Rowmotion Orbits of Trapezoid Posets

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Abstract

Rowmotion is an invertible operator on the order ideals of a poset which has been extensively studied and is well understood for the rectangle poset. In this paper, we show that rowmotion is equivariant with respect to a bijection of Hamaker, Patrias, Pechenik and Williams between order ideals of rectangle and trapezoid posets, thereby affirming a conjecture of Hopkins that the rectangle and trapezoid posets have the same rowmotion orbit structures for order ideals. Our main tools in proving this are K-jeu-de-taquin and (weak) K-Knuth equivalence of increasing tableaux. We define *almost minimal tableaux* as a family of tableaux naturally arising from order ideals and show that for any partition λ, the almost minimal tableaux of shape λ are in different (weak) K-Knuth equivalence classes.

Mathematics Subject Classifications: 05E99, 06A07

1 Introduction

Rowmotion, denoted Row, is an invertible operator on the order ideals of any partially ordered set. For an order ideal I, $\mathrm{Row}(I)$ is the order ideal generated by the minimal

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elements of the complement of I. Rowmotion was first introduced by Brouwer and Schrijver [3] and has been extensively studied by many different authors, including [9, 11, 5, 15, 25]. The name 'rowmotion' is due to Striker and Williams [25]. For more history on rowmotion, see [26, Section 7.1].

We are interested in the action of rowmotion on the following two particular posets:

- the *rectangle poset* $\mathscr{R}(a,b) := \{(i,j) \in \mathbb{Z}^2 : 1 \leqslant i \leqslant a, 1 \leqslant j \leqslant b\}$, and
- the *trapezoid poset* $\mathscr{T}(a,b) := \{(i,j) \in \mathbb{Z}^2 : 1 \leqslant i \leqslant a, i \leqslant j \leqslant a+b-i\}$

for some fixed $a \leqslant b$, where the partial order is induced from the natural order on $\mathbb{Z}^2$. Figure 1 gives the Hasse diagrams of two posets $\mathscr{R}(a,b)$ and $\mathscr{T}(a,b)$ for $(a,b) = (3,4)$.

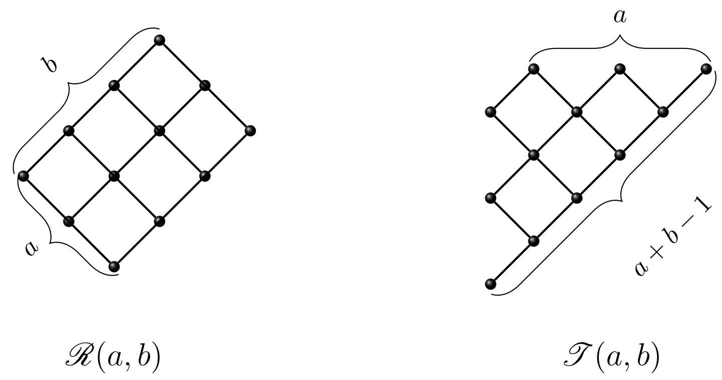


Figure 1: The rectangle poset $\mathscr{R}(3,4)$ and trapezoid poset $\mathscr{T}(3,4)$.

The posets $\mathscr{R}(a,b)$ and $\mathscr{T}(a,b)$ are remarkably similar to each other. In 1983, Proctor [18] proved that $\mathscr{R}(a,b)$ and $\mathscr{T}(a,b)$ have the same *order polynomial*, which implies in particular that they have the same number of order ideals. Since then, many different bijections between the set of order ideals of these two posets have been discovered [24, 10, 6, 14]. Recently, Sam Hopkins [13] made a series of conjectures that generalize these known similarities, including [13, Conjecture 4.9.1] which states that $\mathscr{R}(a,b)$ and $\mathscr{T}(a,b)$ have the same rowmotion orbit structures.

Our main result is a proof of this conjecture. In particular, we show that the infusion involution of [14, Section 6.2.3], which gives a bijection between order ideals of $\mathscr{R}(a,b)$ and $\mathscr{T}(a,b)$, is equivariant with respect to rowmotion. See Section 3.1 for a full definition of this bijection, which we denote by K-Inf.

Theorem 1. *For any order ideal $\mathcal{I}$ of the rectangle poset $\mathscr{R}(a,b)$, we have*

$$K\text{-Inf} \circ \mathrm{Row}(\mathcal{I}) = \mathrm{Row} \circ K\text{-Inf}(\mathcal{I}).$$

Remark 2. In [13], Hopkins conjectured in greater generality that the $\mathcal{P}$*-partitions* of $\mathscr{R}(a,b)$ and $\mathscr{T}(a,b)$ have the same orbit structures under *piecewise-linear rowmotion* [16]. Although the bijection K-Inf is defined for general $\mathcal{P}$-partitions, it fails to commute with piecewise-linear rowmotion on $\mathcal{P}$-partitions of height greater than 1. Our Theorem 1 gives the commutative property for $\mathcal{P}$-partitions of height 1, which are the same as order ideals.

The orbit structure of the rectangle poset $\mathscr{R}(a,b)$ is completely understood: Propp and Roby (expanding upon a remark of Stanley [23] and with further input from Thomas) explained that the action of rowmotion on the rectangle is in equivariant bijection with the action of cyclic rotation on binary words with a 0's and b 1's [17, Proposition 26]. They called this the "Stanley-Thomas word" correspondence, and used the correspondence to deduce various other nice properties of rowmotion on the rectangle such as the *cyclic sieving phenomenon* [21] and *homomesy* [17].

Example 3. Rowmotion on $\mathscr{R}(2,2)$ and $\mathscr{T}(2,2)$ gives rise to one orbit of size 4 and one orbit of size 2. Figure 2 shows the orbits of rowmotion on $\mathscr{R}(2,2)$ and $\mathscr{T}(2,2)$, where each box depicts two order ideals of $\mathscr{R}(2,2)$ and $\mathscr{T}(2,2)$ that are in bijection under K-Inf, as well as the corresponding Stanley-Thomas word which rotates cyclically in each orbit.

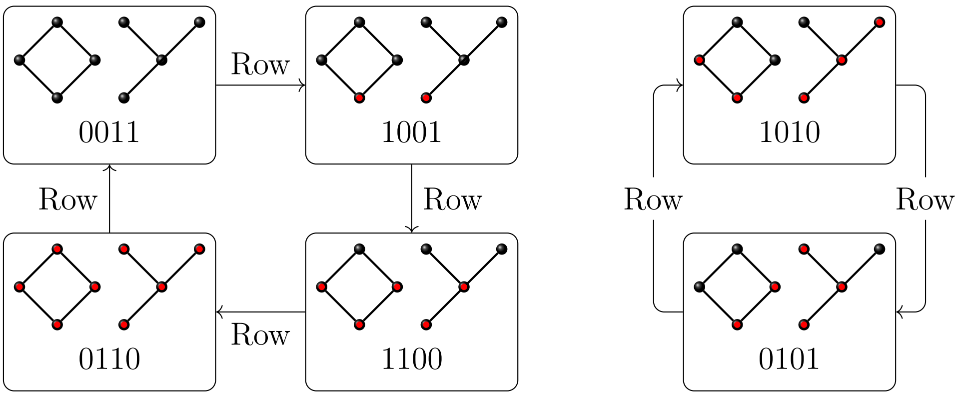


Figure 2: Orbits of Row on $\mathscr{R}(2,2)$ and $\mathscr{T}(2,2)$, where each block contains an order ideal $\mathcal{I}$ of $\mathscr{R}(2,2)$, the ideal K-Inf$(\mathcal{I})$ of $\mathscr{T}(2,2)$, and the Stanley-Thomas word of $\mathcal{I}$.

