Weakly linked embeddings of pairs of complete graphs in $\mathbb{R}^3$

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Abstract

Let G and H be disjoint embeddings of complete graphs K_m and K_n in $\mathbb{R}^3$ such that some cycle in G links a cycle in H with non-zero linking number. We say that G and H are *weakly linked* if the absolute value of the linking number of any cycle in G with a cycle in H is 0 or 1. Our main result is an algebraic characterisation of when a pair of disjointly embedded complete graphs is weakly linked.

As a step towards this result, we show that if G and H are weakly linked, then each contains either a vertex common to all triangles linking the other or a triangle which shares an edge with all triangles linking the other. All families of weakly linked pairs of embedded complete graphs are then characterised by which of these two cases holds in each complete graph.

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1 Introduction

The study of linked cycles within an embedded graph began in 1983 with Conway and Gordon's [1] and Sachs' [8] result that every embedding of K_6 in $\mathbb{R}^3$ contains a pair of triangles with non-zero linking number. Any graph with this property is said to be *intrinsically linked*. In the same paper, Sachs showed that each of the seven graphs in the Petersen family is intrinsically linked and no minor of any of them is intrinsically linked. Then, in 1995, Robertson, Seymour, and Thomas [7] proved that these seven graphs are the only graphs which are minor minimal with respect to being intrinsically linked. Since then, many results have been obtained about intrinsic linking of graphs.

In this paper, we explore how pairs of cycles in disjointly embedded complete graphs in $\mathbb{R}^3$ can be linked. We consider linking from a purely algebraic point of view. Thus we say that disjoint simple closed curves C and D are *linked* if and only if their pairwise linking number $\mathrm{Lk}(C, D)$ is non-zero. We introduce the following definitions.

Definition 1. We say that disjointly embedded simple closed curves C and D in $\mathbb{R}^3$ are *strongly linked* if $|\mathrm{Lk}(C, D)| \geqslant 2$, and *weakly linked* if $|\mathrm{Lk}(C, D)| = 1$.

Definition 2. We say that disjointly embedded graphs G and H in $\mathbb{R}^3$ are *strongly linked* if some cycle in G strongly links a cycle in H; and *weakly linked* if some cycle in G links a cycle in H, but no cycle in G strongly links any cycle in H.

Our main result is a characterisation of all weakly linked embeddings of $G \cong K_m$ and $H \cong K_n$ in terms of the pairwise linking numbers between triangles in G and triangles in H. Since any cycle in a complete graph can be decomposed as a sum of triangles, this completely determines all pairwise linking numbers between cycles in G and cycles in H.

We build our results in stages as follows. In Section 2, we prove Theorem 6, which characterises weak linking between a simple closed curve and an embedded complete graph K_n. Since the complete graph K_3 is a cycle, this also characterises weak linking of K_m and K_n when $\min\{m, n\} = 3$. In Section 3, we prove Theorem 14, which characterises weak linking between a theta curve (i.e., a graph with two vertices joined by three edges, homeomorphic to the Greek letter Θ) and a complete graph K_n. Next, in Section 4, we prove Theorem 16, which characterises weak linking of K_4 and K_n, for $n \geqslant 4$. In Section 5, we prove Theorem 20, which is a technical result needed for our characterisation of weakly linked embeddings of K_m and K_n. Finally in Section 6, we prove the following dichotomy.

Theorem 3 (Theorem 23 paraphrased)**.** *Let $m \geqslant 5$ and $n \geqslant 4$, and suppose that $G \cong K_m$ and $H \cong K_n$ are weakly linked in $\mathbb{R}^3$. Then exactly one of the following holds:*

1. *There is a vertex p of G common to all triangles of G linking H ("G contains a common vertex").*

2. *There is a triangle T^* in G such that a triangle $T \neq T^*$ of G links H if and only if it shares an edge with T^* ("G contains a common triangle").*

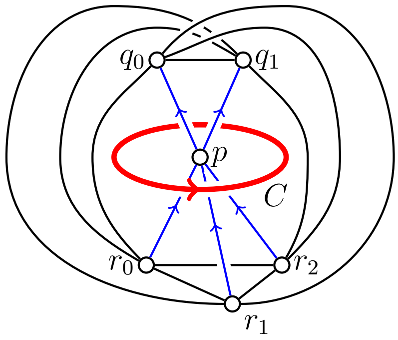

Figure 1: An embedding of $K_6 = \langle p, q_0, q_1, r_0, r_1, r_2 \rangle$ and a curve C such that C links K_6 in the star $p|q_0q_1|r_0r_1r_2$.

Then in Theorem 24, we characterise weak linking between G and H when at least one of G and H contains no vertex common to all triangles linking the other; while in Theorem 26, we characterise weak linking when both G and H contain a vertex common to all triangles linking the other. We conclude the paper with a brief discussion of the problem of determing the least n such that every embedding of K_n in $\mathbb{R}^3$ contains a pair of disjoint cycles that are strong linked.

The concept of a *star* (defined below) will play a key role in our results.

Definition 4. Let $\big(\{p\}, O, I\big)$ be an ordered partition of the vertices of K_n, where $O = \{q_1, \ldots, q_k\}$ and $I = \{r_1, \ldots, r_\ell\}$. The *star pOI* consists of all oriented triangles of the form pqr, where $q \in O$ and $r \in I$. We also express the star pOI as $p|q_1 \cdots q_k|r_1 \cdots r_\ell$.

The vertex p is said to be the *apex* of the star. A star pOI is *proper* if neither O nor I is a singleton, and *improper* otherwise. Note that the improper stars $p\{q\}I$ and $qI\{p\}$ are equal. We also refer to an improper star $p\{q\}I$ as a *fan* with *axis pq*.

If $\Sigma = pOI$ is a star, then we define $-\Sigma$ to be the star $-\Sigma = pIO$. We say that $-\Sigma$ is obtained by reversing the orientation of Σ.

Definition 5. Let $n \geqslant 3$, and let C be an oriented simple closed curve disjoint from an embedding of K_n in $\mathbb{R}^3$. We say that C *links K_n in the star pOI* if for all oriented triangles T of K_n we have

$$\operatorname{Lk}(C, T) = \begin{cases} +1 & \text{if } T \text{ is a triangle of the star } pOI, \\ -1 & \text{if } -T \text{ is a triangle of the star } pOI, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 1 shows an example of an embedding of $K_6 = \langle p, q_0, q_1, r_0, r_1, r_2 \rangle$ and a curve C which links K_6 in the star $p|q_0q_1|r_0r_1r_2$. The vertex p together with all incident edges (shown in blue) form a star, hence the name.

Unoriented stars with $\min\{|O|, |I|\} \leqslant 2$ were previously used by Flapan, Naimi, and Pommersheim [4] and Drummond-Cole and O'Donnol [2] to study intrinsically n-linked graphs. In particular, a graph G is said to be *intrinsically n-linked* or InL if every embedding of G in $\mathbb{R}^3$ contains a non-split link of n-components. Flapan, Naimi, and Pommersheim used stars to prove that K_{10} is I3L; and then Drummond-Cole and O'Donnol used them to show that for every $n \geqslant 2$, $K_{\lfloor \frac{7}{2}n \rfloor}$ is InL.

2 Weak linking of a simple closed curve with K_n

The main result of this section is the following theorem, which shows that weak linking between a simple closed curve and a complete graph can be characterised in terms of stars.

Theorem 6. *Let $n \geqslant 3$, and let C be an oriented simple closed curve disjoint from an embedding of K_n in $\mathbb{R}^3$ such that C links some cycle of K_n. Then C weakly links K_n if and only if C links K_n in a star.*

The case $n = 3$ is immediate, using any vertex as the apex and the remaining two vertices as O and I. For $n \geqslant 4$ we prove Theorem 6 as a series of lemmas, beginning with the "if" direction in Lemma 7:

Lemma 7. *Let C be an oriented cycle disjoint from an embedding of K_n in $\mathbb{R}^3$. If C links K_n in a star, then C weakly links K_n.*

Proof. Suppose that C links the star pOI in K_n, and let $D = v_0v_1 \cdots v_{k-1}$ be a k-cycle in K_n. We first show that if D does not contain p, then D does not link C.

To do this, decompose D as the sum of the triangles $T_i = v_0v_iv_{i+1}$, for $1 \leqslant i \leqslant k-2$, so that in the homology group $H_1(\mathbb{R}^3 - C)$ we have

$$[D] = \sum_{i=1}^{k-2} [T_i].$$

Then since C links K_n in the star pOI, and D does not contain p, we have $[T_i] = 0$ for all i. Therefore $[D] = 0$, showing that D does not link C.

Suppose now that D does contain p. Since C links K_n in the star pOI, C does not strongly link any triangle in K_n. Thus we may assume that $k \geqslant 4$. Assume without loss of generality that $v_0 = p$, and let $T = v_0v_1v_{k-1}$ and $D' = v_1v_2 \cdots v_{k-1}$. Then T is a triangle, D' is a $(k-1)$-cycle, and $D = T + D'$ as 1-chains in K_n. The cycle D' does not contain p so by the previous paragraph, in $H_1(\mathbb{R}^3 - C)$ we have

$$[D] = [T] + [D'] = [T] \in \{0, \pm 1\}.$$

Therefore D does not strongly link C. Since C links K_n, it follows that C weakly links K_n, as required. □

In order to prove the "only if" direction of Theorem 6, we first prove the case $n = 4$ in Lemma 8; then we use Lemma 8 to prove the case $n = 5$ in Lemma 9; then finally we use Lemma 9 to prove the case $n \geqslant 6$ in Lemma 10.

Lemma 8. *Let C be an oriented simple closed curve which weakly links an embedding of K_4 in $\mathbb{R}^3$. Then C links K_4 in a fan.*

Proof. Let $K_4 = \langle v_0, v_1, v_2, v_3 \rangle$, and let

$$C_0 = v_1v_2v_3, \qquad C_1 = v_3v_2v_0,$$
$$C_2 = v_0v_1v_3, \qquad C_3 = v_2v_1v_0.$$

Then as 1-chains in K_4 we have

$$C_0 + C_1 + C_2 + C_3 = 0,$$

and for $i \neq j$ the sum $C_i + C_j$ is a 4-cycle in K_4.

In the homology group $H_1(\mathbb{R}^3 - C)$ we have

$$[C_0] + [C_1] + [C_2] + [C_3] = 0,$$

with each $[C_i] \in \{0, \pm 1\}$ and some $[C_i] \neq 0$. If there exist $i \neq j$ such that $[C_i] = [C_j] \neq 0$, then $[C_i + C_j] = 2[C_i] \neq 0$, and C strongly links the four cycle $C_i + C_j$, contrary to hypothesis. So it must be the case that one term is equal to $+1$, one term is equal to -1, and the other two are zero. After relabelling the vertices and reorienting C (if necessary), we may assume that

$$[C_0] = [C_1] = 0, \qquad [C_2] = -[C_3] = 1.$$

Thus we let $O = \{v_1\}$, $I = \{v_2, v_3\}$, and see that C links K_4 in the fan $v_0\{v_1\}I$. □

Lemma 9. *Let C be an oriented simple closed curve which weakly links an embedding of K_5 in $\mathbb{R}^3$. Then C links K_5 in a star.*

Proof. Since any cycle that links C can be broken into triangles, there must be at least one triangle in K_5 that links C. First we suppose that there is some edge pq common to all triangles which link C. Let $K_5 = \langle p, q, r_0, r_1, r_2 \rangle$, and assume without loss of generality that $\mathrm{Lk}(C, pqr_0) = +1$. We claim that C links K_5 in the star $p|q|r_0r_1r_2$.

To see this, we apply Lemma 8 to $K_4 = \langle p, q, r_0, r_i \rangle$ for $i = 1, 2$. By Lemma 8, C links K_4 in a star. This star must be the fan with axis pq because we know $\mathrm{Lk}(C, pqr_0) = +1$ and pq is common to all triangles in K_5 linking C. Therefore for $i = 1, 2$, the triangle pqr_i in $K_4 = \langle p, q, r_0, r_i \rangle$ links C with linking number $+1$. Thus every triangle in the star $p|q|r_0r_1r_2$ in K_5 links C positively; every triangle in the star $q|p|r_0r_1r_2$ links C negatively; and since every triangle that links C contains pq, no other triangle can link C. It follows that C links K_5 in the star $p|q|r_0r_1r_2$ as claimed, completing the proof in this case.

Suppose now that there is no edge of K_5 common to all triangles linking C. Since any two triangles in K_5 must share at least one vertex, this implies there exist triangles $T_0 = pq_0r_0$ and $T_1 = pq_1r_1$ such that $T_0 \cap T_1 = \{p\}$ and $\mathrm{Lk}(C, T_0) = \mathrm{Lk}(C, T_1) = +1$. We show that C links K_5 in the star $\Sigma = p|q_0q_1|r_0r_1$.

In what follows homology classes are taken with respect to $H_1(\mathbb{R}^3 - C)$, and subscripts are taken modulo 2. We begin by showing that the two remaining triangles pq_0r_1 and pq_1r_0 of Σ link C with linking number $+1$. To see that $[pq_ir_{i+1}] = +1$ for $i = 0, 1$ consider the 5-cycle $D = pr_iq_ir_{i+1}q_{i+1}$. We have

$$\begin{aligned}[D] = [pr_iq_ir_{i+1}q_{i+1}] &= [pr_iq_i] + [pq_ir_{i+1}] + [pr_{i+1}q_{i+1}] \\ &= [pq_ir_{i+1}] - 2,\end{aligned}$$

so we must have $[pq_ir_{i+1}] = +1$ because otherwise either D or pq_ir_{i+1} would strongly link C.

We next show that $[pq_0q_1] = [pr_0r_1] = 0$. Recall that q_i and r_i were chosen so that $[pq_0r_0] = +1$ and $[pq_1r_1] = +1$. Suppose that $[pq_iq_{i+1}] = +1$ for some $i \in \{0, 1\}$. Then letting $D = pq_iq_{i+1}r_{i+1}$ we have

$$[D] = [pq_iq_{i+1}r_{i+1}] = [pq_iq_{i+1}] + [pq_{i+1}r_{i+1}] = +2.$$

Similarly, if $[pr_ir_{i+1}] = +1$ for some i, then letting $D = pq_ir_ir_{i+1}$ we have

$$[D] = [pq_ir_ir_{i+1}] = [pq_ir_i] + [pr_ir_{i+1}] = +2.$$

In either case, some cycle in K_5 would strongly link C, contrary to hypothesis. Since i can be either 0 or 1, we must have $[pq_0q_1] = [pr_0r_1] = 0$.

