$ for $j \leqslant k-2$, and $n + 2k - 2$ for $j = k-1$.

Several more variants and extensions, in particular over positive characteristic, have appeared in the literature both before and after Alon and Füredi [AF93]–Jamison [Jam77], Brouwer [BS78], Ball [Bal00], Zanella [Zan02], Ball and Serra [BS09], Blokhuis [BBS10], and Bishnoi, Boyadzhiyska, Das, and Mészáros [BBDM21], to quote a few. For a detailed history of the hyperplane covering problems as well as the polynomial method, see, for instance, the nice introduction in [BBDM21].

1.3 Finite-degree Z-closures and Z*-closures, and polynomial covering problems

The *finite-degree Zariski closure* was defined by Nie and Wang [NW15] towards obtaining a better understanding of the applications of the polynomial method to combinatorial geometry. This is a closure operator[3] and has been studied implicitly even earlier (see, for instance, Wei [Wei91], Heijnen and Pellikaan [HP98], Keevash and Sudakov [KS05], and Ben-Eliezer, Hod and Lovett [BHL12]). We will abbreviate the term 'Zariski closure' by 'Z-closure'.

1.3.1 Finite-degree Z-closures and Z*-closures

Let G be a uniform grid. For any $d \in [0, N]$ and any $S \subseteq G$, the degree-d Z-closure of S, denoted by $\text{Z-cl}_{G,d}(S)$, is defined to be the common zero set, in G, of all polynomials that vanish on S, and have degree at most d.[4] In the case $G = \{0,1\}^n$, we will instead use the notation $\text{Z-cl}_{n,d}(S)$.

The finite-degree Z-closures are relevant to us in the context of our polynomial covering problems. We are interested in polynomial covering problems that impose conditions on weight-determined sets, and thus, we are interested in finite-degree Z-closures of weight-determined sets. Intriguingly, the finite-degree Z-closure of a weight-determined set need not be weight-determined.

For instance, consider the SU^2 grid $G = [0,2]^3$. In this case, we have $N = 6$ and $T_{6,3} = [0,2] \cup [4,6]$. We have $\underline{3} = \{(2,1,0),(1,2,0),(0,2,1),(0,1,2),(2,0,1),(1,0,2),(1,1,1)\}$. Consider

$$P(X_1, X_2, X_3) = X_1X_2(X_1 - X_2) + X_2X_3(X_2 - X_3) - X_1X_3(X_1 - X_3).$$

Clearly $\deg P = 3$. It is easy to check that $P|_{\underline{T_{6,3}}} = 0$. Further, we get

$$P(2,1,0) = P(0,2,1) = P(1,0,2) = 2 \quad \text{and} \quad P(1,2,0) = P(0,1,2) = P(2,0,1) = -2.$$

So $a \notin \text{Z-cl}_{G,3}(\underline{T_{6,3}})$, for all $a \in \underline{3}$, $a \neq (1,1,1)$. Further, consider any $Q(X_1, X_2, X_3) \in \mathbb{R}[X_1, X_2, X_3]$ such that $Q|_{\underline{T_{6,3}}} = 0$ and $\deg Q \leqslant 3$. Let

$$R(X_1, X_2, X_3) = Q(X_1, X_2, X_3)(X_1^2 + X_2^2 + X_3^2 - 5).$$

So $\deg R = 5$. Then we have $R|_{\underline{T_{6,3}}} = 0$, and $R(a) = 0$, for all $a \in \underline{3}$, $a \neq (1,1,1)$. Thus $R(x) = 0$, for all $x \in G$, $x \neq (1,1,1)$. If $R(1,1,1) \neq 0$, then by Theorem 1, we have $\deg R \geqslant 6$, which is not true. So $R(1,1,1) = 0$, which implies $Q(1,1,1) = 0$. Thus $(1,1,1) \in \text{Z-cl}_{G,3}(\underline{T_{6,3}})$. Hence we have $\underline{3} \cap \text{Z-cl}_{G,3}(\underline{T_{6,3}}) \neq \emptyset$ but $\underline{3} \not\subseteq \text{Z-cl}_{G,3}(\underline{T_{6,3}})$, which implies $\text{Z-cl}_{G,3}(\underline{T_{6,3}})$ is not weight-determined.

We will circumvent this issue by introducing a new closure operator, defined exclusively for weight-determined sets. Let G be a uniform grid. For any $d \in [0, N]$ and $E \subseteq [0, N]$,

[3] A closure operator on a set system $\mathcal{F}$ (over a ground set) is any map $\text{cl} : \mathcal{F} \to \mathcal{F}$ satisfying: (i) $A \subseteq \text{cl}(A), \forall A \in \mathcal{F}$, (ii) $\text{cl}(A) \subseteq \text{cl}(B), \forall A, B \in \mathcal{F}, A \subseteq B$, and (iii) $\text{cl}(\text{cl}(A)) = \text{cl}(A), \forall A \in \mathcal{F}$. This is a well-studied set operator. See, for instance, Birkhoff [Bir73, Chapter V, Section 1] for an introduction.

[4] The set of all polynomials that vanish on S and have degree at most d is a vector space over $\mathbb{R}$. Note that here we include the zero polynomial in the set. To facilitate this, we adopt the convention that the degree of the zero polynomial is $-\infty$.

we define the degree-d Z*-closure of $\underline{E}$, denoted by $\text{Z*-cl}_{G,d}(\underline{E})$, to be the maximal weight-determined set contained in $\text{Z-cl}_{G,d}(\underline{E})$. In other words, $\text{Z-cl}_{G,d}(\underline{E})$ is defined by the implications:

$$\begin{aligned} &\underline{j} \subseteq \text{Z-cl}_{G,d}(\underline{E}) \implies \underline{j} \subseteq \text{Z*-cl}_{G,d}(\underline{E}), \\ \text{and} \quad &\underline{j} \not\subseteq \text{Z-cl}_{G,d}(\underline{E}) \implies \underline{j} \cap \text{Z*-cl}_{G,d}(\underline{E}) = \emptyset. \end{aligned}$$

It follows easily that $\text{Z*-cl}_{G,d}$ is a closure operator[5]; we will prove this in Section 2.

Notation. By definition, it is clear the finite-degree Z*-closure is a weight-determined set. So, whenever convenient, we will use our identification of weight-determined sets with subsets of $[0, N]$ while describing these closures. For $E \subseteq [0, N]$, the notation $\text{Z*-cl}_{G,d}(\underline{E})$ will denote the Z*-closure as a subset of G, and the notation $\text{Z*-cl}_{G,d}(E)$ will denote the Z*-closure as a subset of $[0, N]$. Similar 'double notations' would also apply to Z-closures of symmetric sets of the Boolean cube.

The relevance of the finite-degree Z*-closures to our polynomial covering problems is captured by the following simple lemma, which is quite immediate from the definitions.

Lemma 6. *Let G be a uniform grid. For any $E \subsetneq [0, N]$,*

(a) $\mathsf{PC}_G(E) = \min\{d \in [0, N] : \text{Z*-cl}_{G,d}(E) \neq [0, N]\}$.

(b) $\mathsf{PPC}_G(E) = \min\{d \in [0, N] : \text{Z*-cl}_{G,d}(E) = E\}$.

1.3.2 Combinatorial characterization of finite-degree Z*-closures

We will proceed to give a combinatorial characterization of finite-degree Z*-closures. We need a couple of set operators to proceed.

Two set operators. Fix $M \in \mathbb{Z}^+$ and for every $d \in [0, M]$, define a set operator $L_{M,d} : 2^{[0,M]} \to 2^{[0,M]}$ as follows. Let $E = \{t_1 < \cdots < t_s\} \subseteq [0, M]$. Define

$$L_{M,d}(E) = \begin{cases} E & \text{if } |E| \leqslant d, \\ [0, t_{s-d}] \cup E \cup [t_{d+1}, M] & \text{if } |E| \geqslant d+1. \end{cases}$$

We are interested in iterated applications of the operator $L_{M,d}$, and in an obvious way, for any $k \in \mathbb{N}$, we define the operator $L_{M,d}^{k+1} = L_{M,d} \circ L_{M,d}^k$, where $L_{M,d}^0$ denotes the identity operator. Clearly, for every $d \in [0, M]$ and $E \subseteq [0, M]$, we have the chain

$$E = L_{M,d}^0(E) \subseteq L_{M,d}(E) \subseteq L_{M,d}^2(E) \subseteq \cdots$$

So for every $d \in [0, M]$, define the set operator $\overline{L}_{M,d} : 2^{[0,M]} \to 2^{[0,M]}$ as

$$\overline{L}_{M,d}(E) = \bigcup_{k \geqslant 0} L_{M,d}^k(E), \quad \text{for all } E \subseteq [0, M].$$

We are now ready to state our main theorem: our combinatorial characterization of finite-degree Z*-closures of weight-determined sets of an SU^2 grid.

[5] $\text{Z*-cl}_{G,d}$ is a closure operator on the family of all weight-determined sets of G.

Theorem 7. *Let G be an SU^2 grid. For every $d \in [0, N]$ and $E \subseteq [0, N]$,*

$$\text{Z*-cl}_{G,d}(E) = \overline{L}_{N,d}(E).$$

Also, we can characterize when the finite-degree Z*-closure of a weight-determined set E is equal to E or $[0, N]$.

Proposition 8. *Let G be an SU^2 grid. For every $d \in [0, N]$ and $E \subseteq [0, N]$,*

(a) $\overline{L}_{N,d}(E) \neq [0, N]$ if and only if E is d-admitting.

(b) $\overline{L}_{N,d}(E) = E$ if and only if $T_{N,|E|-d} \subseteq E$.

This would complete the proof of Theorem 4, thus completing our solution to Problem 2 (b) for SU^2 grids.

1.4 Finite-degree h-closures and h*-closures, and hyperplane covering problems

To better understand the difference between the hyperplane and polynomial covering problems, we introduce another new closure operator, which we call the *finite-degree* h-*closure*, defined using polynomials representing hyperplane covers. Let $\mathscr{H}_n$ be the set of all polynomials in $\mathbb{R}[\mathbb{X}] := \mathbb{R}[X_1, \ldots, X_n]$ which are products of polynomials of degree at most 1. Let G be a uniform grid. For any $d \in [0, N]$ and any $S \subseteq G$, we define the degree-d h-closure of S, denoted by $\text{h-cl}_{G,d}(S)$, to be the common zero set, in G, of all polynomials in $\mathscr{H}_n$ that vanish on S and have degree at most d. By definition, it is clear that $\text{Z-cl}_{G,d}(S) \subseteq \text{h-cl}_{G,d}(S)$. In the case $G = \{0, 1\}^n$, we will instead use the notation $\text{h-cl}_{n,d}(S)$.

Note that we do not know if the finite-degree hyperplane closure of every weight-determined set of G is weight-determined. So, akin to the definition of finite-degree Z*-closures, for any $d \in [0, N]$ and $E \subseteq [0, N]$, we define the degree-d h*-closure of $\underline{E}$, denoted by $\text{h*-cl}_{G,d}(\underline{E})$, to be the maximal weight-determined set contained in $\text{h-cl}_{G,d}(\underline{E})$. These closures are relevant to us in the context of our hyperplane covering problems due to the following observation, which is immediate from the definitions.

Observation 9. *Let G be a uniform grid. For any $E \subsetneq [0, N]$,*

(a) $\mathsf{HC}_G(E) = \min\{d \in [0, N] : \text{h*-cl}_{G,d}(\underline{E}) \neq G\}$.

(b) $\mathsf{PHC}_G(E) \geqslant \min\{d \in [0, N] : \text{h*-cl}_{G,d}(\underline{E}) = \underline{E}\}$.

