There does not exist a strongly regular graph with parameters $(1911, 270, 105, 27)$

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Abstract

In this paper we show that there does not exist a strongly regular graph with parameters $(1911, 270, 105, 27)$.

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1 Introduction

In this paper all the graphs are finite, undirected and simple. For definitions, we do not define, see [2]. Recall that a *s*trongly regular graph with parameters (n, k, λ, μ) is a k-regular graph on n vertices such that two distinct vertices have λ, respectively μ, common neighbours when they are adjacent, respectively non-adjacent.

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Let $H(a,t)$ be the graph with $1+a+t$ vertices, consisting of a complete graph K_{a+t} and a vertex adjacent to exactly a vertices of K_{a+t}.

In [6], Greaves, Koolen and Park obtained the following lemma.

Lemma 1. *Let G be a graph with smallest eigenvalue $\theta = \theta_{\min}(G)$. Assume that G contains an induced $H(a,t)$. Then*

$$(a-\theta(\theta+1))(t-(\theta+1)^2) \leqslant (\theta(\theta+1))^2 \tag{1}$$

holds.

Using Lemma 1, Greaves et al. [6] derived a method restricting the order of maximal cliques in a strongly regular graph. They showed the following result.

Lemma 2 (cf. [6, Lemma 3.7])**.** *Let G be a strongly regular graph with parameters (n,k,λ,μ) having smallest eigenvalue $-m$. Let C be a maximal clique of G with order γ.*

If $\mu > m(m-1)$ and $\gamma > \frac{\mu^2}{\mu-m(m-1)} - m + 1$, then

$$((\gamma+m-3)(k-\gamma+1)-2(\gamma-1)(\lambda-\gamma+2))^2-(k-\gamma+1)^2(\gamma+m-1)(\gamma-(m-1)(4m-1)) \geqslant 0. \tag{2}$$

We denote the polynomial on the left hand side of the Inequality (2) by $M_G(\gamma)$.

Now we recall the Delsarte bound for strongly regular graphs. Let G be a strongly regular graph with parameters (n,k,λ,μ) and smallest eigenvalue $-m$. Let C be a clique of G with order γ. Then

$$\gamma \leqslant 1+\frac{k}{m}. \tag{3}$$

The Inequality (3) is called the *Delsarte bound.*

So, if $M_G(1+\frac{k}{m}) < 0$, then we can improve the Delsarte bound using Lemma 2.

In this paper we extend the method of Greaves et al., by considering two large maximal cliques that intersect in many vertices. Although we are not able to enlarge the forbidden interval as given in Lemma 2, in general, we will show the following result:

Theorem 3. *There does not exist a strongly regular graph with parameters $(1911, 270, 105, 27)$.*

To put this result in context, we now discuss a result of Sims. Sims showed the following result:

Theorem 4 (cf. [3, Theorems 8.6.3, 8.6.4])**.** *Let $m \geqslant 2$ be an integer. There exists a constant $N(m) > 0$ such that any primitive strongly regular graph with parameters (n,k,λ,μ) and smallest eigenvalue $-m$ satisfies either $n \leqslant N(m)$ or $\mu \in \{m(m-1), m^2\}$.*

For $m = 3$, the largest open case of a set of feasible parameters of a primitive strongly regular graph with smallest eigenvalue -3 and $\mu \notin \{6, 9\}$ was $(1911, 270, 105, 27)$, and we show in this paper that it does not exist. On the other hand, it is known that $N(3) \geqslant 276$, as there exist many strongly regular graphs with parameters $(276, 135, 78, 54)$, see [3, Section 8.10.1]. On this moment, there are twelve cases of parameter sets of putative primitive strongly regular graphs with smallest eigenvalue -3, $n > 276$ and $\mu \notin \{6, 9\}$ which are still open. They are in Table 1 above (cf. [7]).

(n, k, λ, μ)	$\theta_0, [\theta_1]^{m(\theta_1)}, [\theta_2]^{m(\theta_2)}$	(n, k, λ, μ)	$\theta_0, [\theta_1]^{m(\theta_1)}, [\theta_2]^{m(\theta_2)}$
$(288, 105, 52, 30)$	$105, [25]^{27}, [-3]^{260}$	$(476, 133, 60, 28)$	$133, [35]^{34}, [-3]^{441}$
$(300, 117, 60, 36)$	$117, [27]^{26}, [-3]^{273}$	$(540, 147, 66, 30)$	$147, [39]^{35}, [-3]^{504}$
$(351, 140, 73, 44)$	$140, [32]^{26}, [-3]^{324}$	$(550, 162, 75, 36)$	$162, [42]^{33}, [-3]^{516}$
$(375, 102, 45, 21)$	$102, [27]^{34}, [-3]^{340}$	$(575, 112, 45, 16)$	$112, [32]^{46}, [-3]^{528}$
$(405, 132, 63, 33)$	$132, [33]^{30}, [-3]^{374}$	$(703, 182, 81, 35)$	$182, [49]^{37}, [-3]^{665}$
$(441, 88, 35, 13)$	$88, [25]^{44}, [-3]^{396}$	$(1344, 221, 88, 26)$	$221, [65]^{56}, [-3]^{1287}$