To complete the proof that C links K_5 in the star $\Sigma = p|q_0q_1|r_0r_1$, it remains to show that C links no triangle in $K_4 = \langle q_0, q_1, r_0, r_1\rangle$. Suppose to the contrary that it does. Then it must positively link some triangle of the form $q_iq_{i+1}r_j$ or $q_ir_jr_{j+1}$. In the first case, letting $D = q_iq_{i+1}r_jp$ we have

$$[D] = [q_iq_{i+1}r_jp] = [q_iq_{i+1}r_j] + [q_{i+1}r_jp] = +2;$$

and in the second, letting $D = q_ir_jr_{j+1}p$ we similarly get

$$[D] = [q_ir_jr_{j+1}p] = [q_ir_jr_{j+1}] + [q_ir_{j+1}p] = +2.$$

In either case some cycle in K_5 strongly links C, contrary to hypothesis. So no triangle in $K_4 = \langle q_0, q_1, r_0, r_1\rangle$ can link C, and we conclude that C links K_5 in the star $\Sigma = p|q_0q_1|r_0r_1$. This completes the proof. □

Lemma 10. *Let $n \geqslant 6$, and let C be an oriented simple closed curve which weakly links an embedding of K_n in $\mathbb{R}^3$. Then C links K_n in a star.*

To prove Lemma 10 we will use Lemma 11, which is a special case of a lemma proved in Flapan [3].

Lemma 11 (Triple link implies strong link)**.** *Let $L \cup Z \cup W$ be a 3-component link in $\mathbb{R}^3$, such that Z and W are cycles belonging to an embedding of K_n in $\mathbb{R}^3$. Suppose that $\mathrm{Lk}(L, Z) \neq 0 \neq \mathrm{Lk}(L, W)$. Then K_n contains a cycle which strongly links L.*

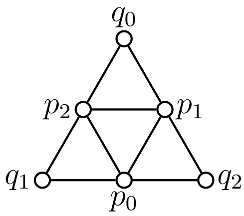

Figure 2: Three triangles $p_0p_1q_2$, $p_0q_1p_2$, $q_0p_1p_2$ that pairwise intersect but share no common vertex.

Proof. In Lemma 1 of [3] the component L is also assumed to be a cycle belonging to K_n, but this hypothesis plays no role in the proof and can be omitted. If either Z or W strongly links L then we are done. Otherwise, we may orient Z and W such that $\mathrm{Lk}(L, Z) = \mathrm{Lk}(L, W) = 1$, and apply [3, Lemma 1] to obtain a cycle J in K_n with at least 6 vertices such that for some orientation of J, we have $\mathrm{Lk}(L, J) \geqslant 2$. □

Proof of Lemma 10. Since C links some cycle of K_n, it links some triangle. Since $n > 5$, this triangle lies in a K_5 subgraph which links C. Thus by Lemma 9, C links a star in this K_5. It follows that C links at least three triangles in K_n. If there are two disjoint triangles in K_n which link C, then by Lemma 11 C strongly links some cycle of K_n. So we assume that no pair of triangles that link C are disjoint.

Now we show that there is a vertex p in K_n such that every triangle in K_n that links C contains p. Suppose this is not the case. Since no pair of triangles that link C are disjoint, there must exist three triangles linking C which pairwise intersect but don't all share a common vertex. We know by Lemma 9 that the set of triangles in a K_5 which link C must all share at least one common vertex. Thus the three triangles which pairwise intersect but don't have a common vertex must use at least 6 vertices. If any pair of them shared a common edge, it would only require 5 vertices; and if they used more than 6 vertices, there would be a pair that did not share a vertex. Thus we have the situation illustrated in Figure 2, with C linking the triangles $p_0p_1q_2$, $p_0q_1p_2$, and $q_0p_1p_2$.

Suppose without loss of generality that $\mathrm{Lk}(C, p_0p_1q_2) = +1$ (re-orienting C, if necessary), and consider the K_5-subgraph $H = \langle p_0, p_1, p_2, q_1, q_2\rangle$. Since $p_0p_1q_2$ and $p_0q_1p_2$ both link C and are contained in H, C must link H in a proper star with apex p_0. This star must be either $p_0|p_1q_1|p_2q_2$ (if $\mathrm{Lk}(C, p_0q_1p_2) = +1$), or $p_0|p_1p_2|q_1q_2$ (if $\mathrm{Lk}(C, p_0q_1p_2) = -1$). Hence either $p_0p_1p_2$ and $p_0q_1q_2$ both link C, or $p_0p_1q_1$ and $p_0p_2q_2$ both link C.

If $p_0p_1p_2$ and $p_0q_1q_2$ both link C, then the triangles $p_0q_1q_2$ and $q_0p_1p_2$ in K_n would be disjoint triangles which both link C. Hence by Lemma 11, there would be a cycle in K_n strongly linking C. As this is contrary to hypothesis, we must have both $p_0p_1q_1$ and $p_0p_2q_2$ linking C instead. Then in the K_5-subgraph $H' = \langle p_0, p_1, p_2, q_0, q_1\rangle$ we have at least the three triangles $p_0q_1p_2$, $q_0p_1p_2$, and $p_0p_1q_1$ linking C. But this is impossible by Lemma 9 since C must link H' in a star, which means that there is a vertex common to all triangles in H' which link C. Thus, in fact, there must be some vertex p in K_n such that every triangle in K_n that links C contains p.

We next show that every vertex s of K_n belongs to some triangle in K_n linking C. Indeed, let pqr be a triangle in K_n linking C, and consider the K_4-subgraph $\langle p, q, r, s \rangle$. By Lemma 8 this K_4-subgraph must link C in a fan, so either pqs or prs must link C.

We're now ready to partition the vertices of $K_n - \{p\}$ into sets O and I as required by the theorem. Let

$$O = \{q \in K_n : [pqr] = 1 \text{ for some } r \in K_n\},$$
$$I = \{r \in K_n : [pqr] = 1 \text{ for some } q \in K_n\},$$

where homology classes are taken with respect to $H_1(\mathbb{R}^3 - C)$. By the previous paragraph every vertex of $K_n - \{p\}$ belongs to $O \cup I$. We claim that moreover $O \cap I = \varnothing$, so that $\{O, I\}$ is in fact a partition of the vertices of $K_n - \{p\}$. To see this, suppose that $r \in O \cap I$. Then there are vertices q and s such that $[pqr] = [prs] = 1$. But this would imply that

$$[pqrs] = [pqr] + [prs] = 2,$$

and hence the square $pqrs$ would strongly link C. Thus $O \cap I = \varnothing$.

Given $q \in O$ and $r \in I$, we must show that C links pqr. Now by definition of O and I, there are vertices $s \in I$ and $t \in O$ such that $[pqs] = [ptr] = 1$. If $s = r$ or $t = q$, then C does link pqr as required. Otherwise p, q, r, s, t are all distinct so $H = \langle p, q, r, s, t \rangle$ is a K_5-subgraph. By Lemma 9, C must link H in the star $p|qt|rs$, and so $[pqr] = 1$, as required.

Finally, to show that C only links triangles of the form pqr with $q \in O$ and $r \in I$, we consider triangles in K_n which are not of this form. Since $\big\{\{p\}, O, I\big\}$ partitions the vertices of K_n, such a triangle must have one of the following forms:

- pq_1q_2 with $q_1, q_2 \in O$, which cannot link C since that would imply that q_1 or q_2 belongs to $O \cap I$.
- pr_1r_2 with $r_1, r_2 \in I$, which cannot link C since that would imply that r_1 or r_2 belongs to $O \cap I$.
- xyz with $p \notin \{x, y, z\}$, which cannot link C because we showed that every triangle that links C contains p.

So we are done. □

Taken together, Lemmas 8, 9, and 10 complete the proof of Theorem 6. By Lemma 11, if a cycle C weakly links a complete graph K_n then any two cycles in K_n that link C must intersect. We extend this result to a pair of weakly linked complete graphs as follows.

Theorem 12. *Let $m, n \geqslant 3$, and suppose that $G \cong K_m$ and $H \cong K_n$ are weakly linked graphs in $\mathbb{R}^3$. If C_1 and C_2 are cycles in G that link H, then there is a vertex p of G that belongs to both C_1 and C_2.*

To prove Theorem 12, we first prove the following lemma.

Lemma 13. *Let C_1 and C_2 be oriented simple closed curves which weakly link an embedding of K_n in $\mathbb{R}^3$. Then there is a cycle of length at most 4 in K_n that links both C_1 and C_2.*

Proof. By Theorem 6, C_1 and C_2 each link K_n in a star. Let $q_1O_1I_1$ be the star that links C_1, and let $q_2O_2I_2$ be the star that links C_2. We show as follows that there is a cycle in K_n that links both C_1 and C_2.

We first suppose that $q_1 \neq q_2$. Since $\{\{q_1\}, O_1, I_1\}$ and $\{\{q_2\}, O_2, I_2\}$ each partition K_n, we can switch the orientations on C_1 and C_2 (if necessary) so that $q_1 \in I_2$ and $q_2 \in O_1$. Hence for all $r \in I_1$, we have $\mathrm{Lk}(C_1, rq_1q_2) = 1$; and for every $r \in O_2$, we have $\mathrm{Lk}(C_2, rq_1q_2) = 1$. If there is some $r \in I_1 \cap O_2$, then the triangle q_1q_2r links both C_1 and C_2.

So we assume that $I_1 \cap O_2 = \varnothing$. Since $\{\{q_1\}, O_1, I_1\}$ and $\{\{q_2\}, O_2, I_2\}$ are partitions of K_n with $q_1 \in I_2$ and $q_2 \in O_1$, it follows that we must have $O_2 \subseteq O_1$ and $I_1 \subseteq I_2$. Let $x \in I_1 \subseteq I_2$, $y \in O_2 \subseteq O_1$, and consider the square xq_1q_2y. In $H_1(\mathbb{R} - C_1)$ we have

$$[xq_1q_2y] = [xq_1y] + [yq_1q_2] = 1 + 0 = 1,$$

because $q_2 \in O_1$; and in $H_1(\mathbb{R} - C_2)$ we have

$$[xq_1q_2y] = [xq_1q_2] + [xq_2y] = 0 + 1 = 1,$$

because $q_1 \in I_2$. Thus C_1 and C_2 both link the square xq_1q_2y.

Next suppose that $q_1 = q_2$. If $O_2 \subseteq O_1$, then by analogy with our above argument $I_1 \subseteq I_2$. In this case, if $x \in O_2$ and $y \in I_1$, then q_1xy links both C_1 and C_2. Thus we assume that $O_2 \nsubseteq O_1$, and similarly that $O_1 \nsubseteq O_2$. It follows that there is some vertex $x \in O_2 \cap I_1$ and some vertex $y \in I_2 \cap O_1$. But now xyq_1 links both C_1 and C_2. □

Proof of Theorem 12. By hypothesis C_1 and C_2 both weakly link H, so by Lemma 13 there is a cycle D in H which links both C_1 and C_2. Theorem 12 now follows from either Lemma 11 or Theorem 6 applied to the cycle D and the complete graph G: for instance, by Theorem 6 D links G in a star pOI, and then p must belong to C_i for $i = 1, 2$, because otherwise C_i does not link D. □

3 Weak linking of a Θ curve with K_n

Theorem 14. *Let Θ be a theta curve with oriented cycles C_1, C_2, C_3 such that*

$$[C_1] + [C_2] + [C_3] = 0$$

in $H_1(\Theta)$. Let $n \geqslant 3$, and suppose that Θ and K_n are weakly linked graphs in $\mathbb{R}^3$. Then exactly one of the following cases holds:

(A1) There is a vertex p of K_n common to all triangles linking a curve in Θ. Then there are pairwise disjoint sets I_1, I_2, I_3 (at most one empty) such that $I_1 \cup I_2 \cup I_3 =$

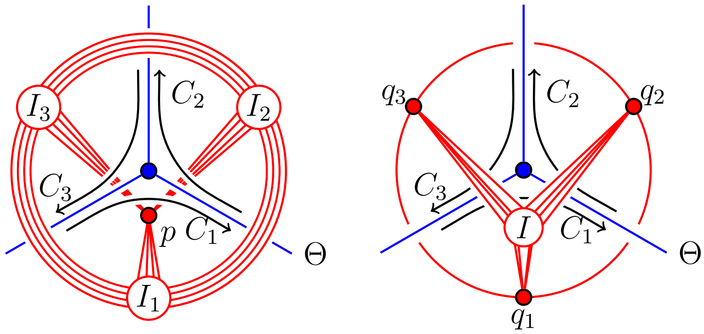

Figure 3: Cases (A1) (left) and (A2) (right) of Theorem 14. The second vertex of Θ is placed at infinity.

$K_n - \{p\}$, and after reversing the orientation of Θ (if necessary), each C_i links the star pO_iI_i in K_n, where

$$O_1 = I_2 \cup I_3, \qquad O_2 = I_1 \cup I_3, \qquad O_3 = I_1 \cup I_2.$$

(A2) There is no vertex of K_n common to all triangles linking a curve in Θ. Then $n \geqslant 5$ and there are distinct vertices p_1, p_2, p_3 in K_n such that, after reversing the orientation of Θ (if necessary), each C_i links the star p_iO_iI in K_n, where

$$O_1 = \{p_2, p_3\}, \qquad O_2 = \{p_1, p_3\}, \qquad O_3 = \{p_1, p_2\},$$

and $I = K_n - \{p_1, p_2, p_3\}$.

Figure 3 illustrates the two cases. The loops C_1, C_2 and C_3 link stars as given in the theorem. It then follows from Theorem 6 that these embeddings are weakly linked because every cycle in Θ links a star in K_n.