1.4.1 The Boolean cube setting: characterizing h-closures

We will characterize the finite-degree h-closures of all symmetric sets of the Boolean cube for all degrees; in fact, we make the intriguing observation that these coincide with the finite-degree Z-closures.

Notation. It is easy to see that the finite-degree h-closure of a symmetric set of a Boolean cube is symmetric. So $\text{h-cl}_{n,d}(\underline{E}) = \text{h*-cl}_{n,d}(\underline{E})$, for all $E \subseteq [0,n]$, $d \in [0,n]$. So once again, we will use the identification between symmetric sets of $\{0,1\}^n$ and subsets of $[0,n]$. For $E \subseteq [0,n]$, the notation $\text{h-cl}_{n,d}(\underline{E})$ will denote the h-closure as a subset of $\{0,1\}^n$, and the notation $\text{h-cl}_{n,d}(E)$ will denote the h-closure as a subset of $[0,n]$.

We already have Theorem 7 that characterizes the finite-degree Z*-closures. Further, since the finite-degree Z-closure of a symmetric set is symmetric, we have $\text{Z-cl}_{n,d}(\underline{E}) = \text{Z*-cl}_{n,d}(\underline{E})$, for all $E \subseteq [0,n]$, $d \in [0,n]$. With an additional observation, we will conclude the following.

Theorem 10. *For every $d \in [0,n]$ and $E \subseteq [0,n]$,*

$$\text{h-cl}_{n,d}(E) = \text{h*-cl}_{n,d}(E) = \text{Z-cl}_{n,d}(E) = \text{Z*-cl}_{n,d}(E) = \overline{L}_{n,d}(E).$$

Further, using Observation 9, Theorem 10 and a tight construction of hyperplane cover, we will prove Theorem 5, our solution to Problem 2 (a) in the Boolean cube setting.

It must be noted that for larger uniform grids, for a weight-determined set, the finite-degree h-closure and Z-closure need not be equal. For instance, let $G = [0,2] \times [0,2]$ and consider $T_{4,2} = [0,1] \cup [3,4]$. Owing to the fact that affine hyperplanes in $\mathbb{R}^2$ are lines, we get $\text{h-cl}_{G,2}(\underline{T_{4,2}}) = G$. Further, let $P(X_1, X_2) = X_1^2 - X_1X_2 + X_2^2 - X_1 - X_2$. Then obviously $\deg P = 2$, and we can check that $P|_{\underline{T_{4,2}}} = 0$. Note that $G \setminus \underline{T_{4,2}} = \underline{2} = \{(2,0),(1,1),(0,2)\}$. We have $P(2,0) = 2$, $P(1,1) = -1$, $P(0,2) = 2$. Thus $\underline{2} \cap \text{Z-cl}_{G,2}(\underline{T_{4,2}}) = \emptyset$, that is, $\text{Z-cl}_{G,2}(\underline{T_{4,2}}) = \underline{T_{4,2}}$.

We do not yet know how to approach the finite-degree hyperplane closures for larger uniform grids. We, therefore, have the following open questions.

Open Problem 11. Let $G \neq \{0,1\}^n$ be a uniform (or SU^2) grid.

(a) Characterize $\text{h-cl}_{G,d}(\underline{E})$ and $\text{h*-cl}_{n,d}(\underline{E})$, for all $E \subseteq [0,N]$.

(b) Solve Problem 2 (a), that is, determine $\mathsf{HC}_G(E)$ and $\mathsf{PHC}_G(E)$, for all $E \subsetneq [0,N]$.

1.4.2 The Boolean cube setting: connection with the affine Hilbert function

Now for any subset $A \subseteq \mathbb{R}^n$, let $V(A)$ denote the vector space of all functions $A \to \mathbb{R}$. For $d \geqslant 0$, let $V_d(A)$ denote the subspace of all functions that admit a polynomial representation with degree at most d. The affine Hilbert function of A is defined by $\mathrm{H}_d(A) = \dim V_d(A)$, $d \geqslant 0$. This is a well-studied object in the literature. (See, for instance, Cox, Little and O'Shea [CLO15, Chapter 9, Section 3] for an introduction.)

Let us fix some notations. Consider the Boolean cube $\{0,1\}^n$. Let $d \in [0,n]$. For any $E \subseteq [0,n]$, let $r_d(E) = |E \setminus [0,d]|$, $\ell_d(E) = |[0,d] \setminus E|$. Further, denote the enumerations $E \setminus [0,d] = \{j_1^+ < \cdots < j_{r_d(E)}^+\}$ and $[0,d] \setminus E = \{j_{\ell_d(E)}^- < \cdots < j_1^-\}$. Bernasconi and Egidi [BE99] characterized the affine Hilbert functions of all symmetric sets of the Boolean cube.

Theorem 12 ([BE99]). *Consider the Boolean cube* $\{0,1\}^n$. *For any* $d \in [0,n]$ *and* $E \subseteq [0,n]$,

$$\mathrm{H}_d(E) = \sum_{j \in E \cap [0,d]} \binom{n}{j} + \sum_{t=1}^{\min\{r_d(E), \ell_d(E)\}} \min\left\{ \binom{n}{j_t^+}, \binom{n}{j_t^-} \right\}.$$

The following fact is folklore and follows easily from the definitions. (See, for instance, Nie and Wang [NW15, Proposition 5.2] for a proof.) It connects affine Hilbert functions with finite-degree Z-closures.

Fact 13 (Folklore, [NW15]). *Let* $d \in [0,n]$ *and* $A \subseteq \{0,1\}^n$. *If* $A \subseteq B \subseteq \text{Z-cl}_{n,d}(A)$, *then* $\mathrm{H}_d(A) = \mathrm{H}_d(B)$. *In particular,* $\mathrm{H}_d(A) = \mathrm{H}_d(\text{Z-cl}_{n,d}(A))$.

We remark that using Fact 13 and the result of Bernasconi and Egidi (Theorem 12), we could obtain Theorem 10, that is, our combinatorial characterization of finite-degree Z*-closures (as well as finite-degree Z-closures and h-closures) of symmetric sets of the Boolean cube. However, our arguments, in fact, prove Theorem 7, that is, our proof works over general uniform grids and could also be regarded as being more combinatorial.

Linear time algorithm. We give a linear time algorithm (Algorithm 1) to compute $\overline{L}_{N,d}(E)$, for any $d \in [0,N]$, $E \subseteq [0,N]$. Thus, the finite-degree Z*-closures (Z-closures in the Boolean cube setting) can be computed in linear time using our characterization. However, it is unclear if the finite-degree Z-closures can be computed in linear time, in the Boolean cube setting, using Fact 13 and the result of Bernasconi and Egidi (Theorem 12).

1.5 Other applications

We believe that the combinatorial characterization in Theorem 7 might also be of independent interest. Indeed, we give a couple of other applications.

1.5.1 An alternate proof of a lemma by Alon et al. (1988)

Alon, Bergmann, Coppersmith and Odlyzko [ABCO88] obtained a tight lower bound for a *balancing problem* on the Boolean cube $\{-1,1\}^n$. Their proof is via the polynomial method, using the following lemma.

Lemma 14 ([ABCO88]). *Let* $n \in \mathbb{Z}^+$ *be even and* $f(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ *represent a nonzero function on* $\{-1,1\}^n$ *such that either of the following conditions is true.*

- $f(x) = 0$, *for all* $x \in \{-1,1\}^n$ *having an even number of* $i \in [n]$ *such that* $x_i = -1$.
- $f(x) = 0$, *for all* $x \in \{-1,1\}^n$ *having an odd number of* $i \in [n]$ *such that* $x_i = -1$.

Then $\deg f \geqslant n/2$.

For $i \in \{0,1\}$, let $E_i = \{j \in [0,n] : j \equiv i \pmod 2\}$. We will observe that the above lemma is equivalent to the following proposition, thus giving us an alternate proof.

Proposition 15. *If* $n \in \mathbb{Z}^+$ *is even, then* $\overline{L}_{n,n/2-1}(E_0) = \overline{L}_{n,n/2-1}(E_1) = [0,n]$.

1.5.2 Certifying degrees of weight-determined sets

Let G be a uniform grid. We say a polynomial $f(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ is a certifying polynomial for a subset $S \subseteq G$ if $f \in V(G)$ is nonconstant, but $f|_S$ is constant. (Thus, S is necessarily a proper subset.) The notion of a certifying polynomial (in the Boolean cube setting, for Boolean functions and Boolean circuits) was studied by Kopparty and Srinivasan [KS18] to prove lower bounds for a certain class of Boolean circuits. Variants of this notion have appeared in theoretical computer science, specifically in complexity theory literature, in the works of Aspnes, Beigel, Furst and Rudich [ABFR93], Green [Gre00], and Alekhinovich and Razborov [AR01]. Certifying polynomials have also appeared in the context of cryptography in Carlet, Dalai, Gupta and Maitra [CDGM06].

For a subset $S \subseteq G$, the certifying degree of S, denoted by cert-deg(S), is defined to be the smallest $d \in [0, N]$ such that S has a certifying polynomial with degree at most d. We will determine the certifying degrees of all weight-determined sets, when G is an SU^2 grid. In fact, we will get the following.

Theorem 16. *Let G be an SU^2 grid. For any proper weight-determined set $E \subsetneq [0, N]$,*

$$\text{cert-deg}(E) = \mathsf{PC}_G(E) = \min\{d \in [0, N] : E \textit{ is } d\textit{-admitting}\}.$$

We will conclude our work by considering a third variant of our covering problems – the *exact covering problem.* As for the other covering problems, we will define (in Section 6) an *exact hyperplane (polynomial) cover* for a weight-determined set $\underline{E}$, $E \subsetneq [0, N]$ in a uniform grid G, and consider the minimum size (degree) of an exact hyperplane (polynomial) cover for $\underline{E}$. Let us denote these by $\mathsf{EHC}_G(E)$ and $\mathsf{EPC}_G(E)$ respectively. (In the Boolean cube setting, we will denote these by $\mathsf{EHC}_n(E)$ and $\mathsf{EPC}_n(E)$ respectively.) We will see some partial results and conjecture a solution in the Boolean cube setting.

We can quickly indicate the progress made in our work via the following tables. In this work, we have obtained combinatorial characterizations of the quantities in the colored cells in the following tables. Further, the cells with the same color have the same characterization, that is, although the corresponding questions are different, the answers are the same. Characterizations of the quantities in the uncolored (white) cells is open.

Cover	Nontrivial	Proper	Exact
Hyperplane	HC_n	PHC_n	EHC_n
Polynomial	PC_n	PPC_n	EPC_n

Table 1: For the Boolean cube $\{0, 1\}^n$

Digression 1: Why consider only uniform grids? It is easy to see that all the results in this work that hold for a uniform grid G would also hold for the image of G under any invertible affine transformation of $\mathbb{R}^n$. A typical grid that is genuinely nonuniform would be of the form $H = \{t_0^{(1)} < \cdots < t_{k_1-1}^{(1)}\} \times \cdots \times \{t_0^{(n)} < \cdots < t_{k_n-1}^{(n)}\}$,

Cover	Nontrivial	Proper	Exact
Hyperplane	HC_G	PHC_G	EHC_G
Polynomial	PC_G	PPC_G	EPC_G

Table 2: For an SU^2 grid G

where the differences $t_{i+1}^{(j)} - t_i^{(j)}$, $i \in [0, k_j - 2]$ are not equal, for some $j \in [n]$. In this case, an obvious way to define the weight of an element $x \in H$ would be $\mathrm{wt}(x) := \sum_{j \in [n]} s_j$, where $x_j = t_{s_j}^{(j)}$ for all $j \in [n]$.[6] Unlike in the uniform setting, it is no longer true that every weight-determined set $\underline{w}$, $w \in [0, N]$ is contained in the zero set of the linear polynomial $\sum_{j \in [n]} X_j - w$. This has undesirable ripple effects; for instance, the finite-degree Z*-closures of weight-determined sets depend on the weights of the points, as well as on the coordinates of the points in the set.

