Trees contained in every orientation of a graph

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Abstract

For every graph G, let $t(G)$ denote the largest integer t such that every oriented tree of order t appears in every orientation of G. In 1980, Burr conjectured that $t(G) \geqslant 1 + \chi(G)/2$ for all G, and showed that $t(G) \geqslant 1 + \lfloor\sqrt{\chi(G)}\rfloor$; this bound remains the state of the art, apart from the multiplicative constant. We present an elementary argument that improves this bound whenever G has somewhat large chromatic number, showing that $t(G) \geqslant \lfloor \chi(G)/\log_2 v(G)\rfloor$ for all G.

Mathematics Subject Classifications: 05C05, 05C20

1 Introduction

If G is a graph and D is an oriented graph, we write $G \to D$ to mean that every orientation of G contains D, and write $t(G)$ to denote the largest integer t such that $G \to T$ for every oriented tree T of order t. Note that $t(G)$ is at most the chromatic number $\chi(G)$, since if we colour G properly with 'colours' $\{1, \ldots, k\}$, then we may direct each edge towards its endvertex of greater colour, obtaining an orientation of G in which each *directed path* $v_1 \to \cdots \to v_t$ contains at most one vertex of each colour, so $t \leqslant k$. Over forty years ago, Burr [4] proved that $t(G) \geqslant 1 + \lfloor\sqrt{\chi(G)}\rfloor$ for every graph G. He conjectured the following.

Conjecture 1 ([4])**.** If G is a graph, then $t(G) \geqslant 1 + \chi(G)/2$.

The current best lower bound for Conjecture 1, which improves Burr's result by a multiplicative constant, was obtained by Addario-Berry, Havet, Reed, Sales and Thomassé [1].

Theorem 2 ([1])**.** *If G is a graph, then $t(G) \geqslant 1/2 + \sqrt{2\chi(G) - 7/4}$.*

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We establish a lower bound for $t(G)$ in terms of both the chromatic number $\chi(H)$ and the order $v(H)$ of $H \subseteq G$. Our bound supersedes Theorem 2 when $\chi(G) > (2\log_2 v(G))^2$, which is the case for many graphs (see Section 3). Our proof is inductive, and was inspired by an argument of Wormald [16] (the strategy was also explicitly suggested in [1]). We define $e(G) \coloneqq |E(G)|$.

Theorem 3. *If G is a graph and $e(G) > 0$, then* $t(G) \geqslant \max\limits_{\substack{H\subseteq G\\ e(H)>0}} \left\lfloor \dfrac{\chi(H)}{\log_2 v(H)} \right\rfloor$.

2 Proofs

All *digraphs* we consider are *oriented graphs*, i.e., they are obtained replacing each edge of an undirected simple graph by precisely one *arc* (ordered pair) with the same elements. For every digraph D and every $S \subseteq V(D)$, we call $N_D^-(S) \coloneqq \{x : (x,s) \in E(D), s \in S\}$ the set of *inneighbours* of S; the set of *outneighbours* of S is $N_D^+(S) \coloneqq \{x : (s,x) \in E(D), s \in S\}$. We call S *dominating* if $S \cup N_D^+(S) = V(D)$, and *anti-dominating* if $S \cup N_D^-(S) = V(D)$. If $v \in V(D)$, we write $N_D^-(v)$ and $N_D^+(v)$ to denote $N_D^-(\{v\})$ and $N_D^+(\{v\})$, respectively. We omit subscripts when they are clear from context. We call S *stable* if the digraph $D[S]$ induced by S contains no arcs. An *in-leaf* is a vertex with precisely one outneighbour and no inneighbours; an *out-leaf* is defined analogously.

Theorem 3 follows from two simple lemmas.