Proof. Let D be a cycle in K_n linking some cycle C_i in Θ. Then in the homology group $H_1(\mathbb{R}^3 - D)$ we have

$$[C_1] + [C_2] + [C_3] = 0.$$

Since there is no strong link between Θ and K_n, every $[C_i]$ is ± 1 or 0. Since some term is non-zero, each of the three possible values must occur exactly once in the sum. Thus, D must link exactly two of the C_i, one positively, and one negatively. We will use this fact repeatedly.

Without loss of generality C_1 links some cycle in K_n. Since C_1 does not strongly link any cycle in K_n, by Theorem 6 it links some star $p_1O_1I_1$. Let p_1qr be a triangle in $p_1O_1I_1$. Then p_1qr links exactly one of C_2 and C_3. So without loss of generality we may assume that C_2 links p_1qr, and therefore C_2 links a star $p_2O_2I_2$. We consider two cases, according to whether or not there is a vertex p common to all triangles in K_n linking either C_1 or C_2.

r	q: $O_1 \cap O_2$	$O_1 \cap I_2$	$I_1 \cap O_2$	$I_1 \cap I_2$
$O_1 \cap O_2$	$(0,0,0)$	$(0,-1,+1)$	$(-1,0,+1)$	$(-1,-1,+2)$
$O_1 \cap I_2$	$(0,+1,-1)$	$(0,0,0)$	$(-1,+1,0)$	$(-1,0,+1)$
$I_1 \cap O_2$	$(+1,0,-1)$	$(+1,-1,0)$	$(0,0,0)$	$(0,-1,+1)$
$I_1 \cap I_2$	$(+1,+1,-2)$	$(+1,0,-1)$	$(0,+1,-1)$	$(0,0,0)$

Table 1: The triples $\big(\mathrm{Lk}(pqr, C_1), \mathrm{Lk}(pqr, C_2), \mathrm{Lk}(pqr, C_3)\big)$ in Case 1 of the proof of Theorem 14.

Case 1: All triangles linking C_1 or C_2 share a vertex

Suppose that there is a vertex p common to all triangles linking either C_1 or C_2. Then we may choose the stars linking C_1 and C_2 so that $p_1 = p_2 = p$. Moreover, since any triangle linking C_3 must also link either C_1 or C_2, if C_3 also links some triangle it contains the vertex p. Hence in this case, C_3 must link a star of the form pO_3I_3.

Observe that $O_1 \cap O_2, O_1 \cap I_2, I_1 \cap O_2, I_1 \cap I_2$ are disjoint sets with union $X = K_n - \{p\}$. Given a triangle pqr in K_n we have

$$\mathrm{Lk}(pqr, C_3) = -\mathrm{Lk}(pqr, C_1) - \mathrm{Lk}(pqr, C_2),$$

so the ordered triple

$$\big(\mathrm{Lk}(pqr, C_1), \mathrm{Lk}(pqr, C_2), \mathrm{Lk}(pqr, C_3)\big)$$

is completely determined by the sets in $O_1 \cap O_2, O_1 \cap I_2, I_1 \cap O_2, I_1 \cap I_2$ that q and r belong to. Calculating these triples we obtain Table 1.

If $O_1 \cap O_2$ and $I_1 \cap I_2$ are both nonempty then by Table 1 C_3 strongly links some cycle in K_n. Since this is contrary to our hypothesis, at least one of these intersections must be empty. Reversing the orientation of Θ switches the roles of O_i and I_i for each i, so after doing this (if necessary), without loss of generality we may assume that $I_1 \cap I_2 = \varnothing$. Table 1 then shows that $\mathrm{Lk}(pqr, C_3) = +1$ precisely when $q \in (O_1 \cap I_2) \cup (I_1 \cap O_2)$ and $r \in O_1 \cap O_2$. Observe that the conditions $I_1 \cap I_2 = \varnothing$ and $O_1 \cup I_1 = O_2 \cup I_2 = X$ together imply $I_1 \subseteq O_2$, $I_2 \subseteq O_1$. We therefore have

$$O_3 = I_1 \cup I_2, \qquad\qquad I_3 = O_1 \cap O_2,$$

and we see that I_1, I_2, I_3 are pairwise disjoint sets with union X. This implies that we also have

$$O_1 = I_2 \cup I_3, \qquad\qquad O_2 = I_1 \cup I_3,$$

and it follows that C_1, C_2 and C_3 link K_n in stars as given by (A1).

Case 2: Triangles linking C_1 or C_2 do not all share a vertex

Every triangle linking C_1 contains p_1, and every triangle linking C_2 contains p_2, so if these triangles do not all share a common vertex we must have $p_1 \neq p_2$. Reversing the orientation of Θ exchanges the roles of O_1 and I_1, so without loss of generality we may assume that $p_2 \in O_1$. At the beginning of the proof, we assumed that without loss of generality there is a triangle T in the star $p_1O_1I_1$ linking C_2. Now since every triangle linking C_2 contains p_2, T must have the form p_1p_2r for some $r \in I_1$. Then

$$\mathrm{Lk}(p_2p_1r, C_2) = -\mathrm{Lk}(p_2p_1r, C_1) = \mathrm{Lk}(p_1p_2r, C_1) = +1. \tag{1}$$

It follows that we also must have $p_1 \in O_2$.

Now there must be some triangle in K_n linking C_1 that does not contain p_2, because otherwise p_2 would be common to all triangles linking C_1 or C_2. Let $T_1 = p_1q_1r_1 \in p_1O_1I_1$ be such a triangle. Then T_1 does not link C_2 because it does not contain p_2, so it must link C_3 instead. By Theorem 6 C_3 links a star $p_3O_3I_3$ in K_n, where $p_3 \in T_1$. If $p_3 = p_1$ then this vertex would be common to all triangles linking C_1 or C_3. Since every triangle linking C_2 also links either C_1 or C_3, this would give us a vertex common to all triangles linking C_1 or C_2, which is contrary to the hypothesis of this case. So $p_3 \neq p_1$. We also have $p_3 \neq p_2$, because $p_2 \notin T_1$. So the vertices p_1, p_2 and p_3 are distinct.

Suppose now that $p_3 = r_1$, so that $p_3 \in I_1$. Then there must be some vertex $r \in I_1$ such that $r \neq r_1$, because otherwise p_3 would be common to all triangles linking C_1 or C_3, and hence to all triangles linking C_1 or C_2. Consider the triangle p_1q_1r. This triangle links C_1, because it belongs to the star $p_1O_1I_1$, but it does not link either C_2 or C_3, because it does not contain either p_2 or p_3. This is a contradiction, so we must have $p_3 = q_1 \in O_1$. The argument of equation (1) then gives $p_1 \in O_3$.

There must also be some triangle $T_2 = p_2q_2r_2 \in p_2O_2I_2$ that does not contain p_1, because otherwise p_1 is common to all triangles linking C_1 or C_2. Arguing as above we conclude that $p_3 \in O_2$, and $p_2 \in O_3$. We now have

$$\{p_2, p_3\} \subseteq O_1, \qquad \{p_1, p_3\} \subseteq O_2, \qquad \{p_1, p_2\} \subseteq O_3. \tag{2}$$

Suppose that there is some $q \in O_1$ such that $p_2 \neq q \neq p_3$. Then for any $r \in I_1$ the triangle p_1qr links C_1, because it belongs to the star $p_1O_1I_1$, but it does not link C_2 or C_3 because it does not contain p_2 or p_3. This is a contradiction, so we must have $O_1 = \{p_2, p_3\}$. By the same reasoning the other inclusions in (2) must also be equalities, and we obtain finally

$$O_1 = \{p_2, p_3\}, \qquad O_2 = \{p_1, p_3\}, \qquad O_3 = \{p_1, p_2\},$$

and hence

$$I_1 = I_2 = I_3 = I = K_n - \{p_1, p_2, p_3\}$$

as claimed. To conclude we note that we must have $n \geqslant 5$, because if $n = 4$ we would have $I = \{r\}$, and the vertex r would be common to all triangles linking a cycle in Θ. $\square$

Corollary 15. *Let $n \geqslant 4$. With notation as in Theorem 14, let Θ and K_n be disjointly embedded in $\mathbb{R}^3$ such that*

1. *C_1 links a fan in K_n with axis pq, and*
2. *C_2 links a star in K_n that is either a fan with axis xy disjoint from pq, or a proper star with apex x disjoint from pq.*

Then C_3 strongly links some cycle in K_n.

Proof. Since $n \geqslant 4$ the only vertices common to all triangles linking C_1 are p and q, and the only vertices common to all triangles linking C_2 are either x and y (if C_2 links a fan), or x alone (if C_2 links a proper star). By hypothesis there is therefore no vertex common to all triangles linking Θ, and so if Θ does not strongly link K_5 we must be in Case (A2) of Theorem 14. But in Case (A2) of the theorem the stars of C_1, C_2 and C_3 are all proper, contradicting the fact that C_1 links a fan. Thus in fact neither case holds, so some cycle in Θ strongly links a cycle in K_n. Both C_1 and C_2 link stars, so it is C_3 that strongly links K_n. □

4 Weak linking of K_4 with K_n

Let $G = \langle p_0, p_1, p_2, p_3 \rangle \cong K_4$, and let

$$C_0 = p_1p_2p_3, \qquad C_2 = p_0p_1p_3,$$
$$C_1 = p_3p_2p_0, \qquad C_3 = p_2p_1p_0.$$

With these orientations we have

$$[C_0] + [C_1] + [C_2] + [C_3] = 0$$

in $H_1(K_4; \mathbb{Z})$, and for any $i \neq j$ the chain $C_i + C_j$ represents a 4-cycle. We use C_1, C_2, C_3, and C_4 in Theorem 16.

Theorem 16. *Let $n \geqslant 4$, and suppose that $G \cong K_4$ and $H \cong K_n$ are weakly linked graphs in $\mathbb{R}^3$. Then exactly one of the following holds:*

(B1) There is a vertex q of H common to all triangles linking a curve in G. Then there are pairwise disjoint sets I_0, I_1, I_2, I_3 (at most two of them empty) such that $I_0 \cup I_1 \cup I_2 \cup I_3 = H - \{q\}$, and after reversing the orientation of $\mathbb{R}^3$ (if necessary), each C_i links the star qO_iI_i in K_n, where

$$O_i = H - \{q\} - I_i.$$

(B2) There is no vertex of H common to all triangles linking a curve in G. Then $n \geqslant 5$ and there are distinct vertices q_1, q_2, q_3 in H such that, after relabelling the vertices of

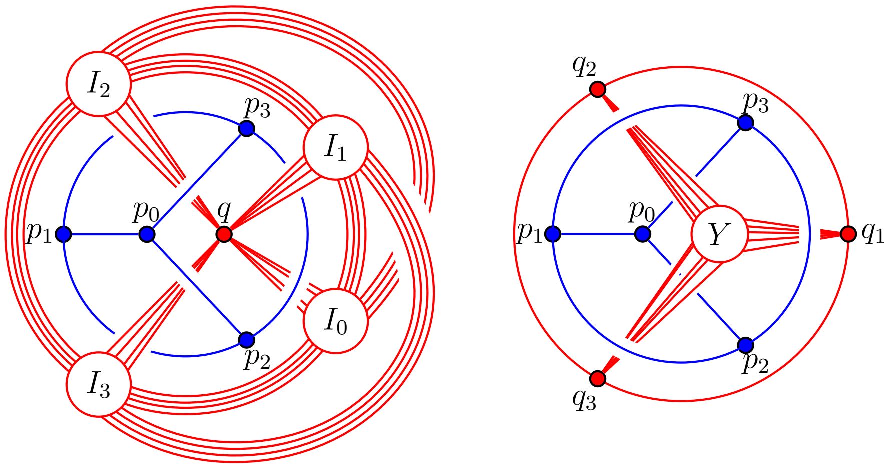

Figure 4: Embeddings of $G \cong K_4$ (blue) and $H \cong K_n$ (red) realising Cases (B1) (left) and (B2) (right) of Theorem 16.

G and reversing orientation of $\mathbb{R}^3$ (if necessary), C_0 does not link H and C_1, C_2, C_3 link H in the stars

$$q_1\{q_2, q_3\}I, \qquad q_2\{q_1, q_3\}I, \qquad q_3\{q_1, q_2\}I,$$

where $I = H - \{q_1, q_2, q_3\}$. In particular, the vertex p_0 of G is common to all triangles of G linking H; and a triangle T of H links G if and only if it shares exactly one edge with $T^* = q_1q_2q_3$.

Embeddings realising (B1) and (B2) are illustrated in Figure 4. These embeddings belong to the families of embeddings realising Theorem 24, which we prove are weakly linked in Theorem 25.

Proof. We consider the two cases in turn.

Case 1: All triangles in H linking G share a common vertex q.

Then each triangle C_i in G links a star qO_iI_i in H, where we allow the possibility that O_i or I_i is empty to cover the case where C_i doesn't link H.

Suppose that C_i and C_j both link H. If $O_i \cap O_j \neq \varnothing \neq I_i \cap I_j$, then we may choose $x \in O_i \cap O_j$ and $y \in I_i \cap I_j$. Then C_i and C_j both link the triangle $qxy \in H$ with linking number 1, and consequently the square $C_i + C_j$ strongly links qxy. So we must have either $O_i \cap O_j = \varnothing$, or $I_i \cap I_j = \varnothing$. Reversing the orientation of $\mathbb{R}^3$ exchanges the roles of O_ℓ and I_ℓ for all ℓ, so after doing this (if necessary) we may assume that $I_i \cap I_j = \varnothing$. This implies $I_i \subseteq O_j$ and $I_j \subseteq O_i$, because $\{O_\ell, I_\ell\}$ is a partition of $H - \{q\}$ for each ℓ.