As an example, let $G = \{0, 1, 2\}^2$ and $H = \{0, 1, 3\}^2$. Both grids have the same dimensions and $N = 4$. In G, we have $\underline{2} = \{(2, 0), (1, 1), (0, 2)\}$, and further $\mathcal{Z}_G(X_1 + X_2 - 2) = \underline{2}$. So $\mathrm{Z}^*\text{-}\mathrm{cl}_{G,1}(\{2\}) = \{2\}$. In H, we have $\underline{2} = \{(3, 0), (1, 1), (0, 3)\}$. If $\underline{2} \subseteq \mathcal{Z}_H(P)$, for $P(X_1, X_2) = aX_1 + bX_2 + c$, $a, b, c \in \mathbb{R}$, then we get the linear equations

$$3a + c = 0, \quad a + b = 0, \quad 3b + c = 0,$$

which implies $a = b = c = 0$, that is, $P(X_1, X_2) = 0$. So $\mathrm{Z}^*\text{-}\mathrm{cl}_{H,1}(\{2\}) = [0, 4]$.

If the grid is uniform, however, the finite-degree Z*-closures of weight-determined sets depend only on the weights of the points in the set. So the upshot is that the 'uniform' condition on grids ensures the setting is nice enough for our tools, tricks and techniques to work well, and give neat results.

Digression 2: What happens over fields other than $\mathbb{R}$? Consider the Boolean cube $\{0, 1\}^n$, as well as any uniform grid $G \subseteq \mathbb{N}^n$. A crucial observation that enables our algebraic arguments (over $\{0, 1\}^n$ as well as over G) to go through is that for any valid $w \in \mathbb{N}$, the zero set of the linear polynomial $\sum_{i \in [n]} X_i - w$ is exactly $\underline{w}$ (in $\{0, 1\}^n$ as well as in G). This is true over any field of characteristic zero. Indeed, all the results in this paper hold true over any field of characteristic zero, without any change in the arguments. Further, it is easy to see that the above mentioned property is also true, and therefore all the results in this paper also hold true, over any field with large positive characteristic p: we require $p > n$ over $\{0, 1\}^n$, and $p > N$ over G. However, our results do not extend to fields with small positive characteristic.

Organization of the paper. In Section 2, we will look at some preliminaries concerning the different closure operators of interest to us. In Section 3, we will prove our combinatorial characterization of Z*-closures of weight-determined sets of an SU^2 grid,

[6] This definition of weight is very natural, if one appeals to lattice theory. Indeed, H is a lattice w.r.t. a suitable partial order with wt, as defined, being the rank function of the lattice.

and then obtain Theorem 4, our more refined solution to Problem 2 (b) for SU^2 grids. In Section 4, we will see that the h-closures coincide with the Z-closures and Z*-closures, for all symmetric sets of the Boolean cube. We also obtain Theorem 5, our solution to Problem 2 (a) in the Boolean cube setting. In Section 5, we consider further applications of our combinatorial characterization of finite-degree Z*-closures from Section 3: (i) an alternate proof of a lemma by Alon et al. (1988) in the context of *balancing problems*, and (ii) a characterization of the *certifying degrees* of weight-determined sets. We conclude in Section 6 by introducing a third variant of our covering problems. We will discuss some partial results and conjecture a solution in the Boolean cube setting.

2 Preliminaries

We begin by recalling some definitions. Consider the uniform grid $G = [0, k_1 - 1] \times \cdots \times [0, k_n - 1]$ with $N = \sum_{i \in [n]}(k_i - 1)$. We will require the following important result by Alon [Alo99], which is, in fact, stronger than the result of Alon and Füredi (Theorem 1 and Theorem 3).

Theorem 17 (Combinatorial Nullstellensatz [Alo99, Theorem 2]).[7] *If $P(\mathbb{X}) \in \mathrm{span}_{\mathbb{R}}\{\mathbb{X}^\gamma : \gamma \in G\}$ is a nonzero polynomial, then there exists $a \in G$ such that $P(a) \neq 0$.*

We adopt the convention that $\deg(0) = -\infty$, where 0 denotes the zero polynomial. For a subset $I \subseteq \mathbb{R}[\mathbb{X}]$, let $\mathcal{Z}(I) = \{a \in G : f(a) = 0,\ \text{for all } f(\mathbb{X}) \in I\}$. Further, for any $d \in [0, N]$ and $I \subseteq \mathbb{R}[\mathbb{X}]$, we denote $I_d = \{f(\mathbb{X}) \in I : \deg f \leqslant d\}$. Also, for any subset $S \subseteq G$, let $\mathcal{I}(I, S) = \{f(\mathbb{X}) \in I : f(a) = 0,\ \text{for all } a \in S\}$, and denote $\mathcal{I}(S) = \mathcal{I}(\mathbb{R}[\mathbb{X}], S)$.

For any $d \in [0, N]$ and $S \subseteq G$, the degree-d Z-closure of S is defined as $\text{Z-cl}_{G,d}(S) = \mathcal{Z}(\mathcal{I}(\mathbb{R}[\mathbb{X}]_d, S))$.

When $G = \{0,1\}^n$, it follows that the finite-degree Z-closure of a symmetric set is symmetric. However, for a general uniform grid G, the finite-degree Z-closure of a weight-determined set need not be weight-determined. We recall the one-to-one correspondence between weight-determined sets of G and subsets of $[0, N]$. Every subset $E \subseteq [0, N]$ corresponds to the weight-determined set $\underline{E} = \{x \in G : \mathrm{wt}(x) \in E\}$. For convenience, we will identify E with $\underline{E}$. We therefore consider a new notion of closure exclusiely for weight-determined sets. Let G be a uniform grid. For $d \in [0, N]$ and $E \subseteq [0, N]$, we define the degree-d Z*-closure of $\underline{E}$ as

$$\text{Z*-cl}_{G,d}(\underline{E}) = \bigcup_{\substack{j \in [0,N] \\ \underline{j} \subseteq \text{Z-cl}_{G,d}(\underline{E})}} \underline{j}.$$

The following properties of finite-degree Z*-closures are similar to that of finite-degree Z-closures, and follow quickly from the definition.

[7] In fact, Theorem 17 is true over any field, assuming G is embedded in the corresponding affine space.

Proposition 18. *Let G be a uniform grid and $d \in [0, N]$.*

(a) $\mathrm{Z}^*\text{-cl}_{G,d}(\underline{E})$ *is weight-determined, for all* $E \subseteq [0, N]$.

(b) $\mathrm{Z}^*\text{-cl}_{G,d}$ *is a closure operator.*

(c) $\mathrm{Z}^*\text{-cl}_{G,d+1}(\underline{E}) \subseteq \mathrm{Z}^*\text{-cl}_{G,d}(\underline{E})$, *for all* $E \subseteq [0, N]$.

Proof. (a) By definition, if $x \in \mathrm{Z}^*\text{-cl}_{G,d}(\underline{E})$, $x \in \underline{j}$, then $\underline{j} \subseteq \mathrm{Z}^*\text{-cl}_{G,d}(\underline{E})$. So $\mathrm{Z}^*\text{-cl}_{G,d}(\underline{E})$ is weight-determined.

(b) Note that, as a set operator, we have $\mathrm{Z}^*\text{-cl}_{G,d} : \mathsf{W}(G) \to \mathsf{W}(G)$, where $\mathsf{W}(G)$ denotes the collection of all weight-determined sets of G. We observe the following.

 (i) Clearly $\underline{E} \subseteq \mathrm{Z}^*\text{-cl}_{G,d}(\underline{E})$, for all $E \subseteq [0, N]$.

 (ii) Now consider any $A \subseteq B \subseteq [0, N]$. We already have $\mathrm{Z}\text{-cl}_{G,d}(\underline{A}) \subseteq \mathrm{Z}\text{-cl}_{G,d}(\underline{B})$. So for $j \in [0, N]$, if $\underline{j} \subseteq \mathrm{Z}\text{-cl}_{G,d}(\underline{A})$, then obviously $\underline{j} \subseteq \mathrm{Z}\text{-cl}_{G,d}(\underline{B})$. Thus $\mathrm{Z}^*\text{-cl}_{G,d}(\underline{A}) \subseteq \mathrm{Z}^*\text{-cl}_{G,d}(\underline{B})$.

 (iii) Further, for any $E \subseteq [0, N]$, since $\mathrm{Z}\text{-cl}_{G,d}(\mathrm{Z}\text{-cl}_{G,d}(\underline{E})) = \mathrm{Z}\text{-cl}_{G,d}(\underline{E})$, we have in fact $\mathrm{Z}^*\text{-cl}_{G,d}(\mathrm{Z}\text{-cl}_{G,d}(\underline{E})) = \mathrm{Z}^*\text{-cl}_{G,d}(\underline{E})$. So we have, using (i) and (ii),

$$\mathrm{Z}^*\text{-cl}_{G,d}(\underline{E}) \subseteq \mathrm{Z}^*\text{-cl}_{G,d}(\mathrm{Z}^*\text{-cl}_{G,d}(\underline{E})) \subseteq \mathrm{Z}^*\text{-cl}_{G,d}(\mathrm{Z}\text{-cl}_{G,d}(\underline{E})) = \mathrm{Z}^*\text{-cl}_{G,d}(\underline{E}).$$

Thus $\mathrm{Z}^*\text{-cl}_{G,d}$ is a closure operator.

(c) Consider any $E \subseteq [0, N]$. We already have $\mathrm{Z}\text{-cl}_{G,d+1}(\underline{E}) \subseteq \mathrm{Z}\text{-cl}_{G,d}(\underline{E})$. So for $j \in [0, N]$, if $\underline{j} \subseteq \mathrm{Z}\text{-cl}_{G,d+1}(\underline{E})$, then obviously $\underline{j} \subseteq \mathrm{Z}\text{-cl}_{G,d}(\underline{E})$. Thus $\mathrm{Z}^*\text{-cl}_{G,d+1}(\underline{E}) \subseteq \mathrm{Z}^*\text{-cl}_{G,d}(\underline{E})$. ☐

For our results, we are most interested in the setting where the uniform grid G is strictly unimodal (SU^2), that is, we have

$$\begin{bmatrix} G \\ 0 \end{bmatrix} < \cdots < \begin{bmatrix} G \\ \lfloor N/2 \rfloor \end{bmatrix} = \begin{bmatrix} G \\ \lceil N/2 \rceil \end{bmatrix} > \cdots > \begin{bmatrix} G \\ N \end{bmatrix}.$$

There is a simple characterization of SU^2 grids given by Dhand [Dha14].

Theorem 19 ([Dha14])**.** *A uniform grid G is strictly unimodal if and only if*

$$2 \max_{i \in [n]} (k_i - 1) \leqslant \sum_{i \in [n]} (k_i - 1) + 1.$$

We will also need the following abbreviation. For any $a_1 \in \{0,1\}^{n_1}$, $a_2 \in \{0,1\}^{n_2}, \ldots,$ $a_k \in \{0,1\}^{n_k}$, the vector

$$\big(\underbrace{a_1, \ldots, a_1}_{i_1}, \underbrace{a_2, \ldots, a_2}_{i_2}, \ldots, \underbrace{a_k, \ldots, a_k}_{i_k}\big) \in \{0,1\}^{n_1 i_1 + n_2 i_2 + \cdots + n_k i_k}$$

will be abbreviated as $a_1^{i_1} a_2^{i_2} \cdots a_k^{i_k}$.

3 Finite-degree Z-closures and Z*-closures, and our polynomial covering problems

In this section, we will obtain our combinatorial characterization of finite-degree Z*-closures of weight-determined sets, and then proceed to solve Problem 2. We begin with some simple results. Let G be a uniform grid. The following fact is folklore and follows from, for instance, the Footprint bound (see Cox, Little and O'Shea [CLO15, Chapter 5, Section 3, Proposition 4]).