Our pair of posets $(\mathscr{R}(a,b), \mathscr{T}(a,b))$ is the first of three *minuscule doppelgänger pairs*

$$\left\{\left(\Lambda_{\mathrm{Gr}(a,a+b)}, \Phi^+_{B_{a,a+b}}\right), \left(\Lambda_{\mathrm{OG}(6,12)}, \Phi^+_{H_3}\right), \left(\Lambda_{\mathbb{Q}^{2n}}, \Phi^+_{I_2(2n)}\right)\right\}$$

for which the bijection K-Inf is defined[1]. The first of each pair is a *minuscule poset* on some root system, while the second is a poset of positive roots of another root system. It can be checked explicitly that K-Inf commutes with rowmotion for the other two pairs, hence Theorem 1 holds for all doppelgänger pairs.

Remark 4. A non-bijective proof that rowmotion orbits are the same for the last two pairs can be obtained by combining the result of [20, 22] for the minuscule posets and that of [7] for posets of positive roots.

In the case of $a = b$, the works of Reiner [19] and Armstrong-Stump-Thomas [1] give a different bijection between $\mathscr{R}(a,a)$ and $\mathscr{T}(a,a)$ which is known to commute with rowmotion. In particular, [1] gives an equivariant bijection between order ideals of $\mathscr{T}(a,a)$

[1]See [14, Figure 1] for pictures of other doppelgänger pairs.

under rowmotion and *non-crossing partitions of type B* under *Kreweras complementation*, while [19] gives an equivariant bijection between these non-crossing partitions and the Stanley-Thomas words of the order ideals of $\mathscr{R}(a, a)$.

Although our main theorem concerns elementary combinatorial objects and actions, some sophisticated tools from algebra and geometry underlie our proofs. In [2], Buch, Kresch, Shimozono, Tamvakis and Yong introduced the Hecke insertion algorithm as a K-theoretic analogue of the Schensted insertion algorithm. The Hecke insertion algorithm produces tableaux whose entries are strictly increasing along columns and rows. Such tableaux are called *increasing tableaux* and are discussed in Section 2.2. Thomas and Yong [27] introduced a K-theoretic version of Schützenberger's jeu-de-taquin operation for increasing tableaux, from which the bijection K-Inf is defined. We review these operations in Section 3.1.

We say a tableau is *almost minimal* if its entries are at most 1 larger than the rank of the entry. One can realize an order ideal of a rectangle poset (resp. trapezoid poset) as an almost minimal ordinary (resp. shifted) tableau (see Definition 7). The bijection K-Inf then applies a sequence of K-jeu-de-taquin slides turning the ordinary (rectangle) tableau into a shifted (trapezoid) tableau. Furthermore, rowmotion on order ideals can then be realized as the inverse of K-promotion (see Definition 14) on almost increasing tableaux (see Lemma 15). As such, we are able to establish our proof in a tableau-theoretic way by showing that K-Pro commutes with K-Inf for almost minimal tableaux.

Remark 5. In analogy to Remark 2, K-Pro does not commute with K-Inf for increasing tableaux that are not almost minimal. This is further elaborated in Remark 33.

The main ingredient for our proof is the *(weak) K-Knuth equivalence relation* of Buch and Samuel [4], which determines whether two (shifted, resp.) ordinary tableaux are K-jeu-de-taquin equivalent based on their reading words. In particular, we establish the following theorem.

Theorem 6. *For non-skew increasing tableaux:*

- *Almost minimal ordinary tableaux of the same shape are in separate K-Knuth equivalence classes.*
- *Almost minimal shifted tableaux of the same shape are in separate weak K-Knuth equivalence classes.*

The plan of the paper is as follows. In Section 2, we review the basics of posets, tableaux and rowmotion. Section 3 surveys the K-jeu-de-taquin theory and the bijection K-Inf of [14]. Section 4 is devoted to the K-Knuth equivalence relations and a proof of Theorem 6. Finally, in Section 5, we prove Theorem 1.

2 Preliminaries

2.1 Partially ordered sets

Given a poset $\mathcal{P}$, we denote covering relations by $x \lessdot y$. A *chain* in a poset $\mathcal{P}$ is a totally ordered subset of $\mathcal{P}$. We say that $\mathcal{P}$ is *graded* if all maximal (by inclusion) chains in $\mathcal{P}$

have the same size. We say that $\mathcal{P}$ is *ranked* if there exists a rank function $\operatorname{rank} : \mathcal{P} \to \mathbb{Z}$ satisfying $\operatorname{rank}(y) = \operatorname{rank}(x) + 1$ whenever $x \lessdot y$. We assume all rank functions are normalized so that $\min\{\operatorname{rank}(p) : p \in \mathcal{P}\} = 1$, in which case a rank function is unique if it exists. Graded posets are always ranked. The posets $\mathscr{R}(a, b)$ and $\mathscr{T}(a, b)$ are examples of graded posets, where the rank of an element (i, j) is $i + j - 1$. An *order ideal* I of $\mathcal{P}$ is a subset of $\mathcal{P}$ such that if $x \in I$ and $y \leqslant x$ in $\mathcal{P}$, then $y \in I$ as well. We denote the set of order ideals of $\mathcal{P}$ by $J(\mathcal{P})$.

2.2 Young diagrams and tableaux

Throughout this section let Λ denote either the positive orthant $\mathbb{N} \times \mathbb{N}$ with partial order $(i_1, j_1) \leqslant (i_2, j_2)$ if $i_1 \leqslant i_2$ and $j_1 \leqslant j_2$, or the subset of the positive orthant $\{(i, j) \in \mathbb{N}^2 | i \geqslant j\}$ with the induced order. We refer to elements of Λ as boxes and refer to the set theoretic difference of finite order ideals $\mu \subsetneq \lambda$ of Λ as a *shape*. We call a shape λ/μ a *skew shape* when $\mu \neq \emptyset$ and a *straight shape* when $\mu = \emptyset$.

A *partition* λ is a sequence $(\lambda_1, \dots, \lambda_m)$ of nonnegative integers with $\lambda_1 \geqslant \cdots \geqslant \lambda_m$. Associated to a partition is its *(ordinary) Young diagram*, which is the shape that has λ_i consecutive boxes in a row starting at $(1, i)$ for $i = 1, \dots, m$. A *strict partition* λ is a sequence $(\lambda_1, \dots, \lambda_m)$ of nonnegative integers with $\lambda_1 > \cdots > \lambda_m$. The *shifted Young diagram* associated to the strict partition λ is define similarly to its ordinary Young diagram. It has λ_i consecutive boxes in a row starting at (i, i) for $i = 1, \dots, m$. A *filling* of shape $\lambda/\mu \subset \Lambda$ is a function $f : \lambda/\mu \to X$ for some set X. An *increasing tableaux* of shape $\lambda/\mu \subset \Lambda$ is a function $T : \lambda/\mu \to S$ for a partially ordered set S such that $T(x) < T(y)$ whenever $x < y$. As all tableaux we consider will be increasing, we will drop the adjective increasing from now on.