We claim now that if C_k also links H, then $I_i \cap I_k = I_j \cap I_k = \varnothing$. Suppose to the contrary that $I_i \cap I_k$ is nonempty. Then we must have $O_i \cap O_k = \varnothing$, by the previous paragraph, and arguing as above we must have $O_i \subseteq I_k$, and $O_k \subseteq I_i$. But then $I_j \subseteq O_i \subseteq I_k$ and $O_k \subseteq I_i \subseteq O_j$, so $I_j \cap I_k \neq \varnothing \neq O_j \cap O_k$, giving us a strong link. We must therefore have $I_a \cap I_b = \varnothing$ whenever C_a and C_b link H, and we extend this to hold for all a and b by setting $I_\ell = \varnothing$, $O_\ell = H - \{p\}$ if C_ℓ does not link H. Note that at most two of the I_ℓ can be empty, because any triangle in H linking G must link it in exactly two triangles, one positively and one negatively.

To complete the proof in this case we must show that $I_0 \cup I_1 \cup I_2 \cup I_3 = H - \{q\}$. Let $x \in H - \{q\}$. If $x \notin I_i$ then we necessarily have $x \in O_i$. Then qxy links C_i with linking number $+1$ for some y, so it must link some C_k with linking number -1. Then $qyx \in qO_kI_k$, and we conclude that $x \in I_k$.

Case 2: There is no vertex common to all triangles in H linking G.

First we show that some triangle C_i of G must link a proper star in H.

Suppose to the contrary that every triangle of G that links H links it in a fan. We may suppose that some triangle C_i links H in the fan with axis xy. By assumption x does not belong to every triangle of H linking G, so some C_j links a triangle $T_1 \subseteq H$ that does not contain x. Then C_j links a fan in H, and since $C_i \cup C_j$ is a theta curve the axis of this fan must meet xy, by Corollary 15. Thus C_j links H in a fan with axis $\pm yz$, for some $z \neq x$. Now since y is not common to all triangles of H linking G, some cycle C_k must link a triangle T_2 that does contain y. Then C_k links H in a fan, and by Corollary 15 the axis of this fan must meet both xy and yz. The axis must therefore be $\pm zx$. But now the triangle $xyz \subseteq H$ links all three triangles C_i, C_j, C_k, contradicting the fact that it must link G in a star, which contains exactly two triangles. So some triangle in G must link a proper star in H. Note that this immediately implies $n \geqslant 5$.

Without loss of generality C_1 links a proper star with apex q_1. By assumption q_1 is not common to all triangles of H linking G, so without loss of generality C_2 links a triangle T that does not contain q_1. Now $C_1 \cup C_2$ is a theta curve, and since q_1 is the only vertex common to all triangles linking C_1, and some triangle linking C_2 does not contain q_1, there is no vertex common to all triangles of H linking C_1 or C_2. We must therefore be in Case (A2) of Theorem 14, so there are vertices q_2 and q_3 such that (after reversing orientation of $\mathbb{R}^3$, if necessary) C_1, C_2 and $-(C_1 + C_2)$ link H in the stars

$$\begin{aligned} \Sigma_{C_1} &= q_1\{q_2, q_3\}I, \\ \Sigma_{C_2} &= q_2\{q_1, q_3\}I, \\ \Sigma_{-(C_1+C_2)} &= q_3\{q_1, q_2\}I, \end{aligned}$$

where $I = H - \{q_1, q_2, q_3\}$.

We now consider C_0 and C_3. At least one of them must link H, because otherwise $C_0 + C_3 = -(C_1 + C_2)$ would not, a contradiction. After relabelling the vertices of G (if necessary) we may therefore assume that C_3 links H. We will show under these conditions that C_0 does not link H. To do this we use the fact that C_0, C_3 and $-(C_0+C_3) = C_1+C_2$ form a theta curve, with $-(C_0+C_3)$ linking H in the star $-\Sigma_{-(C_1+C_2)} = q_3I\{q_1, q_2\}$.

We first show that C_3 must link H in a star with apex q_3. Suppose to the contrary that q_3 is not common to all triangles linking C_3. If C_3 links H in a star with apex q_1 then by Case (A2) of Theorem 14 it must link H in the star $q_1I\{q_2, q_3\}$; but then for any $r \in I$ both C_2 and C_3 link the triangle q_2q_1r with linking number $+1$, and it follows that the square $C_2 + C_3$ strongly links H. The same argument shows that C_3 cannot link a star with apex q_2, so suppose finally that C_3 links H in a star with apex $q_4 \in I$. Then by Theorem 14 Case (A2) it must be the case that $|I| = 2$, so $n = 5$ and there is a vertex q_5 such that C_3, C_0 and $-(C_0 + C_3)$ link H in the stars

$$\begin{aligned}\Sigma_{C_0} &= q_5\{q_3, q_4\}\{q_1, q_2\},\\ \Sigma_{C_3} &= q_4\{q_3, q_5\}\{q_1, q_2\},\\ \Sigma_{-(C_0+C_3)} &= q_3\{q_4, q_5\}\{q_1, q_2\}.\end{aligned}$$

Observe now that $C_1 \cup C_3$ is a theta curve, and the stars

$$\begin{aligned}\Sigma_{C_1} &= q_1\{q_2, q_3\}\{q_4, q_5\},\\ \Sigma_{C_3} &= q_4\{q_3, q_5\}\{q_1, q_2\}\end{aligned}$$

don't satisfy (A2). It follows that the square $C_1 + C_3$ must strongly link H: for example, it strongly links the square $q_1q_2q_4q_5$ in H. We conclude that C_3 must link H in a star $\Sigma_{C_3} = q_3O_3I_3$ with apex q_3, as claimed.

We now use the fact that both $C_1 \cup C_3$ and $C_2 \cup C_3$ are theta curves. The vertices q_1 and q_2 cannot both be common to every triangle in H linking C_3, because then the only triangle in H linking C_3 would be $q_1q_2q_3$. So suppose without loss of generality that q_1 is not common to every triangle in H linking C_3. Then $C_1 \cup C_3$ must satisfy Case (A2) of Theorem 14. However, the only star with apex q_3 that can satisfy (A2) together with Σ_{C_1} is $q_3\{q_1, q_2\}I = \Sigma_{-(C_1+C_2)}$, so it must be the case that $\Sigma_{C_3} = q_3\{q_1, q_2\}I$ too. But then

$$\Sigma_{C_0+C_3} = \Sigma_{-(C_1+C_2)} = \Sigma_{C_3},$$

and it follows that C_0 does not link H. This completes the proof. ☐

Remark 17. We may express the linking between G and H in Case (B2) of Theorem 16 in terms of stars in G as follows.

Notice that a triangle of H links G if and only if it has the form $\pm q_iq_jy$, for $i \neq j$ and $y \in I$. Consider the triangle q_3q_2y. This is linked positively by the triangles $C_3 = p_1p_0p_2$ of G and $-C_2 = p_1p_0p_3$ of G, and hence links G in the star $p_1\{p_0\}\{p_2, p_3\}$. Considering the triangles yq_1q_3 and q_2q_1y in turn, we find that the triangles $\pm q_iq_jy$ link G according to the following stars:

$$\begin{aligned}\Sigma_{q_3q_2y} &= p_1\{p_0\}\{p_2, p_3\},\\ \Sigma_{yq_1q_3} &= p_2\{p_0\}\{p_1, p_3\},\\ \Sigma_{q_2q_1y} &= p_3\{p_0\}\{p_1, p_2\}.\end{aligned}$$

5 Stars with no common apex

A key step in our characterisation of weakly linked embeddings of $G \cong K_m$ and $H \cong K_n$ is an analysis of the way in which two stars in a weakly linked embedding can meet. In Theorem 20, we analyse a pair of stars that do not share a common apex. We begin with the following definitions.

Definition 18. Let $\Sigma_1 = p_1 O_1 I_1, \Sigma_2 = p_2 O_2 I_2$ be stars in K_m, with $m \geqslant 4$.

1. If $p_1 \neq p_2$, then Σ_1 and Σ_2 are *mutually oriented* if $p_1 \in O_2$ and $p_2 \in O_1$.
2. If there is a vertex p of K_m such that Σ_1, Σ_2 may be expressed in the form $\Sigma_i = pO_i' I_i'$ for each i, then Σ_1 and Σ_2 have *a common apex*. Otherwise, we say that Σ_1 and Σ_2 have *no common apex*.

Remark 19. Suppose that $p_1 \neq p_2$. Then $\varepsilon_1\Sigma_1, \varepsilon_2\Sigma_2$ are mutually oriented for a unique choice of signs $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$. Furthermore, if Σ_1 and Σ_2 are mutually oriented, then they have a common apex if and only if one of the following holds:

1. Σ_1 is a fan with axis p_1p_2 (so that $\Sigma_1 = p_1\{p_2\}I_1$ may be expressed in the form $\Sigma_1 = p_2 I_1 \{p_1\}$).
2. Σ_2 is a fan with axis p_2p_1 (so that $\Sigma_2 = p_2\{p_1\}I_2$ may be expressed in the form $\Sigma_2 = p_1 I_2 \{p_2\}$).
3. There exists a vertex p in K_m such that Σ_i is a fan with axis pp_i for each i (so that $\Sigma_i = p_i O_i \{p\}$ may be expressed in the form $\Sigma_i = p\{p_i\}O_i$ for each i).

Theorem 20 (Stars with no common apex)**.** *Let m and n be positive integers with $m \geqslant 5$. Suppose that $G \cong K_m$ and $H \cong K_n$ are weakly linked graphs in $\mathbb{R}^3$, and T_1 and T_2 are triangles in H linking G in stars $\Sigma_1 = p_1O_1I_1$ and $\Sigma_2 = p_2O_2I_2$, respectively, which have no common apex. Then (after possibly re-orienting T_1 and T_2 so that Σ_1 and Σ_2 are mutually oriented) precisely one of the following holds:*

(C1) There is a vertex p_3 distinct from p_1, p_2 such that $I_1 = \{p_3\}$ and $O_2 = \{p_1, p_3\}$.

(C2) There is a vertex p_3 distinct from p_1, p_2 such that $I_2 = \{p_3\}$ and $O_1 = \{p_2, p_3\}$.

(C3) There is a vertex p_3 distinct from p_1, p_2 such that $O_1 = \{p_2, p_3\}$ and $O_2 = \{p_1, p_3\}$.

Thus, disregarding orientations, $\Sigma_1 \cup \Sigma_2$ consists of all triangles sharing an edge with $T^* = p_1p_2p_3$, with the sole exception of T^* itself in (C3)*.*

Remark 21. Note that if stars Σ_1 and Σ_2 satisfy one of (C1)–(C3), then at least one of them must be a proper star. Consequently, if triangles T_1 and T_2 in H link fans in G then the axes of the fans must intersect.

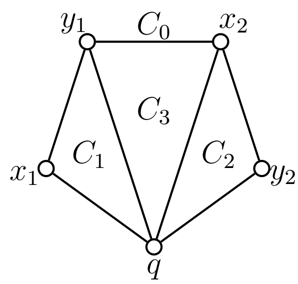

Figure 5: The cycles C_1, C_2 and C_3 in the proof of Proposition 22.

Theorem 20 turns out to be easier to prove for $m \geqslant 6$ than for $m = 5$. However, for the sake of space, we present a single proof for $m \geqslant 5$. The proof breaks into two cases, according to the way in which T_1 and T_2 intersect. The case where they intersect in an edge was addressed by Theorem 14, and the case where they intersect in a single vertex is addressed by Proposition 22.

Proposition 22. *Let m and n be positive integers with $m \geqslant 5$. Suppose that $G \cong K_m$ and $H \cong K_n$ are weakly linked graphs in $\mathbb{R}^3$. Let p_1, p_2 be vertices of G such that $p_1 \neq p_2$, and let C_1 and C_2 be cycles in H that intersect in a single vertex q, and link G in mutually oriented stars $\Sigma_1 = p_1O_1I_1$ and $\Sigma_2 = p_2O_2I_2$, respectively. If Σ_1 and Σ_2 have no common apex, then they are described by one of conditions* (C1)–(C3)*.*

Proof. Without loss of generality we assume that C_1 and C_2 are triangles. Let $C_1 = qx_1y_1$, $C_2 = qx_2y_2$, and set $C_3 = qy_1x_2$, $C_0 = qy_2x_2y_1x_1$, so that

$$\sum_{i=0}^{3}[C_i] = 0$$

in $H_1(H)$ (see Figure 5). In addition, let Θ_1, Θ_2 be the theta curves $\Theta_1 = C_1 \cup C_3$, $\Theta_2 = C_2 \cup C_3$.

If C_3 does not link G, then $C_2' = C_2 + C_3 = qy_1x_2y_2$ links G in Σ_2. The cycles C_0, C_1 and C_2' together form a theta curve, and since Σ_1 and Σ_2 have no common apex we are in Case (A2) of Theorem 14. Since also Σ_1 and Σ_2 are mutually oriented, it follows that Case (C3) above holds.

Suppose then that C_3 links G in a star Σ_3. We consider cases, according to which case of Theorem 14 is satisfied by each of Θ_1 and Θ_2.

Case 1: Both Θ_1 and Θ_2 satisfy Case (A2) of Theorem 14

Then Σ_3 has no common apex with Σ_1 or Σ_2. Let $\Sigma_3 = p_3O_3I_3$, where $p_3 \neq p_1, p_2$.

By (A2) applied to Θ_1 there are two possible ways in which Σ_1 and Σ_3 can meet: either $O_1 = O_3$, $p_1 \in I_3$ and $p_3 \in I_1$; or $I_1 = I_3$, $p_1 \in O_3$ and $p_3 \in O_1$. Likewise, by (A2) applied to Θ_2 there are two possible ways in which Σ_2 and Σ_3 can meet: either $O_2 = O_3$, $p_2 \in I_3$ and $p_3 \in I_2$; or $I_2 = I_3$, $p_2 \in O_3$ and $p_3 \in O_2$. If $O_1 = O_3$ and $p_1 \in I_3$ then (since $p_1 \in O_2$) both $O_2 = O_3$ and $I_2 = I_3$ are impossible, so it must be the case that $I_1 = I_3$ and $p_1 \in O_3$. By the same argument it must also be the case that $I_2 = I_3$ and $p_2 \in O_3$. Then by (A2) it follows that

$$O_1 = \{p_2, p_3\}, \qquad O_2 = \{p_1, p_3\}, \qquad O_3 = \{p_1, p_2\},$$

and $I_1 = I_2 = I_3 = G - \{p_1, p_2, p_3\}$. This shows that (C3) above holds.