Fact 20 (Folklore, [CLO15]). *Let G be a uniform grid. Then $\text{Z*-cl}_{G,d}([0,r]) = [0,N]$, for any $r, d \in [0,N]$, $r \geqslant d$.*

The following result is elementary.

Proposition 21. *If $E \subseteq [0,N]$ with $|E| \leqslant d$, then $\text{Z*-cl}_{G,d}(E) = E$.*

Proof. Clearly $E \subseteq \text{Z*-cl}_{G,d}(E)$. Also the polynomial $P(\mathbb{X}) := \prod_{t\in E}\left(\sum_{i=1}^n X_i - t\right)$ satisfies $\deg P = |E| \leqslant d$, $P|_{\underline{E}} = 0$ and $P|_{\underline{j}} \neq 0$, for all $j \notin E$. Thus $\text{Z*-cl}_{G,d}(E) \subseteq E$. ☐

3.1 Two main lemmas

We require two main lemmas to obtain our characterization; let us prove them here.

Our first lemma holds over any uniform grid and identifies a collection of 'layers' which are certain to lie in the finite-degree Z*-closures of weight-determined sets.

Lemma 22 (Closure Builder Lemma). *Let $d \in [0,N]$ and $E \subseteq [0,N]$ with $|E| \geqslant d+1$. Then*

$$[0, \min E] \cup [\max E, N] \subseteq \text{Z*-cl}_{G,d}(E).$$

Proof. Let $m = \min E$, $M = \max E$. Consider any $j \in [0, m-1]$. Suppose $j \notin \text{Z*-cl}_{G,d}(E)$. This means $\underline{j} \not\subseteq \text{Z-cl}_{G,d}(\underline{E})$. So there exists $a \in \underline{j}$ and $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ such that $\deg P \leqslant d$, $P|_{\underline{E}} = 0$ and $P(a) = 1$. Define

$$Q(\mathbb{X}) = P(\mathbb{X}) \cdot \left(\prod_{i\in[n]} \prod_{s\in[0,a_i-1]} (X_i - s)\right)\left(\prod_{t\in[j+1,N]\setminus E}\left(\sum_{i=1}^n X_i - t\right)\right).$$

Then we have $\deg Q \leqslant d + j + (N - j - |E|) \leqslant N-1$. Also, $Q|_{\underline{t}} = 0$, for all $t \in [0,N]$, $t \neq j$. Further, $Q(x) = 0$ if $x \in \underline{j}$, $x \neq a$, and $Q(a) = a_1! \cdots a_n! \cdot \prod_{t\in[j+1,N]\setminus E}(j-t) \neq 0$. So by Theorem 3, $\deg Q \geqslant N$, a contradiction. Thus $j \in \text{Z*-cl}_{G,d}(E)$. Hence we conclude that $[0,m] \subseteq \text{Z*-cl}_{G,d}(E)$. Similarly, we can conclude that $[M,N] \subseteq \text{Z*-cl}_{G,d}(E)$. ☐

The following observation will be important for further discussion, and is easy to check. Recall the set operators $L_{n,d}$, $d \in [0,n]$ from Subsection 1.3.

Observation 23. *Let G be a uniform grid. For $d \in [0, N]$ and $E = \{t_1 < \cdots < t_s\} \subseteq [0, N]$, if $|E| \geqslant d + 1$, then*

$$L_{N,d}(E) = [0, t_{s-d}] \cup E \cup [t_{d+1}, N] = \bigcup_{A \in \binom{E}{d+1}} \big([0, \min A] \cup A \cup [\max A, n]\big).$$

Consequently, $\overline{L}_{N,d}(E) \subseteq \mathrm{Z}^*\text{-}\mathrm{cl}_{G,d}(E)$.

Thus $L_{n,d}(E)$ represents repeated application of the Closure Builder Lemma 22 to $(d+1)$-subsets of E.

Our second lemma characterizes the finite-degree Z*-closure of $T_{N,i}$, $i \in [0, N]$ in an SU^2 grid.

Lemma 24. *Let G be an SU^2 grid. For every $i \in [0, N]$,*

$$\mathrm{Z}^*\text{-}\mathrm{cl}_{G,d}(T_{N,i}) = \begin{cases} T_{N,i} & \text{if } i \leqslant d, \\ [0, N] & \text{if } i > d. \end{cases}$$

Proof. Note that

$$T_{N,\lfloor N/2 \rfloor} = \begin{cases} [0, N] \setminus \{N/2\} & \text{if } N \text{ is even}, \\ [0, N] \setminus \{\lfloor N/2 \rfloor, \lfloor N/2 \rfloor + 1\} & \text{if } N \text{ is odd}. \end{cases}$$

So if $i \geqslant \lfloor N/2 \rfloor + 1$, then obviously $T_{N,i} = [0, N]$, and so we are trivially done. So assume $i \leqslant \lfloor N/2 \rfloor$. If $i > d$, then $[0, d] \subseteq T_{N,i}$, and so by Closure Builder Lemma 22, we conclude that $\mathrm{Z}\text{-}\mathrm{cl}_{G,d}(T_{N,i}) = [0, N]$.

Now also assume $i \leqslant d$. Clearly $T_{N,i} \subseteq \mathrm{Z}^*\text{-}\mathrm{cl}_{G,d}(T_{N,i})$. Let $T'_{N,i} = [0, i-1] \cup \{N-i+1\}$. Again by Closure Builder Lemma 22 and Proposition 18 (b), we get $\mathrm{Z}^*\text{-}\mathrm{cl}_{G,i}(T_{N,i}) = \mathrm{Z}^*\text{-}\mathrm{cl}_{G,i}(T'_{N,i})$. Also by Proposition 18 (c), we have $\mathrm{Z}^*\text{-}\mathrm{cl}_{G,d}(T_{N,i}) \subseteq \mathrm{Z}^*\text{-}\mathrm{cl}_{G,i}(T_{N,i})$. So it is enough to prove that $\mathrm{Z}^*\text{-}\mathrm{cl}_{G,i}(T_{N,i}) = T_{N,i}$.

Suppose $i \in \mathrm{Z}^*\text{-}\mathrm{cl}_{G,i}(T_{N,i})$. This means $[0, i] \subseteq \mathrm{Z}^*\text{-}\mathrm{cl}_{G,i}(T_{N,i})$. But then by Lemma 22, and Proposition 18 (b) and (c), we have $[i, N] \subseteq \mathrm{Z}^*\text{-}\mathrm{cl}_{G,i}([0, i]) \subseteq \mathrm{Z}^*\text{-}\mathrm{cl}_{G,i}(\mathrm{Z}^*\text{-}\mathrm{cl}_{G,i}(T_{N,i}))$. This means $\mathrm{Z}^*\text{-}\mathrm{cl}_{G,i}(T_{N,i}) = [0, N]$, that is, $\mathrm{Z}\text{-}\mathrm{cl}_{G,i}(\underline{T_{N,i}}) = G$. Thus we get $\mathrm{Z}\text{-}\mathrm{cl}_{G,i}(\underline{T'_{N,i}}) = G$. Now consider the linear system

$$\sum_{\gamma \in \underline{[0,i]}} c_\gamma \mathbb{X}^\gamma(a) = 0, \quad a \in \underline{T'_{N,i}}.$$

Note that in this system, the variables are c_γ, $\gamma \in \underline{[0, i]}$, and the constraints are indexed by $a \in \underline{T'_{N,i}}$. So the number of variables is $|\underline{[0, i]}| = \left[{G \atop [0,i]}\right] = \left[{G \atop [0,i-1]}\right] + \left[{G \atop i}\right]$, and the number of constraints is $|\underline{T'_{N,i}}| = |\underline{[0, i-1]}| + |\underline{N-i+1}| = \left[{G \atop [0,i-1]}\right] + \left[{G \atop N-i+1}\right]$. Since G is SU^2, we have $\left[{G \atop i}\right] > \left[{G \atop i-1}\right] = \left[{G \atop N-i+1}\right]$, and therefore this system has a nontrivial solution. Thus, there exists a nonzero polynomial $P(\mathbb{X}) \in \mathrm{span}_{\mathbb{R}}\{\mathbb{X}^\gamma : \gamma \in \underline{[0, i]}\}$ such that $P|_{\underline{T'_{N,i}}} = 0$.

By the Combinatorial Nullstellensatz (Theorem 17), we then get $P \neq 0$ in $V(G)$, that is, $P(a) \neq 0$ for some $a \in G$. (Also, obviously by the spanning set description, we get $\deg P \leqslant i$.) This means $a \notin \text{Z-cl}_{G,i}(\underline{T'_{N,i}})$, which implies $\text{wt}(a) \notin \text{Z}^*\text{-cl}_{G,i}(T'_{N,i})$. So $\text{Z}^*\text{-cl}_{G,i}(T'_{N,i}) \neq G$, a contradiction.

So we get $i \notin \text{Z}^*\text{-cl}_{G,i}(T_{N,i})$. Similarly, we can conclude that $N - i \notin \text{Z}^*\text{-cl}_{G,i}(T_{N,i})$. Now consider any $j \in [i+1, N-i-1]$. If $j \in \text{Z}^*\text{-cl}_{G,i}(T_{N,i})$, then by the Closure Builder Lemma 22 and Observation 23, we get $[0, j] \subseteq \text{Z}^*\text{-cl}_{G,i}(T_{N,i})$, which implies $i \in \text{Z}^*\text{-cl}_{G,i}(T_{N,i})$, a contradiction. Thus we have proven that $[i, N-i] \cap \text{Z}^*\text{-cl}_{G,i}(T_{N,i}) = \emptyset$, that is, $\text{Z}^*\text{-cl}_{G,i}(T_{N,i}) = T_{N,i}$. This completes the proof. $\square$

3.2 The main theorem

We will now prove Theorem 7, our combinatorial characterization of finite-degreee Z*-closures of all weight-determined sets in an SU^2 grid. We restate Theorem 7 for convenience.

Theorem 7. *Let G be an SU^2 grid. For every $d \in [0, N]$ and $E \subseteq [0, N]$,*

$$\text{Z}^*\text{-cl}_{G,d}(E) = \overline{L}_{N,d}(E).$$

Proof. Fix a $d \in [0, N]$ and consider any $E \subseteq [0, N]$. If $|E| \leqslant d$, then by Proposition 21 and the definition of $\overline{L}_{N,d}$, we have $\overline{L}_{N,d}(E) = \text{Z}^*\text{-cl}_{G,d}(E) = E$.

So now assume that $|E| \geqslant d+1$. By the Closure Builder Lemma 22 and Observation 23, we already have $\overline{L}_{N,d}(E) \subseteq \text{Z}^*\text{-cl}_{G,d}(E)$. Now consider any $j \notin \overline{L}_{N,d}(E)$. In particular, $\overline{L}_{N,d}(E) \neq [0, N]$. We will prove that $j \notin \text{Z}^*\text{-cl}_{G,d}(E)$. Let

$$i_0 = \max\{i \in [0, \lfloor N/2 \rfloor] : T_{N,i} \subseteq \overline{L}_{N,d}(E)\}.$$

Note that

$$T_{N,\lfloor N/2 \rfloor} = \begin{cases} [0, N] \setminus \{N/2\} & \text{if } N \text{ is even,} \\ [0, N] \setminus \{\lfloor N/2 \rfloor, \lfloor N/2 \rfloor + 1\} & \text{if } N \text{ is odd.} \end{cases}$$

So clearly $T_{N,i_0} = [0, i_0 - 1] \cup [N - i_0 + 1, N] \neq [0, N]$. If $i_0 \geqslant d + 1$, then there exists $k \in \mathbb{Z}^+$ such that $[0, d] \subseteq [0, i_0 - 1] \subseteq L^k_{N,d}(E)$. Then by Observation 23, we have $L^{k+1}_{N,d}(E) = [0, N]$, that is, $\overline{L}_{N,d}(E) = [0, N]$, a contradiction. So we have $i_0 \leqslant d$.