In this paragraph and elsewhere when not otherwise specified, S is assumed to be $\{1 < 2 < \cdots\}$. Given a shape $\lambda/\mu \subset \Lambda$, we denote by $\mathrm{IT}^S(\lambda/\mu)$ the set of all tableaux of the given shape with entries in S. We refer to these as *ordinary tableaux* when Λ is the positive orthant, and as *shifted tableaux* when $\Lambda = \{(i, j) \in \mathbb{N}^2 \mid i \geqslant j\}$. Figure 3 gives examples of an ordinary and a shifted tableau.

$$
\begin{array}{|c|c|cc}
\hline
4 & 5 & \multicolumn{1}{c}{} & \\
\hline
2 & 4 & \multicolumn{1}{c|}{6} & \\
\hline
1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\
\hline
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & \boxed{6} & \boxed{7} & \\
 & \boxed{3} & \boxed{5} & \boxed{6} & \\
\boxed{1} & \boxed{2} & \boxed{4} & \boxed{5} & \boxed{7}
\end{array}
$$

Figure 3: An ordinary tableau of shape $(4, 3, 2)$ and a shifted tableau of shape $(5, 3, 2)$.

Every shape naturally defines a poset consisting of its boxes with partial order induced from Λ. In this way, we may apply all the poset theoretic concepts from Section 2.1 to Young diagrams. We may now talk about *poset maps* which are fillings $f : \lambda/\mu \to S$ for a partially ordered set S which respect the partial order of λ/μ (i.e. $f(x) \leqslant f(y)$ whenever $x \leqslant y$). The rank function on Λ descends to a rank function on any Young diagram, thus we may speak of the rank of a box in a Young diagram (where we always subtract the appropriate amount so the minimal rank of a box in a Young diagram is 1).

Observe that the rectangle poset $\mathscr{R}(a,b)$ is the same as the ordinary Young diagram $\lambda = b^a$ and the trapezoid poset $\mathscr{T}(a,b)$ is the same as the shifted Young diagram $\lambda = (a+b-1, a+b-3, \ldots, b-a+1)$. Figure 4 gives an example of this identification for $(a,b) = (3,5)$.

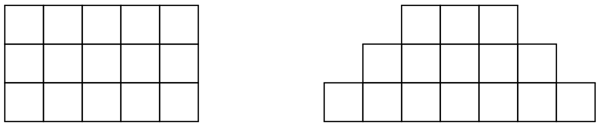


Figure 4: The posets $\mathscr{R}(3,5)$ and $\mathscr{T}(3,5)$, where the boxes are poset elements and edges in the Hasse diagram are replaced with adjacency relations.

From now on, we will depict all posets as Young diagrams rather than Hasse diagrams, and will depict order ideals as $\{0,1\}$-poset maps of these Young diagrams. By abuse of notation, we denote by $\mathrm{IT}^\ell(\mathscr{R}(a,b))$ the set of all ordinary tableaux with S as the totally ordered set $1 < 2 < \cdots < r$ whose shape is the one corresponding to the rectangle poset $\mathscr{R}(a,b)$, and similarly we write $\mathrm{IT}^\ell(\mathscr{T}(a,b))$ for the trapezoid poset.

Given a shape λ/μ (either ordinary or shifted), the *minimal tableau* of λ/μ is the tableau T with $T(s) = \mathrm{rank}(s)$ for all boxes $s \in \lambda/\mu$. In analogy to this, we make the following definition.

Definition 7 (Almost minimal tableaux)**.** Given an ordinary or shifted shape λ/μ, an *almost minimal tableau* T of shape λ/μ is a tableau such that $T(s) - \mathrm{rank}(s) \in \{0,1\}$ for any box $s \in \lambda/\mu$.

For both ordinary and shifted case, there is a bijection between $J(\lambda/\mu)$ and the set of almost minimal tableaux of shape λ/μ obtained by adding rank to a $\{0,1\}$-poset map. This gives the following lemma.

Lemma 8. *For any shape λ/μ graded of length ℓ, we have a natural bijection*

$$\Psi : J(\lambda/\mu) \to \mathrm{IT}^{\ell+1}(\lambda/\mu),$$

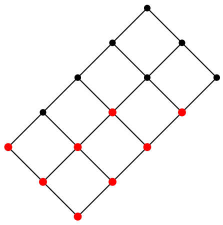
obtained by adding rank to every ideal, viewed as a $\{0,1\}$-poset map.

In particular, we have bijections between $J(\mathscr{R}(a,b))$ and $\mathrm{IT}^{a+b}(\mathscr{R}(a,b))$, and also between $J(\mathscr{T}(a,b)))$ and $\mathrm{IT}^{a+b}(\mathscr{T}(a,b)))$. As a result, order ideals and almost increasing tableaux (of both standard and shifted shapes) will be treated as freely interchangeable objects.

2.3 Rowmotion

Definition 9. Let $\mathcal{P}$ be a poset, and $I \in J(\mathcal{P})$ an order ideal of $\mathcal{P}$. Then the rowmotion of I, denoted $\mathrm{Row}(I)$, is the order ideal of $\mathcal{P}$ generated by the minimal elements that are not in I, i.e.

$$\mathrm{Row}(I) = \langle a \in \mathcal{P} : a \in \min\{\mathcal{P} \setminus I\} \rangle.$$



0	1	1	1	1
0	0	0	1	1
0	0	0	0	1

3	5	6	7	8
2	3	4	6	7
1	2	3	4	6

Figure 5: Left: an order ideal of the poset $\mathscr{R}(3,5)$ identified by the color red. Middle: the corresponding $\{0,1\}$-poset map. Right: the corresponding almost minimal tableau.

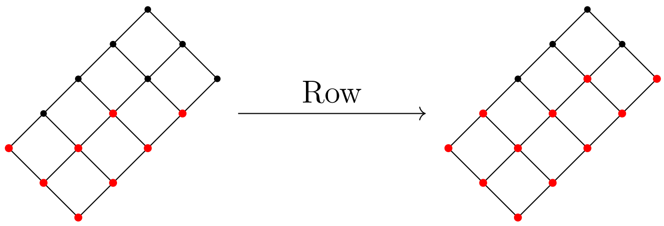


Figure 6: Rowmotion acting on order ideals.

The invertibility of Row is clear from the following alternative definition of Cameron and Fon-der-Flaass [5].

Proposition 10. *For $p \in \mathcal{P}$ and order ideal I, we define the toggle operation of p on I as follows.*

$$\tau_p(I) = \begin{cases} I \cup p & \text{if } p \notin I \text{ and } I \cup p \in J(\mathcal{P}), \\ I \setminus p & \text{if } p \in I \text{ and } I \setminus p \in J(\mathcal{P}), \\ I & \text{otherwise.} \end{cases}$$

Then rowmotion is performed by toggles "row by row" [2] from the largest to smallest, i.e.

$$\mathrm{Row}(I) = \tau_{p_1} \circ \tau_{p_{n-1}} \circ \cdots \circ \tau_{p_n}(I)$$

where $p_1 \leqslant \cdots \leqslant p_n$ is any linear extension of the poset $\mathcal{P}$.