Lemma 4. *Each oriented graph D has a stable set S such that $\left|N^+(S)\right| \geqslant \left|V(D) \setminus S\right|/2$.*

Proof. Let $D_1 \coloneqq D$. For each $i = 1, 2, \dots$ proceed as follows. Fix $v_i \in V(D_i)$ such that

$$\left|N_{D_i}^+(v_i)\right| \geqslant \left|N_{D_i}^-(v_i)\right| \tag{1}$$

(such v_i exists by averaging), and form $D_{i+1} \coloneqq D_i - \big(\{v_i\} \cup N_{D_i}^-(v_i) \cup N_{D_i}^+(v_i)\big)$ from D_i by removing v_i and its neighbours. Note that $S_i \coloneqq \{v_1, v_2, \dots, v_i\}$ is a stable set of D, and $v(D_1), v(D_2), \dots$ strictly decreases until $v(D_m) = 0$ for some integer m. By (1), we may take $S \coloneqq S_{m-1}$. $\square$

Applying Lemma 4 repeatedly, we obtain the next result. The *chromatic number* of a digraph is the chromatic number of its underlying (undirected) graph.

Lemma 5. *Each oriented graph D with $e(D) > 0$ contains a dominating set S such that $D[S]$ is acyclic and $\chi\big(D[S]\big) \leqslant \log_2 v(D)$.*

Proof. We argue by induction on the order n of D. The result is trivial if $n = 2$, so we may suppose $n > 2$. By Lemma 4, there exists a maximal stable set $I \subseteq V(D)$ such that $|N^-(I) \setminus N^+(I)| \leqslant (n - |I|)/2$. Let $D' \coloneqq D - \big(I \cup N^+(I)\big) = D\big[N^-(I) \setminus N^+(I)\big]$. By induction, there exists a dominating set $S' \subseteq V(D')$ of D' such that $D[S']$ is acyclic and $\chi(D[S']) = \chi(D'[S']) \leqslant \log_2(v(D')) \leqslant \log_2(n/2)$.

Let $S \coloneqq D[I \cup S']$. Clearly $\chi(D[S]) \leqslant 1 + \chi(D[S']) \leqslant \log_2 n$ and S is a dominating set of D. Note that $S' \subseteq V(D') = N^-(I) \setminus N^+(I)$, so each edge between S' and I is directed from S' to I. We conclude that $D[S]$ is acyclic (since $D[S']$ is acyclic and I is stable). $\square$

Note that Lemma 5 holds if we replace dominating by anti-dominating.

Proof of Theorem 3. Let D be an oriented graph and fix $H \subseteq D$ attaining the maximum value of $t \coloneqq \lfloor \chi(H)/\log_2 v(H) \rfloor$. Let $n \coloneqq v(H)$ and let T be an oriented tree of order t. We will show that H contains a copy of T. The proof is by induction on t. Note that $n \geqslant 2$ and $t \geqslant 1$. If $t \in \{1, 2\}$, then the result follows trivially since $\chi(H) \geqslant t \log_2 n \geqslant t$. For the induction step, suppose that $t \geqslant 3$ and that T contains an out-leaf v (respectively, in-leaf). By Lemma 5, H contains an anti-dominating set S (respectively, dominating) such that $\chi(H[S]) \leqslant \log_2(n)$. Let $H' \coloneqq H - S$ and $n' \coloneqq v(H') = n - |S|$. Since $t \geqslant 3$, we have $\chi(H) \geqslant 3 \log_2 n$. Therefore $n' \geqslant \chi(H') \geqslant \chi(H) - \chi(H[S]) \geqslant 2 \log_2 n \geqslant 2$, so $e(H') > 0$. By induction, H' contains every oriented tree of order

$$\left\lfloor \frac{\chi(H')}{\log_2(n')} \right\rfloor \geqslant \left\lfloor \frac{\chi(H) - \log_2 n}{\log_2(n - |S|)} \right\rfloor \geqslant \left\lfloor \frac{\chi(H) - \log_2 n}{\log_2 n} \right\rfloor \geqslant t - 1,$$

so H' contains a copy of $T - v$. Since each vertex of H' has an outneighbour (respectively, inneighbour) in S, it follows that H contains a copy of T. $\square$

3 Concluding remarks

A simple first moment calculation (see, e.g., [12]) shows that for every positive ε and sufficiently large k, a typical orientation of any k-chromatic graph G contains every oriented tree of order $(1 - \varepsilon)k/\log k$. In this note we improve this, showing that 'typical' can be replaced by 'every' whenever $\chi(G) > v(G)^{1-\varepsilon}$, a condition that holds for many graphs.