Case 2: Θ_1 satisfies Case (A1) **of Theorem 14, Θ_2 satisfies Case** (A2)

Then Σ_1 and Σ_3 have a common apex, and Σ_2 and Σ_3 are proper stars with no common apex.

Suppose first that p_1 cannot be chosen as the common apex of Σ_1 and Σ_3. Then we can write $\Sigma_1 = p_3 O_1' I_1'$, $\Sigma_3 = p_3 O_3 I_3$, for some $p_3 \neq p_1, p_2$. Since either p_1 or p_3 can be chosen as the apex of Σ_1, this star must be a fan with axis $\pm p_1 p_3$ and therefore one of O_1 and I_1 must be equal to $\{p_3\}$. But $p_2 \in O_1$ by hypothesis, so it must be I_1 that is equal to $\{p_3\}$; that is, $\Sigma_1 = p_1 O_1 \{p_3\} = p_3 \{p_1\} O_1$ and we have $O_1' = \{p_1\}$, $I_1' = O_1$.

The star Σ_3 is proper, so $|I_3| \geqslant 2$ and hence I_3 contains some $r \neq p_1, p_3$. Then r must belong to $I_1' = O_1 = G - \{p_1, p_3\}$ also, which implies $I_1' \cap I_3 \neq \varnothing$. It now follows by (A1) applied to Θ_1 that $O_1' \cap O_3$ must be empty, and therefore $p_1 \in I_3$. Now recall that $p_1 \in O_2$. This means that $O_2 \neq O_3$ and $I_2 \neq I_3$, in contradiction with Case (A2). We conclude that p_1 must be the common apex of Σ_1 and Σ_3.

Accordingly, let $\Sigma_3 = p_1 O_3 I_3$ and consider $\Theta_2 = C_2 \cup C_3$. Since the apex p_1 of Σ_3 belongs to O_2, by (A2) we must have $I_2 = I_3$ and $O_2 = \{p_1, p_3\}$, $O_3 = \{p_2, p_3\}$ for some $p_3 \in G$. Then since $O_1 \cap O_3$ contains p_2 it is nonempty, so by (A1) applied to Θ_1 we must have $I_1 \cap I_3 = \varnothing$. But $I_3 = G - \{p_1, p_2, p_3\}$ and $p_1, p_2 \notin I_1$, so the only possibility is $I_1 = \{p_3\}$. We conclude that (C1) holds.

Case 3: Θ_1 satisfies Case (A2) **of Theorem 14, Θ_2 satisfies Case** (A1)

Reversing the roles of Θ_1 and Θ_2 in Case 2, we conclude that Σ_1 and Σ_2 satisfy (C2).

Case 4: Both Θ_1 and Θ_2 satisfy Case (A1) **of Theorem 14**

Then Σ_1 and Σ_2 have a common apex with Σ_3. Since they do not share a common apex with each other, there must be vertices r_1, r_2 in G such that Σ_3 is a fan with axis $\pm r_1 r_2$, and r_i can be chosen as the apex of Σ_i for $i = 1, 2$.

Suppose first that $r_i = p_i$ for $i = 1, 2$. Without loss of generality we may assume that Σ_3 has axis $r_1 r_2 = p_1 p_2$; that is, $\Sigma_3 = p_1 \{p_2\} I_3$, where $I_3 = G - \{p_1, p_2\}$. Choose $p_3 \in I_1$. Then $p_3 \in I_3$ also, so triangle $T = p_1 p_2 p_3$ belongs to both Σ_1 and Σ_3. This means that $C_1' = C_1 + C_3$ strongly links T, a contradiction.

It must therefore be the case that $r_i \neq p_i$ for some i. Suppose without loss of generality that $r_1 \neq p_1$. Let $p_3 = r_1$, and let $\Sigma_1 = p_3 O_1' I_1'$, $\Sigma_3 = p_3 O_3 I_3$. Then as in the second paragraph of Case 2 above we must have $\Sigma_1 = p_3\{p_1\}O_1$; that is, $I_1 = \{p_3\}$, $O'_1 = \{p_1\}$, $I'_1 = O_1 = G - \{p_1, p_3\}$.

By (A1) applied to Θ_1 at least one of $O'_1 \cap O_3$ and $I'_1 \cap I_3$ must be empty. The star Σ_3 is a fan with axis $\pm p_3 r_2$, so one of O_3 and I_3 must equal $\{r_2\}$. If I_3 were equal to $\{r_2\}$ then (since p_1, p_3 and r_2 are distinct) we would have $r_2 \in I'_1 \cap I_3$ and $p_1 \in O'_1 \cap O_3$, contradicting the fact that at least one of $O'_1 \cap O_3$ and $I'_1 \cap I_3$ must be empty. So it must instead be the case that $O_3 = \{r_2\}$, and therefore $\Sigma_3 = p_3\{r_2\}I_3$ for $I_3 = G - \{r_2, p_3\}$. It now follows from (A1) that $C'_1 = C_1 + C_3$ links the proper star $\Sigma_4 = p_3\{p_1, r_2\}I_4$, for $I_4 = G - \{p_1, p_3, r_2\}$.

We now consider the theta curve $\Theta = C'_1 \cup C_2$. If it were the case that $r_2 \neq p_2$ then Σ_2 would be a fan with axis $\pm p_2 r_2$ disjoint from the apex p_3 of Σ_4, which is impossible by Corollary 15 applied to Θ. So we must instead have $r_2 = p_2$, giving $\Sigma_3 = p_3\{p_2\}I_3$, $\Sigma_4 = p_3\{p_1, p_2\}I$ for $I = G - \{p_1, p_2, p_3\}$.

Observe now that C_3 positively links the triangle $T = p_3p_2p_1$. Recall that $\Sigma_2 = p_2O_2I_2$, with $p_1 \in O_2$. If it were the case that $p_3 \in I_2$ then C_2 would also positively link T, and then $C'_2 = C_2 + C_3$ would strongly link T. We must therefore have $p_3 \in O_2$ instead, and hence $\{p_1, p_3\} \subseteq O_2$. It follows that Σ_2 and Σ_4 cannot have a common apex, so by Theorem 14 applied to Θ, O_2 must exactly equal $\{p_1, p_3\}$. We already have $I_1 = \{p_3\}$, so this shows that (C1) holds. This completes the proof. ☐

Proof of Theorem 20. Suppose that Σ_1 and Σ_2 have no common apex, and re-orient T_1 and T_2 (if necessary) so that Σ_1 and Σ_2 are mutually oriented. By Theorem 12 the triangles T_1, T_2 must intersect. If T_1 and T_2 meet in an edge, then $T_1 \cup T_2$ forms a theta curve so Σ_1 and Σ_2 are described by Theorem 14. Since they have no common apex they must satisfy condition (A2), which co-incides with condition (C3). Otherwise, T_1 and T_2 meet in a single vertex and the result follows from Proposition 22. ☐

6 Our main results

We are now ready to complete our characterisation of weakly linked embeddings of $G \cong K_m$ and $H \cong K_n$. The first step is to prove the common vertex or common triangle dichotomy of Theorem 3, restated here as Theorem 23. This leads to two cases: one of G and H contains a common triangle (see Section 6.1), or both contain a common vertex (see Sections 6.2 and 6.3). In each case, we first determine the possible patterns of linking numbers; exhibit embeddings realising them; and then prove that our embeddings are weakly linked.

Theorem 23. *Let $m \geqslant 5$ and $n \geqslant 4$. Suppose that $G \cong K_m$ and $H \cong K_n$ are weakly linked graphs in $\mathbb{R}^3$. If there is no vertex of G common to all triangles of G linking H, then there is a triangle T^* in G such that a triangle $T \neq T^*$ of G links H if and only if it shares an edge with T^*.*

Proof. Let $\mathcal{T}_G$ be the set of oriented triangles in G that link H, and let $\mathcal{T}_H$ be the set of oriented triangles in H that link G. By Theorem 6 each triangle T in $\mathcal{T}_H$ links G in a star

Σ_T, and we let

$$\mathcal{S}_G = \{\Sigma_T : T \in \mathcal{T}_H\}.$$

We claim that

$$\mathcal{T}_G = \bigcup_{\Sigma \in \mathcal{S}_G} \Sigma.$$

Indeed, any triangle $S \in \Sigma_T \subseteq \mathcal{S}_G$ links the triangle T of H, so belongs to $\mathcal{T}_G$; and conversely, any triangle $S \in \mathcal{T}_G$ must positively link some triangle $T \in \mathcal{T}_H$ (for example, by subdividing a cycle D in H linking S, or because S must link H in a star Σ_S), and consequently belongs to Σ_T.

First suppose that there is no proper star in $\mathcal{S}_G$. Then every star in $\mathcal{S}_G$ is a fan, and by Remark 21 the axes of any two such fans must intersect. Choose vertices p_1, p_2 in G such that the fan with axis p_1p_2 belongs to $\mathcal{S}_G$, and note that the axis of any other fan in $\mathcal{S}_G$ must contain either p_1 or p_2. Since p_1 is not common to all triangles linking H, there must be a vertex p_3 of G such that the fan with axis p_2p_3 belongs to $\mathcal{S}_G$; and since also p_2 is not common to all triangles linking H, there must be a vertex p_4 of G such that the fan with axis p_4p_1 belongs to $\mathcal{S}_G$. But then the fan axes p_2p_3 and p_4p_1 are disjoint unless $p_3 = p_4$. It follows that we must have $p_3 = p_4$, and then $\mathcal{S}_G$ contains precisely the fans with axes $\pm p_1p_2$, $\pm p_2p_3$ and $\pm p_3p_1$. The triangle $T^* = p_1p_2p_3$ therefore satisfies the conclusion of the theorem.

Now suppose that there is $T_1 \in \mathcal{T}_H$ such that $\Sigma_{T_1} = p_1O_1I_1$ is a proper star. By assumption p_1 is not common to all triangles in G linking H, so there is a triangle $T_2 \in \mathcal{T}_H$ such that $\Sigma_{T_2} = p_2O_2I_2$ has no common apex with Σ_{T_1}. Without loss of generality we may assume that Σ_{T_1} and Σ_{T_2} are mutually oriented, and then by Theorem 20 there is a vertex p_3 of G such that $O_1 = \{p_2, p_3\}$, and either $O_2 = \{p_1, p_3\}$, or Σ_{T_2} is a fan with axis p_3p_2. Let $T^* = p_1p_2p_3$, $I^* = G - \{p_1, p_2, p_3\}$, and for $i, j \in \{1, 2, 3\}$ define $I_{ij} = G - \{p_i, p_j\}$. We claim that, up to orientation, every star in $\mathcal{S}_G$ is equal to one of

$$\Sigma_1 = p_1\{p_2, p_3\}I^*, \qquad \Sigma_2 = p_2\{p_3, p_1\}I^*, \qquad \Sigma_3 = p_3\{p_1, p_2\}I^*,$$
$$\Sigma_{12} = p_1\{p_2\}I_{12}, \qquad \Sigma_{23} = p_2\{p_3\}I_{23}, \qquad \Sigma_{31} = p_3\{p_1\}I_{31}.$$

It would then follow that T^* satisfies the conclusion of the theorem. Note that under these conditions T^* links H if and only if one of the stars Σ_{ij} belongs to $\mathcal{S}_G$.

So far we have Σ_{T_1} equal to Σ_1, and Σ_{T_2} equal to either Σ_2 or $-\Sigma_{23}$. The case $m \geqslant 6$ is simpler than the case $m = 5$, so we will assume for now that $m \geqslant 6$ and address the case $m = 5$ later. Under the assumption $m \geqslant 6$ we have $|I^*| \geqslant 3$. If $\Sigma = pOI \in \mathcal{S}_G$ is a proper star with $p \neq p_1$, then Σ must be one of $\pm\Sigma_2, \pm\Sigma_3$, by Theorem 20 applied to Σ and $\Sigma_{T_1} = \Sigma_1$. In addition, if $\Sigma = pOI \in \mathcal{S}_G$ is a proper star with $p = p_1$, then Σ must be $\pm\Sigma_1$, by Theorem 20 applied to Σ and Σ_{T_2}, regardless of whether Σ_{T_2} is equal to Σ_2 or $-\Sigma_{23}$. We conclude that, up to orientation, when $m \geqslant 6$ the only proper stars that can belong to $\mathcal{S}_G$ are Σ_1, Σ_2 and Σ_3.

Still assuming $m \geqslant 6$, if $\Sigma \in \mathcal{S}_G$ is a fan with axis pq disjoint from p_1 then by Theorem 20 we must have $\Sigma = \pm\Sigma_{23}$. The axis of any other fan in $\mathcal{S}_G$ must therefore meet p_1. Regardless of whether Σ_{T_2} is equal to Σ_2 or $-\Sigma_{23}$, up to orientation the only other fan axes possible are p_1p_2 and p_3p_1, giving us Σ_{12} or Σ_{31}: if $\Sigma_2 \in \mathcal{S}_G$ then p_3p_1 is the only fan axis disjoint from p_2 satisfying Theorem 20 with respect to Σ_2, and p_1p_2 is the only axis meeting both p_1 and p_2; while if $-\Sigma_{23} \in \mathcal{S}_G$ then any axis must meet both p_1 and p_2p_3. Thus, the only possible stars in $\mathcal{S}_G$ are those listed above, and the theorem is proved for $m \geqslant 6$.

We turn now to the case $m = 5$. Then $|I^*| = 2$, and we let $I^* = \{p_4, p_5\}$. The additional difficulty that arises in this case is that the fan with axis p_4p_5 and the proper stars $p_4\{p_2, p_3\}\{p_1, p_5\}$ and $p_5\{p_2, p_3\}\{p_1, p_4\}$ also satisfy Theorem 20 applied to Σ_1.

Suppose first that every proper star in $\mathcal{S}_G$ has apex p_1. Then Σ_{T_2} is equal to $-\Sigma_{23}$. The only proper stars with apex p_1 that satisfy Theorem 20 with respect to Σ_{23} are $\pm\Sigma_1$, so there can be no other proper star in $\mathcal{S}_G$. The axis of any other fan must meet p_2p_3; by Theorem 20 applied to Σ_1 the only possibilities are $\pm\Sigma_{12}$ and $\pm\Sigma_{31}$. Thus T^* satisfies the required conditions.

