Size and structure of large (s,t)-union intersecting families

Ali Taherkhani

Department of Mathematics
Institute for Advanced Studies in Basic Sciences (IASBS)
Zanjan 45137-66731, Iran

and

School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran, Iran

ali.taherkhani@iasbs.ac.ir

Submitted: Jun 11, 2021; Accepted: Mar 5, 2022; Published: May 6, 2022

Abstract

A family $\mathcal{F}$ of k-sets on an n-set X is said to be an (s,t)-union intersecting family if for any $A_1,\ldots,A_{s+t}$ in this family, we have $(\cup_{i=1}^{s} A_i) \cap \left(\cup_{i=1}^{t} A_{i+s}\right) \neq \varnothing$. The celebrated Erdős-Ko-Rado theorem determines the size and structure of the largest intersecting (or $(1,1)$-union intersecting) family. Also, the Hilton-Milner theorem determines the size and structure of the second largest $(1,1)$-union intersecting family of k-sets. In this paper, for $t \geqslant s \geqslant 1$ and sufficiently large n, we find out the size and structure of some large and maximal (s,t)-union intersecting families. Our results are nontrivial extensions of some recent generalizations of the Erdős-Ko-Rado theorem such as the Han and Kohayakawa theorem [Proc. Amer. Math. Soc. 145 (2017), pp. 73–87] which finds the structure of the third largest intersecting family, the Kostochka and Mubayi theorem [Proc. Amer. Math. Soc. 145 (2017), pp. 2311–2321], and the more recent Kupavskii's theorem [arXiv:1810.009202018 (2018)] whose both results determine the size and structure of the ith largest intersecting family of k-sets for $i \leqslant k+1$. In particular, when $s=1$, we confirm a conjecture of Alishahi and Taherkhani [J. Combin. Theory Ser. A 159 (2018), pp. 269–282]. As another consequence, our result provides some stability results related to the famous Erdős matching conjecture.

Mathematics Subject Classifications: 05D05

1 Introduction and Main Results

1.1 Erdős-Ko-Rado theorem and its generalization

Let n and k be two positive integers such that $n \geqslant k$. The symbol $[n]$ stands for the set $\{1, \dots, n\}$ and the symbol $[k, n]$ stands for the set $[n] \setminus [k-1]$. The family of all k-element subsets (or k-sets) of $[n]$ is denoted by $\binom{[n]}{k}$. In this paper, we only consider families which consist of k-sets on $[n]$. A family $\mathcal{F}$ is said to be *intersecting* if the intersection of every two members of $\mathcal{F}$ is non-empty. If all members of $\mathcal{F}$ contain a fixed element of $[n]$, then it is clear that $\mathcal{F}$ is an intersecting family which is called a *star* or a *trivial* family. For each $i \in [n]$, the family $\mathcal{S}_i \stackrel{\text{def}}{=} \{A \in \binom{[n]}{k} | i \in A\}$ is a maximal star. Also, the following two families are well-known examples of intersecting families. Let B be a k-set of $[n]$ such that $1 \notin B$. Define

$$\mathcal{HM} \stackrel{\text{def}}{=} \{A | \, 1 \in A, \; A \cap B \neq \varnothing\} \cup \{B\}$$

and

$$\mathcal{HM}' \stackrel{\text{def}}{=} \{A | \, |A \cap \{1,2,3\}| \geqslant 2\}.$$

Note that for $2 \leqslant k \leqslant 3$, we have $|\,\mathcal{HM}\,| = |\,\mathcal{HM}'\,|$ and if $n > 2k$ and $k \geqslant 4$, then $|\,\mathcal{HM}\,| > |\,\mathcal{HM}'\,|$.

The well-known Erdős-Ko-Rado theorem [9] states that every intersecting family of $\binom{[n]}{k}$ has cardinality at most $\binom{n-1}{k-1}$ provided that $n \geqslant 2k$; moreover, if $n > 2k$, then the only intersecting families of this cardinality are maximal stars.

As a generalization of the Erdős-Ko-Rado theorem, Hilton and Milner [24] proved a useful and interesting stability result. They showed that for $n > 2k$ the maximum possible size of a nontrivial intersecting family $\mathcal{F}$ of $\binom{[n]}{k}$ is $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. Furthermore, equality is possible only for a family $\mathcal{F}$ which is isomorphic to $\mathcal{HM}$ or $\mathcal{HM}'$, the latter can hold only for $k \leqslant 3$.

A family $\mathcal{F}$ is called a *Hilton-Milner family* if $\mathcal{F}$ is isomorphic to a subfamily of $\mathcal{HM}$ for some k or it is isomorphic to a subfamily of $\mathcal{HM}'$ for $k \in \{2, 3\}$.

There also exist some other interesting extensions of Erdős-Ko-Rado and Hilton-Milner theorems in the literature (e.g. [1, 2, 5, 6, 12, 14, 15, 17, 19–23, 27, 29, 31–33, 35]).

Beyond the Hilton-Milner theorem, it was shown by Hilton and Milner [24] that the maximum size of a nontrivial intersecting family which is not a Hilton-Milner family is at most $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2} + 2$. In fact they proved the following interesting result (see [23, 24]).

Theorem A. [24] *Let n, k, and s be positive integers with $\min\{3, s\} \leqslant k \leqslant \frac{n}{2}$ and let $\mathcal{F} = \{A_1, \dots, A_m\}$ be an intersecting family of k-sets on $[n]$. If for any $S \subset [m]$ with $|S| > m - s$, we have $\cap_{i \in S} A_i = \varnothing$, then*

$$m \leqslant \begin{cases} \binom{n-1}{k-1} - \binom{n-k}{k-1} + n - k & \text{if } 2 < k \leqslant s+2, \\ \\ \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-s}{k-s-1} + s & \text{if } k \leqslant 2 \text{ or } k \geqslant s+2. \end{cases} \qquad (1)$$

Moreover, the bounds in Inequality (1) are the best possible.

Recently, Han and Kohayakawa [23] gave a different and simpler proof of Theorem A. Moreover, they characterized all extremal families achieving the bounds in (1). In this regard they introduced the following construction.

Definition 1. Let i be a nonnegative integer. For any $(i+1)$-set $J \subset [n]$ with $1 \in J$ and any $(k-1)$-set $E \subset [n] \setminus J$, define the family $\mathcal{J}_i$ as follows,

$$\mathcal{J}_i \stackrel{\text{def}}{=} \{A : E \subset A,\ A \cap J \neq \varnothing\} \cup \{A : J \subset A\} \cup \{A : 1 \in A,\ A \cap E \neq \varnothing\}.$$

Note that $\mathcal{J}_0 = \mathcal{S}_1$, $\mathcal{J}_1 = \mathcal{HM}$, $|\,\mathcal{J}_i\,| = \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-i}{k-i-1} + i$, and $|\,\mathcal{J}_i \setminus \mathcal{S}_1\,| = i$.

Theorem B. [23] *Let n, k be positive integers with $3 \leqslant k < \frac{n}{2}$ and let $\mathcal{F}$ be an intersecting family of k-sets on $[n]$. Assume that $\mathcal{F}$ is neither a star nor a Hilton-Milner family. Then $|\,\mathcal{F}\,| \leqslant |\,\mathcal{J}_2\,|$. Moreover, for $k \geqslant 5$, equality holds if and only if $\mathcal{F}$ is isomorphic to $\mathcal{J}_2$.*

Definition 2. For $i \leqslant k$ let us define the family $\mathcal{F}_i$ of $\binom{[n]}{k}$ as follows,

$$\mathcal{F}_i \stackrel{\text{def}}{=} [2, k+1] \cup [i+1, k+i] \cup \{A : 1 \in A, A \cap [2, k+1] \neq \varnothing, A \cap [i+1, k+i] \neq \varnothing\}.$$

In [29], Kostochka and Mubayi proved that the size of an intersecting family which is neither a star nor is contained in $\mathcal{J}_i$, for $i \in \{1, \ldots, k-1, n-k\}$, is at most $|\,\mathcal{F}_3\,|$ for $k \geqslant 5$ and sufficiently large $n = n(k)$. Also, more recently Kupavskii [32] extended this result and showed that the same result holds when $5 \leqslant k < \frac{n}{2}$.

Theorem C. [32] *Let n, k be positive integers with $5 \leqslant k < \frac{n}{2}$ and let $\mathcal{F}$ be an intersecting family of k-sets on $[n]$ with $|\,\mathcal{F}\,| > |\,\mathcal{F}_3\,|$. Then $\mathcal{F} \subseteq \mathcal{J}_i$ for $i \in \{0, 1, \ldots k-1, n-k\}$.*

1.2 G-free subgraphs of Kneser graphs and (s,t)-union intersecting families

Let $n \geqslant 2k$. The *Kneser graph* $\mathrm{KG}_{n,k}$ is a graph whose vertex set is $\binom{[n]}{k}$ where two vertices are adjacent if their corresponding sets are disjoint. From another point of view, the Erdős-Ko-Rado theorem [9] determines the maximum independent sets of Kneser graphs. Recalling the fact that an independent set in a graph G is a subset of vertices containing no subgraph isomorphic to K_2, the following question was asked in [1].

"Given a graph G, how large a family $\mathcal{F} \subseteq \binom{[n]}{k}$ must be chosen to guarantee that $\mathrm{KG}_{n,k}[\mathcal{F}]$ has some subgraph isomorphic to G? What is the structure of the largest subset $\mathcal{F} \subseteq \binom{[n]}{k}$ for which $\mathrm{KG}_{n,k}[\mathcal{F}]$ has no subgraph isomorphic to G?"

This problem has already been investigated for some special cases. In particular, if $G = K_2$, the answer is the Erdős-Ko-Rado theorem and if $G = K_{1,t}$ or $G = K_{s,t}$, the question has been studied in [1, 20] and [1, 28], respectively. If $G = K_{r+1}$, the question is equivalent to the famous Erdős matching conjecture [7].

In [1], Alishahi and the author determined the size and structure of a family $\mathcal{F}$ of k-sets on $[n]$ with maximum size such that the induced subgraph $\mathrm{KG}_{n,k}[\mathcal{F}]$ is G-free provided that n is sufficiently large. They showed that

$$|\,\mathcal{F}\,| \leqslant \binom{n}{k} - \binom{n - \chi(G) + 1}{k} + \eta(G) - 1.$$

where $\chi(G)$ is the chromatic number and $\eta(G)$ is the minimum possible size of a color class of G over all possible proper $\chi(G)$-colorings of G.