3 K-jeu-de-taquin and the Bijection K-Inf

In this section we describe the bijection K-Inf between $J(\mathscr{R}(a,b))$ and $J(\mathscr{T}(a,b))$. The construction is based on the *K-jeu-de-taquin* slides of Thomas and Yong [27].

[2] Here 'row' refers to a rank of a poset, which is not a row but a diagonal in a Young diagram notation.

3.1 K-jeu-de-taquin theory for increasing tableaux

Definition 11. Call two boxes s, s' *adjacent* if s covers s' or s' covers s. We define the *swap* of two entries a, b in a filling f to be the filling $\mathrm{swap}_{a,b}(f)$ such that for all $x \in \mathcal{P}$:

$$\mathrm{swap}_{a,b}(f)(x) = \begin{cases} a & \text{if } f(x) = b \text{ and there exists } y \text{ adjacent to } x \text{ such that } f(y) = a, \\ b & \text{if } f(x) = a \text{ and there exists } y \text{ adjacent to } x \text{ such that } f(y) = b, \\ f(x) & \text{otherwise.} \end{cases}$$

Next, we can describe K-jeu-de-taquin as a sequence of swaps.

Definition 12. Let $T : \lambda/\mu \to \{1 < 2 < \cdots < \ell\}$ be an increasing tableau. Let C be some subset of maximal elements in μ and define the filling $T \cup C : \lambda/\mu \cup C \to \{\bullet < 1 < 2 < \cdots < \ell\}$ by

$$(T \cup C)(x) = \begin{cases} T(x) & x \notin \mu \\ \bullet & x \in C \end{cases}.$$

The *K-jeu-de-taquin forward slide* of C is the restriction of $\left(\prod_{b=1}^{\ell} \mathrm{swap}_{\bullet,b}\right)(T \cup C)$ to $[1, \ell]$. The *K-jeu-de-taquin reverse slide* of a subset of minimal elements C' in Λ/λ is defined similarly by

$$(T \cup C')(x) := \begin{cases} T(x) & x \notin \mu \\ \bullet & x \in C' \end{cases},$$

and $\widehat{K\text{-Jdt}_{C'}}(T)$ is the restriction of $\left(\prod_{b=\ell}^{1} \mathrm{swap}_{b,\bullet}\right)(T \cup C')$ to $[1, \ell]$. Both of the above are commonly referred to as K-Jdt *slides*.

We now define the bijection K-Inf for the case of rectangle and trapezoid. For the definition of K-Inf for other doppelgänger pairs, see [14, Section 6].

Definition 13. Given an increasing tableau $T \in \mathrm{IT}^{\ell}(\mathscr{R}(a, b))$ of the rectangle, one obtains an increasing tableau $K\text{-Inf}(T) \in \mathrm{IT}^{\ell}(\mathscr{T}(a, b))$ as follows:

1) Realize T as a skew shifted tableau T' by shifting it $a - 1$ boxes to the right.
2) Let S be the minimal shifted tableau of shape $(a-1, a-2, \ldots, 1)$, where we overline the entries of S to distinguish them from the entries of T. Then

$$K\text{-Inf}(T) = K\text{-Jdt}_{S^{-1}(\overline{1})} \circ K\text{-Jdt}_{S^{-1}(\overline{2})} \circ \cdots \circ K\text{-Jdt}_{S^{-1}(\overline{2a-3})}(T').$$

In other words, we continually perform K-Jdt forward slides with C as all maximal elements of the skew part until the resulting shifted tableau is straight.

In [14], Hamaker, Patrias, Pechenik and Williams proved that K-Inf is indeed a bijection between $\mathrm{IT}^{\ell}(\mathscr{R}(a, b))$ and $\mathrm{IT}^{\ell}(\mathscr{T}(a, b))$. Recall from Lemma 8 that almost minimal tableaux are the same as order ideals up to the bijection Ψ, hence in the case of $\ell = a + b$, we can view K-Inf as a bijection from $J(\mathscr{R}(a, b))$ to $J(\mathscr{T}(a, b))$. See Figure 7 for an example.

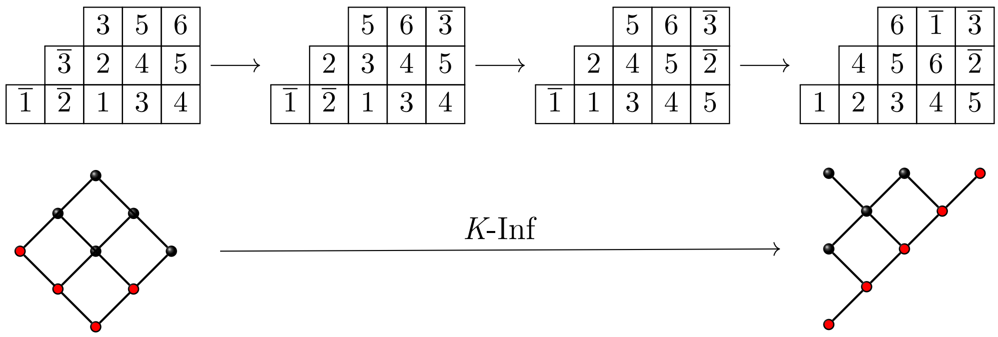


Figure 7: The map K-Inf for increasing tableaux $I \in \mathrm{IT}^6(\mathscr{R}(3,3))$ and the corresponding increasing tableau $\Psi(T) \in J(\mathscr{R}(3,3))$.

3.2 Rowmotion via K-jeu-de-taquin

We can describe rowmotion as a composition of K-Jdt slides. This is done in [8] under the name K-promotion, which is the K-theoretic analogue of Schützenberger's promotion action on standard Young tableaux.

Definition 14 (K-promotion)**.** Fix an ordinary or shifted shape λ/μ. For an increasing tableau $T \in \mathrm{IT}^\ell(\lambda/\mu)$, the *$K$-promotion* of T is a tableau $K\text{-Pro}(T) \in \mathrm{IT}^\ell(\lambda/\mu)$ defined as follows.

1) Remove all entries of 1 from T and subtract 1 from all other entries of T. This gives a tableau T' of shape $\lambda/(\mu \cup T^{-1}(1))$ with $T'(s) = T(s) - 1$ for all $s \in \lambda/(\mu \cup T^{-1}(1))$.
2) Send $T' \mapsto K\text{-Jdt}_{T^{-1}(1)}(T')$. This is called *$K$-rectifying* T' within λ/μ. Call the resulting tableau T''.
3) Add entries of ℓ to T'' such that the resulting tableau is of shape λ/μ.

In the case where there does not exist a 1 in the tableau, K-promotion simply decrements each entry by 1.

Built upon the bijection Ψ described in Lemma 8, the following lemma further relates rowmotion on order ideals with K-promotion on increasing tableaux.