By definition of i_0, it is clear that either $i_0 \notin \overline{L}_{N,d}(E)$ or $N - i_0 \notin \overline{L}_{N,d}(E)$. Without loss of generality, suppose $i_0 \notin \overline{L}_{N,d}(E)$. Now we have $j \in [i_0, N - i_0]$. By Lemma 24, $\text{Z}^*\text{-cl}_{G,i_0}(T_{N,i_0}) = T_{N,i_0}$, and so $j \notin \text{Z}^*\text{-cl}_{G,i_0}(T_{N,i_0})$. This means $\underline{j} \nsubseteq \text{Z-cl}_{G,i_0}(T_{N,i_0})$. So there exists $x_0 \in \underline{j}$ and a nonzero $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ satisfying $\deg P \leqslant i_0$, $P|_{\underline{T_{N,i_0}}} = 0$ and $P(x_0) = 1$.

Let us now show that $|E \cap [i_0, N - i_0]| \leqslant d - i_0$. On the contrary, suppose $|E \cap [i_0, N - i_0]| \geqslant d - i_0 + 1$. Then

$$\big|[N - i_0 + 1, N] \cup \big(E \cap [i_0, N - i_0]\big)\big| \geqslant d + 1.$$

Let $E^+ = [N - i_0 + 1, N] \cup \big(E \cap [i_0, N - i_0]\big)$. Note that since $T_{N,i_0} \subseteq \overline{L}_{N,d}(E)$, we have $E^+ \subseteq \overline{L}_{N,d}(E)$. Since $|E^+| \geqslant d + 1$, by definition of $\overline{L}_{N,d}$, we have $[0, \min E^+] \subseteq \overline{L}_{N,d}(E)$. But we have $i_0 \in [0, \min E^+]$. So in particular, we have $i_0 \in \overline{L}_{N,d}(E)$, which is a contradiction since we assumed $i_0 \notin \overline{L}_{N,d}(E)$. Hence $|E \cap [i_0, N - i_0]| \leqslant d - i_0$.

Now define

$$Q(\mathbb{X}) = \prod_{t \in E \cap [i_0, N-i_0]} \left(\sum_{i=1}^{n} X_i - t \right).$$

Then $\deg Q \leqslant d - i_0$ and $Q(x_0) \neq 0$. This gives $\deg PQ \leqslant d$, $PQ|_{\underline{E}} = 0$ and $PQ(x_0) \neq 0$, that is, $PQ|_{\underline{j}} \neq 0$. So $\underline{j} \not\subseteq$ Z-cl$_{G,d}(E)$, which implies $j \notin$ Z*-cl$_{G,d}(E)$. This completes the proof. ☐

3.3 Computing the finite-degree Z*-closures efficiently

The characterization in Theorem 7 gives a simple algorithm to compute Z*-cl$_{G,d}(E)$, for any $d \in [0, N]$, $E \subseteq [0, N]$, in time $O(N)$ (linear time). We only consider the complexity of computing the finite-degree Z*-closures modulo the bit complexity of representing the weight-determined sets in the uniform grid G; accomodating the bit complexity will only multiply our bound by a poly$(\log N)$ factor. We will now describe the algorithm.

We will need to consider *shifted* versions of the set operators $L_{N,d}$ and $\overline{L}_{N,d}$, defined for subsets of intervals other than $[0, N]$. Fix any $a, b \in \mathbb{Z}$, $a \leqslant b$. For every $d \in [0, b - a]$, define $L_{[a,b],d} : 2^{[a,b]} \to 2^{[a,b]}$ and $\overline{L}_{[a,b],d} : 2^{[a,b]} \to 2^{[a,b]}$ as

$$L_{[a,b],d}(E) = L_{b-a,d}(E - a) + a \quad \text{and} \quad \overline{L}_{[a,b],d}(E) = \overline{L}_{b-a,d}(E - a) + a, \quad \text{for all } E \subseteq [a, b].$$

Immediately, we have the following analogue of Observation 23.

Observation 25. *Let $a, b \in \mathbb{Z}$, $a \leqslant b$. For any $d \in [0, b - a]$ and $E = \{t_1 < \cdots < t_s\} \subseteq [a, b]$, if $|E| \geqslant d + 1$, then*

$$L_{[a,b],d}(E) = [a, t_{s-d}] \cup E \cup [t_{d+1}, b] = \bigcup_{A \in \binom{E}{d+1}} \big([a, \min A] \cup A \cup [\max A, b]\big).$$

The following properties of the above set operators will enable us to compute them in linear time.

Proposition 26. *Let $a, b \in \mathbb{Z}$, $a \leqslant b$.*

(a) $\overline{L}_{[a,b],0}(\emptyset) = \emptyset$, and $\overline{L}_{[a,b],0}(E) = [a, b]$, for all $\emptyset \neq E \subseteq [a, b]$.

(b) $\overline{L}_{[a,b],b-a}(E) = E$, for all $E \subseteq [a, b]$.

(c) If $b - a \geqslant 2$, then for any $d \in [b - a]$,

$$\overline{L}_{[a,b],d}(E) = \begin{cases} E & \text{if } |E| \leqslant d, \\ \{a, b\} \sqcup \overline{L}_{[a+1,b-1],d-1}(E \setminus \{a, b\}) & \text{if } |E| \geqslant d + 1. \end{cases}$$

Proof. Items (a) and (b) are evident from the definitions. For Item (c), the claim is clear if $|E| \leqslant d$, again by the definitions.

So now suppose $|E| \geqslant d+1$. It is then immediate that $\{a,b\} \subseteq \overline{L}_{[a,b],d}(E)$ and so $\overline{L}_{[a,b],d}(E) = \overline{L}_{[a,b],d}(\{a,b\} \cup E)$. So let $E' = \{a,b\} \cup E$. We only need to show that $\overline{L}_{[a,b],d}(E') = \{a,b\} \sqcup \overline{L}_{[a+1,b-1],d-1}(E' \setminus \{a,b\})$, since $\overline{L}_{[a,b],d}(E') = \overline{L}_{[a,b],d}(E)$ and $E' \setminus \{a,b\} = E \setminus \{a,b\}$.

If $|E'| = d+1$, then clearly, $\overline{L}_{[a,b],d}(E') = E' = \{a,b\} \sqcup \overline{L}_{[a+1,b-1],d-1}(E' \setminus \{a,b\})$. Now assume $|E'| \geqslant d+2$. It is enough to prove $L^k_{[a,b],d}(E') = \{a,b\} \sqcup L^k_{[a+1,b-1],d-1}(E' \setminus \{a,b\})$, for all $k \in \mathbb{N}$. We will show this by induction on k. The case $k=0$ is obvious, since $L^0_{[a,b],d}$ is the identity operator. Now suppose the claim is true for some $k \in \mathbb{N}$. Let $L^k_{[a,b],d}(E') = \{t_1 < \ldots < t_s\}$. So we get

$$\begin{aligned} L^{k+1}_{[a,b],d}(E') &= [a, t_{s-d}] \cup L^k_{[a,b],d}(E') \cup [t_{d+1}, b] \\ &= [a, t_{s-d}] \cup L^k_{[a+1,b-1],d-1}(E' \setminus \{a,b\}) \cup [t_{d+1}, b], \end{aligned} \tag{1}$$

where the first equality is by the definition of $L_{[a,b],d}$, and the second equality is by the induction hypothesis. But we already have $t_1 = a$, $t_s = b$, and $L^k_{[a+1,b-1],d-1}(E' \setminus \{a,b\}) = \{t_2 < \cdots < t_{s-1}\}$. So

$$L^{k+1}_{[a+1,b-1],d-1}(E' \setminus \{a,b\}) = [a+1, t_{s-d}] \cup L^k_{[a+1,b-1],d-1}(E' \setminus \{a,b\}) \cup [t_{d+1}, b-1]. \tag{2}$$

So from (1) and (2), we get

$$L^{k+1}_{[a,b],d}(E') = \{a,b\} \sqcup L^{k+1}_{[a+1,b-1],d-1}(E' \setminus \{a,b\}).$$

This completes the proof. ◻

We then get a straightforward linear time recursive algorithm to compute $\overline{L}_{[a,b],d}$, $d \in [0, b-a]$. The base case of the recursion appeals to Proposition 26 (a) and (b), and the recursive step appeals to Proposition 26 (c). The linear run-time is obvious since it is easy to see that there exists a constant $C > 0$ such that for any $a, b \in \mathbb{Z}$, $a \leqslant b$ and $d \in [0, b-a]$, $E \subseteq [a,b]$, (i) $|E|$ can be computed in time at most $C(b-a)$, and further (ii) appealing to Observation 25, $L_{[a,b],d}(E)$ can be computed in time at most $C(b-a)$. So for inputs $d \in [0,N]$ and $E \subseteq [0,N]$, the algorithm computes $\overline{L}_{N,d}(E)$ in time $O(N)$.

In Algorithm 1 we state the pseudocode for this algorithm. Here we assume that the input $E \subseteq [a,b]$ is given by its indicator vector $(e_a, \ldots, e_b) \in \{0,1\}^{[a,b]} = \{0,1\}^{b-a+1}$, and $w \coloneqq \sum_{t \in [a,b]} e_t = |E|$ is already computed in time at most $C(b-a)$. Note that Proposition 26 is also the proof of correctness of this algorithm.

3.4 Solving our polynomial covering problems

Let us gather the work done so far to solve our polynomial covering problems (Problem 2 (b)). Let us begin by proving Lemma 6 that relates our polynomial covering problems with the finite-degree Z*-closures.

Algorithm 1: `L-bar`: Computing $\overline{L}_{[a,b],d}$

Input: $a, b \in \mathbb{N}$, $a \leqslant b$; $d \in [0, b-a]$; $(e_a, \ldots, e_b) \in \{0,1\}^{[a,b]}$; $w \coloneqq \sum_{t \in [a,b]} e_t$

Output: $(f_a, \ldots, f_b) \in \{0,1\}^{[a,b]}$ such that if $E \coloneqq \{j \in [a,b] : e_j = 1\}$, then $F \coloneqq \{j \in [a,b] : f_j = 1\}$ satisfies $F = \overline{L}_{[a,b],d}(E)$.

if $b = a$ **then**
  return (e_a)
else if $b = a+1$ **then**
  if $d = 0$ **then**
    if $(e_a, e_{a+1}) = (0,0)$ **then**
      return $(0,0)$
    else
      return $(1,1)$
  else
    return (e_a, e_{a+1})
else if $d = b - a$ **or** $w \leqslant d$ **then**
  return $(e_a, \ldots, e_b)$
else
  return $\big(1, \texttt{L-bar}\big(a+1, b-1, d-1, (e_{a+1}, \ldots, e_{b-1}), w - (e_a + e_b)\big), 1\big)$

Proof of Lemma 6. Let G be a uniform grid and consider $E \subsetneq [0, N]$.