Let s and t be two positive integers such that $t \geqslant s$. A family $\mathcal{F}$ of k-sets on $[n]$ is said to be an (s,t)-*union intersecting family* if for any subfamily $\{A_1, A_2, \ldots, A_{s+t}\}$ of $\mathcal{F}$,

$$\left(\bigcup_{i=1}^{s} A_i\right) \cap \left(\bigcup_{i=1}^{t} A_{s+i}\right) \neq \varnothing.$$

It is straightforward to see that a family $\mathcal{F}$ is an (s,t)-union intersecting family if and only if $\mathrm{KG}_{n,k}[\mathcal{F}]$ is $K_{s,t}$-free. As a generalization of the Erdős-Ko-Rado theorem Katona and Nagy [28] showed that for sufficiently large n, any (s,t)-union intersecting family has cardinality at most $\binom{n-1}{k-1} + s - 1$. Alishahi and the author improved this result, and moreover, characterized the extremal cases in [1]. Also, in [1] an asymptotic Hilton-Milner-type stability theorem was proved for an (s,t)-union intersecting family of k-sets on $[n]$. More recently, an explicit extension of this result is proved by Gerbner, Methuku, Nagy, Patkós, and Vizer [21]. They show that for $2 \leqslant s \leqslant t$, the size of an (s,t)-union intersecting family of k-sets on $[n]$, which is not isomorphic to a subfamily of

$$\mathcal{S}_1 \cup \{F_j |\, 1 \leqslant j \leqslant s-1,\ 1 \notin F_j\}$$

for some $F_1, \ldots, F_{s-1}$, is at most $\binom{n-1}{k-1} - \binom{n-sk-1}{k-1} + s + t - 1$ and characterize the largest one. In fact, they prove that a Hilton-Milner-type theorem for an (s,t)-union intersecting family is true when $t \geqslant s \geqslant 2$ and n is sufficiently large.

Note that the first largest (s,t)-union intersecting family is the union of the star $\mathcal{S}_1$ and $s-1$ other k-sets. For $i \geqslant 2$, we say $\mathcal{F}$ is the ith largest (s,t)-union intersecting family, if $\mathcal{F}$ is a maximal (s,t)-union intersecting subfamily of $\binom{[n]}{k}$ and is not contained in the jth largest (s,t)-union intersecting family for every $j \leqslant i-1$. The Hilton-Milner theorem determines the size and structure of the second $(1,1)$-union intersecting family. Also, Han and Kohayakawa in [23] characterize the size and structure of the third $(1,1)$-union intersecting family. For sufficiently large n, Kostochka and Mubayi in [29] and Kupavskii in [32] find the size and structure of the ith $(1,1)$-union intersecting family when $i \leqslant k+1$. In this regard, for sufficiently large n, Gerbner et al. in [21] determine the size and structure of the second largest (s,t)-union intersecting family when $t \geqslant s \geqslant 2$. Motivated by the mentioned results, one may naturally ask the following question.

Question 3. What are the size and structure of the ith largest (s,t)-union intersecting family?

For a family $\mathcal{F}$ and an integer $r \geqslant 2$, let $\ell_r(\mathcal{F})$ denote the minimum number m such that by removing m sets from $\mathcal{F}$, the resulting family has no r pairwise disjoint sets. For simplicity of notation, let $\ell(\mathcal{F}) \overset{\text{def}}{=} \ell_2(\mathcal{F})$. Also, Question 3 has a close relationship with the next question.

Question 4. What are the size and structure of the largest (s,t)-union intersecting family with $\ell(\mathcal{F}) \geqslant s + \beta$?

It is worth mentioning that each family $\mathcal{F}$ with $\ell(\mathcal{F}) \leqslant s-1$ is (s,t)-union intersecting and the largest (s,t)-union intersecting family

$$\mathcal{F} \stackrel{\text{def}}{=} \mathcal{S}_1 \cup \{A_i | \, 1 \leqslant i \leqslant s-1, \, 1 \notin A_i\}$$

has $\ell(\mathcal{F}) = s-1$. Gerbner et al. in [21], as their main result, determine the size and structure of the largest (s,t)-union intersecting family with $\ell(\mathcal{F}) \geqslant s$, when $t \geqslant s \geqslant 2$ and n is sufficiently large. By using the Hilton-Milner theorem and their result, one can verify that the second largest (s,t)-union intersecting family must have $\ell(\mathcal{F}) \geqslant s$. In fact, the next theorem determines the second largest (s,t)-union intersecting family.

Theorem D. [21] *For any $2 \leqslant s \leqslant t$ and k there exists $N = N(s,t,k)$ such that if $n \geqslant N$ and $\mathcal{F}$ is a family with $\ell(\mathcal{F}) \geqslant s$ and $\mathrm{KG}_{n,k}[\mathcal{F}]$ is $K_{s,t}$-free, then we have*

$$|\,\mathcal{F}\,| \leqslant \binom{n-1}{k-1} - \binom{n-sk-1}{k-1} + s + t - 1.$$

Moreover, equality holds if and only if $\mathcal{F}$ is isomorphic to some $\mathcal{F}_{s,t}$ which is defined as follows,

$$\mathcal{F}_{s,t} \stackrel{\text{def}}{=} \{A : 1 \in A, A \cap [2, sk+1] \neq \varnothing\} \cup \{A_1, \ldots, A_s\} \cup \{F_1, \ldots, F_{t-1}\}$$

where $A_i \stackrel{\text{def}}{=} [(i-1)k+2, ik+1]$ for each $1 \leqslant i \leqslant s$, and for each $j \leqslant t-1$, we have $1 \in F_j$ and $F_j \cap [2, sk+1] = \varnothing$.

Motivated by the mentioned results and questions, in this paper, we try to determine the structure and size of an (s,t)-union intersecting family with maximum size when $\ell(\mathcal{F}) \geqslant s+\beta$ and n is sufficiently large. To state our main results, we need the following definitions.

Definition 5. Let n, k, s, and β be fixed nonnegative integers. Let $A_1, \ldots, A_{s+\beta}$ be $s+\beta$ pairwise distinct k-sets on $[n]$ such that $1 \notin \cup_{i=1}^{s+\beta} A_i$. Define $\mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s)$ as follows

$$\mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s) \stackrel{\text{def}}{=} \{A \in \mathcal{S}_1 \mid A \,\text{is disjoint from at most}\, s-1 \,\text{of}\, A_i\text{'s}\}.$$

Also, define

$$T(A_1, \ldots, A_{s+\beta} : s) \stackrel{\text{def}}{=} \{x | \,\text{there exist distinct}\, i_1, i_2, \ldots, i_{\beta+1} \,\text{such that}\, x \in \cap_{j=1}^{\beta+1} A_{i_j}\}.$$

Note that when $\beta = 0$, we have $T(A_1, \ldots, A_s : s) = \cup_{i=1}^{s} A_i$ and $\mathcal{S}_1(A_1, \ldots, A_s : s)$ is equal to $\mathcal{S}_1 \setminus \{A : A \cap (\cup_{i=1}^{s} A_i) = \varnothing\}$. Also, when $s = 1$ the family $\mathcal{S}_1(A_1, \ldots, A_{1+\beta} : 1)$ is equal to $\mathcal{S}_1 \setminus \{A | \, A \cap A_i = \varnothing$ for some $1 \leqslant i \leqslant \beta+1\}$ and $T(A_1, \ldots, A_{1+\beta} : 1) = \cap_{i=1}^{1+\beta} A_i$.

Definition 6. Let k, s, and β be fixed nonnegative integers. If $\lfloor \frac{(s+\beta)k}{\beta+1} \rfloor > k$, define $\hat{\beta} \stackrel{\text{def}}{=} \hat{\beta}(k, s, \beta)$ as the largest positive integer such that $\lfloor \frac{(s+\beta)k}{\beta+1} \rfloor = \lfloor \frac{(s+\hat{\beta})k}{\hat{\beta}+1} \rfloor$; else if $\lfloor \frac{(s+\beta)k}{\beta+1} \rfloor = k$, define $\hat{\beta} \stackrel{\text{def}}{=} \beta$.

Now, we are in a position to state our first result.

Theorem 7. *For any nonnegative integers $k \geqslant 3, t \geqslant s \geqslant 1$, and β, there exists $n(s,t,k,\beta)$ such that for all $n \geqslant n(s,t,k,\beta)$ we have the following: if $\mathcal{F}$ is an (s,t)-union intersecting family on $[n]$ such that $\ell(\mathcal{F}) \geqslant s+\beta$, then*

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n - \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor - 1}{k-1} + s + t + \hat{\beta} - 1.$$

Equality holds if and only if there exist pairwise distinct k-sets $A_1, \ldots, A_{s+\hat{\beta}}$ and $F_1, \ldots, F_{t-1}$ such that

1. $1 \notin \bigcup_{i=1}^{s+\hat{\beta}} A_i$,
2. $|T(A_1, \ldots, A_{s+\hat{\beta}} : s)| = \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor$,
3. *for each* $i \leqslant t-1$, $F_i \in \mathcal{S}_1 \setminus \mathcal{S}_1(A_1, \ldots, A_{s+\hat{\beta}} : s)$, *and*
4. *the family* $\{A_1, \ldots, A_{s+\hat{\beta}}, F_1, \ldots, F_{t-1}\}$ *is an (s,t)-union intersecting family*

and $\mathcal{F}$ is isomorphic to $\mathcal{S}_1(A_1, \ldots, A_{s+\hat{\beta}} : s) \cup \{A_1, \ldots, A_{s+\hat{\beta}}\} \cup \{F_1, \ldots, F_{t-1}\}$.

It is worth mentioning that Theorem D follows from Theorem 7 by choosing $\beta = 0$ and $s \geqslant 2$. By applying the previous theorem and using some properties of $T(A_1, \ldots, A_{s+\beta} : s)$, we can find out the jth largest (s,t)-union intersecting family for some j's. We provide a more detailed analysis in our remarks proceeding the proof of Theorem 7.

In [1], it was shown that if $\mathcal{F}$ is a $(1,t)$-union intersecting family of $\binom{[n]}{k}$ with at least $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + (t-1)\binom{2k-1}{k-1} + t$ members, then it is contained in some star $\mathcal{S}_i$ for sufficiently large n. Moreover, as an extension of the Hilton-Milner theorem, it is posed as a conjecture in [1] that for sufficiently large n one can replace the term $(t-1)\binom{2k-1}{k-1}$ by 1. This conjecture is one of our motivations for this study. The conjecture follows from Theorem 7 by choosing $s = 1$ and $\beta = 0$.

Concerning our next result when $s = 1, t \geqslant 1$, and $\beta \leqslant k-3$, motivated by Theorems B and C and the mentioned conjecture, we determine the maximum size and structure of a $(1,t)$-union intersecting family $\mathcal{F}$ with $\ell(\mathcal{F}) \geqslant 1+\beta$. Note that when $s = 1$ and $\beta \geqslant 1$, Theorem 7 does not give a sharp bound for maximum size of $(1,t)$-union intersecting families. This result leads us to determine the ith largest $(1,t)$-union intersecting families where $i \leqslant k-2$.

Before stating the next result we need to introduce the following construction.

Definition 8. Let $i \leqslant k-1$ be a nonnegative integer. For any $(i+1)$-set $J = \{1, x_1, \ldots, x_i\}$ of $[n]$ and any $(k-1)$-set $E \subset [n] \setminus J$. Let $A_1, \ldots, A_i$ be i pairwise distinct k-subsets on $[n] \setminus \{1\}$ such that $\cap_{j=1}^{i} A_j = E$ and $A_j \setminus E = \{x_j\}$ for each $j \leqslant i$ define $\mathcal{J}_i^{1,t}$ as follows

$$\mathcal{J}_i^{1,t} \stackrel{\text{def}}{=} \mathcal{S}_1(A_1, \ldots, A_i : 1) \cup \{A_1, \ldots, A_i\} \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_i,$$

where $\mathcal{B}_j$, for $j \leqslant i$, defined as follows

$$\mathcal{B}_j \stackrel{\text{def}}{=} \{B_p : p \in [t-1],\, |B_p| = k, B_p \cap E = \varnothing,\ J \setminus B_p = \{x_j\}\}.$$

Notice that $\mathcal{J}_i^{1,1}$ isomorphic to $\mathcal{J}_i$ and $\mathcal{J}_i = \mathcal{S}_1(A_1,\dots,A_i : 1) \cup \{A_1,\dots,A_i\}$. Since $\mathcal{B}_j$'s in the definition of $\mathcal{J}_i^{1,t}$ are pairwise disjoint. Therefore, $|\,\mathcal{J}_i^{1,t}\,| = |\,\mathcal{J}_i\,| + i(t-1)$.