Lemma 15. *Let λ/μ be an ordinary or shifted shape, graded of length ℓ. Then the bijection Ψ defined in Lemma 8 is an equivariant bijection between K-promotion on $\mathrm{IT}^{\ell+1}(\lambda/\mu)$ and inverse rowmotion on $J(\lambda/\mu)$.*

In [8, Theorem 4.4], Dilks, Pechenik and Striker prove this lemma in the case λ/μ is a rectangle. We will use their proof technique to show this for all shapes λ/μ. Central to their proof are the *K-Bender-Knuth involutions*, defined on any tableau $T \in \mathrm{IT}^\ell(\lambda/\mu)$ or $T \in \mathrm{IT}^\ell(\lambda/\mu)$ as

$$K\text{-BK}_i(T)(x) := \begin{cases} i & \text{if } f(x) = i+1 \text{ and } f(y) \neq i \text{ for all } y < x, \\ i+1 & \text{if } f(x) = i \text{ and } f(y) \neq i+1 \text{ for all } y > x, \\ f(x) & \text{otherwise.} \end{cases}$$

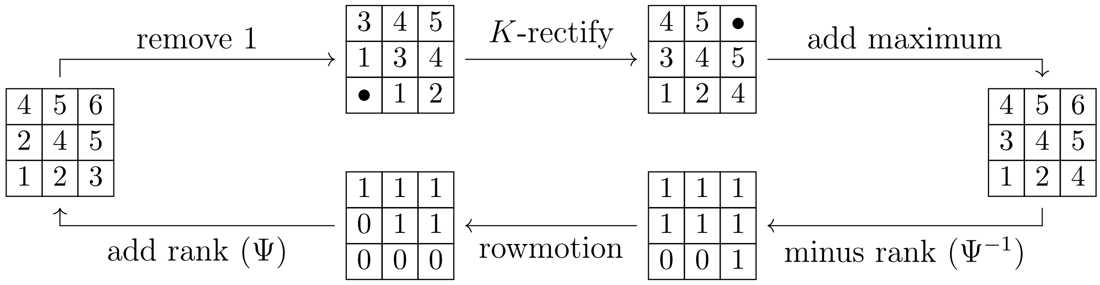


Figure 8: Inverse of rowmotion is in equivariant bijection with K-promotion.

[8, Proposition 2.5] then shows that

$$K\text{-Pro}(T) = K\text{-BK}_{\ell-1} \circ K\text{-BK}_{\ell-2} \circ \cdots \circ K\text{-BK}_1(T).$$

For a graded poset, define τ_i to be product of all toggles τ_p where p has rank i. From the alternate description of rowmotion in terms of toggles (see Proposition 10), inverse rowmotion on λ/μ can be written as $\text{Row}^{-1} = \tau_r \circ \tau_{r-1} \circ \cdots \circ \tau_1$.

Proof. By [8, Proposition 2.5] and the toggle description of rowmotion, it suffices to show that the bijection Ψ from Lemma 8 commutes with each τ_i and K-BK$_i$. In other words, we need to show that the following diagram commutes for $1 \leqslant i \leqslant r$

$$\begin{tikzcd} J(\lambda/\mu) \arrow[r, "\Psi"] \arrow[d, "\tau_i"] & \mathrm{IT}^{\ell+1}(\lambda/\mu) \arrow[d, "K\text{-BK}_i"] \\ J(\lambda/\mu) \arrow[r, "\Psi"] & \mathrm{IT}^{\ell+1}(\lambda/\mu) \end{tikzcd}$$

Viewing order ideals as $\{0,1\}$-poset maps and Ψ as adding rank, the above diagram says that for any $\{0,1\}$-poset map f on λ/μ and any box $s \in \lambda/\mu$,

$$(\Psi \circ \tau_i)(f)(s) = (K\text{-BK}_i \circ \Psi)(f)(s).$$

- If $\Psi(f)(s) \notin \{i, i+1\}$: since $\text{rank}(s) \neq i$, we have $\tau_i(f)(s) = f(s)$. Since $\Psi(f)(s) \notin \{i, i+1\}$, $(K\text{-BK}_i \circ \Psi)(f)(s) = \Psi(f)(s)$.

- If $\text{rank}(s) = i-1$ and $f(s) = 1$: since $\text{rank}(s) \neq i$, we have $\tau_i(f)(s) = f(s)$. For any box s' covering s, $f(s') = 1$ and $\Psi(T)(s') = i+1$, thus $(K\text{-BK}_i \circ \Psi)(f)(s) = \Psi(f)(s)$.

- If $\text{rank}(s) = i$ and $f(s) = 1$: if $f(s') = 1$ for some box s' covered by s, then $\tau_i(f)(s) = 1$ and $(K\text{-BK}_i \circ \Psi)(f)(s) = i+1$. Otherwise $\tau_i(f)(s) = 0$ and $(K\text{-BK}_i \circ \Psi)(f)(s) = i$.

- If $\text{rank}(s) = i$ and $f(s) = 0$: if $f(s') = 0$ for some box s' covering s, then $\tau_i(f)(s) = 0$ and $(K\text{-BK}_i \circ \Psi)(f)(s) = i$. Otherwise, $\tau_i(f)(s) = 1$ and $(K\text{-BK}_i \circ \Psi)(f)(s) = i+1$.

- If $\mathrm{rank}(s) = i+1$ and $f(s) = 0$: since $\mathrm{rank}(s) \neq i$, $\tau_i(f)(s) = f(s)$. For any box s' covered by s, $f(s') = 0$ and $\Psi(T)(s') = i$, thus $(K\text{-BK}_i \circ \Psi)(f)(s) = \Psi(f)(s)$. $\square$

Remark 16. As in [8, Theorem 4.4], the above argument can be extended to show that Ψ is an intertwining operator between inverse rowmotion on $J(\lambda/\mu \times \{1, 2, \ldots, m\})$ and K-promotion on $\mathrm{IT}^{\ell+m}(\lambda/\mu)$ (resp. $\mathrm{IT}^{\ell+m}(\lambda/\mu)$) for an ordinary (resp. shifted) shape λ/μ which is graded of length ℓ.

Note that for $m > 1$, rowmotion on $J(\lambda/\mu \times \{1, 2, \ldots, m\})$ is *different* from piecewise-linear rowmotion on P-partitions of λ/μ of size m (whose definition can be found in [13]). In particular, K-promotion is not equivariant with piecewise-linear rowmotion for general P-partitions of λ/μ.

4 *K*-Knuth and weak *K*-Knuth Equivalence

We have shown that K-Inf and Row can be described in terms of K-jeu-de-taquin slides. In this section, we will introduce invariants of K-jeu-de-taquin and use these to prove Theorem 6. The main tools for proving Theorem 1 are investigated in this section, including Theorem 6 and an alternate characterization of the bijection K-Inf on order ideals.

4.1 *K*-Jdt equivalence for ordinary and shifted tableaux

Using the forward and reverse K-Jdt slides in Definition 12, we may define an equivalence relation on tableaux in a fixed orthant Λ, where two tableaux T, T' are considered K-Jdt *equivalent* if T can be reached from T' by a series of K-Jdt slides. We will be concerned with K-Jdt equivalence of ordinary and shifted tableaux. In particular, in the bijection K-Inf, the realization T' of a tableau T of the rectangle as a skew shifted tableau is K-Jdt equivalent to K-Inf(T) since K-Inf can be written as a composition of K-Jdt slides.