(a) We have the following equivalences.

$$\begin{aligned} & d' = \mathsf{PC}_G(E) \\ \iff\ & \text{There exists } P(\mathbb{X}) \in \mathcal{I}(\underline{E})_{d'} \text{ such that } \mathcal{Z}_G(P) \neq G \\ & \quad \text{and for every } Q(\mathbb{X}) \in \mathcal{I}(\underline{E})_{d'-1},\ \text{we have } \mathcal{Z}_G(P) = G \\ \iff\ & \text{Z-cl}_{G,d'}(\underline{E}) \neq G \text{ and } \text{Z-cl}_{G,d'-1}(\underline{E}) = G \\ \iff\ & \text{Z*-cl}_{G,d'}(E) \neq [0,N] \text{ and } \text{Z*-cl}_{G,d'-1}(E) = [0,N] \\ \iff\ & d' = \min\{d \in [0,N] : \text{Z*-cl}_{G,d'}(E) \neq [0,N]\}. \end{aligned}$$

(b) Let $d' = \mathsf{PPC}_G(E)$ and $d'' = \min\{d \in [0,N] : \text{Z*-cl}_{G,d}(E) = E\}$. We have the following implications.

$$\begin{aligned} & d' = \mathsf{PPC}_G(E) \\ \implies\ & \text{There exists } P(\mathbb{X}) \in \mathcal{I}(\underline{E})_{d'} \text{ such that } \underline{j} \not\subseteq \mathcal{Z}_G(P),\ \text{for every } j \in [0,N] \setminus E \\ \implies\ & \underline{j} \not\subseteq \text{Z-cl}_{G,d'}(\underline{E}),\ \text{for every } j \in [0,N] \setminus E \\ \implies\ & j \notin \text{Z*-cl}_{G,d'}(E),\ \text{for every } j \in [0,N] \setminus E,\ \text{that is, } \text{Z*-cl}_{G,d'}(E) = E \\ \implies\ & d'' \leqslant d'. \end{aligned}$$

Now since $d'' = \min\{d \in [0,N] : \text{Z*-cl}_{G,d}(E) = E\}$, we have $\underline{j} \not\subseteq \text{Z-cl}_{G,d''}(\underline{E})$, for every $j \in [0,N] \setminus E$. For every $j \in [0,N] \setminus E$, choose $a^{(j)} \in \underline{j} \setminus \text{Z-cl}_{G,d''}(\underline{E})$; so there exists $P_j(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ such that $\deg P_j \leqslant d''$, $P_j|_{\underline{E}} = 0$ and $P_j(a^{(j)}) = 1$. Then we can choose $\beta_j \in \mathbb{R}$, $j \in [0,N] \setminus E$ such that the polynomial $P(\mathbb{X}) := \sum_{j \in [0,N] \setminus E} \beta_j P_j(\mathbb{X})$ satisfies $\deg P \leqslant d''$, $P|_{\underline{E}} = 0$ and $P(a^{(j)}) \neq 0$, for all $j \in [0,N] \setminus E$. This implies $d' \leqslant d''$, and therefore completes the proof. $\square$

Let us now proceed to prove Proposition 8. We require the following lemma.

Lemma 27. *Let $d \in [0,N]$, $i \in [0,d]$. Then $E \subseteq [0,N]$ is (d,i)-admitting if and only if $L_{N,d}(E)$ is (d,i)-admitting.*

Proof. Fix any $d \in [0,N]$, $i \in [0,d]$ and $E \subseteq [0,N]$. We clearly have the following implications.

$$\begin{aligned}
&L_{N,d}(E) \text{ is } (d,i)\text{-admitting} \\
\implies\quad & L_{N,d}(E) \cup T_{N,i} \neq [0,N] \text{ and } |L_{N,d}(E) \setminus T_{N,i}| \leqslant d-i \\
\implies\quad & E \cup T_{N,i} \subseteq L_{N,d}(E) \cup T_{N,i} \neq [0,N] \text{ and } |E \setminus T_{N,i}| \leqslant |L_{N,d}(E) \setminus T_{N,i}| \leqslant d-i \\
\implies\quad & E \text{ is } (d,i)\text{-admitting.}
\end{aligned}$$

Conversely suppose E is (d,i)-admitting. If $|E| \leqslant d$, then $L_{N,d}(E) = E$ and we are done. So now assume $|E| \geqslant d+1$. We have $E \cup T_{N,i} \neq [0,N]$ and $|E \setminus T_{N,i}| \leqslant d-i$. This implies

$$\begin{aligned}
&|E \cap [0,N-i]| \leqslant |[0,i-1] \cup (E \setminus T_{N,i})| \leqslant d, &(3)\\
\text{and}\quad &|E \cap [i,N]| \leqslant |[N-i+1,N] \cup (E \setminus T_{N,i})| \leqslant d. &(4)
\end{aligned}$$

Enumerate $E = \{t_1 < \cdots < t_s\}$. Then $L_{N,d}(E) = [0,t_{s-d}] \cup E \cup [t_{d+1,N}]$, since $|E| \geqslant d+1$. Further, Inequality (3) implies $t_{d+1} \geqslant N-i+1$, and Inequality (4) implies $t_{s-d} \leqslant i-1$. So we get

(a) $L_{N,d}(E) \subseteq [0,i-1] \cup E \cup [N-i+1] = E \cup T_{N,i} \neq [0,N]$, and

(b) $L_{N,d}(E) \cap [i,N-i] = E \cap [i,N-i]$, which gives $|L_{N,d}(E) \setminus T_{N,i}| = |E \setminus T_{N,i}| \leqslant d-i$.

Thus $L_{N,d}(E)$ is (d,i)-admitting. $\square$

We are now ready to prove Proposition 8.

Proof of Proposition 8. (a) We need to prove that $\overline{L}_{N,d}(E) \neq [0,N]$ if and only if E is d-admitting.

Suppose $\overline{L}_{N,d}(E) \neq [0,N]$. Let

$$i_0 = \max\{i \in [0, \lfloor N/2 \rfloor] : T_{N,i} \subseteq \overline{L}_{N,d}(E)\}.$$

If $i_0 \geqslant d+1$, then there exists $k \in \mathbb{Z}^+$ such that $[0,d] \subseteq [0,i_0-1] \subseteq L^k_{N,d}(E)$. Then by definition of $L_{N,d}$ and Lemma 22, we have $L^{k+1}_{N,d}(E) = [0,N]$, thus implying

$\overline{L}_{N,d}(E) = [0,N]$, a contradiction. So we have $i_0 \leqslant d$. Also, without loss of generality, let $i_0 \notin \overline{L}_{N,d}(E)$. Clearly $E \cup T_{N,i_0} \subseteq \overline{L}_{N,d}(E) \neq [0,N]$.

Now suppose $|E \setminus T_{N,i_0}| \geqslant d - i_0 + 1$, then $|E \cap [i_0, N]| \geqslant d+1$. Let $m = \min(E \cap [i_0,N]) \geqslant i_0$. By Observation 23, we get $[0,m] \subseteq \overline{L}_{N,d}(E)$; in particular, we get $i_0 \in \overline{L}_{N,d}(E)$, which is a contradiction. So $|E \setminus T_{N,i_0}| \leqslant d - i_0$. Thus E is (d,i_0)-admitting, and hence E is d-admitting.

Conversely, suppose E is d-admitting. Let $k_0 \in \mathbb{Z}^+$ be the least such that $\overline{L}_{N,d}(E) = L_{N,d}^{k_0}(E)$. So applying Lemma 27 precisely k_0 times, we conclude that $\overline{L}_{N,d}(E)$ is d-admitting. So there exists $i \in [0,d]$ such that $\overline{L}_{N,d}(E) \cup T_{N,i} \neq [0,N]$, which implies $\overline{L}_{N,d}(E) \neq [0,N]$.

(b) We need to prove that $\overline{L}_{N,d}(E) = E$ if and only if $T_{N,|E|-d} \subseteq E$. Firstly, note that by definition of $\overline{L}_{N,d}$, we have $\overline{L}_{N,d}(E) = E$ if and only if $L_{N,d}(E) = E$. Secondly, note that the assertion is vacuously true if $|E| \leqslant d$. So now assume $|E| \geqslant d+1$. Let $E = \{t_1 < \cdots < t_s\}$.

Now suppose $L_{N,d}(E) = E$. Let $E' = \{t_1, \ldots, t_{d+1}\}$. By definition of $L_{N,d}$, we have $[0,t_1] \cup E' \cup [t_{d+1}, N] \subseteq L_{N,d}(E') \subseteq \overline{L}_{N,d}(E) = E$. So $\{t_{d+1}, \ldots, t_s\} = [t_{d+1}, N]$, which implies $N - t_{d+1} + 1 = s - d$, that is, $t_{d+1} = N - s + d + 1 = N - (|E| - d) + 1$. Thus $[N - (|E| - d) + 1, N] \subseteq E$. Further, let $E'' = \{t_{s-d}, \ldots, t_s\}$. Again, by definition of $L_{N,d}$, we have $[0, t_{s-d}] \cup E'' \cup [t_s, N] \subseteq L_{N,d}(E'') \subseteq \overline{L}_{N,d}(E) = E$. So $\{t_1, \ldots, t_{s-d}\} = [0, t_{s-d}]$, which implies $t_{s-d} + 1 = s - d$, that is, $t_{s-d} = s - d - 1 = (|E| - d) - 1$. Thus $[0, (|E| - d) - 1] \subseteq E$. Hence $T_{N,|E|-d} \subseteq E$.

Conversely, suppose $T_{N,|E|-d} \subseteq E$. So $[0, (|E|-d)-1] = \{t_1, \ldots, t_{s-d}\}$ and $[N - (|E| - d) + 1, N] = \{t_{d+1}, \ldots, t_s\}$, that is, $t_{s-d} = (|E| - d) - 1$ and $t_{d+1} = N - (|E| - d) + 1$. Hence $L_{N,d}(E) = [0, t_{s-d}] \cup E \cup [t_{d+1}, N] = E$. $\square$

We have thus proved Theorem 4, which is our solution to Problem 2 (b) for SU^2 grids.

4 Finite-degree h-closures and our hyperplane covering problems

We introduce another new closure operator, defined using polynomials representing hyperplane covers. Let

$$\mathscr{H}_n = \left\{ \prod_{i=1}^{k} \ell_i(\mathbb{X}) : k \in \mathbb{N};\ \ell_i(\mathbb{X}) \in \mathbb{R}[\mathbb{X}] \text{ and } \deg \ell_i \leqslant 1, \text{ for all } i \in [k] \right\}.$$

Let G be a uniform grid. For any $d \in [0,N]$ and $S \subseteq G$, we define the degree-d h-closure of S as $\text{h-cl}_{G,d}(S) = \mathcal{Z}(\mathcal{I}(\mathscr{H}_n, S)_d)$. We will focus on the case of the Boolean cube. It is immediate that the finite-degree h-closure of a symmetric set is symmetric, and so we will use our indentification of symmetric sets of $\{0,1\}^n$ with subsets of $[0,n]$.

Our main result in this section is a characterization of finite-degree h-closures of all symmetric sets of the Boolean cube. In fact, we prove that these coincide with the finite-degree Z-closures (and Z*-closures). Let us first show that Lemma 22 and Lemma 24 have analogues for finite-degree hyperplane closures.

Lemma 28. *(a)* (Closure Builder Lemma) *Let $d \in [0,n]$ and $E \subseteq [0,n]$ with $|E| \geqslant d+1$. Then*

$$[0, \min E] \cup [\max E, n] \subseteq \text{h-cl}_{n,d}(E).$$

(b) For every $i \in [0,N]$,

$$\text{h-cl}_{n,d}(T_{n,i}) = \begin{cases} T_{n,i} & \text{if } i \leqslant d, \\ [0,n] & \text{if } i > d. \end{cases}$$

Proof. (a) The proof is similar to that of Lemma 22; instead of considering polynomials in $\mathbb{R}[\mathbb{X}]$, we just need to consider polynomials in $\mathscr{H}_n$ throughout.