For $s = 1$ we can state a strong improvement of Theorem 7 as follows.

Theorem 9. *For any nonnegative integers $k \geqslant 5, t \geqslant 1$, and $\gamma = 1 + \beta \leqslant k-2$, there exists $n(k,t,\gamma)$ such that for all $n \geqslant n(k,t,\gamma)$ we have the following: if $\mathcal{F}$ is a $(1,t)$-union intersecting family on $[n]$ such that $\ell(\mathcal{F}) \geqslant \gamma$, then*

$$|\,\mathcal{F}\,| \leqslant \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-\gamma}{k-\gamma-1} + \gamma t.$$

Equality holds if and only if $\mathcal{F}$ is isomorphic to $\mathcal{J}_\gamma^{1,t}$.

1.3 Stability results for the Erdős matching conjecture and its generalization

The Erdős matching conjecture is one of the famous open problems in extremal set theory. It states that for $n \geqslant (r+1)k$, the size of the largest subset $\mathcal{F} \subseteq \binom{[n]}{k}$ for which $\mathrm{KG}_{n,k}[\mathcal{F}]$ has no copy of K_{r+1} is $\max\{\binom{(r+1)k-1}{k}, \binom{n}{k} - \binom{n-r}{k}\}$. In recent years, this conjecture has received considerable attention (e.g. [4, 7, 8, 16, 19, 25, 34]). Improving the earlier results, Frankl [15] confirmed the conjecture for $n \geqslant (2r+1)k - r$; moreover, he determined the structure of the extremal cases in this range. Frankl and Kupavskii [18] proved a Hilton-Milner-type stability theorem for the Erdős matching conjecture for $n \geqslant (2 + o_r(1))(r+1)k$ as a significant improvement of a classical result due to Bollobás, Daykin and Erdős [4].

Hereafter, we will focus on complete multipartite graphs $K_{s_1,s_2,\cdots,s_{r+1}}$ as a forbidden subgraph. We show that the previous results for (s,t)-union intersecting family can be extended to $K_{s_1,s_2,\cdots,s_{r+1}}$-free subgraph of Kneser graphs instead of $K_{s,t}$-free subgraphs of Kneser graphs as nontrivial extensions of the Erdős matching conjecture. In this regard, Gerbner et al. [21] show that a generalization of Theorem D holds when $\mathrm{KG}_{n,k}[\mathcal{F}]$ is $K_{s_1,s_2,\cdots,s_{r+1}}$-free when $s_1 \geqslant \cdots \geqslant s_{r+1} \geqslant 2$. They determine the size and structure of the second largest family $\mathcal{F}$ on $[n]$ such that $\mathrm{KG}_{n,k}[\mathcal{F}]$ is $K_{s_1,s_2,\ldots,s_{r+1}}$-free, where $s_{r+1} \geqslant 2$ for sufficiently large n. Before stating their result, we need an extension of the construction of Definition 5.

Definition 10. Let n, k, s, and β be positive integers. Let $A_1, \dots, A_{s+\beta}$ be $s+\beta$ pairwise distinct k-sets on $[n]$ such that $[r] \bigcap (\cup_{i=1}^{s+\beta} A_i) = \varnothing$. Define $\mathcal{S}_r^{[r-1]}(A_1,\dots,A_{s+\beta} : s)$ as follows

$$\{A \in \mathcal{S}_r \mid A \cap [r-1] = \varnothing \text{ and } A \text{ is disjoint from at most } s-1 \text{ of } A_i\text{'s}\}.$$

Note that the family $\mathcal{S}_r(A_1,\dots,A_{s+\beta} : s)$ in Definition 5 is a special case of Definition 10 when $r = 1$.

We are able to prove an analog of the previous theorem by using the Erdős-Stone-Simonovits theorem and Theorem 7.

Theorem 11. *For any nonnegative integers $k \geqslant 3, s_1 \geqslant \cdots \geqslant s_{r+1} \geqslant 1$, and β, there exists $n(s_1, \ldots, s_{r+1}, k, \beta)$ such that for all $n \geqslant n(s_1, \ldots, s_{r+1}, k, \beta)$ we have the following: if $\mathrm{KG}_{n,k}[\mathcal{F}]$ is $K_{(s_1,\ldots,s_{r+1})}$-free such that $\ell_{r+1}(\mathcal{F}) \geqslant s_{r+1} + \beta$, then*

$$|\,\mathcal{F}\,| \leqslant \binom{n}{k} - \binom{n-r}{k} - \binom{n - \lfloor \frac{(s_{r+1}+\beta)k}{\beta+1} \rfloor - r}{k-1} + s_r + s_{r+1} + \hat{\beta} - 1.$$

Equality holds if and only if there exist $s_{r+1} + \hat{\beta}$ pairwise distinct k-sets $A_1, \ldots, A_{s_{r+1}+\hat{\beta}}$ such that

- $[r] \bigcap (\bigcup_{i=1}^{s_{r+1}+\hat{\beta}} A_i) = \varnothing$,
- $|T(A_1, \ldots, A_{s_{r+1}+\hat{\beta}} : s_{r+1})| = \lfloor \frac{(s_{r+1}+\beta)k}{\beta+1} \rfloor$,
- *for each $i \leqslant s_r - 1$, $F_i \in \mathcal{S}_r \setminus \mathcal{S}_r^{[r-1]}(A_1, \ldots, A_{s_{r+1}+\hat{\beta}} : s_{r+1})$ and $F_i \cap [r-1] = \varnothing$, and*
- *the family $\{A_1, \ldots, A_{s_{r+1}+\hat{\beta}}, F_1, \ldots, F_{s_r-1}\}$ is an (s_{r+1}, s_r)-union intersecting family and*

$\mathcal{F}$ is isomorphic to

$$\bigcup_{i=1}^{r-1} \mathcal{S}_i \cup \mathcal{S}_r^{[r-1]}(A_1, \ldots, A_{s_{r+1}+\hat{\beta}} : s_{r+1}) \cup \{A_1, \ldots, A_{s_{r+1}+\hat{\beta}}\} \cup \{F_1, \ldots, F_{s_r-1}\}.$$

When $s_{r+1} = 1$ same as Theorem 9 we are able to prove a stronger result than Theorem 11, which yields a new stability result for Erdős matching conjecture for sufficiently large n.

Definition 12. Let $i \leqslant k-1$ be a nonnegative integer. For any $(i+r)$-set $J = \{1, \ldots, r, x_1, \ldots, x_i\}$ of $[n]$ and any $(k-1)$-set $E \subset [n] \setminus J$. Let $A_1, \ldots, A_i$ be i pairwise distinct k-subsets on $[n] \setminus [r]$ such that $\cap_{j=1}^{i} A_j = E$ and $A_j \setminus E = \{x_j\}$ for each $j \leqslant i$ define $\mathcal{J}_{i,r}^{1,t}$ as follows

$$\mathcal{J}_{i,r}^{1,t} \stackrel{\text{def}}{=} \bigcup_{q=1}^{r-1} \mathcal{S}_q \cup \mathcal{S}_r^{[r-1]}(A_1, \ldots, A_i : 1) \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_i$$

where $\mathcal{B}_j$, for $j \leqslant i$ defined as follows,

$$\mathcal{B}_j \stackrel{\text{def}}{=} \{B_p : p \in [t-1], |B_p| = k, B_p \cap E = \varnothing,\ J \setminus B_p = \{1, \ldots, r-1, x_j\}\}.$$

Notice that $\mathcal{J}_{i,1}^{1,t}$ is isomorphic to $\mathcal{J}_{i}^{1,t}$. Now we are in a position to state a stability result related to Erdős matching conjecture provided that n is sufficiently large.

Theorem 13. *For any nonnegative integers $k \geqslant 5, s_1 \geqslant \cdots \geqslant s_r \geqslant 1$, and $\gamma(= 1+\beta) \leqslant k-2$, there exists $n(s_1, \ldots, s_r, k, \gamma)$ such that for all $n \geqslant n(s_1, \ldots, s_r, k, \gamma)$ we have the following: if $\mathrm{KG}_{n,k}[\mathcal{F}]$ is $K_{(s_1,\ldots,s_r,1)}$-free such that $\ell_{r+1}(\mathcal{F}) \geqslant \gamma$, then*

$$|\,\mathcal{F}\,| \leqslant \binom{n}{k} - \binom{n-r}{k} - \binom{n-k-r+1}{k-1} + \binom{n-k-r-\gamma+1}{k-\gamma-1} + \gamma t.$$

Equality holds if and only if $\mathcal{F}$ is isomorphic to $\mathcal{J}_{\gamma,r}^{1,s_r}$

2 Proofs

Before the proof of Theorem 7, let us state an interesting lemma from [21]. Here we show that a strong generalization of Lemma A is true.

Lemma A. [21] *Let $s \leqslant t$ and let $A_1, A_2, \ldots, A_{s+1}$ be k-sets on $[n]$ such that $1 \notin \cup_{i=1}^{s+1} A_i$. Suppose that $\mathcal{F}'$ is a subfamily of $\mathcal{S}_1$ such that for $\mathcal{F} = \mathcal{F}' \cup \{A_1, A_2, \ldots, A_{s+1}\}$ the induced subgraph of $\mathrm{KG}_{n,k}[\mathcal{F}]$ is $K_{s,t}$-free. There exists $n_0 = n(k, s, t)$ such that if $n \geqslant n_0$ holds, then we have*

$$|\mathcal{F}| \leqslant \binom{n-1}{k-1} - \binom{n - \lfloor \frac{(s+1)k}{2} \rfloor - 1}{k-1} + (s+1)t.$$

The next lemma provides an interesting and useful generalization of Lemma A. I believe that Lemma 14 independently will be a useful result and will have more applications.

Lemma 14. *Let k, s, and β be fixed nonnegative integers. Let $A_1, A_2, \ldots, A_{s+\beta}$ be pairwise distinct k-sets on $[n]$ such that $1 \notin \cup_{i=1}^{s+\beta} A_i$. Then, there exists $n(k, s, \beta)$ such that for all $n \geqslant n(k, s, \beta)$ we have the following:*

(a) $\binom{n-1}{k-1} - \binom{n-|T(A_1,\ldots,A_{s+\beta}:s)|-1}{k-1} \leqslant |\mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s)|$.