Proposition 17. *T' and K-Inf(T) are K-Jdt equivalent shifted tableaux.*

For an increasing tableau $T : \lambda/\mu \to \{1 < 2 < \cdots < \ell\}$, define the tableau $T_{[a,b]}$ as the restriction of T to $T^{-1}([a, b]) \subseteq \lambda/\mu$. A useful consequence of the way we perform the swaps is that if we restrict two K-Jdt equivalent tableaux T, T' to the same interval $[a, b]$, then a slight modification of the K-Jdt slides which change T into T' will change $T_{[a,b]}$ into $T'_{[a,b]}$. Specifically,

Lemma 18. [4, Lemma 3.3] *If T and T' are K-Jdt equivalent, then $T_{[a,b]}$ and $T'_{[a,b]}$ are K-Jdt equivalent.*

In [4], Buch and Samuel show that K-Jdt equivalence for ordinary and shifted tableaux can be described by K-Knuth and weak K-Knuth equivalence relations of their reading words, respectively.

Definition 19. The *row reading word* of a tableau T is the word obtained by reading the rows of T from top to bottom, and from left to right within each row.

$$\begin{array}{|c|c|c|}\hline 4&5&6\\\hline 2&4&5\\\hline 1&2&3\\\hline\end{array}$$ has row reading word 456245123, and $$\begin{array}{ccccc} & & \boxed{6} & & \\ & \boxed{3} & \boxed{4} & \boxed{5} & \\ \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{6}\end{array}$$ has row reading word 634512346.

Figure 9: Row reading words for ordinary and shifted increasing tableaux.

Theorem 20. [4, Theorem 6.2] *Ordinary tableaux T, T' are K-*Jdt *equivalent if and only if their row reading words are K-Knuth equivalent, where K-Knuth equivalence is the symmetric transitive closure of the following relations:*

- *$uaav \equiv uav$ for integers a and words u, v,*
- *$uabav \equiv ubabv$ for integers a, b and words u, v,*
- *$uabcv \equiv uacbv$ for integers $b < a < c$ and words u, v,*
- *$uabcv \equiv ubacv$ for integers $a < c < b$ and words u, v.*

Theorem 21. [4, Theorem 7.8] *Shifted tableaux T, T' are K-*Jdt *equivalent if and only if their row reading words are weakly K-Knuth equivalent, where weak K-Knuth equivalence is the symmetric transitive closure of the relations of K-Knuth equivalence and the following relation:*

- *$abv \equiv bav$ for integers a, b and word v.*

In light of the above theorems, we say two ordinary tableaux are *K-Knuth equivalent* if their row reading words are K-Knuth equivalent and we say two shifted tableaux are *weak K-Knuth equivalent* if their row reading words are weak K-Knuth equivalent.

The reader may notice that K-Knuth and weak K-Knuth equivalence are similar and hence suspect that K-Jdt equivalences of ordinary and shifted tableaux are related. This is true. For a shifted tableau T, we may construct an ordinary tableau by reflecting T across the diagonal. To make this precise, we define T^2 to be the ordinary tableau with boxes $(i, j), (j, i)$ for boxes (i, j) of T, where

$$T^2(i,j) := \begin{cases} T(i,j) & i \geqslant j \\ T(j,i) & i < j \end{cases}.$$

Proposition 22. [4, Proposition 7.1] *If T and T' are K-*Jdt *equivalent shifted tableaux, then T^2 and $(T')^2$ are K-*Jdt *equivalent ordinary tableaux.*

4.2 Hecke permutations

While K-Knuth and weak K-Knuth equivalence of row reading words completely describe K-Jdt equivalence of ordinary and shifted tableaux, these equivalences can be difficult to work with. Buch and Samuel [4] introduce a simpler, yet cruder invariant of K-Jdt on ordinary tableaux called a *Hecke permutation*. The *Hecke product* of a permutation u and a simple transposition $s_i = (i, i+1)$ is denoted $u \cdot s_i$ with

$$u \cdot s_i = \begin{cases} u & \text{if } u(i) > u(i+1) \\ us_i & \text{if } u(i) < u(i+1) \end{cases}.$$

Definition 23. [4] The *Hecke permutation* of an ordinary tableau T with reading word $u = a_1a_2a_3 \dots a_k$ is the Hecke product

$$s_{a_k} \cdot (s_{a_{k-1}} \cdot (s_{a_{k-2}} \cdots (s_{a_2} \cdot s_{a_1}) \cdots)),$$

which is a permutation on $\max(a_1, a_2, \dots, a_k) + 1$ elements. We will denote this permutation by $w(T)$ or $w(u)$, and consider it in one-line notation.

If the reading words u and u' of two ordinary tableaux are K-Knuth equivalent, then we have $w(u) = w(u')$, but the converse need not be true. In particular, this implies

Corollary 24. [4, Corollary 6.5] *The Hecke permutation of an ordinary tableau is invariant under K-*Jdt *slides.*

4.3 Minimal ideals and (weak) K-Knuth equivalence

In this section, we will prove Theorem 6 by showing that for a given shape, the (weak) K-Knuth equivalence class of an almost minimal tableau is determined by its Hecke permutation. We begin with the following definition.

Definition 25. Given an increasing tableau T of shape λ/μ (either ordinary or shifted), its *minimal ideal* is the set of boxes s such that $T(s) - \mathrm{rank}(s) = 0$. This set is an order ideal of the poset λ/μ. For convenience, we will denote the minimal ideal of an ordinary tableau T by I_0 and that of a shifted tableau T' by I'_0.

Note that an almost minimal tableau T is determined by the restriction to its minimal ideal, since all entries outside of the ideal must have the maximum value allowed. We will show how the minimal ideal I_0 of an ordinary tableau T determines its Hecke permutation $w(T)$.

Lemma 26. *Let T be an ordinary tableau and $T_{>r}$ be the tableau T without its bottommost r rows. For any i, we have*

$$w(T)^{-1}(i) \geqslant w(T_{>r})^{-1}(i) - r.$$

Proof. Let $m = w(T_{>r})^{-1}(i)$. Each time we compute the Hecke product of w with a transposition s_n, only the n-th and $(n+1)$-st entries of w are changed. Since each row of T is increasing, the position of i can only decrease at most once, after which all terms in the reading word will be greater than or equal to m. But the entry $m-1$ only appears at most once in the r-th row, and thus

$$w(T_{>r-1})^{-1}(i) \geqslant w(T_{>r})^{-1}(i) - 1.$$

The lemma now follows by induction. □

In the following lemma, T_i denotes the i-th row of the tableau T.

Lemma 27. *Let I_0 be the minimal ideal of a straight ordinary tableau T.*

(i) *If* $|T_i \cap I_0| < |T_{i-1} \cap I_0|$, *then* $w(T)^{-1}(i) = |T_i \cap I_0| + 1$;

(ii) *If* $|T_i \cap I_0| = |T_{i-1} \cap I_0|$, *then* $w(T)^{-1}(i) > |T_i \cap I_0| + 1$.