(b) We only need to consider $i \leqslant \min\{d, \lfloor n/2 \rfloor\}$. The other case can be argued exactly as in the proof of Lemma 24. Now consider the polynomial

$$P(\mathbb{X}) = (X_1 - X_2) \cdots (X_{2i-1} - X_{2i}).$$

Clearly $\deg P = i \leqslant d$. For any $x \in \underline{j}$, where $j \in [0, i-1]$, there exists $t \in [i]$ such that $x_{2t-1} = x_{2t} = 0$; this gives $P(x) = 0$. For any $x \in \underline{j}$, where $j \in [N-i+1, N]$, there exists $t \in [i]$ such that $x_{2t-1} = x_{2t} = 1$; this gives $P(x) = 0$. So $P|_{\underline{T_{n,i}}} = 0$. Now consider any $j \in [i, n-i]$. Let $x^{(j)} = (10)^i 1^{j-i} 0^{n-j-i} \in \underline{j}$. Then we have $P(x^{(j)}) = 1$. This implies $j \notin \text{h-cl}_{n,d}(T_{n,i})$. Hence $\text{h-cl}_{n,d}(T_{n,i}) = T_{n,i}$. ☐

We can now prove Theorem 10.

Proof of Theorem 10. The proof is similar to that of Theorem 7. Instead of considering polynomials in $\mathbb{R}[\mathbb{X}]$, we need to consider polynomials in $\mathscr{H}_n$ throughout. In addition, we need to replace Lemma 22 and Lemma 24 with Lemma 28 (a) and (b) respectively, throughout. ☐

By Observation 9 (a), Theorem 10 and Proposition 8 (a), we have proved Theorem 5 (a).

Observation 29. *From the proof of Lemma 28 (b), showing* $\text{h-cl}_{n,d}(T_{n,i}) = T_{n,i}$ *for $i \leqslant d$, we can infer the stronger statement: For $i, d \in [0,n]$, $i \leqslant d$, there exists $P(\mathbb{X}) \in \mathscr{H}_n$ such that $P|_{\underline{T_{n,i}}} = 0$ and $P|_{\underline{j}} \neq 0$, for every $j \in [i, n-i]$.*

By Observation 9 (b), Theorem 10, Proposition 8 (b) and Observation 29, we have proved Theorem 5 (b). This completes our solution to Problem 2 (a) in the Boolean cube setting.

5 Other applications

Our combinatorial characterization of finite-degree Z*-closures of weight-determined sets in SU^2 grids (Theorem 7) may also be interesting in its own right. Indeed, we will consider two other applications in this section.

5.1 An alternate proof of a lemma by Alon et al. (1988)

Consider the following simple fact; the proof is obvious.

Fact 30. *Let G be a uniform grid, $\theta : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible affine linear transformation and $\mathcal{G} = \theta(G)$. Note that θ induces an obvious invertible affine linear transformation $\theta : \mathbb{R}[\mathbb{X}] \to \mathbb{R}[\mathbb{X}]$ (by abuse of notation).*

(a) $\mathcal{Z}_{\mathcal{G}}(I) = \theta(\mathcal{Z}_G(\theta^{-1}(I)))$, for every $I \subseteq \mathbb{R}[\mathbb{X}]$.

(b) Note that for any $d \in [0, N]$, we have $\theta(\mathbb{R}[\mathbb{X}]_d) = \mathbb{R}[\mathbb{X}]_d$. So by Item (a),

$$\mathcal{Z}_{\mathcal{G}}(\mathbb{R}[\mathbb{X}]_d) = \theta(\mathcal{Z}_G(\mathbb{R}[\mathbb{X}]_d)), \quad \textit{for every } d \in [0, N].$$

Now suppose we represent the Boolean cube as $\{-1,1\}^n$. In this case, we define the Hamming weight of $x \in \{-1,1\}^n$ as $|x| = |\{i \in [n] : x_i = -1\}|$. By Fact 30, we can therefore *port* all our results to the setting of the Boolean cube $\{-1,1\}^n$ by considering $\theta : \{0,1\}^n \to \{-1,1\}^n$ defined as $\theta(x) = 1^n - 2x$, $x \in \{0,1\}^n$.

Appealing to Fact 30, Lemma 14 states that $\text{Z*-cl}_{n,n/2-1}(E_0) = \text{Z*-cl}_{n,n/2-1}(E_1) = [0, n]$. This is equivalent to Proposition 15, by Theorem 7. In fact, we can prove the following slightly more general statement; Proposition 15 is a special case.

Proposition 31. *Let G be a uniform grid and $m \in [N]$. Let $E_{m,i} = \{j \in [0,N] : j \equiv i \pmod{m}\}$, $i \in [0, m-1]$. Then $\overline{L}_{N,\lfloor N/m \rfloor -1}(E_{m,i}) = [0, N]$, for all $i \in [0, m-1]$.*

For simplicity, let us just prove Proposition 31 for the case of N being an even positive integer, and $m = 2$. The general case can be proven along similar lines.

Proof of Proposition 31 (when N is even and $m = 2$). Let $N = 2k$, $k \in \mathbb{Z}^+$. Since $m = 2$, we need to prove that $\overline{L}_{2k,k-1}(E_{2,0}) = \overline{L}_{2k,k-1}(E_{2,1}) = [0, 2k]$. Let us prove that $\overline{L}_{2k,k-1}(E_{2,0}) = [0, 2k]$; the other claim can be proved in an analogous way.

Our argument is an illustration of Algorithm 1, by using Proposition 26. Recall the set operators $L_{[a,b],d} : 2^{[a,b]} \to 2^{[a,b]}$ and $\overline{L}_{[a,b],d} : 2^{[a,b]} \to 2^{[a,b]}$, for $a, b \in \mathbb{Z}$, $a \leqslant b$ and $d \in [0, b-a]$, defined in Subsection 3.3. So $\overline{L}_{2k,k-1}(E_{2,0}) = \overline{L}_{[0,2k],k-1}(E_{2,0})$. For $i \in [k]$, let $F_i = E_{2,0} \cap [k-i, k+i]$. It is easy to see that for each $i \in [k]$, we have

$$|F_i| = \begin{cases} i & \text{if } i \text{ is odd,} \\ i+1 & \text{if } i \text{ is even.} \end{cases}$$

We will prove, by induction, that $\overline{L}_{[k-i,k+i],i-1}(F_i) = [k-i, k+i]$, for all $i \in [k]$. By Proposition 26 (a), we get the base case as $\overline{L}_{[k-1,k+1],0}(F_1) = [k-1, k+1]$. Now assume $\overline{L}_{[k-i,k+i],i-1}(F_i) = [k-i, k+1]$, for some $i \in [k-1]$. Note that we have $F_i = F_{i+1} \setminus \{k-i-1, k+i+1\}$. So by Proposition 26 (c) and the induction hypothesis, we get

$$\overline{L}_{[k-i-1,k+i+1],i}(F_{i+1}) = \{k-i-1, k+i+1\} \cup \overline{L}_{[k-i,k+i],i-1}(F_i) = [k-i-1, k+i+1].$$

This completes the proof. $\square$

5.2 Certifying degrees of weight-determined sets

Recall that for a uniform grid G and subset $S \subseteq G$, the certifying degree cert-deg(S) is defined to be the smallest $d \in [0, N]$ such that S has a certifying polynomial with degree at most d. By this definition, we observe that

$$\text{cert-deg}(S) = \min\{d \in [0, N] : \text{Z-cl}_{G,d}(S) \neq G\}.$$

Thus for any weight-determined set $\underline{E}$, $E \subsetneq [0, N]$, we get

$$\text{cert-deg}(\underline{E}) = \min\{d \in [0, N] : \text{Z-cl}_{G,d}(\underline{E}) \neq G\} = \min\{d \in [0, N] : \text{Z*-cl}_{G,d}(E) \neq [0, N]\},$$

since $\text{Z-cl}_{G,d}(\underline{E}) = G$ if and only if $\text{Z*-cl}_{G,d}(E) = [0, N]$. Consider any symmetric subset $E \subsetneq [0, n]$. It then follows immediately from Lemma 6, Theorem 7 and Proposition 8 that if G is an SU^2 grid, then for any $E \subsetneq [0, N]$,

$$\text{cert-deg}(\underline{E}) = \mathsf{PC}_G(E) = \min\{d \in [0, N] : E \text{ is } d\text{-admitting}\}.$$

This proves Theorem 16.

6 A third variant: the exact covering problem

Our third covering problem is quite an intuitive variant of the hyperplane and polynomial covering problems, given the *nontrivial* and *proper* covering versions that we have considered so far. However, we have more questions than answers about this third variant. Let G be a uniform grid. Consider a weight-determined set $\underline{E}$, where $E \subsetneq [0, N]$. We say

- a family of hyperplanes $\mathcal{H}$ in $\mathbb{R}^n$ is an exact hyperplane cover of $\underline{E}$ if $\underline{E} = \left(\bigcup_{H \in \mathcal{H}} H\right) \cap G$.
- a polynomial $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ is an exact polynomial cover of $\underline{E}$ if $\underline{E} = \mathcal{Z}_G(P)$.

Let $\mathsf{EHC}_G(E)$ and $\mathsf{EPC}_G(E)$ denote the minimum size of an exact hyperplane cover and the minimum degree of an exact polynomial cover respectively, for a weight-determined set $\underline{E}$, $E \subsetneq [0, N]$. In the case $G = \{0, 1\}^n$, we will instead use the notations $\mathsf{EHC}_n(E)$ and $\mathsf{EPC}_n(E)$.

We first note that EPC_G can be characterized in terms of the finite-degree Z-closures. Contrast this with Lemma 6 which characterizes PPC_G in terms of the finite-degree Z*-closures.

Proposition 32. *Let G be a uniform grid. For any $E \subsetneq [0,N]$,*

$$\mathsf{EPC}_G(E) = \min\{d \in [0,N] : \text{Z-cl}_{G,d}(\underline{E}) = \underline{E}\}.$$

Proof of Proposition 32. Let $d' = \mathsf{EPC}_G(E)$ and $d'' = \min\{d \in [0,N] : \text{Z-cl}_{G,d}(\underline{E}) = \underline{E}\}$. There exists a polynomial $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ such that $\deg P = d'$, $P|_{\underline{E}} = 0$ and $P(a) \neq 0$, for all $a \in G \setminus \underline{E}$. This implies $\text{Z-cl}_{G,d'}(\underline{E}) = \underline{E}$, and so $d'' \leqslant d'$.

Further, for every $a \in G \setminus \underline{E}$, there exists $Q_a(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ such that $\deg Q_a \leqslant d''$, $Q_a|_{\underline{E}} = 0$ and $Q_a(a) = 1$. We can then choose scalars $\beta_a \in \mathbb{R}$, $a \in G \setminus \underline{E}$ such that the polynomial $Q(\mathbb{X}) \coloneqq \sum_{a \in G \setminus \underline{E}} \beta_a P_a(\mathbb{X})$ satisfies $\deg P \leqslant d''$, $P|_{\underline{E}} = 0$ and $P(a) \neq 0$, for all $a \in G \setminus \underline{E}$. So $d' \leqslant d''$, and this completes the proof. $\square$

However, we do not have a further characterization of the finite-degree Z-closures of weight-determined sets. In the Boolean cube setting, however, since the finite-degree Z-closures and Z*-closures coincide for all symmetric sets, we immediately get the following by appealing to Theorem 4, Theorem 5, Theorem 7 and Theorem 10.

Corollary 33. *Consider the Boolean cube $\{0,1\}^n$. For any $E \subsetneq [0,n]$,*

$$\mathsf{EPC}_n(E) = \mathsf{PPC}_n(E) = \mathsf{PHC}_n(E) = |E| - \max\{i \in [0,n] : T_{n,i} \subseteq E\}.$$

Further, characterizing $\mathsf{EHC}_n(E)$ for $E \subsetneq [0,n]$ seems to be even more difficult. We have the following partial results.

Proposition 34. *Consider the Boolean cube $\{0,1\}^n$, and any $E \subsetneq [0,n]$.*

(a) If $T_{n,1} \nsubseteq E$, then $\mathsf{EHC}_n(E) = |E|$.