(b) $|\mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s)| \leqslant \binom{n-1}{k-1} - \binom{n-\lfloor \frac{(s+\beta)k}{\beta+1} \rfloor - 1}{k-1}$ *and equality holds if and only if*

$$|T(A_1, \ldots, A_{s+\beta} : s)| = \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor.$$

In particular, if $|T(A_1, \ldots, A_{s+\beta} : s)| < \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor$, then $|\mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s)|$ is at most

$$\binom{n-1}{k-1} - \binom{n - |T(A_1, \ldots, A_{s+\beta} : s)| - 1}{k-1} + \binom{s+\beta}{1+\beta} 2^{k(\beta+1)} \binom{n - |T(A_1, \ldots, A_{s+\beta} : s)| - 3}{k-3}.$$

(c) For $s = 1$, we have $|\mathcal{S}_1(A_1, \ldots, A_{1+\beta} : 1)| \leqslant \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-\beta-1}{k-\beta-2}$. Moreover for $\beta \geqslant 1$, equality holds if and only if $|T(A_1, \ldots, A_{1+\beta} : 1)| = k - 1$.

In particular, if $|T(A_1, \ldots, A_{1+\beta} : 1)| < k-1$, then $|\mathcal{S}_1(A_1, \ldots, A_{1+\beta} : 1)|$ is at most

$$\binom{n-1}{k-1} - \binom{n - |T(A_1, \ldots, A_{1+\beta} : 1)| - 1}{k-1} + 2^{k(\beta+1)} \binom{n - |T(A_1, \ldots, A_{1+\beta} : 1)| - 3}{k-3}.$$

Proof. For abbreviation, let $T(A_1, \ldots, A_{s+\beta} : s) = T_\beta$. For the proof of (a), let $1 \in A$. If $A \cap T_\beta \neq \varnothing$, then A is disjoint from at most $s-1$ sets of $A_1, A_2, \ldots, A_{s+\beta}$. Therefore, $\binom{n-1}{k-1} - \binom{n-|T_\beta|-1}{k-1} \leqslant |\mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s)|$.

Now we prove (b). One can check that $|T_\beta| \leqslant \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor$. Assume that $|T_\beta| < \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor$. Let $A \in \mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s)$. Therefore, A intersects at least $\beta+1$ of $A_1, \ldots, A_{s+\beta}$.

We have two possibilities for A. Either $A \cap T_\beta \neq \varnothing$ or $A \cap T_\beta = \varnothing$ and A intersects at least $\beta + 1$ of $A_1, \ldots, A_{s+\beta}$. The number of members in $\mathcal{S}_1$ which meet T_β is equal to $\binom{n-1}{k-1} - \binom{n-|T_\beta|-1}{k-1}$. The number of k-sets in $\mathcal{S}_1$, which intersect at least $\beta+1$ of $A_1, \ldots, A_{s+\beta}$ and have no common element with T_β, is at most

$$\sum_{i_1,\ldots,i_{\beta+1} \in [s+\beta]} \; \sum_{\varnothing \neq B_1 \subseteq A_{i_1} \setminus T_\beta} \cdots \sum_{\varnothing \neq B_{\beta+1} \subseteq A_{i_{\beta+1}} \setminus T_\beta} \binom{n - |T_\beta| - |\cup_{i=1}^{\beta+1} B_i| - 1}{k - |\cup_{i=1}^{\beta+1} B_i| - 1} \tag{2}$$

which is at most

$$\begin{aligned} &\sum_{i_1,\ldots,i_{\beta+1} \in [s+\beta]} \prod_{j=1}^{\beta+1} 2^{|A_{i_j} \setminus T_\beta|} \binom{n - |T_\beta| - 3}{k - 3} \\ &\leqslant \binom{s+\beta}{1+\beta} 2^{k(\beta+1)} \binom{n - |T_\beta| - 3}{k-3} \end{aligned}$$

if $\beta \geqslant 1$ and is 0 if $\beta = 0$. Therefore, if $\beta = 0$, then

$$|\,\mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s)| \leqslant \binom{n-1}{k-1} - \binom{n - |T_\beta| - 1}{k-1} \tag{3}$$

and if $\beta \geqslant 1$, then

$$|\,\mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s)| \leqslant \binom{n-1}{k-1} - \binom{n - |T_\beta| - 1}{k-1} + \binom{s+\beta}{1+\beta} 2^{k(\beta+1)} \binom{n - |T_\beta| - 3}{k-3}.$$

Then, $|\,\mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s)|$ is at most

$$\begin{aligned} &\binom{n-1}{k-1} - \binom{n - \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor - 1}{k-1} - \sum_{i=|T_\beta|+1}^{\lfloor \frac{(s+\beta)k}{\beta+1} \rfloor} \binom{n-i-1}{k-2} + \binom{s+\beta}{1+\beta} 2^{k(\beta+1)} \binom{n - |T_\beta| - 3}{k-3} \\ &< \binom{n-1}{k-1} - \binom{n - \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor - 1}{k-1}, \end{aligned}$$

provided that n is sufficiently large.

Now assume that we have the equality $|\,\mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s)| = \binom{n-1}{k-1} - \binom{n - \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor - 1}{k-1}$. By contradiction assume that $|T_\beta| < \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor$. Using the same reasoning one may verify that when n is sufficiently large, then $|\,\mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s)|$ is less than $\binom{n-1}{k-1} - \binom{n - \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor - 1}{k-1}$ which is not possible.

Now suppose that $|T_\beta| = \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor$. To prove the last part of (b), it suffices to show that

$$\mathcal{S}_1 \setminus \mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s) = \{A | 1 \in A,\ A \cap T_\beta = \varnothing\}.$$

From the division algorithm, we know that $(s+\beta)k = \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor (\beta+1) + r$ where $0 \leqslant r \leqslant \beta$. Since $|T_\beta| = \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor$, there are at most $r \leqslant \beta$ elements in $\cup_{i \in [s+\beta]} A_i$ which are not in

T_β. Therefore, there exist $1 \leqslant i_1 < \ldots < i_s \leqslant s+\beta$ such that $A_{i_1} \cup \cdots \cup A_{i_s} \subseteq T_\beta$. On the other hand, for every $1 \leqslant j_1 < \ldots < j_s \leqslant s+\beta$, we have $T_\beta \subseteq A_{j_1} \cup \cdots \cup A_{j_s}$. Therefore, $T_\beta = A_{i_1} \cup \cdots \cup A_{i_s}$. Assume that $1 \in A$ and $A \cap T_\beta = \varnothing$. Hence, $A \cap (A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_s}) = \varnothing$. Therefore, A is disjoint from at least s sets of $A_1, A_2, \ldots, A_{s+\beta}$ and consequently $A \in \mathcal{S}_1 \setminus \mathcal{S}_1(A_1, \ldots, A_{s+\beta} : s)$. If $1 \in A$ and A is disjoint from at least s sets of $A_1, A_2, \ldots, A_{s+\beta}$, then it is clear that each element of A appears in at most β of A_i's and hence we have $A \cap T_\beta = \varnothing$.

For the proof of (c), if $|T_\beta| \leqslant k-2$, then the proof is the same as the first part of (b). Hence, we may assume that $|T_\beta|$ is $k-1$ or k. Note that when $s=1$, $T_\beta = \cap_{i=1}^{1+\beta} A_i$. If $|T_\beta| = k$, then β must be equal to 0 and consequently $|\,\mathcal{S}_1(A_1 : 1)| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1}$. Thus, we may assume that $|T_\beta| = |\cap_{i=1}^{1+\beta} A_i| = k-1$ and $\beta \geqslant 1$. Then, there exist $\beta+1$ elements in $[n]$, say $x_1, \ldots, x_{\beta+1}$, such that $A_j \setminus T_\beta = \{x_j\}$. Let $A \in \mathcal{S}_1(A_1, \ldots, A_{1+\beta} : 1)$. Therefore, A intersects each of $A_1, \ldots, A_{1+\beta}$. We have two possibilities for A. Either $A \cap T_\beta \neq \varnothing$ or $A \cap T_\beta = \varnothing$ and A intersects all of $A_1, \ldots, A_{1+\beta}$. There are $\binom{n-1}{k-1} - \binom{n-k}{k-1}$ members in $\mathcal{S}_1$ such that $A \cap T_\beta \neq \varnothing$. The number of k-sets in $\mathcal{S}_1$, which intersect all of $A_1, \ldots, A_{1+\beta}$ and have no common element with T_β, is equal to $\binom{n-k-\beta-1}{k-\beta-2}$. Therefore,

$$|\,\mathcal{S}_1(A_1, \ldots, A_{1+\beta} : 1)| = \binom{n-1}{k-1} - \binom{n-|T_\beta|-1}{k-1} + \binom{n-k-\beta-1}{k-\beta-2}.$$

Note that when $\beta \geqslant k-1$, we have $\binom{n-k-\beta-1}{k-\beta-2} = 0$. ■

In the proof of Theorem 7 in addition to Lemma 14, we will use the following two results. The first one is a result on the number of edges of a $K_{s,t}$-free graph, which is a classical theorem by Kővari, Sós, and Turán [30]. The second one is a result on the number of disjoint pairs in a family $\mathcal{F}$ of k-sets by Balogh, Bollobás, and Narayanan [2].

Theorem E. [30] *For any two positive integers $s \leqslant t$, if G is a $K_{s,t}$-free graph with n vertices, then the number of edges of G is at most $(\frac{1}{2}+o(1))(t-1)^{\frac{1}{s}} n^{2-\frac{1}{s}}$.*

Lemma B. [2] *Let $\mathcal{F}$ be a family k-sets on $[n]$. Then the number of disjoint pairs in $\mathcal{F}$ is at least $\frac{\ell(\mathcal{F})^2}{2\binom{2k}{k}}$.*

For an intersecting family $\mathcal{F}'$ on $[n]$, its maximum degree $\Delta(\mathcal{F}')$ is the maximum number of elements of $\mathcal{F}'$ containing any particular element of $[n]$, i.e., $\Delta(\mathcal{F}') \overset{\text{def}}{=} \max_{i\in[n]} |\,\mathcal{F}' \cap \mathcal{S}_i\,|$.

Proof of Theorem 7. Let $\mathcal{F}$ be an (s,t)-union intersecting family of $\binom{[n]}{k}$ with $\ell(\mathcal{F}) \geqslant s+\beta$ and cardinality

$$M \overset{\text{def}}{=} \binom{n-1}{k-1} - \binom{n - \lfloor\frac{(s+\beta)k}{\beta+1}\rfloor - 1}{k-1} + t - 1 + s + \hat{\beta}.$$

We consider the following three cases.

(i) $\ell(\mathcal{F}) = s + \beta'$ *where* $\beta \leqslant \beta' \leqslant \hat{\beta}$.