Proof. We first analyze the difference between $w(T_{>r-1})$ and $w(T_{>r})$ for a row index r. Note that $T_{>r}$ has entries greater than r, hence the first r positions of $w(T_{>r})$ are unchanged by the Hecke product and are equal to $1, 2, \dots, r$. Let $s = |T_r \cap I_0|$. If $s > 0$, then the first s entries in T_r will be $r, r+1, \dots, r+s-1$. Taking the Hecke product of these entries will change $w(T_{>r})$ as follows

$$(1, \dots, r-1, r, a_{r+1}, a_{r+2}, \dots, a_{r+s}, \dots) \mapsto (1, \dots, r-1, a_{r+1}, a_{r+2} \dots, a_{r+s}, r, \dots).$$

Further Hecke products with higher entries of T_r will not change the first $r+s$ entries. Thus, we have that

$$w(T_{>r-1})^{-1}(r) = |T_r \cap I_0| + r, \tag{1}$$

and for any integer a, if

$$r < w(T_{>r})^{-1}(a) \leqslant |T_r \cap I_0| + r \tag{2}$$

then

$$w(T_{>r-1})^{-1}(a) = w(T_{>r})^{-1}(a) - 1. \tag{3}$$

When $|T_r \cap I_0| = 0$, there is no r in the tableaux $T_{>r-1}$ and Equation (1) still holds, while Equation (2) is never satisfied hence Equation (3) is vacuously true.

(i) We will prove by induction that for all $j \leqslant i$

$$w(T_{>j-1})^{-1}(i) = |T_i \cap I_0| + j,$$

where the desired equality is the case $j = 1$. Our induction here is on j, with the base case as $j = i$ which follows from Equation (1). For the inductive step, suppose for $j \leqslant i$

$$w(T_{>j-1})^{-1}(i) = |T_i \cap I_0| + j.$$

By our assumption $|T_i \cap I_0| < |T_{i-1} \cap I_0|$ and since I_0 is an ideal, we get $|T_{i-1} \cap I_0| \leqslant |T_{j-1} \cap I_0|$. Thus we have

$$j - 1 < |T_i \cap I_0| + j \leqslant |T_{i-1} \cap I_0| + j - 1 \leqslant |T_{j-1} \cap I_0| + j - 1.$$

Our inductive assumption then implies Equation (2) holds for $r = j-1$, $a = i$ and thus Equation (3) finish our inductive step.

(ii) Equation (1) implies

$$w(T_{>i-1})^{-1}(i) = |T_i \cap I_0| + i.$$

The $(i-1)$-th row of T will skip the value $i - 1 + |T_{i-1} \cap I_0|$, and using our assumption $|T_i \cap I_0| = |T_{i-1} \cap I_0|$ this value is equal to $w(T_{>i-1})^{-1}(i) - 1$. Thus

$$w(T_{>i-2})^{-1}(i) \geqslant w(T_{>i-1})^{-1}(i),$$

and by Lemma 26, we conclude that

$$w(T)^{-1}(i) \geqslant |T_i \cap I_0| + 2. \qquad \square$$

Theorem 28 (ordinary shapes)**.** *Let T and T' be* K-Jdt *equivalent straight ordinary tableaux with minimal ideals I_0 and I_0'. Then $I_0 = I_0'$.*

Proof. Suppose that $I_0 \neq I_0'$, and let r be the first row where I_0 and I_0' differ. Then $|T_{r-1} \cap I_0| = |T'_{r-1} \cap I'_0|$, so either $|T_r \cap I_0| < |T_{r-1} \cap I_0|$ or $|T'_r \cap I'_0| < |T'_{r-1} \cap I'_0|$. In any case, Lemma 27 implies that

$$w(T)^{-1}(r) \neq w(T')^{-1}(r).$$

Therefore the Hecke permutations of T and T' differ. Since by Corollary 24, Hecke permutations are invariant under K-Jdt slides for ordinary tableaux, we conclude that T and T' are not K-Jdt equivalent. □

To extend the result to shifted tableaux, we will use the connection between K-Jdt of the shifted tableau T and K-Jdt of the ordinary doubled tableau T^2. Note that in an ordinary tableau, $\mathrm{rank}(a, b) = \mathrm{rank}(b, a)$. It follows for a straight shifted tableau T, for any box $s = (a, b) \in T^2$, one has $T^2(s) - \mathrm{rank}(s) = T^2[(b, a)] - \mathrm{rank}(b, a)$. Thus T is almost minimal if and only if T^2 is almost minimal, and two minimal ideals of tableaux T and T' are equal if and only if the minimal ideals of T^2 and $(T')^2$ are equal. The above result on ordinary shapes combined with these observations and Proposition 22 imply the following.

Theorem 29 (shifted shapes)**.** *Let T and T' be* K-Jdt *equivalence straight shifted tableaux with minimal ideals I_0 and I_0'. Then $I_0 = I_0'$.*

Since almost minimal tableaux (of either ordinary or shifted shape) are completely described by their shape and their minimal ideal, we obtain Theorem 6, restated below.

Theorem 6.

- *For any partition λ, all almost minimal ordinary tableaux of shape λ are in separate K-Knuth equivalence classes.*
- *For any strict partition λ, all almost minimal shifted tableaux of shape λ are in separate weak K-Knuth equivalence classes.*

Remark 30. A *unique rectification target* is a straight tableau T such that it is the only straight tableau in its K-Jdt equivalence class. Unique rectification targets are crucial to the K-theoretic origins of K-Jdt in [27]. They have been further studied in [4] and [12], the former of which showed that minimal tableaux are unique rectification targets. Similar to [12, Proposition 2.43], Theorem 28 describes an invariant of K-Jdt rectifications and could have nice implications for unique rectification targets. One might ask if almost minimal tableaux are unique rectification targets but this is not always the case, see [12, Example 7.4].

The last ingredient we will need to prove that K-infusion commutes with rowmotion on order ideals is the following corollary.

Corollary 31. *Let T, T' be two almost minimal (ordinary or shifted) tableaux of the same shape, graded of length ℓ. Then $T_{[1,\ell]}$ is* K-Jdt *equivalent to $T'_{[1,\ell]}$ if and only if $T = T'$.*

Proof. ($\Leftarrow$) If $T = T'$, then $T_{[1,\ell]} = T'_{[1,\ell]}$.
($\Rightarrow$) Suppose $T_{[1,\ell]}$ is K-Jdt equivalent to $T'_{[1,\ell]}$. Let I_0 be the minimal ideal of T and I'_0 the minimal ideal of T'. Notice that I_0, I'_0 are also the minimal ideals of $T_{[1,\ell]}$ and $T'_{[1,\ell]}$, respectively. Then by Theorem 28, $I_0 = I'_0$. Since almost minimal tableaux are completely determined by their shape and their ideal, $T = T'$. □

Recall from Proposition 17 that the bijection K-Inf : $\mathrm{IT}^\ell(\mathscr{R}(a,b)) \to \mathrm{IT}^\ell(\mathscr{T}(a,b))$ preserves K-Jdt equivalence (and hence weak K-Knuth equivalence). As described in Lemma 8, we may restrict K-Inf to almost minimal tableaux, which represent order ideals, to get a bijection K-Inf : $J(\mathscr{R}(a,b)) \to J(\mathscr{T}(a,b))$. We then have the following corollary.