Suppose finally then that there is a proper star in $\mathcal{S}_H$ with apex not equal to p_1. We could have chosen this star as Σ_{T_2}, so without loss of generality we may assume that $\Sigma_{T_2} = \Sigma_2$. Any other proper star in $\mathcal{S}_G$ has no common apex with at least one of Σ_1 and Σ_2, and so must satisfy Theorem 20 with respect to one or both of Σ_1 and Σ_2. Up to orientation, the stars that are compatible with Σ_1 are

$$p_2|p_1p_3|p_4p_5, \qquad p_4|p_2p_3|p_1p_5,$$
$$p_3|p_1p_2|p_4p_5, \qquad p_5|p_2p_3|p_1p_4;$$

while those that are compatible with Σ_2 are

$$p_1|p_2p_3|p_4p_5, \qquad p_4|p_1p_3|p_2p_5,$$
$$p_3|p_1p_2|p_4p_5, \qquad p_5|p_1p_3|p_2p_4.$$

The only star that appears on both lists is $p_3\{p_1, p_2\}\{p_4, p_5\} = \Sigma_3$, so we conclude that up to orientation the only proper stars that can belong to $\mathcal{S}_G$ are Σ_1, Σ_2 and Σ_3.

By Theorem 20 applied to each of Σ_1 and Σ_2, if there is a fan other than $\pm\Sigma_{12}$, $\pm\Sigma_{23}$ and $\pm\Sigma_{31}$ in $\mathcal{S}_G$ then it must have axis $\pm p_4p_5$. So suppose that there is a triangle in H linking the fan $\Sigma_{45} = p_4|p_5|p_1p_2p_3$. Then this is in fact the only fan in $\mathcal{S}_G$, because Σ_{12}, Σ_{23} and Σ_{31} all have axes disjoint from p_4p_5. So up to orientation, the only stars that can belong to $\mathcal{S}_G$ are Σ_i for $i = 1, 2, 3$ and Σ_{45}. We show that this is impossible.

Label the triangles of the K_4 subgraph $K = \langle p_1, p_2, p_4, p_5\rangle$ such that

$$C_1 = p_2p_4p_5, \qquad C_2 = p_5p_4p_1,$$
$$C_4 = p_1p_2p_5, \qquad C_5 = p_4p_2p_1.$$

Observe that each of these triangles belongs to at least one of $\pm\Sigma_1$, $\pm\Sigma_2$ and $\pm\Sigma_{45}$, and so links H. Since every triangle of K links H we must be in case (B1) of Theorem 16. It follows that there is a vertex q of H, a sign $\varepsilon \in \{\pm 1\}$, and pairwise disjoint subsets J_1, J_2, J_4, J_5 of $H - \{q\}$ such that C_i links H in the star $\varepsilon q P_i J_i$ for $i = 1, 2, 4, 5$, where

$$P_i = H - \{q\} - J_i.$$

Moreover, J_i must be nonempty for each i, because otherwise C_i does not link H. We may therefore choose $x_i \in J_i$ for $i = 2, 5$, to get a triangle qx_2x_5 in H that links both C_2 and C_5 in K. But this is impossible, because none of the stars that can belong to $\mathcal{S}_G$ contains both of these triangles, so no triangle in H can link both C_2 and C_5. It follows that the fan with axis p_4p_5 cannot belong to $\mathcal{S}_G$, and the theorem is proved. $\square$

6.1 Embeddings with a common triangle

In this section we analyse the case where at least one of G and H contains no vertex common to all triangles linking the other. Without loss of generality we may assume that this is G, and then by Theorem 23 there is a triangle T^* in G such that a triangle $T \neq T^*$ in G links H if and only if T shares an edge with T^*.

Theorem 24 shows that there are two possible patterns of linking numbers, according to whether or not H contains a vertex common to all triangles linking G. We exhibit embeddings realising these in Figure 6, and then prove that our embeddings are weakly linked in Theorem 25.

Theorem 24. *Let $m, n \geqslant 5$, and let $G \cong K_m$ and $H \cong K_n$ be weakly linked graphs in $\mathbb{R}^3$. Suppose that there is a triangle $T^* = p_1p_2p_3$ in G such that a triangle $T \neq T^*$ in G links H if and only if T shares an edge with T^*. Let $X = G - \{p_1, p_2, p_3\}$, and for each $x \in X$ let*

$$T_0(x) = T^* = p_1p_2p_3, \qquad T_1(x) = p_3p_2x, \qquad T_2(x) = xp_1p_3, \qquad T_3(x) = p_2p_1x.$$

Then exactly one of the following holds:

(D1) There is a vertex q of H common to all triangles of H linking G. Then there are pairwise disjoint sets I_0, I_1, I_2, I_3 such that $I_0 \cup I_1 \cup I_2 \cup I_3 = H - \{q\}$, and after reversing the orientation of $\mathbb{R}^3$ (if necessary), for each $x \in X$ and $0 \leqslant i \leqslant 4$ the triangle $T_i(x)$ links H in the star qO_iI_i, where

$$O_i = H - \{q\} - I_i.$$

Moreover, I_i is nonempty for $1 \leqslant i \leqslant 3$, and I_0 is nonempty if and only if T^* links H.

(D2) There is no vertex of H common to all triangles of H linking G. Then T^* does not link H, and there is a triangle $U^* = q_1q_2q_3$ of H such that a triangle U of H links G if and only if U shares exactly one edge with U^*. Let $Y = H - \{q_1, q_2, q_3\}$, and for each $y \in Y$ let

$$U_1(y) = q_3q_2y, \qquad U_2(y) = yq_1q_3, \qquad U_3(y) = q_2q_1y.$$

Then after relabelling the p_i, q_i and reversing orientation of $\mathbb{R}^3$ (if necessary),

(a) *for each* $x \in X$ *the triangles* $T_1(x), T_2(x), T_3(x)$ *link* H *in the stars*

$$q_1\{q_2, q_3\}Y, \qquad q_2\{q_1, q_3\}Y, \qquad q_3\{q_1, q_2\}Y;$$

and

(b) *for each* $y \in Y$ *the triangles* $U_1(y), U_2(y), U_3(y)$ *link* G *in the stars*

$$p_1X\{p_2, p_3\}, \qquad p_2X\{p_1, p_3\}, \qquad p_3X\{p_1, p_2\}.$$

Proof. Let $x_1, x_2 \in X$. We begin by showing that

$$\mathrm{Lk}(T_i(x_1), D) = \mathrm{Lk}(T_i(x_2), D)$$

for $1 \leqslant i \leqslant 3$ and all cycles D in H.

By symmetry, we may assume without loss of generality that $i = 1$, so that $T_i(x_1) = T_1(x_1) = p_3p_2x_1$, $T_i(x_2) = T_1(x_2) = p_3p_2x_2$. Consider the 4-cycle $C = p_2x_1p_3x_2$ in G. As a 1-chain we have

$$C = x_1p_3x_2 + x_2p_2x_1.$$

The triangles $x_1p_3x_2$, $x_2p_2x_1$ have no edge in common with T^*, so by hypothesis they do not link H. Hence in $H_1(\mathbb{R}^3 - D)$ we have

$$[C] = [x_1p_3x_2] + [x_2p_2x_1] = 0 + 0 = 0.$$

On the other hand, we may also write

$$C = p_2x_1p_3 + p_3x_2p_2 = T_1(x_1) - T_2(x_2),$$

and therefore

$$[T_1(x_1)] - [T_2(x_2)] = [C] = 0.$$

It follows that $\mathrm{Lk}(T_1(x_1), D) = \mathrm{Lk}(T_1(x_2), D)$ as claimed.

Fix $x \in X$. Since no triangle contained in X links H by hypothesis, it follows from the above that the linking between G and H is completely determined by the linking between $G' = \langle x, p_1, p_2, p_3\rangle$ and H. Since $G' \cong K_4$, this is given by Theorem 16, with x in the role of p_0; that is, with $C_i = T_i(x)$ for $0 \leqslant i \leqslant 3$.

If case (B1) of Theorem 16 holds, then (since $T_i(x')$ must link H in the same star as $T_i(x)$ for $1 \leqslant i \leqslant 3$ and all $x' \in X$) G links H according to Case (D1) above. We note that I_i is necessarily nonempty for $1 \leqslant i \leqslant 3$, because $T_i(x)$ links H for all $x \in X$ by hypothesis. Moreover, T^* links H if and only I_0 is nonempty, as given.

If Case (B2) of Theorem 16 holds, then G links H according to (D2a). To obtain part (D2b), we replace $\{p_0\}$ with X in the stars given in Remark 17. □

Theorem 25. *The embeddings of Figure 6 realising Cases* (D1) *and* (D2) *of Theorem 24 are weakly linked.*

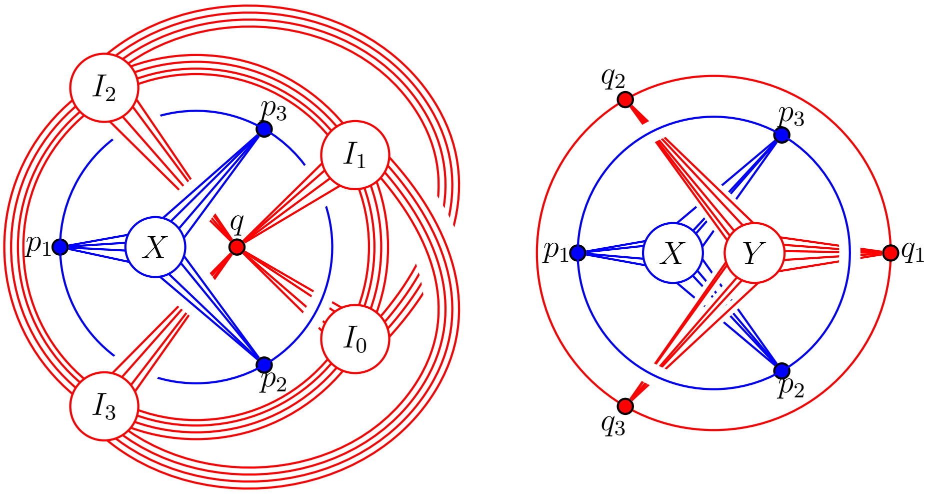

Figure 6: Embeddings of $G \cong K_m$ (blue) and $H \cong K_n$ (red) realising Cases (D1) (left) and (D2) (right) of Theorem 24.

Proof. Let T^* be the triangle $p_1p_2p_3$, and let G' be G minus the three edges p_1p_2, p_2p_3 and p_3p_1 of T^*. Then there is a 2-sphere separating G' from H, so G' does not link H. Therefore any cycle C in G that links H must use at least one edge belonging to T^*.

If C uses all three edges of T^*, then we necessarily have $C = \pm T^*$. In the embedding of Figure 6 (left) T^* does not link H, and in the embedding of Figure 6 (right) T^* links H in the star qO_0I_0, where $O_0 = I_1 \cup I_2 \cup I_3$. In either case C does not strongly link H, so we may assume in what follows that C uses at most two edges of T^*.

The edges of T^* on C must occur consecutively, so we may decompose C as the concatenation PQ, where P is a path in T^* and Q is a path in G'. By symmetry, we may assume without loss of generality that P begins at p_1 and ends at p_2 (travelling anticlockwise if it has length 1, and clockwise via p_3 if it has length 2). Let x_0 be the first vertex of X on Q, and let R, $\bar{R}$ be the paths $p_2x_0p_1$, $p_1x_0p_2$, respectively. Then we may decompose C as $C = C_1 + C_2$, where C_1 is the concatenation PR and C_2 is the concatenation $\bar{R}Q$ (when $Q = R$ we have simply $C = C_1$). Then C_2 does not link H, because it is a cycle in G', so for any cycle D in H we have

$$\mathrm{Lk}(C, D) = \mathrm{Lk}(C_1, D).$$

To complete the proof we check that C_1 does not strongly link H, by verifying that it links a star in H.

If the path P has length 1 (that is, if it is simply the edge p_1p_2) then $C_1 = p_1p_2x_0$ is the triangle $-T_3(x_0)$. In either embedding this links H in a star: the star qI_3O_3 in Figure 6 (left), and the star $q_3Y\{q_1, q_2\}$ in Figure 6 (right). On the other hand, if P has length 2 (that is, if P is the path $p_1p_3p_2$), then C_1 links the star $q(I_1 \cup I_2)(I_0 \cup I_3)$ in

Figure 6 (left), and the star $q_3Y\{q_1, q_2\}$ in Figure 6 (right). In all cases C_1 links a star in H, so G and H are not strongly linked. □

6.2 Embeddings with a common vertex in both G and H

In this section we analyse the case where there is a vertex in each graph common to all triangles linking the other. The linking between the two graphs is described by Theorem 26, and we exhibit a weakly linked embedding realising it in Figure 7.

Theorem 26. *Let $m, n \geqslant 5$, and let $G \cong K_m$ and $H \cong K_n$ be weakly linked graphs in $\mathbb{R}^3$. Suppose that there is a vertex p of G common to all triangles of G linking H, and a vertex q of H common to all triangles of H linking G. Then for some $2 \leqslant \ell \leqslant \min\{m, n\} - 1$, there exists*

- *a partition $\mathcal{X} = \{X_0, X_1, \ldots, X_{\ell-1}\}$ of $G' = G - \{p\}$, such that the triangle pxy of G links H if and only if x and y belong to different parts of $\mathcal{X}$; and*
- *a partition $\mathcal{Y} = \{Y_0, Y_1, \ldots, Y_{\ell-1}\}$ of $H' = H - \{q\}$, such that the triangle quv of H links G if and only if u and v belong to different parts of $\mathcal{Y}$.*

Moreover:

1. *If $x_j \in X_j$, $x_k \in X_k$ for $j < k$, then px_jx_k links H in the star $qO_{jk}I_{jk}$, where*
$$O_{jk} = \bigcup_{i=j}^{k-1} Y_i, \qquad I_{jk} = H' - O_{jk} = \left(\bigcup_{i=0}^{j-1} Y_i\right) \cup \left(\bigcup_{i=k}^{\ell-1} Y_i\right).$$
2. *If $y_j \in Y_j$, $y_k \in Y_k$ for $j < k$, then qy_jy_k links G in the star $pP_{jk}J_{jk}$, where*
$$J_{jk} = \bigcup_{i=j+1}^{k} X_i, \qquad P_{jk} = G' - J_{jk} = \left(\bigcup_{i=0}^{j} X_i\right) \cup \left(\bigcup_{i=k+1}^{\ell-1} X_i\right).$$

An embedding realising the linking of Theorem 26 is described in Construction 40, and the case $\ell = 5$ is illustrated in Figure 7. Note that the partitions $\mathcal{X}$ and $\mathcal{Y}$ are circularly rather than linearly ordered.