This implies that there exist $A_1, \ldots, A_{s+\beta'}$ in $\mathcal{F}$ such that $\mathcal{F}' = \mathcal{F} \setminus \{A_1, \ldots, A_{s+\beta'}\}$ is an intersecting family. Without loss of generality assume that $\Delta(\mathcal{F}')$ has the maximum possible value and also $\Delta(\mathcal{F}') = |\,\mathcal{F}' \cap \mathcal{S}_1\,|$. Therefore, $|\,\mathcal{F}'\,|$ is equal to

$$\binom{n-1}{k-1} - \binom{n - \lfloor\frac{(s+\beta)k}{\beta+1}\rfloor - 1}{k-1} + t - 1 + \hat{\beta} - \beta'.$$

First we show that for each $i \leqslant s+\beta'$, $1 \notin A_i$. If $\mathcal{F}' \subseteq \mathcal{S}_1$, then by the minimality of $\ell(\mathcal{F})$, each A_i must be disjoint from at least one member of $\mathcal{F}' \subseteq \mathcal{S}_1$, so $1 \notin \cup_{i=1}^{s+\beta'} A_i$. If $\mathcal{F}' \nsubseteq \mathcal{S}_1$, then by the Hilton-Milner theorem, we conclude that $|\,\mathcal{F}'\,| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. Consequently, there exists a unique $B \in \mathcal{F}'$ such that $\mathcal{F}' \setminus \{B\} \subseteq \mathcal{S}_1$ and moreover, we must have $t = 2$, $\lfloor\frac{(s+\beta')k}{\beta'+1}\rfloor = k$ and $\beta' = \hat{\beta}$. If there is A_i such that $1 \in A_i$, by the minimality of $\ell(\mathcal{F})$, A_i must be disjoint from B. Define $\mathcal{F}'' = (\mathcal{F}' \setminus \{B\}) \cup \{A_i\}$. Hence, $|\,\mathcal{F}'\,| = |\,\mathcal{F}''\,|$ and $\Delta(\mathcal{F}'') = \Delta(\mathcal{F}') + 1$ which contradicts with the fact that $\Delta(\mathcal{F}')$ has the maximum possible value. Then, $1 \notin \cup_{i=1}^{s+\beta'} A_i$. We now consider the following three subcases.

(a) $\mathcal{F}' \subseteq \mathcal{S}_1$ *and* $|T(A_1, A_2, \ldots, A_{s+\beta'} : s)| = \lfloor\frac{(s+\beta')k}{\beta'+1}\rfloor$.

Since $\beta \leqslant \beta' \leqslant \hat{\beta}$, by the definition of $\hat{\beta}$ we have $\lfloor\frac{(s+\beta')k}{\beta'+1}\rfloor = \lfloor\frac{(s+\beta)k}{\beta+1}\rfloor$. In view of the last part of the proof of Lemma 14 (b), there are $1 \leqslant i_1 < \ldots < i_s \leqslant s+\beta'$ such that $T(A_1, \ldots, A_{s+\beta'} : s) = A_{i_1} \cup \cdots \cup A_{i_s}$. Also, note that for every $1 \leqslant j_1 < \ldots < j_s \leqslant s+\beta'$, we have

$$A_{i_1} \cup \cdots \cup A_{i_s} = T(A_1, \ldots, A_{s+\beta'} : s) \subseteq A_{j_1} \cup \cdots \cup A_{j_s}.$$

From this fact and since $\mathcal{F}$ is an (s,t)-union intersecting family, the number of elements of $\mathcal{F}'$ which can be disjoint from $\cup_{\ell=1}^{s} A_{j_\ell}$ for some s sets such as $A_{j_1}, \ldots, A_{j_s}$ of A_i's is at most $t-1$, say $F_1, \ldots, F_{t-1}$. Therefore, $\mathcal{F}' \subseteq \mathcal{S}_1(A_1, \ldots, A_{s+\beta'} : s) \cup \{F_1, \ldots, F_{t-1}\}$. Thus, by applying Lemma 14 (b), we obtain

$$|\,\mathcal{F}'\,| \leqslant \binom{n-1}{k-1} - \binom{n - \lfloor\frac{(s+\beta')k}{\beta'+1}\rfloor - 1}{k-1} + t - 1$$

and consequently $\beta' = \hat{\beta}$. Therefore,

$$|\,\mathcal{F}\,| \leqslant \binom{n-1}{k-1} - \binom{n - \lfloor\frac{(s+\beta)k}{\beta+1}\rfloor - 1}{k-1} + t - 1 + s + \beta'$$

and equality holds if and only if $\mathcal{F}$ is isomorphic to

$$\mathcal{S}_1(A_1, \ldots, A_{s+\beta'} : s) \cup \{A_1, \ldots, A_{s+\beta'}\} \cup \{F_1, \ldots, F_{t-1}\}$$

such that $|T(A_1, \ldots, A_{s+\beta'} : s)| = \lfloor\frac{(s+\beta')k}{\beta'+1}\rfloor$, $F_i \in \mathcal{S}_1 \setminus \mathcal{S}_1(A_1, \ldots, A_{s+\beta'} : s)$, and the family $\{A_1, \ldots, A_{s+\beta'}, F_1, \ldots, F_{t-1}\}$ is an (s,t)-union intersecting family.

(b) $\mathcal{F}' \nsubseteq \mathcal{S}_1$ *and* $|T(A_1, A_2, \dots, A_{s+\beta'} : s)| = \lfloor \frac{(s+\beta')k}{\beta'+1} \rfloor$.

As $\mathcal{F}' \nsubseteq \mathcal{S}_1$, there exists a k-set $B \in \mathcal{F}'$ such that $\mathcal{F}' \setminus \{B\} \subseteq \mathcal{S}_1$ and we have $t = 2$, $\lfloor \frac{(s+\beta')k}{\beta'+1} \rfloor = k$, and $\beta' = \hat{\beta}$. Since $|T(A_1, A_2, \dots, A_{s+\beta'} : s)| = \lfloor \frac{(s+\beta')k}{\beta'+1} \rfloor = k$, in view of the last part of the proof of Lemma 14 (b), there are $1 \leqslant i_1 < \dots < i_s \leqslant s+\beta'$ such that $T(A_1, \dots, A_{s+\beta'} : s) = A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_s}$. As $|T(A_1, \dots, A_{s+\beta'} : s)| = k$, s must be equal to 1. Therefore $|T(A_1, \dots, A_{1+\beta'} : 1)| = k$. Since $T(A_1, \dots, A_{1+\beta'} : 1) = \cap_{i=1}^{1+\beta'} A_i$, we obtain $\beta' = 0$. As $t = 2$, $s = 1$, and $\mathcal{F}$ is (s,t)-union intersecting, there is a unique $B_1 \in \mathcal{F}'$ that $A_1 \cap B_1 = \varnothing$. One can check that $\mathcal{F}' \setminus \{B, B_1\} \subseteq \mathcal{S}_1(A_1, B : 1)$. Therefore, $|\mathcal{F}'| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2} + 2$, a contradiction.

(c) $|T(A_1, \dots, A_{s+\beta'} : s)| < \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor$.

There is at most one member $B \in \mathcal{F}'$ such that $\mathcal{F}' \setminus \{B\} \subseteq \mathcal{S}_1$. Since $\mathcal{F}$ is an (s,t)-union intersecting family, every s sets of A_i's such as $A_{i_1}, \dots, A_{i_s}$ are disjoint from at most $t-1$ elemnts in $\mathcal{F}'$. Therefore,

$$|\mathcal{F}' \setminus \{B\}| \leqslant |\mathcal{S}_1(A_1, \dots, A_{s+\beta'} : s)| + \binom{s+\beta'}{s}(t-1).$$

Now by applying Lemma 14 (b), we obtain $|\mathcal{F}'|$ is at most

$$\binom{n-1}{k-1} - \binom{n - |T(A_1, \dots, A_{s+\beta'} : s)| - 1}{k-1} + \binom{s+\beta'}{1+\beta'} 2^{k(\beta'+1)} \binom{n - |T(A_1, \dots, A_{s+\beta'} : s)| - 3}{k-3} + \binom{s+\beta'}{s}(t-1) + 1.$$

Since $|T(A_1, \dots, A_{s+\beta'} : s)| < \lfloor \frac{(s+\beta')k}{\beta'+1} \rfloor$ and $k \geqslant 3$, one can check that

$$|\mathcal{F}| < \binom{n-1}{k-1} - \binom{n - \lfloor \frac{(s+\beta')k}{\beta'+1} \rfloor - 1}{k-1}$$

provided that n is sufficiently large, which is not possible.

(ii) $s + \hat{\beta} + 1 \leqslant \ell(\mathcal{F}) \leqslant M^{1-\frac{1}{3s}}$.

Let $\mathcal{F}'$ be a largest intersecting family of $\mathcal{F}$. Hence, $|\mathcal{F}'|$ is at least

$$\binom{n-1}{k-1} - \binom{n - \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor - 1}{k-1} - M^{1-\frac{1}{3s}}.$$

By the definition of M, we have $M^{1-\frac{1}{3s}} = o(n^{k-2})$. Since $\lfloor \frac{(s+\beta)k}{\beta+1} \rfloor \geqslant k$ and $M^{1-\frac{1}{3s}} = o(n^{k-2})$, if n is sufficiently large, then we have

$$|\mathcal{F}'| > \binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2} + 2.$$

By using Theorem B, $\mathcal{F}'$ is a star or a Hilton-Milner family. Therefore, without loss of generality we may assume that there exists at most one $B \in \mathcal{F}'$ such that $\mathcal{F}' \setminus \{B\}$ is a subfamily $\mathcal{S}_1$.

First assume that $\lfloor \frac{(s+\beta)k}{\beta+1} \rfloor \geqslant k+1$. By applying Lemma 14 (b) for $\mathcal{F}' \setminus \{B\}$ and one of $s+\hat{\beta}+1$ sets of $\mathcal{F} \setminus \mathcal{F}'$, we obtain

$$|\,\mathcal{F}'\,| \leqslant \binom{n-1}{k-1} - \binom{n - \lfloor \frac{(s+\hat{\beta}+1)k}{\hat{\beta}+2} \rfloor - 1}{k-1} + \binom{s+\hat{\beta}+1}{s}(t-1)+1.$$

Hence,

$$|\,\mathcal{F}\,| \leqslant \binom{n-1}{k-1} - \binom{n - \lfloor \frac{(s+\hat{\beta}+1)k}{\hat{\beta}+2} \rfloor - 1}{k-1} + \binom{s+\hat{\beta}+1}{s}(t-1) + M^{1-\frac{1}{3s}} + 1.$$

Note that $\binom{n-\lfloor \frac{(s+\hat{\beta}+1)k}{\hat{\beta}+2} \rfloor - 1}{k-1} - \binom{n-\lfloor \frac{(s+\hat{\beta})k}{\hat{\beta}+1} \rfloor - 1}{k-1} = \binom{n-\lfloor \frac{(s+\hat{\beta})k}{\hat{\beta}+1} \rfloor - 1}{k-2}$. Therefore, $|\,\mathcal{F}\,|$ is at most

$$\binom{n-1}{k-1} - \binom{n - \lfloor \frac{(s+\hat{\beta})k}{\hat{\beta}+1} \rfloor - 1}{k-1} - \binom{n - \lfloor \frac{(s+\hat{\beta})k}{\hat{\beta}+1} \rfloor - 1}{k-2} + \binom{s+\hat{\beta}+1}{s}(t-1) + M^{1-\frac{1}{3s}} + 1.$$

This concludes that for sufficiently large n, $|\,\mathcal{F}\,|$ is less than M, a contradiction.