(b) If $n \geqslant 2$ and $T_{n,1} \subseteq E$, $T_{n,2} \nsubseteq E$, then $\mathsf{EHC}_n(E) = |E| - 1$.

(c) If $n \geqslant 4$, then $\mathsf{EHC}_n(T_{n,2}) = 2$.

Proof. (a) Clearly $\mathsf{EHC}_n(E) \geqslant \mathsf{PPC}_n(E) = |E| - \max\{i \in [0,n] : T_{n,i} \subseteq E\} = |E|$, since $T_{n,1} \nsubseteq E$. Further, the hyperplane cover $\{H_t : t \in E\}$, where $H_t(\mathbb{X}) \coloneqq \sum_{i \in [n]} X_i - t$, $t \in E$, is an exact cover of $\underline{E}$ having size $|E|$.

(b) Again clearly $\mathsf{EHC}_n(E) \geqslant \mathsf{PPC}_n(E) = |E| - \max\{i \in [0,n] : T_{n,i} \subseteq E\} = |E| - 1$, since $T_{n,1} \subseteq E$, $T_{n,2} \nsubseteq E$. Further, we can choose scalars $a_i \in \mathbb{R}$, $i \in [n]$ such that $H_0(\mathbb{X}) \coloneqq \sum_{i \in [n]} a_i X_i$ satisfies $\mathcal{Z}_G(H_0) = \{0^n, 1^n\}$. So the hyperplane cover $\{H_0\} \cup \{H_t : t \in E \setminus T_{n,1}\}$, where $H_t(\mathbb{X}) \coloneqq \sum_{i \in [n]} X_i - t$, $t \in E \setminus T_{n,1}$, is an exact cover of $\underline{E}$ having size $|E| - 1$.

(c) Obviously $\mathsf{EHC}_n(T_{n,2}) \geqslant \mathsf{PPC}_n(T_{n,2}) = 2$. Consider the hyperplane cover $\{H_0, H_1\}$, where

$$H_0(\mathbb{X}) \coloneqq \; -(n-2)X_2 + \sum_{i \in [3,n]} X_i \quad \text{and} \quad H_1(\mathbb{X}) \coloneqq \; -(n-3)X_1 + \sum_{i \in [2,n]} X_i - 1.$$

It is clear that $\{0^n, 1^n, 10^{n-1}, 01^{n-1}\} \subseteq \mathcal{Z}(H_0)$, and $\{0^j10^{n-j-1}, 1^j01^{n-j-1}\} \subseteq \mathcal{Z}(H_1)$, for every $j \in [2, n]$. Let $x \in \underline{[2, n-2]}$. If $x_2 = 0$ then $H_0(x) = \sum_{i\in[3,n]} x_i \geqslant 1$, and if $x_2 = 1$ then $H_0(x) = -(n-2) + \sum_{i\in[3,n]} x_i \leqslant -1$. Further, if $x_1 = 0$ then $H_1(x) = \sum_{i\in[2,n]} x_i - 1 \geqslant 1$, and if $x_1 = 1$ then $H_1(x) = -(n-3) + \sum_{i\in[2,n]} x_i - 1 \leqslant -1$. Thus $\{H_0, H_1\}$ is an exact hyperplane cover of $\underline{T_{n,2}}$, with size 2. This implies that $\mathsf{EHC}_n(T_{n,2}) = 2 = |T_{n,2}| - 2$. □

Now assume $n \geqslant 4$, and let $E \subsetneq [0, n]$ such that $T_{n,2} \subseteq E$. Consider $\{H_0, H_1\}$, the exact hyperplane cover of $\underline{T_{n,2}}$, as given in the proof of Proposition 34 (c). Then $\{H_0, H_1\} \cup \{H_j : j \in E \setminus T_{n,2}\}$ is an exact hyperplane cover for $\underline{E}$, with size $|E| - 2$, where $H_j(\mathbb{X}) := \sum_{i\in[n]} X_i - j$, for all $j \in E \setminus T_{n,2}$. So $\mathsf{EHC}_n(E) \leqslant |E| - 2$. We conjecture that this bound is tight, for every $E \subsetneq [0, n]$ such that $T_{n,2} \subseteq E$.

Conjecture 35. For $n \geqslant 4$, consider the Boolean cube $\{0, 1\}^n$. If $E \subsetneq [0, n]$ such that $T_{n,2} \subseteq E$, then $\mathsf{EHC}_n(E) = |E| - 2$.

Finally, we propose the following open question.

Open Problem 36. For a uniform (or SU^2) grid $G \neq \{0, 1\}^n$, determine $\mathsf{EHC}_G(E)$ and $\mathsf{EPC}_n(E)$, for all $E \subsetneq [0, N]$.

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References

[ABCO88] Noga Alon, Ernest E. Bergmann, Don Coppersmith, and Andrew M. Odlyzko. Balancing sets of vectors. *IEEE Transactions on Information Theory*, 34(1):128–130, 1988. https://doi.org/10.1109/18.2610.

[ABFR93] James Aspnes, Richard Beigel, Merrick Furst, and Steven Rudich. The expressive power of voting polynomials. In *Combinatorica*, pages 402–409, 1993. https://doi.org/10.1007/BF01215346.

[AF93] Noga Alon and Zoltán Füredi. Covering the Cube by Affine Hyperplanes. *European Journal of Combinatorics*, 14(2):79 – 83, 1993. https://doi.org/10.1006/eujc.1993.1011.

[AGG+20] James Aaronson, Carla Groenland, Andrzej Grzesik, Bartłomiej Kielak, and Tom Johnston. Exact hyperplane covers for subsets of the hypercube. arXiv:2010.00315, 2020.

[Alo99] Noga Alon. Combinatorial Nullstellensatz. *Combinatorics, Probability and Computing*, 8(1-2):7–29, 1999. https://doi.org/10.1017/S0963548398003411.

[AR01] Michael Alekhnovich and Alexander A. Razborov. Lower bounds for polynomial calculus: non-binomial case. In *Proceedings 42nd IEEE Symposium on Foundations of Computer Science*, pages 190–199, 2001. https://doi.org/10.1109/SFCS.2001.959893.

[Bal00] Simeon Ball. On intersection sets in Desarguesian affine spaces. *European Journal of Combinatorics*, 21(4):441–446, 2000. https://doi.org/10.1006/eujc.2000.0350.

[BBDM21] Anurag Bishnoi, Simona Boyadzhiyska, Shagnik Das, and Tamás Mészáros. Subspace coverings with multiplicities. arXiv:2101.11947, 2021.

[BBS10] Aart Blokhuis, Andries E. Brouwer, and Tamás Szőnyi. Covering all points except one. *Journal of Algebraic Combinatorics*, 32(1):59–66, 2010. https://doi.org/10.1007/s10801-009-0204-1.

[BE99] Anna Bernasconi and Lavinia Egidi. Hilbert function and complexity lower bounds for symmetric Boolean functions. *Inf. Comput.*, 153(1):1–25, August 1999. https://doi.org/10.1006/inco.1999.2798.

[BHL12] Ido Ben-Eliezer, Rani Hod, and Shachar Lovett. Random low-degree polynomials are hard to approximate. *Comput. Complex.*, 21(1):63–81, 2012. https://doi.org/10.1007/s00037-011-0020-6.

[Bir73] Garrett Birkhoff. *Lattice theory*, volume 25. American Mathematical Soc., 1973.

[BS78] Andries E. Brouwer and Alexander Schrijver. The blocking number of an affine space. *Journal of Combinatorial Theory, Series A*, 24(2):251–253, 1978. https://doi.org/10.1016/0097-3165(78)90013-4.

[BS09] Simeon Ball and Oriol Serra. Punctured Combinatorial Nullstellensätze. *Combinatorica*, 29(5):511–522, 2009. https://doi.org/10.1007/s00493-009-2509-z.

[CDGM06] Claude Carlet, Deepak K. Dalai, Kishan C. Gupta, and Subhamoy Maitra. Algebraic immunity for cryptographically significant boolean functions: analysis and construction. *IEEE Transactions on Information Theory*, 52(7):3105–3121, 2006. https://doi.org/10.1109/TIT.2006.876253.

[CH20] Alexander Clifton and Hao Huang. On almost k-covers of hypercubes. *Combinatorica*, 40(4):511–526, 2020. https://doi.org/10.1007/s00493-019-4221-y.

[CLO15] David A. Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms*. Springer, 2015. https://doi.org/10.1007/978-3-319-16721-3.

[Dha14] Vivek Dhand. A combinatorial proof of strict unimodality for q-binomial coefficients. *Discrete Mathematics*, 335:20–24, 2014. https://doi.org/10.1016/j.disc.2014.07.001.

[Gre00] Frederic Green. Complex fourier technique for lower bounds on the mod-m degree. *Computational Complexity*, 9:16–38, 2000. https://doi.org/10.1007/PL00001599.

[HP98] Petra Heijnen and Ruud Pellikaan. Generalized Hamming weights of q-ary Reed-Muller codes. *IEEE Transactions on Information Theory*, 44(1):181–196, 1998. https://doi.org/10.1109/18.651015.

[Jam77] Robert E. Jamison. Covering finite fields with cosets of subspaces. *Journal of Combinatorial Theory, Series A*, 22(3):253–266, 1977. https://doi.org/10.1016/0097-3165(77)90001-2.

[Kom94] Péter Komjáth. Partitions of vector spaces. *Periodica Mathematica Hungarica*, 28(3):187–193, 1994. https://doi.org/10.1007/bf01876338.

[KR12] Géza Kós and Lajos Rónyai. Alon's Nullstellensatz for multisets. *Combinatorica*, 32(5):589–605, 2012. https://doi.org/10.1007/s00493-012-2758-0.

[KS05] Peter Keevash and Benny Sudakov. Set systems with restricted cross-intersections and the minimum rank of inclusion matrices. *SIAM Journal on Discrete Mathematics*, 18(4):713–727, 2005. https://doi.org/10.1137/S0895480103434634.

[KS18] Swastik Kopparty and Srikanth Srinivasan. Certifying polynomials for $AC^0[\oplus]$ circuits, with applications to lower bounds and circuit compression. *Theory of Computing*, 14(1):1–24, 2018. http://doi.org/10.4086/toc.2018.v014a012.

[LR05] Nathan Linial and Jaikumar Radhakrishnan. Essential covers of the cube by hyperplanes. *Journal of Combinatorial Theory, Series A*, 109(2):331–338, 2005. https://doi.org/10.1016/j.jcta.2004.07.012.

[NW15] Zipei Nie and Anthony Y. Wang. Hilbert functions and the finite degree Zariski closure in finite field combinatorial geometry. *Journal of Combinatorial Theory, Series A*, 134:196–220, 2015. https://doi.org/10.1016/j.jcta.2015.03.011.

[Sax13] David Saxton. Essential positive covers of the cube. *Journal of Combinatorial Theory, Series A*, 120(5):971–975, 2013. https://doi.org/10.1016/j.jcta.2013.02.002.

[SW20] Lisa Sauermann and Yuval Wigderson. Polynomials that vanish to high order on most of the hypercube. arXiv:2010.00077, 2020.

[Wei91] Victor K. Wei. Generalized Hamming weights for linear codes. *IEEE Transactions on Information Theory*, 37(5):1412–1418, 1991. https://doi.org/10.1109/18.133259.

[YY21] Gal Yehuda and Amir Yehudayoff. A lower bound for essential covers of the cube. arXiv:2105.13615, 2021.

[Zan02] Corrado Zanella. Intersection sets in $\mathrm{AG}(n, q)$ and a characterization of the hyperbolic quadric in $\mathrm{PG}(3, q)$. *Discrete Mathematics*, 255(1):381–386, 2002. https://doi.org/10.1016/S0012-365X(01)00413-7.