Table 1: List of putative primitive strongly regular graphs with smallest eigenvalue -3 for $n > 276$.

So our main result shows that $N(3) \leqslant 1344$. We believe $N(3) = 276$, and we conjecture:

Conjecture 5. Let G be a primitive strongly regular graph with parameters (n, k, λ, μ) and smallest eigenvalue -3. Then either $\mu \in \{6, 9\}$ or $n \leqslant 276$.

This paper is organized as follows: In the next section we give the preliminaries. In Section 3 we give some properties of a strongly regular graph with parameters $(1911, 270, 105, 27)$. In Section 4, we find large cliques in a strongly regular graph with parameters $(1911, 270, 105, 27)$ and apply the properties given in Section 3 to show the main result.

2 Preliminaries

2.1 Graphs

Let $G = (V(G), E(G))$ be a graph with order $n(G) < \infty$. The *adjacency matrix* $A(G)$ is a square $(n(G) \times n(G))$-matrix, whose rows and columns are indexed by $V(G)$, such that $A(G)_{xy} = 1$ if $xy \in E(G)$, and 0 otherwise. The *eigenvalues* of G are the eigenvalues of its adjacency matrix $A(G)$. The smallest eigenvalue of G is denoted by $\theta_{\min}(G)$. For a connected graph G and two vertices x and y of G, define the *distance* $d(x, y)$ as the length of a shortest path connecting x and y.

The valency $k_G(x)$ of a vertex x of G is the number of neighbours of x, i.e. the number of the vertices $y \in V(G)$ such that $xy \in E(G)$. A graph G is k-regular if $k_G(x) = k$ for all vertices $x \in V(G)$. As mentioned in the introduction, a graph G is *strongly regular* with parameters (n, k, λ, μ) if G has n vertices, is k-regular and any two distinct vertices have exactly λ (resp. μ) common neighbours if they are adjacent (resp. non-adjacent). In this case, we will also write G is an $\mathrm{SRG}(n,k,\lambda,\mu)$. A strongly regular graph G is called *primitive* if G and its complement are both connected. A graph G is *co-edge-regular* with parameters (n,k,μ) if G has n vertices, is k-regular and any two distinct non-adjacent vertices have exactly μ common neighbours. A *clique* is a complete graph.

2.2 Interlacing

If M (resp. N) is a real symmetric $m\times m$ (resp. $n\times n$) matrix with $\theta_1(M)\geqslant\theta_2(M)\geqslant\cdots\geqslant\theta_m(M)$ (resp. $\theta_1(N)\geqslant\theta_2(N)\geqslant\cdots\geqslant\theta_n(N)$) the eigenvalues of M (resp. N) in non-increasing order. Assume $m\leqslant n$. Then we say that the eigenvalues of M *interlace* the eigenvalues of N, if $\theta_{n-m+i}(N)\leqslant\theta_i(M)\leqslant\theta_i(N)$ for $i=1,\ldots,m$.

The following result is a special case of interlacing.

Lemma 6 (cf. [5, Theorem 9.1.1]). *Let B be a real symmetric $n\times n$ matrix and C be a principal submatrix of B of order m, where $m<n$. Then the eigenvalues of C interlace the eigenvalues of B.*

As an easy consequence of Lemma 6, we have the following proposition.

Proposition 7. *Let G be a graph and H a proper induced subgraph of G. Denote by $\theta_{\min}(G)$ (resp. $\theta_{\min}(H)$) the smallest eigenvalue of G (resp. H). Then $\theta_{\min}(G)\leqslant\theta_{\min}(H)$.*

Let $G=(V(G),E(G))$ be a graph and $\pi=\{V_1,\ldots,V_r\}$ be a partition of $V(G)$. For a vertex $x\in V(G)$, define $\beta_j(x)$ as the number of neighbours of x in V_j for $j=1,2,\ldots,r$. Define the *quotient matrix* Q of π as the $(r\times r)$-matrix with entries

$$Q_{i,j}:=\frac{\sum_{x\in V_i}\beta_j(x)}{|V_i|},$$

for $1\leqslant i,j\leqslant r$.