Assume that $\lfloor \frac{(s+\beta)k}{\beta+1} \rfloor = k$. Therefore, $\hat{\beta} = \beta$ and $\lfloor \frac{(s+\beta+1)k}{\beta+2} \rfloor = k$. Take $A_1, \ldots, A_{s+\beta+1}$ in $\mathcal{F} \setminus \mathcal{F}'$. If we have $|T(A_1, \ldots, A_{s+\beta+1} : s)| < k$, then by applying Lemma 14 (b), we obtain $|\,\mathcal{F}' \setminus \{B\}|$ is at most

$$\binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{s+\beta+1}{2+\beta} 2^{k(\beta+2)} \binom{n - |T(A_1, \ldots, A_{s+\beta+1} : s)| - 3}{k-3} + \binom{s+\beta+1}{s}(t-1).$$

This implies that $|\,\mathcal{F}\,|$ is at most

$$\binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-1}{k-2} +$$
$$\binom{s+\beta+1}{2+\beta} 2^{k(\beta+2)} \binom{n - |T(A_1, \ldots, A_{s+\beta+1} : s)| - 3}{k-3} + \binom{s+\beta+1}{s}(t-1) + 1 + M^{1-\frac{1}{3s}},$$

which is less than M when n is sufficiently large, a contradiction.

Assume that $|T(A_1, \ldots, A_{s+\beta+1} : s)| = \lfloor \frac{(s+\beta+1)k}{\beta+2} \rfloor = k$. In view of the last part of the proof of Lemma 14 (b), there are $1 \leqslant i_1 < \ldots < i_s \leqslant s+\beta+1$ such that

$$T(A_1, \ldots, A_{s+\beta+1} : s) = A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_s}.$$

This implies that s must be equal to 1. If $s = 1$, then we have $T(A_1, \ldots, A_{\beta+2} : 1) = \cap_{i=1}^{\beta+2} A_i$ and hence $|T(A_1, \ldots, A_{\beta+2} : 1)| = |\cap_{i=1}^{\beta+2} A_i| \leqslant k-1$ which contradicts with $|T(A_1, \ldots, A_{\beta+2} : 1)| = k$.

(iii) $\ell(\mathcal{F}) > M^{1-\frac{1}{3s}}$.

By Lemma B, we have $e(\mathrm{KG}_{n,k}[\mathcal{F}]) \geqslant \frac{M^{2-\frac{2}{3s}}}{2\binom{2k}{k}}$ and by Theorem E, $\mathcal{F}$ contains a subgraph which is isomorphic to $K_{s,t}$ when n is sufficiently large. ■

Note that perhaps for some k, s, and β there exist no pairwise distinct $A_1, \ldots, A_{s+\beta}$ satisfying Condition (2) in Theorem 7. For example, one may choose $k = 3, s = 3$, and $\beta = 5$. Thus, we have $\lfloor \frac{(s+\beta)k}{\beta+1} \rfloor = 4$. Since $\cup_{i=1}^{8} A_i = T(A_1, \ldots, A_8 : 3)$, if there exist $A_1, \ldots, A_8$ for which $|T(A_1, \ldots, A_8 : 3)| = 4$, then at least two of A_i's must be identical, which is not possible. Therefore, for some k, s, and β there do not exist any $A_1, \ldots, A_{s+\beta}$ such that $|T(A_1, \ldots, A_{s+\beta} : s)| = \lfloor \frac{(s+\beta)k}{\beta+1} \rfloor$. Consequently, as we showed in the proof of Theorem 7, each (s,t)-union intersecting family $\mathcal{F}$ is of size less than $\binom{n-1}{k-1} - \binom{n-\lfloor \frac{(s+\beta)k}{\beta+1} \rfloor -1}{k-1}$.

Here we intend to elaborate on the ith largest (s,t)-union intersecting families for some i. Assume that n is sufficiently large. Let $\{A_1, \ldots, A_s\}$ be s pairwise distinct k-subsets of $[n]$. By Definition 5 we know that $T(A_1, \ldots, A_s : s) = \cup_{i=1}^{s} A_i$. Define

$$\mathcal{L} \stackrel{\text{def}}{=} \mathcal{S}_1(A_1, \ldots, A_s : s) \cup \{A_1, \ldots, A_s\} \cup \{F_1, \ldots, F_{t-1}\}$$

where $F_i \in \mathcal{S}_1 \setminus \mathcal{S}_1(A_1, \ldots, A_s : s)$. By using Inequality (3), one can verify that $|\,\mathcal{L}\,|$ is equal to

$$\binom{n-1}{k-1} - \binom{n - |T(A_1, \ldots, A_s : s)| - 1}{k-1} + s + t - 1.$$

Let $n = n(k,s)$ be sufficiently large and $s \geqslant 2$. If $\lfloor \frac{(s+1)k}{2} \rfloor < |T(A_1, \ldots, A_s : s)| \leqslant sk$, then by using Theorem 7, $\mathcal{L}$ is the ith largest (s,t)-union intersecting family, where $i = sk - |T(A_1, \ldots, A_s : s)| + 2$.

If $|T(A_1, \ldots, A_s : s)| = \lfloor \frac{(s+1)k}{2} \rfloor$, then $|\,\mathcal{L}\,|$ is equal to $\binom{n-1}{k-1} - \binom{n-\lfloor \frac{(s+1)k}{2} \rfloor -1}{k-1} + s + t - 1$. Let $\{A'_1, \ldots, A'_{s+1}\}$ be $s+1$ pairwise distinct k-subsets of $[n]$ such that $T(A'_1, \ldots, A'_{s+1} : s) = \lfloor \frac{(s+1)k}{2} \rfloor$. Define

$$\mathcal{L}' \stackrel{\text{def}}{=} \mathcal{S}_1(A'_1, \ldots, A'_{s+1} : s) \cup \{A'_1, \ldots, A'_{s+1}\} \cup \{F'_1, \ldots, F'_{t-1}\}.$$

We have $|\,\mathcal{L}'\,|$ is equal to $\binom{n-1}{k-1} - \binom{n-\lfloor \frac{(s+1)k}{2} \rfloor -1}{k-1} + s + t$ which is greater than $|\,\mathcal{L}\,|$. Therefore, $\mathcal{L}'$ and $\mathcal{L}$ are the $(\lfloor \frac{(s-1)k}{2} \rfloor + 2)$th and $(\lfloor \frac{(s-1)k}{2} \rfloor + 3)$th largest (s,t)-union intersecting families, respectively.

Now assume that there are two families $\{A_1, \ldots, A_s\}$ and $\{A'_1, \ldots, A'_{s+1}\}$ such that $|T(A_1, \ldots, A_s : s)| = \lfloor \frac{(s+1)k}{2} \rfloor - 1$ and $|T(A'_1, \ldots, A'_{s+1} : s)| = \lfloor \frac{(s+1)k}{2} \rfloor - 1$. If $(s+1)k$ is even, then $2|T(A'_1, \ldots, A'_{s+1} : s)| = (s+1)k - 2$. Therefore, there are at most two members in $\cup_{i=1}^{s+1} A'_i$ such that each of them appears in one of A'_i's. If for each $i \leqslant s+1$ we have $A'_i \subset T(A'_1, \ldots, A'_{s+1} : s)$, in view of Expression (2), we obtain

$$|\,\mathcal{L}'\,| = \binom{n-1}{k-1} - \binom{n - \lfloor \frac{(s+1)k}{2} \rfloor}{k-1} + s + t.$$

If for only one $i \leqslant s+1$ we have $A'_i \not\subseteq T(A'_1, \ldots, A'_{s+1} : s)$, then one can construct an (s,t)-union intersecting family $\mathcal{L}'_1$ with $\ell(\mathcal{L}'_1) = s+1$ and

$$|\,\mathcal{L}'_1\,| = \binom{n-1}{k-1} - \binom{n - \lfloor\frac{(s+1)k}{2}\rfloor}{k-1} + s + t.$$

Now suppose that $A'_i \not\subseteq T(A'_1, \ldots, A'_{s+1} : s)$ and $A'_j \not\subseteq T(A'_1, \ldots, A'_{s+1} : s)$ for exactly two $1 \leqslant i \neq j \leqslant s+1$. In view of Expression (2), one easily sees that the number of elements in $\mathcal{S}_1$ which has no common element with $T(A'_1, \ldots, A'_{s+1} : s)$ and intersects at least two of A'_i's is $\binom{n-|T(A'_1,\ldots,A'_{s+1}:s)|-3}{k-3}$. Therefore, for $0 \leqslant m \leqslant t-1$, one can construct a maximal (s,t)-union family $\mathcal{L}'_{2,m}$ with $\ell(\mathcal{L}'_{2,m}) = s+1$ and

$$|\,\mathcal{L}'_{2,m}\,| = \binom{n-1}{k-1} - \binom{n - \lfloor\frac{(s+1)k}{2}\rfloor}{k-1} + \binom{n - \lfloor\frac{(s+1)k}{2}\rfloor - 2}{k-3} + s + t + m.$$

Therefore, we have some different types (s,t)-union intersecting families with $\ell(\mathcal{F}) = s+1$, $|T(A'_1, \ldots, A'_{s+1} : s)| = \lfloor\frac{(s+1)k}{2}\rfloor - 1$, and different sizes and one type of (s,t)-union intersecting families with $\ell(\mathcal{F}) = s$, $|T(A_1, \ldots, A_s : s)| = \lfloor\frac{(s+1)k}{2}\rfloor - 1$.

If $(s+1)k$ is odd, then $2|T(A'_1, \ldots, A'_{s+1} : s)| = (s+1)k - 3$. Therefore, there are at most three members in $\cup_{i=1}^{s+1} A'_i$ such that each of them appears in one of A'_i's. Using the same discussion as above one can find some different types of (s,t)-union intersecting families with $\ell(\mathcal{F}) = s+1$, $|T(A'_1, \ldots, A'_{s+1} : s)| = \lfloor\frac{(s+1)k}{2}\rfloor - 1$, and different sizes.

In the proof of Theorem 9, we need the following theorem by Frankl [13] and independently Kalai [26] which is a generalization of a classical result due to Bollobás [3].

Theorem F. [13,26] *Let k and ℓ be two positive integers and let $\{(A_1, B_1), \ldots, (A_h, B_h)\}$ be a family of pairs of subsets of an arbitrary set with $|A_i| = k$ and $|B_i| = \ell$ for all $1 \leqslant i \leqslant h$. If $A_i \cap B_i = \varnothing$ for $1 \leqslant i \leqslant h$ and $A_i \cap B_j \neq \varnothing$ for $1 \leqslant i < j \leqslant h$, then $h \leqslant \binom{k+\ell}{k}$.*

For simplicity of notation, for each $1 \leqslant i \leqslant k-1$, define $N_i \overset{\text{def}}{=} \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-i}{k-i-1}$ and for k define $N_k \overset{\text{def}}{=} \binom{n-1}{k-1} - \binom{n-k}{k-1}$. Note that for $1 \leqslant i \leqslant k-1$, we have that $N_{i-1} - N_i = \binom{n-k-i}{k-i} = \Omega(n^{k-i})$.