We will prove Theorem 26 through a series of intermediate results. These will typically be proved under the hypotheses of Theorem 26. To avoid repeating these, unless some other hypothesis is given, *we assume throughout this section that G and H are weakly linked.* Our first step is to get our hands on the partition $\mathcal{X}$, which we will do by defining an equivalence relation $\sim$ on G'. The definition of $\sim$ depends only on the existence of the vertex $p \in G$ common to all triangles linking H, and not on the existence of the vertex $q \in H$ common to all triangles linking G. For full generality we therefore begin by assuming only the existence of p, and postpone introducing the hypothesis of the existence of q. Thus, unless some other hypothesis is given, *we assume throughout this section that there is a vertex p of G common to all triangles of G linking H.*

Definition 27. Let $G' = G - \{p\}$. We define a relation $\sim$ on the vertices of G' by $x \sim y$ if and only if $x = y$, or $x \neq y$ and pxy does not link H.

We prove that $\sim$ is an equivalence relation on G' in Lemma 29 below. We will write $[x]$ for the equivalence class of $x \in G'$ with respect to $\sim$, and $\mathcal{X}$ for $\{[x] : x \in G'\}$, the set of equivalence classes of $\sim$. Note that $|\mathcal{X}| \geqslant 2$, because if $|\mathcal{X}| = 1$ then G does not link H.

To prove Lemma 29 and establish some other properties of $\sim$ we will repeatedly use the following lemma.

Lemma 28. *Let x, y, z be distinct vertices of G', and let D be a cycle of H. Then*

$$[pxy] + [pyz] + [pzx] = 0 \tag{3}$$

holds in $H_1(\mathbb{R}^3 - D)$.

Proof. The triangles pxy, pyz, pzx and zyx satisfy $pxy + pyz + pzx + zyx = 0$ as 1-chains in G, so in $H_1(\mathbb{R}^3 - D)$ we have

$$[pxy] + [pyz] + [pzx] + [zyx] = 0.$$

By assumption p is common to all triangles of G linking H, so zyx does not link H and therefore $[zyx] = 0$. The lemma follows. □

Lemma 29. *The relation $\sim$ on G' of Definition 27 is an equivalence relation.*

Proof. The relation $\sim$ is reflexive by definition, and it is symmetric because $\mathrm{Lk}(pyx, D) = -\mathrm{Lk}(pxy, D)$ for all $x \neq y$ in G' and any cycle D in H. To prove that $\sim$ is transitive, suppose that x, y, z are distinct vertices of G' such that $x \sim y$ and $y \sim z$. Let D be a cycle of H. Then $[pxy] = [pyz] = 0$ in $H_1(\mathbb{R}^3 - D)$, so by Lemma 28 we have

$$[pxz] = [pxy] + [pyz] = 0 + 0 = 0$$

also. Since this holds for any cycle D in H we conclude that pxz does not link H, and therefore $x \sim z$. □

Remark 30. In the embedding of Figure 6 (left), the vertex q is common to all triangles of H linking G. The equivalence classes of the corresponding relation $\sim$ defined on $H' = H - \{q\}$ are the sets I_i, for $0 \leqslant i \leqslant 3$.

Lemma 31. *Let x, $y \in G'$ with $x \sim y$, and let D be a cycle in H. Then*

$$\mathrm{Lk}(pxz, D) = \mathrm{Lk}(pyz, D)$$

for all $z \in G'$ with $z \neq x, y$.

Proof. For any $z \in G'$ we have $[pxy] = 0$ in equation (3), and so $[pxz] = -[pzx] = [pyz]$. □

Lemma 32. *Let x, $y \in G'$. Suppose that there is $z \in G'$ such that*

$$\text{Lk}(pxz, D) = \text{Lk}(pyz, D)$$

for all cycles D in H. Then $x \sim y$.

Proof. Let D be a cycle of H. Applying Lemma 28, in $H_1(\mathbb{R}^3 - D)$ we have

$$[pxy] = [pxz] - [pyz] = 0.$$

Since this holds for any cycle D in H we conclude that pxy does not link H, and therefore $x \sim y$. □

We now introduce the hypothesis of the existence of q. Thus, unless some other hypothesis is given, *we assume throughout the rest of this section that there is a vertex q of H common to all triangles of H linking G.* Let $x, y \in G'$ be such that $x \not\sim y$. Then pxy links H, so by Theorem 6 it links H in a star $qO_{xy}I_{xy}$ with apex q, because q is common to all triangles of H linking G. Note here that $\{O_{xy}, I_{xy}\}$ is a partition of $H' = H - \{q\}$.

Lemma 33. *Let $x, y \in G'$ such that $x \not\sim y$. Then the star $qO_{xy}I_{xy}$ depends only on the equivalence classes of x and y. More precisely, if $x \sim z$ and $y \sim w$, then $O_{xy} = O_{zw}$ and $I_{xy} = I_{zw}$.*

Proof. Since $x \sim z$, by Lemma 31 we have $\text{Lk}(pxy, D) = \text{Lk}(pzy, D)$ for all cycles D in H. It follows that $qO_{zy}I_{zy} = qO_{xy}I_{xy}$. Similarly, since $y \sim w$, we have $qO_{yz}I_{yz} = qO_{wz}I_{wz}$. The result now follows from the fact that if the triangle pab links H in the star qOI, then $pba = -pab$ links H in the star $-qOI = qIO$; that is, $qO_{ba}I_{ba} = qI_{ab}O_{ab}$. □

Our next step is to establish the cyclic ordering of $\mathcal{X}$, the set of equivalence classes of $\sim$. We do this below by introducing a method of cyclically ordering triples of points in G'. This will be well defined on equivalence classes, and we will show that we can use it to cyclically order them.

Let (x, y, z) be an ordered triple of points in G' such that $x \not\sim y \not\sim z \not\sim x$. Consider $K_4 = \langle p, x, y, z \rangle$, with the faces labelled and oriented such that

$$C_0 = zyx, \qquad C_1 = pyz, \qquad C_2 = pxz, \qquad C_3 = pyz.$$

Note that $\sum_i C_i = 0$ as a 1-chain in G. Since q is common to all triangles of H linking K_4 the linking between K_4 and H is described by Case (B1) of Theorem 16. Furthermore xyz does not link H, and the other three triangles all do because $x \not\sim y \not\sim z \not\sim x$, so exactly one of the following holds:

(a) the sets O_{xy}, O_{yz}, O_{zx} are a partition of H', and

$$I_{xy} = O_{zx} \cup O_{yz}, \qquad I_{yz} = O_{xy} \cup O_{zx}, \qquad I_{zx} = O_{yz} \cup O_{xy};$$

or

(b) the sets I_{xy}, I_{yz}, I_{zx} are a partition of H', and

$$O_{xy} = I_{zx} \cup I_{yz}, \qquad O_{yz} = I_{xy} \cup I_{zx}, \qquad O_{zx} = I_{yz} \cup I_{xy}.$$

We define

$$\varepsilon(x, y, z) = \begin{cases} +1 & \text{if case (a) holds,} \\ -1 & \text{if case (b) holds.} \end{cases}$$

By Lemma 33 the value of $\varepsilon(x, y, z)$ depends only on the equivalence classes of x, y and z with respect to $\sim$, so we may define ε on triples of distinct equivalence classes by

$$\varepsilon([x], [y], [z]) = \varepsilon(x, y, z).$$

Observe that

$$\varepsilon(x, y, z) = \varepsilon(y, z, x) = \varepsilon(z, x, y)$$

since cyclically permuting x, y, z does not change the triangles involved; and

$$\varepsilon(x, y, z) = -\varepsilon(y, x, z),$$

since swapping x and y reverses the orientations of all the triangles, and $qO_{ba}I_{ba} = qI_{ab}O_{ab}$ for all $a \not\sim b$.

Remark 34. Observe that if $\varepsilon(x, y, z) = 1$, then

$$O_{xz} = I_{zx} = O_{xy} \cup O_{yz}.$$

Note also that $O_{xy} \cap O_{yz} = \varnothing$, so $\{O_{xy}, O_{yz}\}$ is a partition of O_{xz}.

Lemma 35. *Let x, y, z, w be distinct vertices in G' such that no two belong to the same equivalence class. Suppose that $\varepsilon(x, y, z) = \varepsilon(x, z, w) = 1$. Then $\{O_{xy}, O_{yz}, O_{zw}, O_{wx}\}$ is a partition of H', and $\varepsilon(y, z, w) = \varepsilon(y, w, x) = 1$.*

Proof. Since $\varepsilon(x, y, z) = 1$, the sets O_{xy}, O_{yz}, O_{zx} are a partition of H', and $I_{zx} = O_{yz} \cup O_{xy}$. Likewise O_{xz}, O_{zw}, O_{wx} are a partition of H', and $I_{xz} = O_{wx} \cup O_{zw}$. Then $O_{zx} = I_{xz} = O_{wx} \cup O_{zw}$, and since $\{O_{zx}, I_{zx}\}$ is a partition of G', it must be the case that $\{O_{xy}, O_{yz}, O_{zw}, O_{wx}\}$ is a partition of G' too. In particular, $O_{yz} \cap O_{zw} = O_{wx} \cap O_{xy} = \varnothing$, so the ordered triples (y, z, w) and (y, w, x) must both satisfy case (a) above. ☐

Proposition 36. *Suppose that $|\mathcal{X}| = \ell$. Then there is a bijection $i \mapsto X_i$ from $\{i : 0 \leqslant i \leqslant \ell - 1\}$ to $\mathcal{X}$ such that $\varepsilon(X_i, X_j, X_k) = 1$ for $i \neq j \neq k \neq i$ if and only if the strictly increasing permutation of i, j, k is a cyclic permutation of (i, j, k).*

Proof. Fix $x_0 \in G'$, and let $\mathcal{X}' = \mathcal{X} - \{[x_0]\}$. Define a relation $\preceq$ on $\mathcal{X}'$ by $[y] \preceq [z]$ if and only if $[y] = [z]$, or $[y] \neq [z]$ and $\varepsilon(x_0, y, z) = 1$. We claim that $\preceq$ is a total order on $\mathcal{X}'$.

The relation $\preceq$ is reflexive by definition. To prove that it is antisymmetric, observe that if $[y] \neq [z]$, then exactly one of $\varepsilon(x_0, y, z) = 1$ and $\varepsilon(x_0, z, y) = 1$ holds, so exactly one of $y \preceq z$ and $z \preceq y$ holds. This also shows that the relation $\preceq$ is connex[1], so it only remains to prove that $\preceq$ is transitive. This follows from Lemma 35. Suppose that $y \preceq z$ and $z \preceq w$ for y, z, w belonging to distinct classes. Then $\varepsilon(x_0, y, z) = \varepsilon(x_0, z, w) = 1$, so by Lemma 35 $\varepsilon(y, w, x_0) = 1$. But $\varepsilon(x_0, y, w) = \varepsilon(y, w, x_0)$, so $y \preceq w$.

For $1 \leqslant i \leqslant \ell - 1$ choose $x_i \in G'$ such that $i \mapsto [x_i]$ is an order preserving bijection from $(\{i : 1 \leqslant i \leqslant \ell - 1\}, \leqslant)$ to $(\mathcal{X}', \preceq)$. Let $X_i = [x_i]$ for $0 \leqslant i \leqslant \ell - 1$. Then $i \mapsto X_i$ is a bijection from $\{i : 0 \leqslant i \leqslant \ell - 1\}$ to $\mathcal{X}$, and we claim it satisfies the required condition.

To prove this, it suffices to show that $\varepsilon(x_i, x_j, x_k) = 1$ whenever $i < j < k$. For $[y], [z] \in \mathcal{X}'$ write $[y] \prec [z]$ if $[y] \neq [z]$ and $[y] \preceq [z]$. If $i = 0$ then $\varepsilon(x_0, x_j, x_k) = 1$ by definition of $\preceq$, because $[x_j] \prec [x_k]$ if and only if $j < k$. Otherwise, since $0 < i < j < k$ we have $x_i \prec x_j \prec x_k$, so $\varepsilon(x_0, x_i, x_j) = \varepsilon(x_0, x_j, x_k) = 1$. Then $\varepsilon(x_i, x_j, x_k) = 1$ by Lemma 35, and we are done. □

We now define the sets Y_i of Theorem 26, and establish the structure of the stars $pO_{x_jx_k}I_{x_jx_k}$. As in the proof of Proposition 36, for $0 \leqslant i \leqslant \ell - 1$ choose $x_i \in G'$ such that $X_i = [x_i]$. Let $Y_i = O_{x_ix_{i+1}}$ (subscripts on x taken mod ℓ), and set $\mathcal{Y} = \{Y_i : 0 \leqslant i \leqslant \ell - 1\}$. Then:

Proposition 37. *The set $\mathcal{Y}$ is a partition of H', and if $j < k$ then*

$$O_{x_jx_k} = \bigcup_{i=j}^{k-1} Y_i. \tag{4}$$

Consequently

$$I_{x_jx_k} = H' - O_{x_jx_k} = \left(\bigcup_{i=0}^{j-1} Y_i\right) \cup \left(\bigcup_{i=k}^{\ell-1} Y_i\right).$$

Proof. Each set Y_i is nonempty, because $x_i \not\sim x_{i+1}$ and so $O_{x_ix_{i+1}} \neq \varnothing$. We show that $Y_i \cap Y_j = \varnothing$ if $i \neq j$.