Proposition 8 (cf. [5, Lemma 9.6.1]). *Let $G=(V(G),E(G))$ be a graph and $\pi:=\{V_1,\ldots,V_r\}$ be a partition of $V(G)$. Let Q be the quotient matrix of π. Then the eigenvalues of Q interlace the eigenvalues of G.*

As an easy consequence of Proposition 8 we have the following lemma.

Lemma 9. *Let $G=(V(G),E(G))$ be a graph and $\pi:=\{V_1,\ldots,V_r\}$ be a partition of $V(G)$. Let Q be the quotient matrix of π. Denote by $\theta_{\min}(G)$ (resp. $\theta_{\min}(Q)$) the smallest eigenvalue of G (resp. Q). Then $\theta_{\min}(G)\leqslant\theta_{\min}(Q)$.*

2.3 Terwilliger graphs

A *Terwilliger graph* is a non-complete graph such that, for any two vertices x and y at distance 2, the subgraph induced by common neighbours of x and y forms a clique with order c (for some fixed $c\geqslant 0$).

Lemma 10 (cf. [1, Corollary 1.16.6 (ii)]). *There does not exist a strongly regular Terwilliger graph with parameters (n,k,λ,μ) satisfying $k<50(\mu-1)$.*

2.4 Join of graphs

Let G_1 and G_2 be two graphs such that $V(G_1) \cap V(G_2) = \emptyset$. The *join* of G_1 and G_2, denoted by $G_1 \nabla G_2$, has as vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{\{x_1, x_2\} \mid x_1 \in V(G_1), x_2 \in V(G_2)\}$. The following lemma is a consequence of [2, Section 2.3.1].

Lemma 11. *Let G_i be a k_i-regular graph with n_i vertices, for $i = 1, 2$, such that $V(G_1) \cap V(G_2) = \emptyset$. Then the smallest eigenvalue $\theta_{\min}(G_1 \nabla G_2)$ of the join $G_1 \nabla G_2$ satisfies*

$$\theta_{\min}(G_1 \nabla G_2) = \min\{\theta_{\min}(G_1), \theta_{\min}(G_2), \theta_{\min}(Q)\}$$

where

$$Q = \begin{pmatrix} k_1 & n_2 \\ n_1 & k_2 \end{pmatrix}.$$

The following lemma was inspired by Cao, Koolen, Munemasa, Yoshino [4].

Lemma 12. *Let G be a k-regular graph on n vertices with smallest eigenvalue $\theta_{\min}(G) \leqslant -1$. Consider $K_t \nabla G$ for some positive integer t. Then $\theta_{\min}(K_t \nabla G) = \theta_{\min}(G)$ if and only if*

$$(\theta_{\min}(G) - k)(\theta_{\min}(G) + 1 - t) \geqslant nt.$$

Proof. By Lemma 11, we find $\theta_{\min}(G) = \theta_{\min}(K_t \nabla G)$ if and only if

$$\theta_{\min}(\begin{pmatrix} t-1 & n \\ t & k \end{pmatrix}) \geqslant \theta_{\min}(G)$$

if and only if

$$\det(\begin{pmatrix} t-1-\theta_{\min}(G) & n \\ t & k-\theta_{\min}(G) \end{pmatrix}) \geqslant 0,$$

as $\theta_{\min}(K_t) \geqslant -1$ (because $t \geqslant 1$). This shows the lemma. ◻

3 Some properties of a SRG(1911, 270, 105, 27)

In this section we collect some elementary properties of a strongly regular graph with parameters $(1911, 270, 105, 27)$. First we show that all cliques in such a graph have order at most 32.

Lemma 13. *If a strongly regular graph G with parameters $(1911, 270, 105, 27)$ exists, then any clique in G has order at most 32.*

Proof. Let G be a strongly regular graph with parameters $(1911, 270, 105, 27)$. Then, it has smallest eigenvalue -3. Let C be a maximal clique in G of order γ. If $\gamma > \frac{27^2}{27-6} - 3 + 1 = 32\frac{5}{7}$, then, by Lemma 2, we have

$$M_G(\gamma) = 672\gamma^3 - 80784\gamma^2 + 1468512\gamma + 3277200 \geqslant 0$$

as $c = 27 > 6$. It is easily checked that $M_G(0) > 0$, $M_G(26) < 0$ and $M_G(97) < 0$. This means that $\gamma \geqslant 98$. This gives a contradiction, as the Delsarte bound gives

$$\gamma \leqslant 1 + \frac{k}{m} = 1 + \frac{270}{3} = 91.$$

So we obtain that any clique in G has order at most 32. □

Next we will show that there must exist an induced quadrangle in such a strongly regular graph.