Proof of Theorem 9. First we show that $\ell(\mathcal{F}) \leqslant \binom{2k-1}{k-1}(t-1)$. If $t = 1$, $\mathcal{F}$ is intersecting and hence $\ell(\mathcal{F}) = 0$. Assume that $t \geqslant 2$ and $\mathcal{F}$ is not intersecting. Therefore, there exists some disjoint pair in $\mathcal{F}$. For a k-set A, define $N(A) = \{B \in \binom{[n]}{k} |\, A \cap B = \varnothing\}$. Define $\mathcal{F}_1 = \mathcal{F}$. For each $i \geqslant 2$, if there exists some disjoint pair in $\mathcal{F}_{i-1}$, choose $B_{i-1} \in \mathcal{F}_{i-1}$ and $C_{i-1} \in N(B_{i-1}) \cap \mathcal{F}_{i-1}$ and define $\mathcal{F}_i = \mathcal{F}_{i-1} \setminus (N(B_{i-1})$. Let m be the largest index i for which $\mathcal{F}_i$ contains some disjoint pair. For $m+1 \leqslant j \leqslant 2m$, set $B_j = C_{2m-j+1}$ and $C_j = B_{2m-j+1}$. One may check that the family$\{(B_1, C_1), \ldots, (B_{2m}, C_{2m})\}$ satisfies the condition of Theorem F for $l = k$ and consequently $m \leqslant \binom{2k-1}{k-1}$. Let $\mathcal{N}$ be a subfamily of $\mathcal{F}$ defined as follows

$$\mathcal{N} = \Big\{F \in \mathcal{F}\,|\text{there is some } i \leqslant m \text{ such that } F \cap B_i = \varnothing\Big\}.$$

Since $\mathcal{F}$ is $(1,t)$-union intersecting, one can verify that $|\mathcal{N}| \leqslant m(t-1)$. Note that $\mathcal{F}_{m+1}$ is an intersecting family and $\mathcal{F}$ is disjoint union of $\mathcal{F}_{m+1}$ and $\mathcal{N}$. This yields $\ell(\mathcal{F}) \leqslant |\mathcal{N}| \leqslant \binom{2k-1}{k-1}(t-1)$.

Assume that $|\mathcal{F}| = N_\gamma + \gamma t$. Let $\mathcal{F}^*$ be one of largest intersecting subfamilies of $\mathcal{F}$ such that $\Delta(\mathcal{F}^*)$ has the maximum possible value. Assume that $\mathcal{F} \setminus \mathcal{F}^* = \{A_1, \ldots, A_{\ell(\mathcal{F})}\}$. Therefore, $|\mathcal{F}^*| = |\mathcal{F}| - \ell(\mathcal{F})$. Consider the following three cases.

1. $\ell(\mathcal{F}) = \gamma$ *and* $\mathcal{F}^* \subseteq \mathcal{S}_1$.

   We have $|\mathcal{F}^*| = N_\gamma + \gamma(t-1)$. Since $\ell(\mathcal{F}) = \gamma$ and $\mathcal{F} = \mathcal{F}^* \cup \{A_1, \ldots A_\gamma\}$, each A_j is disjoint from at least one member of $\mathcal{F}^*$ and hence $1 \notin \cup_{j=1}^{\gamma} A_j$. Then

   $$\mathcal{F}^* \setminus (\cup_{j=1}^{\gamma} N(A_j)) \subseteq \mathcal{S}_1(A_1, \ldots, A_\gamma : 1).$$

   Since $\gamma \leqslant k-2$, by applying Lemma 14 (c), we conclude that $|\mathcal{F}^* \setminus (\cup_{j=1}^{\gamma} N(A_j))| \leqslant N_\gamma$. Since $\mathcal{F}$ is $(1,t)$-union intersecting, for each j, A_j is disjoint from at most $t-1$ members of $\mathcal{F}$. As for each j, $|N(A_j) \cap \mathcal{F}| \leqslant t-1$, $|\mathcal{F}| = N_\gamma + \gamma t$, and

   $$\mathcal{F} = \mathcal{F}^* \setminus (\cup_{j=1}^{\gamma} N(A_j)) \cup (\cup_{j=1}^{\gamma} N(A_j) \cap \mathcal{F}) \cup \{A_1, \ldots, A_\gamma\},$$

   we have $\mathcal{F}$ is a disjoint union of

   $$\mathcal{F}^* \setminus (\cup_{j=1}^{\gamma} N(A_j)), N(A_1) \cap \mathcal{F}, \ldots, N(A_\gamma) \cap \mathcal{F}, \text{ and } \{A_1, \ldots, A_\gamma\}.$$

   Moroever, for each j, we have $|N(A_j) \cap \mathcal{F}| = t-1$, $N(A_j) \cap \mathcal{F} \subseteq \mathcal{F}^* \subseteq \mathcal{S}_1$, and $|\mathcal{F}^* \setminus (\cup_{j=1}^{\gamma} N(A_j))| = N_\gamma$. From the last equality and by using Lemma 14 (c), we obtain

   $$\mathcal{F}^* \setminus (\cup_{j=1}^{\gamma} N(A_j)) = \mathcal{S}_1(A_1, \ldots, A_\gamma : 1)$$

   and $|\cap_{j=1}^{\gamma} A_j| = k-1$. By taking $E = \cap_{j=1}^{\gamma} A_j$ and $J = \{1\} \cup (\cup_{j=1}^{\gamma} A_j \setminus E)$ in Definition 1, one can see that $\mathcal{F} \setminus (\cup_{j=1}^{\gamma} N(A_j))$ is isomorphic to $\mathcal{J}_\gamma$. For each $j \leqslant \beta + 1$, by taking $\mathcal{B}_j = N(A_j) \cap \mathcal{F}$ in Definition 8, one can check that $\mathcal{F}$ is isomorphic to $\mathcal{J}_\gamma^{1,t}$. By Theorem C, $\mathcal{F}^*$ is either a star or isomorphic to a subfamily $\mathcal{J}_i$ where $0 \leqslant i \leqslant \gamma - 1$.

2. $\gamma + 1 \leqslant \ell(\mathcal{F}) \leqslant \binom{2k-1}{k-1}(t-1)$ *and* $\mathcal{F}^* \subseteq \mathcal{S}_1$.

   Let $A_1, \ldots, A_{\gamma+1} \in \mathcal{F} \setminus \mathcal{F}^*$. By using minimality of $\ell(\mathcal{F})$, each A_i is disjoint from at least one member of $\mathcal{F}^*$. Therefore, $1 \notin A_i$ for each $i \leqslant \gamma + 1$. Then

   $$\mathcal{F}^* \setminus ((\cup_{i=1}^{\gamma+1} N(A_i)) \subseteq \mathcal{S}_1(A_1, \ldots, A_{\gamma+1} : 1)$$

   and by applying Lemma 14 (c), we obtain $|\mathcal{F}^* \setminus ((\cup_{i=1}^{\gamma+1} N(A_i))| \leqslant N_{\gamma+1}$. Since

   $$\mathcal{F} = (\mathcal{F}^* \setminus (\cup_{j=1}^{\gamma+1} N(A_j)) \cup (\cup_{i=1}^{\gamma+1} N(A_i) \cap \mathcal{F}) \cup \{A_1, \ldots, A_{\ell(\mathcal{F})}\},$$

   we have $|\mathcal{F}| \leqslant N_{\gamma+1} + (\gamma+1)(t-1) + \ell(\mathcal{F}) < N_\gamma$, which is not possible when n is sufficiently large.

3. $\gamma \leqslant \ell(\mathcal{F}) \leqslant \binom{2k-1}{k-1}(t-1)$ *and* $\mathcal{F}^*$ *is not a star.*

By Theorem C, $\mathcal{F}^* \subseteq \mathcal{J}_c$ for some $1 \leqslant c \leqslant \beta+1$. Then, for some $b \leqslant c$, there exist $B_1, \ldots, B_b \in \mathcal{F}^*$ such that $\mathcal{F}^* \setminus \{B_1, \ldots, B_b\} \subseteq \mathcal{S}_1$ and $B_j \notin \mathcal{S}_1$. At most $b-1$ of $A_1, \ldots, A_\gamma$ contain 1; otherwise if for $1 \leqslant j_1 \leqslant \cdots \leqslant j_b \leqslant \gamma$ we have $1 \in \cap_{i=1}^{b} A_{j_i}$, then $\mathcal{F}' = (\mathcal{F}^* \setminus \{B_1, \ldots, B_b\}) \cup \{A_{j_1}, \ldots, A_{j_b}\}$ is an intersecting family with $|\,\mathcal{F}'\,| = |\,\mathcal{F}^*\,|$ and $\Delta(\mathcal{F}') > \Delta(\mathcal{F}^*)$, which contradicts with the fact that $\Delta(\mathcal{F}^*)$ has the maximum possible value. Therefore, without loss of generality we can assume that $A_1, \ldots, A_{b'}$ do not contain 1 for $b' = \gamma + 1 - b$. Hence,

$$\mathcal{F}^* \setminus ((\cup_{j=1}^{b'} N(A_j)) \cup \{B_1, \ldots, B_b\}) \subseteq \mathcal{S}_1(A_1, \ldots, A_{b'}, B_1, \ldots, B_b : 1)$$

and by Lemma 14 (c), we obtain $|\,\mathcal{F}^* \setminus ((\cup_{j=1}^{b'} N(A_j)) \cup \{B_1, \ldots, B_b\})| \leqslant N_{\gamma+1}$. Since

$$\mathcal{F} = (\mathcal{F}^* \setminus \cup_{j=1}^{b'} N(A_j)) \cup (\cup_{j=1}^{b'} N(A_j) \cap \mathcal{F}) \cup \{A_1, \ldots, A_{\ell(\mathcal{F})}\},$$

we obtain $|\,\mathcal{F}\,| \leqslant N_{\gamma+1} + b + b'(t-1) + \ell(\mathcal{F}) < N_\gamma$, which is not possible when n is sufficiently large. ■

It can be seen that the next corollary is a direct consequence of Theorem 9. Notice that we need to apply Theorem C to prove it.

Corollary 15. *Let n, $k \geqslant 5$, $t \geqslant 1$, and $\gamma \leqslant k-2$ be nonnegative integers such that $n = n(k, t, \gamma)$ is sufficiently large. Let $\mathcal{F}$ be a $(1, t)$-union intersecting family that is not isomorphic to a subfamily of $\mathcal{J}_i \cup \mathcal{B}$ where $\mathcal{B} \subseteq \mathcal{S}_1 \setminus \mathcal{J}_i$ and $0 \leqslant i \leqslant \gamma - 1$. Then*

$$|\,\mathcal{F}\,| \leqslant \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-\gamma}{k-\gamma-1} + \gamma t.$$

Equality holds if and only if $\mathcal{F}$ is isomorphic to some $\mathcal{J}_\gamma^{1,t}$.

For the proof of Theorem 11 we need to use the well-known Erdős-Stone-Simonovits theorem [10, 11]. For a given graph G, the *Turán number* $\mathrm{ex}(n, G)$ is defined to be the maximum number of edges in a graph with n vertices containing no subgraph isomorphic to G. The Erdős-Stone-Simonovits theorem asserts that for any graph G with $\chi(G) \geqslant 2$, $\mathrm{ex}(G, n) = (1 - \frac{1}{\chi(G)-1})\binom{n}{2} + o(n^2)$.