Let $n \in \mathbb{N}$ and $p \in (0, 1)$. The *binomial random graph* $G(n, p)$ is obtained from the complete graph K_n by independently deleting each of its edges with probability $1 - p$. Note, for instance, that $G(n, 1/2)$ is uniformly distributed over all labeled graphs of order n. A celebrated result of Bollobás [3] implies that $\chi\big(G(n, 1/2)\big) = \big(1 + \mathrm{o}(1)\big)n/2 \log_2 n$ with probability $1 - \mathrm{o}(1)$ as $n \to \infty$. Hence, by Theorem 3, almost every graph G satisfies

$$\big(1 - \mathrm{o}(1)\big) \frac{\chi(G)}{\log_2 \chi(G)} \leqslant t(G) \leqslant \chi(G).$$

Question 6. What is the typical value $t\big(G(n, 1/2)\big)$?

We remark that embedding arbitrarily oriented oriented trees covering about half the vertices of a tournament was a longstanding open problem, known as Sumner's conjecture, until Kühn, Mycroft and Osthus [10, 11] proved that $K_{2n-2} \to T$ for every oriented tree T of order n (where n is sufficiently large). Together with Theorem 3, this suggests that the difficult cases of Burr's conjecture are graphs which are chromatically sparse (i.e., have low chromatic number relative to their order, but contain no small subgraph which 'witnesses' the chromatic number).

It is natural to attempt to replace $v(H)$ by some smaller function $f_G(H)$ in Theorem 3. This would follow by improving the bound in Lemma 4.

Question 7 ([1])**.** What is the largest integer $f = f(k)$ such that for each digraph D with $\chi(D) = k$ there is $S \subseteq V(D)$ such that $N \coloneqq N_D^+(S) \setminus S$ satisfies $\chi\big(D[N]\big) \geqslant f$?

We note that $f(k) \geqslant k/2-1$. Indeed, every acyclic digraph contains a stable dominating set, and every oriented graph D can be decomposed into edge-disjoint acyclic digraphs A, B; a stable dominating set S' of A is acyclic in D, and either $\chi(D - S') \geqslant k/2$ (take $S = S'$) or $\chi(D[S']) \geqslant k/2$ (take S to be a stable dominating set of $D[S']$).

We mention yet another direction of research. For each digraph H, let $q(H)$ denote the smallest integer q such that $G \to H$ for every graph G with $\chi(G) \geqslant q$. Note that $q(H) < \infty$ if and only if H is a forest, since, as shown by Erdős, there exist graphs with arbitrarily large chromatic number and girth [6]. Conjecture 1 is equivalent to the statement '$q(T) \leqslant 2v(T) - 2$ for every oriented tree T'. The value of $q(T)$ is only known for stars [4], directed paths [7, 8, 13, 15] and paths formed by concatenating two directed paths [2]. More precisely, $q(S)$ is either $2v(S) - 3$ or $2v(S) - 2$ if S is an oriented star, and $q(P) = v(P)$ if P is the concatenation of two directed paths, where we consider paths of order 3 as stars. For more about Conjecture 1, see, e.g., [2, 12, 14].

Question 8. What is the typical value of $q(T)$ if T is a labeled oriented tree of order t chosen uniformly at random?

Dross and Havet proved that $K_n \to T$ for every oriented tree T of order t with k leaves, where $n = \min\{21v(T)/8 - 47/16, v(T) + 144k^2 - 280k + 124\}$ [5]. They also proposed the following strengthening of Burr's conjecture.

Conjecture 9 ([5])**.** If T is an oriented tree with k leaves, then $q(T) \leqslant v(T) + k - 1$.

Conjecture 9 generalizes an earlier analogous conjecture of Havet and Thomassé [9] in the same way that Burr's conjecture generalizes Sumner's. These conjectures suggest that arbitrary tournaments, in a sense, already exhibit all obstacles for embedding oriented trees. We formalise this by making the following conjecture.

Conjecture 10. If T is an oriented tree, then $q(T) = \min\{n \in \mathbb{N} : K_n \to T\}$.

We note that conjectures 1, 9 and 10 hold for oriented stars [4] and also for paths formed by concatenating (up to) two directed paths [2, 7, 8, 13, 15]. Note that the truth of Conjecture 10 would immediately confirm Burr's conjecture for every graph with sufficiently large chromatic number, using the proof of Sumner's conjecture for large tournaments.

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