Lemma 14. *If a strongly regular graph with parameters* $(1911, 270, 105, 27)$ *exists, then it contains an induced quadrangle.*

Proof. Suppose that there exists a strongly regular graph G with parameters $(1911, 270, 105, 27)$, which does not contain any induced quadrangles. Then G is a Terwilliger graph. By Lemma 10, the valency of G is at least 1300, as $\mu = 27$. This is a contradiction, as $k = 270$. This shows the lemma. □

We will need the following consequence of Lemma 1 later in the paper.

Lemma 15. *Let G be a graph with smallest eigenvalue at least -3. Let C be a clique of G with order γ. Let x be a vertex of G that is not in C and has exactly a neighbours in C. Then $a \leqslant a_{\min}$ or $a \geqslant a_{\max}$ where $a_{\min}$ and $a_{\max}$ are as in Table 2.*

γ	29	30	31	32
$a_{\min}$	8	8	7	7
$a_{\max}$	23	24	26	27

Table 2: Values of $a_{\min}$ and $a_{\max}$

Proof. This lemma follows immediately from Lemma 1. □

Now we will show the following restriction on maximal cliques intersecting in many vertices in a strongly regular graph with parameters $(1911, 270, 105, 27)$.

Lemma 16. *Let G be strongly regular graph with parameters* $(1911, 270, 105, 27)$*. Assume there are two distinct maximal cliques C_1 and C_2 such that $|V(C_1)| \geqslant 29$, $|V(C_2)| \geqslant 29$, and $|V(C_1) \cap V(C_2)| \geqslant 22$. Then $|V(C_1)| = |V(C_2)| = 29$ and $|V(C_1) \cap V(C_2)| = 27$.*

Proof. We have that $|V(C_1) \cap V(C_2)| =: t \in \{22, 23, \ldots, 27\}$ as $\mu = 27$.

Assume $t = 22$. Let C_1' (resp. C_2') be a subclique of C_1 (resp. C_2) such that $V(C_1') \supseteq V(C_1) \cap V(C_2)$, $V(C_2') \supseteq V(C_1) \cap V(C_2)$ and $|V(C_1')| = |V(C_2')| = 29$. Let K be the induced subgraph on $V(C_1') \cup V(C_2')$. By Proposition 7, we see that K has smallest eigenvalue at least -3. Let $\pi = \{V(C_1') \cap V(C_2'), (V(C_1') \setminus V(C_2')) \cup (V(C_2') \setminus V(C_1'))\}$ of $V(C_1') \cup V(C_2')$ be a partition of K with quotient matrix

$$Q = \begin{pmatrix} 21 & 14 \\ 22 & \alpha + 6 \end{pmatrix}.$$

By Lemma 9, we see that the smallest eigenvalue of Q is at least -3. This implies that $24\alpha \geqslant 92$, as $\det(Q + 3\mathbf{I}) \geqslant 0$. So, $\alpha \geqslant \frac{23}{6}$. We obtain that there are at least $\lceil \frac{7 \times 23}{6} \rceil = 27$ edges between $V_1 := V(C_1') \setminus V(C_2')$ and $V_2 := V(C_2') \setminus V(C_1')$. Now all vertices of V_1 (resp. V_2) have at most 5 neighbours in V_2 (resp. V_1), as $\mu = 27$.

Consider the bipartite graph B with color classes V_1 and V_2, where $v_1 \in V_1$ is adjacent to $v_2 \in V_2$ if they are adjacent in G. We know that $k_B(v_1) + k_B(v_2) \leqslant 5$ if $v_1 \in V_1$, $v_2 \in V_2$ and $v_1 \not\sim v_2$, as $\mu = 27$. Let B have maximal valency p, and we may assume that $d_B(x) = p$ for a vertex $x \in V_1$ and $p \geqslant 3$. Then the neighbours of x in V_2 have valency at most p in B and the non-neighbours of x in V_2 have valency at most $5 - p$ in B. So B has at most $p^2 + (7 - p)(5 - p)$ edges. This means that B has at most 25 edges, as $p \in \{3, 4, 5\}$. This is a contradiction with the fact that B has to have at least 27 edges. This shows that $t = 22$ is not possible.

In similar fashion, it can be shown that $t \notin \{23, 24, 25, 26\}$.