Proof of Theorem 11. The proof is by induction on r. By Theorem 7, the assertion is true when $r = 1$. Let $r \geqslant 2$. Suppose now that the assertion is true for $r-1$. Also, without loss of generality suppose that

$$|\,\mathcal{F}\,| = \binom{n}{k} - \binom{n-r}{k} - \binom{n - \lfloor \frac{(s_{r+1}+\beta)k}{\beta+1} \rfloor - r}{k-1} + s_r + s_{r+1} + \hat{\beta} - 1.$$

Consider the following cases.

1. $\max_{i\in[n]} |\mathcal{F}\cap\mathcal{S}_i| \leqslant \binom{n-1}{k-1} - \binom{n-\sum_{j=2}^{r+1} s_j k-1}{k-1} + s_1$.

   Then the number of disjoint pair in $\mathcal{F}$ is at least

$$\binom{|\mathcal{F}|}{2} - \sum_{i\in[n]} \binom{|\mathcal{F}\cap\mathcal{S}_i|}{2} \geqslant (1-\frac{1}{r})\binom{|\mathcal{F}|}{2} + o(|\mathcal{F}|^2)$$

   provided that n is large enough. Hence, by the Erdős-Stone-Simonovits theorem $\mathrm{KG}_{n,k}[\mathcal{F}]$ contains some subgraph isomorphic to $K_{s_1,s_2,\ldots,s_{r+1}}$ provided that n is large enough, which is a contradiction.

2. $\max_{i\in[n]} |\mathcal{F}\cap\mathcal{S}_i| > \binom{n-1}{k-1} - \binom{n-\sum_{j=2}^{r+1} s_j k-1}{k-1} + s_1$.

   Without loss of generality assume that $\max_{i\in[n]} |\mathcal{F}\cap\mathcal{S}_i| = |\mathcal{F}\cap\mathcal{S}_n|$. If $\mathcal{S}_n \not\subset \mathcal{F}$, then $|\mathcal{F}\cap\mathcal{S}_n| < \binom{n-1}{k-1}$. Therefore,

$$|\mathcal{F}\setminus\mathcal{S}_n| \geqslant \binom{n-1}{k} - \binom{n-r}{k} - \binom{n-\lfloor\frac{(s_{r+1}+\beta)k}{\beta+1}\rfloor - r}{k-1} + s_r + s_{r+1} + \hat{\beta}.$$

   By induction hypothesis $\mathrm{KG}_{n-1,k}[\mathcal{F}\setminus\mathcal{S}_n]$ contains a copy $K_{s_2,\ldots,s_{r+1}}$. As

$$|\mathcal{F}\cap\mathcal{S}_n| > \binom{n-1}{k-1} - \binom{n-\sum_{j=2}^{r+1} s_j k-1}{k-1} + s_1,$$

   one can greedily pick s_1 sets of $\mathcal{S}_n$ such that constructs a copy of $K_{s_1,s_2,\ldots,s_{r+1}}$ in $\mathrm{KG}_{n,k}[F]$, a contradiction. Therefore, one can assume that $\mathcal{S}_n \subset \mathcal{F}$. Similarly as before $\mathrm{KG}_{n-1,k}[\mathcal{F}\setminus\mathcal{S}_n]$ does not contain any copy of $K_{s_2,\ldots,s_{r+1}}$. Therefore, by induction hypothesis, we have

$$|\mathcal{F}\setminus\mathcal{S}_n| \leqslant \binom{n-1}{k} - \binom{n-r}{k} - \binom{n-\lfloor\frac{(s_{r+1}+\beta)k}{\beta+1}\rfloor - r}{k-1} + s_r + s_{r+1} + \hat{\beta} - 1,$$

   and the equality holds if and only if $\mathcal{F}\setminus\mathcal{S}_n$ is isomorphic to

$$\bigcup_{i=1}^{r-2}(\mathcal{S}_i\setminus\mathcal{S}_n) \cup (\mathcal{S}_{r-1}^{[r-2]}(A_1,A_2,\ldots,A_{s_{r+1}+\hat{\beta}}:s)\setminus\mathcal{S}_n) \cup \{A_1,A_2,\ldots,A_{s_{r+1}+\hat{\beta}}\} \cup \{F_1,\ldots,F_{s_r-1}\}$$

   such that

$$|T(A_1,A_2,\ldots,A_{s_{r+1}+\hat{\beta}})| = \left\lfloor \frac{(s_{r+1}+\beta)k}{\beta+1} \right\rfloor,$$

   $F_i \in \mathcal{S}_{r-1}\setminus\mathcal{S}_{r-1}^{[r-2]}(A_1,A_2,\ldots,A_{s+\hat{\beta}}:s)$, and $F_i\cap[r-2]=\varnothing$ for each i (Note that in this step all families are subfamilies of $\binom{[n-1]}{k}$ because we remove $\mathcal{S}_n$ from $\mathcal{F}$ so we do not meet n.).

   Thus,

$$|\mathcal{F}| \leqslant \binom{n}{k} - \binom{n-r}{k} - \binom{n-\lfloor\frac{(s_{r+1}+\beta)k}{\beta+1}\rfloor - r}{k-1} + s_r + s_{r+1} + \hat{\beta} - 1,$$

and the equality holds if and only if $\mathcal{F}$ is isomorphic to

$$\bigcup_{i=1}^{r-1} \mathcal{S}_i \cup \mathcal{S}_r^{[r-1]}(A_1, A_2, \ldots, A_{s_{r+1}+\hat{\beta}} : s) \cup \{A_1, A_2, \ldots, A_{s_{r+1}+\hat{\beta}}\} \cup \{F_1, \ldots, F_{s_r-1}\}$$

such that $|T(A_1, A_2, \ldots, A_{s_{r+1}+\hat{\beta}})| = \lfloor \frac{(s_{r+1}+\beta)k}{\beta+1} \rfloor$, $F_i \in \mathcal{S}_r \setminus \mathcal{S}_r^{[r-1]}(A_1, A_2, \ldots, A_{s+\hat{\beta}} : s)$, and $F_i \cap [r-1] = \varnothing$ for each i. ■

The proof of Theorem 13 is the same as the proof of Theorem 11.

Acknowledgements

The author is grateful to Meysam Alishahi and Amir Daneshgar for their valuable comments. This research was in part supported by a grant from IPM (No. 98050012).

References

[1] M. Alishahi and A. Taherkhani. Extremal G-free induced subgraphs of Kneser graphs. *J. Combin. Theory Ser. A*, 159:269–282, 2018.

[2] J. Balogh, B. Bollobás, and B. P. Narayanan. Transference for the Erdős–Ko–Rado theorem. *Forum Math. Sigma*, 3:e23, 18, 2015.

[3] B. Bollobás. On generalized graphs. *Acta Math. Acad. Sci. Hungar.*, 16:447–452, 1965.

[4] B. Bollobás, D. E. Daykin, and P. Erdős. Sets of independent edges of a hypergraph. *Quart. J. Math. Oxford Ser. (2)*, 27(105):25–32, 1976.

[5] B. Bollobás, B. P. Narayanan, and A.M. Raigorodskii. On the stability of the Erdős-Ko-Rado theorem. *J. Combin. Theory Ser. A*, 137:64–78, 2016.

[6] M. Deza and P. Frankl. Erdős-Ko-Rado theorem-22 years later. *SIAM Journal on Algebraic Discrete Methods*, 4(4):419–431, 1983.

[7] P. Erdős. A problem on independent r-tuples. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 8:93–95, 1965.

[8] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar*, 10:337–356 (unbound insert), 1959.

[9] P. Erdős, C. Ko, and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12:313–320, 1961.

[10] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar*, 1:51–57, 1966.

[11] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946.

[12] P. Frankl. On intersecting families of finite sets. *J. Combin. Theory Ser. A*, 24(2):146–161, 1978.

[13] P. Frankl. An extremal problem for two families of sets. *European J. Combin.*, 3(2):125–127, 1982.

[14] P. Frankl. An Erdős-Ko-Rado theorem for direct products. *European J. Combin.*, 17(8):727–730, 1996.

[15] P. Frankl. Improved bounds for Erdős' matching conjecture. *J. Combin. Theory Ser. A*, 120(5):1068–1072, 2013.

[16] P. Frankl. On the maximum number of edges in a hypergraph with given matching number. *Discrete Appl. Math.*, 216(part 3):562–581, 2017.

[17] P. Frankl and Z. Füredi. A new short proof of the EKR theorem. *J. Combin. Theory Ser. A*, 119(6):1388 – 1390, 2012.

[18] P. Frankl and A. Kupavskii. Two problems on matchings in set families – in the footsteps of Erdős and Kleitman. *Journal of Combinatorial Theory, Series B*, 138:286 – 313, 2019.

[19] P. Frankl, T. Łuczak, and K. Mieczkowska. On matchings in hypergraphs. *Electron. J. Combin.*, 19(2):#P42, 5, 2012.

[20] D. Gerbner, N. Lemons, C. Palmer, B. Patkós, and V. Szécsi. Almost intersecting families of sets. *SIAM J. Discrete Math.*, 26(4):1657–1669, 2012.

[21] D. Gerbner, A. Methuku, D. Nagy, B. Patkós, and M. Vizer. Stability results for vertex Turán problems in Kneser graphs. *Electron. J. Combin.*, 26(2):#P2.13, 12, 2019.

[22] C. Godsil and K. Meagher. A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations. *European J. Combin.*, 30(2):404–414, 2009.

[23] J. Han and Y. Kohayakawa. The maximum size of a non-trivial intersecting uniform family that is not a subfamily of the Hilton-Milner family. *Proc. Amer. Math. Soc.*, 145(1):73–87, 2017.

[24] A. J. W. Hilton and E. C. Milner. Some intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 18:369–384, 1967.

[25] H. Huang, P.-S. Loh, and B. Sudakov. The size of a hypergraph and its matching number. *Combin. Probab. Comput.*, 21(3):442–450, 2012.

[26] G. Kalai. Intersection patterns of convex sets. *Israel J. Math.*, 48(2-3):161–174, 1984.

[27] G. O. H. Katona. A simple proof of the Erdős-Chao Ko-Rado theorem. *J. Combin. Theory Ser. B*, 13(2):183 – 184, 1972.

[28] G. O. H. Katona and D. T. Nagy. Union-intersecting set systems. *Graphs Combin.*, 31(5):1507–1516, 2015.

[29] A. Kostochka and D. Mubayi. The structure of large intersecting families. *Proc. Amer. Math. Soc.*, 145(6):2311–2321, 2017.

[30] T. Kővari, V. T. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Mathematicae*, 3(1):50–57, 1954.

[31] C. Y. Ku and I. Leader. An Erdős-Ko-Rado theorem for partial permutations. *Discrete Math.*, 306(1):74–86, 2006.

[32] A. Kupavskii. Structure and properties of large intersecting families. arXiv:1810.00920, October 2018.

[33] Y.-S. Li and J. Wang. Erdős-Ko-Rado-type theorems for colored sets. *Electron. J. Combin.*, 14(1):#R1, 9, 2007.

[34] T. Łuczak and K. Mieczkowska. On Erdős' extremal problem on matchings in hypergraphs. *J. Combin. Theory Ser. A*, 124:178–194, 2014.

[35] R. M. Wilson. The exact bound in the Erdős-Ko-Rado theorem. *Combinatorica*, 4(2-3):247–257, 1984.