Note we consider subscripts mod ℓ. Without loss of generality, assume $i < j$. If $j = i + 1$ then $Y_i \cap Y_{i+1} = \varnothing$ follows from $\varepsilon(x_i, x_{i+1}, x_{i+2}) = 1$, so suppose $j > i + 1$. Consider the 4-tuple $(x_i, x_{i+1}, x_j, x_{j+1})$. Then $\varepsilon(x_i, x_{i+1}, x_j) = \varepsilon(x_i, x_j, x_{j+1}) = 1$, so $\{O_{x_ix_{i+1}}, O_{x_{i+1}x_j}, O_{x_jx_{j+1}}, O_{x_{j+1}x_i}\}$ is a partition of H' by Lemma 35. In particular, $Y_i \cap Y_j = O_{x_ix_{i+1}} \cap O_{x_jx_{j+1}} = \varnothing$, as required.

The proof of equation (4) is by induction on k, using Remark 34 for the inductive step. The case $k = j + 1$ holds by definition of Y_j. If the equation is true for some $k > j$, then since $\varepsilon(x_j, x_k, x_{k+1}) = 1$, for $k + 1$ we have

$$O_{x_jx_{k+1}} = O_{x_jx_k} \cup O_{x_kx_{k+1}} = \left(\bigcup_{i=j}^{k-1} Y_i\right) \cup Y_k = \bigcup_{i=j}^{k} Y_i.$$

To complete the proof we must show that $\bigcup_{i=0}^{\ell-1} Y_i = H'$. Given $u \in H'$, consider the triangle $px_{\ell-1}x_0$, which links H in the star $qO_{x_{\ell-1}x_0}I_{x_{\ell-1}x_0}$. If $u \in O_{x_{\ell-1}x_0} = Y_{\ell-1}$ we are done; and otherwise we must have $u \in I_{x_{\ell-1}x_0} = O_{x_0x_{\ell-1}} = \bigcup_{i=0}^{\ell-2} Y_i$. □

[1]A binary relation $\bowtie$ on a set A is *connex* if for all $x, y \in A$, the condition $x \bowtie y$ or $y \bowtie x$ holds.

Since q is common to all triangles of H linking G, as in Definition 27 and Lemma 29 we may define an equivalence relation $\simeq$ on H' by $u \simeq v$ if and only if quv does not link G. We show that $\mathcal{Y}$ is the set of equivalence classes of $\simeq$ on H':

Corollary 38. *The set $\mathcal{Y}$ is the set of equivalence classes of $\simeq$ on H' defined by $u \simeq v$ if and only if quv does not link G.*

Proof. Let $u, v \in H'$, and suppose that $u \in Y_i$, $v \in Y_j$. If $i \neq j$ then $u \in O_{x_i x_{i+1}}$ but $v \notin O_{x_i x_{i+1}}$, so quv links $px_i x_{i+1}$. Therefore $u \not\simeq v$. On the other hand, if $i = j$ then by Proposition 37 u and v belong to the same part of $\{O_{xy}, I_{xy}\}$ for all $x, y \in G'$ with $x \not\sim y$, so quv does not link pxy for any $x, y \in G'$ and therefore $u \simeq v$. □

To complete the proof of Theorem 26, we establish the structure of the stars linked by triangles in H. This is done by re-expressing the linking described by the stars $qO_{x_a x_b} I_{x_a x_b}$ in terms of stars in G.

Proposition 39. *Suppose that $y_j \in Y_j$, $y_k \in Y_k$. If $j < k$ then $qy_j y_k$ links G in the star $pP_{jk} J_{jk}$ in G, where*

$$J_{jk} = \bigcup_{i=j+1}^{k} X_i, \qquad P_{jk} = G' - J_{jk} = \left(\bigcup_{i=0}^{j} X_i \right) \cup \left(\bigcup_{i=k+1}^{\ell-1} X_i \right).$$

Proof. Let $x_a, x_b \in G$ be such that $x_a \in X_a$, $x_b \in X_b$ and $\mathrm{Lk}(px_a x_b, qy_j y_k) = 1$; that is, so that $y_j \in O_{x_a x_b}$ and $y_k \in I_{x_a x_b}$. If $a < b$ then by Proposition 37 we have $y_j \in O_{x_a x_b}$ and $y_k \in I_{x_a x_b}$ if and only if $a \leqslant j < b$ and $b \leqslant k$, so $a \leqslant j < b \leqslant k$. Otherwise, if $b < a$ then by Proposition 37 we have $y_j \in O_{x_a x_b} = I_{x_b x_a}$ and $y_k \in I_{x_a x_b} = O_{x_b x_a}$ if and only if $b \leqslant k < a$ and $j < b$, so $j < b \leqslant k < a$. Thus $\mathrm{Lk}(px_a x_b, qy_j y_k) = 1$ if and only if b belongs to the interval $(j, k]$ and a does not, and the result follows. □

6.3 Realising Theorem 26

We now describe an embedding of G and H in $\mathbb{R}^3$ realising the linking described by Theorem 26. We will use co-ordinates (z, t) for $\mathbb{R}^3$, where $z \in \mathbb{C}$ and $t \in \mathbb{R}$.

Construction 40. Let $\mathcal{X} = \{X_0, X_1, \ldots, X_{\ell-1}\}$, $\mathcal{Y} = \{Y_0, Y_1, \ldots, Y_{\ell-1}\}$ be partitions of G' and H', respectively, where $\ell \geqslant 2$. Let ζ be the (2ℓ)th root of unity $\zeta = e^{\pi i/\ell}$, and choose $\rho \in \mathbb{R}$ such that $\rho < |1 - \zeta|/2$, so that the circles centred on 1 and ζ with radius ρ do not intersect. This choice also ensures that the circles do not contain 0. Place p at $(0, 1)$ and q at $(0, -1)$, and for $0 \leqslant j \leqslant \ell - 1$

- place the points belonging to X_j on the circle in the plane $t = -1$ with centre ζ^{2j} and radius ρ, so that they are equally spaced on this circle; and
- place the points belonging to Y_j on the circle in the plane $t = +1$ with centre ζ^{2j+1} and radius ρ, so that they are equally spaced on this circle.

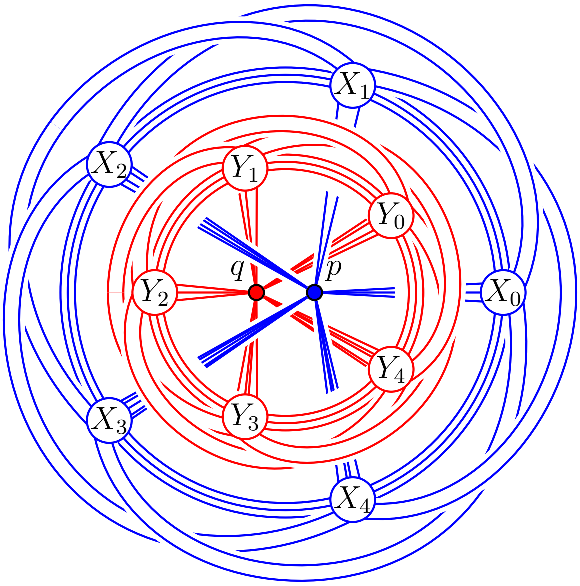

Figure 7: Construction 40 when $\ell = 5$.

Connect p to each vertex $x \in G'$ by a straight line, and similarly connect q to each vertex $y \in H'$ by a straight line. No edge px meets any edge qy, because the projections of these line segments into the plane $t = 0$ meet only at $z = 0$. To complete the embedding, join each pair of vertices in G' by an embedded arc in the half space $t \leqslant -1$, and similarly join each pair of vertices in H' by an embedded arc in the half space $t \geqslant 1$.

Figure 7 illustrates the embedding in the case $\ell = 5$. We show that it realises the linking pattern of Theorem 26 in Proposition 41, and then use Proposition 42 to show that it is in indeed weakly linked in Corollary 43.

Proposition 41. *The embedding of Construction 40 realises the linking pattern of Theorem 26.*

Proof. If p is deleted from G then there is a 2-sphere separating G' from H, so G' does not link H. Similarly, if q is deleted from H then there is a 2-sphere separating H' from G, so H' does not link G. Therefore p is common to all triangles of G linking H, and q is common to all triangles of H linking G. Let $x_j \in X_j$, $x_k \in X_k$, with $j < k$. We show that px_jx_k links H in the star $qO_{jk}I_{jk}$ of Theorem 26. This completely determines the linking between G and H, because by Proposition 39 each triangle quv in H then links G as described in Theorem 26 also.

Let C be the simple closed curve in $\mathbb{R}^3$ consisting of the line segment from $p = (0, 1)$ to $(\zeta^{2j}, -1)$, the arc of the unit circle in the plane $t = -1$ from $(\zeta^{2j}, -1)$ to $(\zeta^{2k}, -1)$ (taken in the positive direction, so that it contains the point $(\zeta^{k+j}, -1)$), followed by the line segment from $(\zeta^{2k}, -1)$ to p. There is an isotopy of $\mathbb{R}^3$ fixing H and deforming px_jx_k

into C, so $\mathrm{Lk}(px_jx_k, D) = \mathrm{Lk}(C, D)$ for all cycles D in H. We show that C links H in the star $qO_{jk}I_{jk}$.

The curve C lies on the cone with apex p that contains the unit circle in the plane $t = -1$. Let F be the portion of this cone bounded by C. Then F is a Seifert surface for C, so we may calculate $\mathrm{Lk}(C, D)$ by counting signed intersections of D with F. The only edges of H which meet F are edges of the form qy_a, with $y_a \in Y_a$ for $j \leqslant a < k$, and all such oriented edges meet F with intersection number $+1$. It follows that a triangle T of H links C if and only if it contains exactly one such edge, and the linking number is $+1$ if and only if T orients the edge from q to y_a. It follows that C, and hence px_jx_k, links H in the star $qO_{jk}I_{jk}$, as required. □

To prove that the embedding of Construction 40 is weakly linked we will use the following proposition.

Proposition 42. *Let $m, n \geqslant 3$, and suppose that $G \cong K_m$ and $H \cong K_n$ are disjointly embedded in $\mathbb{R}^3$ such that*

1. *there is a vertex q of H common to all triangles in H that link G; and*
2. *every triangle in H that links G, links G in a star.*

Then G and H are not strongly linked.

We note that the proposition may be used to give a second proof that the embedding of Figure 6 (left) is weakly linked.

Proof. Let C be a cycle in G, and let $D = v_0v_1 \cdots v_{k-1}$ be a k-cycle in H. We will show that C does not strongly link D. The argument is essentially identical to the proof of Lemma 7.

If q does not belong to D then we decompose D as the sum of the triangles $T_i = v_0v_iv_{i+1}$, for $1 \leqslant i \leqslant k-2$. Since q does not belong to D but is common to all triangles in H linking G we have $\mathrm{Lk}(C, T_i) = 0$ for all i, and thus in the homology group $H_1(\mathbb{R}^3 - C)$ we have

$$[D] = \sum_{i=1}^{k-2} [T_i] = 0.$$

It follows that only cycles in H that contain q can link G.

Now suppose that q belongs to D. By hypothesis and Lemma 7 no triangle in H strongly links G, so we may assume that $k \geqslant 4$. Assume without loss of generality that $v_0 = q$, and let $T = v_0v_1v_{k-1}$, $D' = v_1v_2 \cdots v_{k-1}$. Then T is a triangle, D' is a $(k-1)$-cycle, and $D = T + D'$ as 1-chains in H. The cycle D' does not contain q, so by the previous paragraph in $H_1(\mathbb{R}^3 - C)$ we have

$$[D] = [T] + [D'] = [T] \in \{0, \pm 1\}.$$

Therefore C does not strongly link D. □

Corollary 43. *The embedding of Construction 40 is weakly linked.*

Proof. By Proposition 41 the embedding of Construction 40 realises the linking pattern of Theorem 26, so q is common to every triangle in H linking G, and each triangle in H that links G, links G in a star. Therefore G and H are linked but not strongly linked, by Proposition 42. □

7 Discussion

Our definition of a weakly linked embedding of a pair of graphs G and H excludes from consideration links between disjoint cycles that both lie in G or both lie in H. This is because Flapan [3, Theorem 1] has shown that K_{10} is *intrinsically strongly linked* (ISL), meaning that every embedding of K_{10} in $\mathbb{R}^3$ contains a pair of disjoint cycles that are strongly linked. Thus, if we had included links contained entirely within G or H we would have been limited to $m, n \leqslant 10$. It is not at present known if this upper bound is sharp, and to conclude the paper we briefly discuss the following question, which was the original motivation for the work in this paper.

Question 44. Determine the least n such that K_n is intrinsically strongly linked.

Fleming and Mellor [5, Fig. 9] exhibit an embedding of K_8 that contains only Hopf links, so K_8 is not ISL. It follows that the least n such that K_n is ISL is either 9 or 10. Despite our efforts we have not yet been able to resolve this question by either proving that K_9 is ISL or finding an embedding of K_9 that contains only weak links. We nevertheless make the following conjecture:

Conjecture 45. The complete graph K_9 is the smallest complete graph that is intrinsically strongly linked.

If true, this would show that for complete graphs, being intrinsically strongly linked is a strictly weaker property than being intrinsically triple linked (I3L, meaning every embedding contains a non-split 3-component link; in practice, this typically means a link $L_1 \cup L_2 \cup L_3$ such that $\mathrm{Lk}(L_i, L_{i+1})$ is nonzero for $i = 1, 2$). Flapan [3, Lemma 1] proved that if an embedding of K_n contains a triple link (in the sense given above), then it contains a pair of disjoint cycles that are strongly linked. The fact that K_{10} is ISL then follows from Flapan, Naimi and Pommersheim's proof [4] that K_{10} is I3L. In the same paper they show that K_9 is not I3L, by exhibiting an embedding that contains no triple link. This embedding nevertheless contains a strong link, offering some support for our conjecture. In addition, Naimi and Pavelescu [6] use oriented matroid techniques to show that all linear embeddings of K_9 are triple linked, implying that they are also strongly linked.

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