Corollary 32. *If T is an almost minimal tableau of shape $\mathscr{R}(a,b)$, then K-Inf(T) is the unique almost minimal tableau of shape $\mathscr{T}(a,b)$ that is weak K-Knuth equivalent to T.*

5 Commutation of Rowmotion and *K*-Inf

We have now built up the machinery to prove Theorem 1. Using the definition of rowmotion as inverse K-promotion, we will show that performing the three-step process of K-promotion preserves weak K-Knuth equivalence of order ideals of the rectangle and trapezoid, i.e. if I is weakly K-Knuth equivalent to J, then $\mathrm{Row}^{-1}(I)$ is weakly K-Knuth equivalent to $\mathrm{Row}^{-1}(J)$ (see Figure 10). By Corollary 32 this implies that Row^{-1} commutes with K-Inf and thus Row commutes with K-Inf.

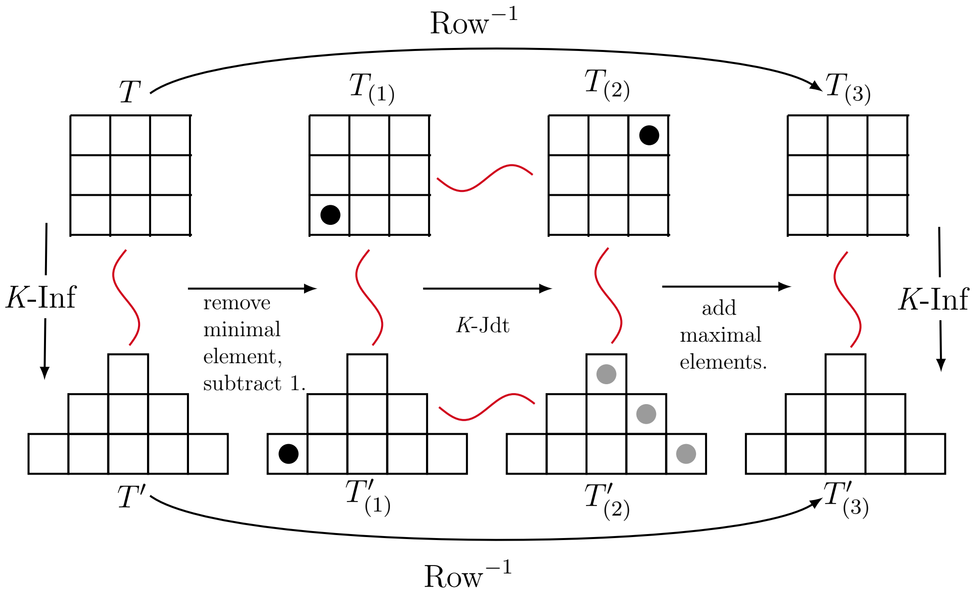


Figure 10: Commutative diagram for proof of Theorem 1. The red squiggles indicate weak K-Knuth equivalence. Note that unlike the case of rectangle posets, the final location(s) of the large dot(s) in $T'_{(2)}$ depends on the order ideal T'.

Proof of Theorem 1. All tableaux in this proof are realized as shifted tableaux. Recall that by Theorem 21, two shifted tableaux are K-Jdt equivalent if and only if their reading words are weak K-Knuth equivalent.

Let T and T' be almost minimal tableaux of the rectangle $\mathscr{R}(a,b)$ and trapezoid $\mathscr{T}(a,b)$ respectively which are K-Jdt equivalent. By Corollary 32, this is equivalent to $K\text{-Inf}(T) = T'$. Let $r = a + b - 1$ be the rank of maximal elements in both posets. Recall that Definition 14 defines K-promotion as a three-step process. Let $T_{(1)}$, $T_{(2)}$, $T_{(3)}$ be the intermediate tableaux obtained by applying K-Pro to T, where the final result $T_{(3)}$ is equal to $\text{Row}^{-1}(T)$ by Lemma 15. Similarly, we define $T'_{(1)}, T'_{(2)}, T'_{(3)}$ for T' and have that $T'_{(3)} = \text{Row}^{-1}(T')$.

By Lemma 18, $T_{[2,r+1]}$ and $T'_{[2,r+1]}$ are K-Jdt equivalent. Thus $T_{(1)}$ and $T'_{(1)}$ are K-Jdt equivalent. Since performing K-Jdt preserves K-Jdt equivalence, $T_{(2)}$ and $T'_{(2)}$ are also K-Jdt equivalent. By Corollary 32, $T_{(3)}$ is K-Jdt equivalent to an almost minimal tableau T^* of the trapezoid. By Lemma 18, $T^*_{[1,r]}$ is K-Jdt equivalent to $(T_{(3)})_{[1,r]} = T_{(2)}$. Since $T_{(3)}$ and T^* have the same shape as $T^*_{[1,r]} = (T'_{(3)})_{[1,r]}$, by corollary 31, we have $T^* = T_{(3)}$. Putting everything together, we get

$$\text{Row}^{-1}(T) = T_{(3)} \overset{K\text{-Jdt}}{\equiv} T^* = T'_{(3)} = \text{Row}^{-1}(T').$$

Finally, Corollary 32 implies that $K\text{-Inf}(\text{Row}^{-1}(T)) = \text{Row}^{-1}(T') = \text{Row}^{-1}(K\text{-Inf}(T))$. The fact that K-Inf commutes with Row^{-1} implies that it commutes with Row as well. $\square$

Remark 33. We end our paper by noting that K-Pro also fails to commute with K-Inf for tableaux that are not almost minimal, even though for any tableau T (of standard or shifted shape), the tableaux $K\text{-Inf} \circ K\text{-Pro}(T)$ and $K\text{-Pro} \circ K\text{-Inf}(T)$ always lie in the same (weak) K-Knuth class and have the same Hecke permutation. See Figure 11 for an example. This is because for non-almost-minimal tableaux, two tableaux in a same (weak) K-Knuth class can have the same shape (i.e. Theorem 6 fails to hold), and therefore our proof technique does not extend to generic tableaux or P-partitions.

It remains an intriguing open question to find a bijection that commutes with piecewise-linear rowmotion on general P-partitions of the rectangle $\mathscr{R}(a,b)$ and trapezoid $\mathscr{T}(a,b)$.

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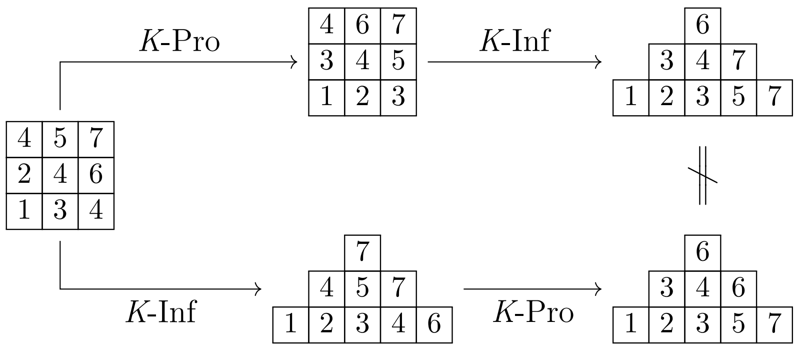

Figure 11: K-Pro fails to commute with K-Inf for non almost-minimal tableaux. Both of the two shifted tableaux on the right have Hecke permutation (14)(26)(58) and lie in the same weak K-Knuth class.

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