Now assume $t = 27$. If $|V(C_1)| \geqslant 30$ and $|V(C_2)| \geqslant 29$, then the quotient matrix Q' of $\pi' = \{V(C_1) \cap V(C_2), V(C_1) \setminus V(C_2), V(C_2) \setminus V(C_1)\}$ satisfies

$$\begin{pmatrix} 26 & t_1 & t_2 \\ 27 & t_1 - 1 & 0 \\ 27 & 0 & t_2 - 1 \end{pmatrix}, \text{ where } t_1 + 27 = |V(C_1)| \text{ and } t_2 + 27 = |V(C_2)|.$$

As the smallest eigenvalue of Q' is at least -3, we obtain that

$$29(t_1 + 2)(t_2 + 2) - 27(t_1(t_2 + 2) + t_2(t_1 + 2)) \geqslant 0.$$

This means

$$-25t_1t_2 + 4(t_1 + t_2) + 116 \geqslant 0,$$

and hence

$$25(t_1 - \frac{4}{25})(t_2 - \frac{4}{25}) < 117.$$

As $t_1 \geqslant 3$ and $t_2 \geqslant 2$ we have $25(t_1 - \frac{4}{25})(t_2 - \frac{4}{25}) > 130$, which gives a contradiction. This shows the lemma. $\square$

4 Proof of Theorem 3

In this section we give a proof of Theorem 3.

Let G be a strongly regular graph with parameters (n, k, λ, μ). Let x be a vertex of G. Let $y_1, y_2, \ldots, y_\ell$ be distinct pairwise non-adjacent neighbours of x, that is, the induced subgraph of G on $\{x, y_1, y_2, \ldots, y_\ell\}$ is a ℓ-claw.

Let $p := |\{(y_i, x', y_j) \mid y_i \sim x' \sim y_j,\ 1 \leqslant i < j \leqslant \ell,\ x \neq x', \text{ and } x \not\sim x'\}|$. For $1 \leqslant i < j \leqslant \ell$, let $C(y_i, y_j) := \{z \in V(G) \mid z \sim x, z \sim y_i, z \sim y_j\}$ and let $c(y_i, y_j) = |C(y_i, y_j)|$. If there is no confusion possible, we will abbreviate $c(y_i, y_j)$ by c_{ij}. We see that $c_{ij} \leqslant \mu - 1$ for all $1 \leqslant i < j \leqslant \ell$. Further, let $m_e = |\{z \sim x \mid z$ is adjacent to exactly e y_i's$\}|$ for $e = 0, 1, \ldots, \ell$.

Then the following equations hold:

$$\sum_{e=0}^{\ell} m_e = k - \ell, \tag{4}$$

$$\sum_{e=0}^{\ell} e m_e = \lambda \ell, \tag{5}$$

$$\sum_{e=0}^{\ell} \binom{e}{2} m_e = \binom{\ell}{2}(\mu - 1) - p, \tag{6}$$

$$\sum_{e=0}^{\ell} \binom{e}{2} m_e = \sum_{1 \leqslant i < j \leqslant \ell} c_{ij}. \tag{7}$$

Combining Equations (4), (5) and (6), we see that

$$0 \leqslant \sum_{e=0}^{\ell} \binom{e-1}{2} m_e = k - (\lambda + 1)\ell + \binom{\ell}{2}(\mu - 1) - p \tag{8}$$

holds.

Now let G be a strongly regular graph with parameters $(1911, 270, 105, 27)$. By Lemma 14, we know that G contains an induced quadrangle say $x \sim y_1 \sim x' \sim y_2$. Let $W = \{w \sim x \mid d(y_1, w) = d(y_2, w) = 2\}$, and let H be the induced subgraph of G on W. Note that the cardinality of W, $|W|$, is at least $270 - 2 \times (105 + 1) = 58$. As G does not contain a clique of order at least 33, by Lemma 13, the graph H is not a complete graph.

We will show that H contains many large cliques.

First we establish the following claim:

Claim 1. The graph H does not contain an independent set of cardinality 3.
Proof. Assume that H contains an independent set of cardinality 3, say $\{w_1, w_2, w_3\}$. Then the subgraph of G induced on $\{x, y_1, y_2, w_1, w_2, w_3\}$ is a 5-claw. Then, by Equation (8), we see that $0 \leqslant 270 - 5 \times 106 + \binom{5}{2} \times 26 - p = -p$, so $p = 0$. This is a contradiction, as the induced subgraph on $\{x, x', y_1, y_2\}$ is a quadrangle. This shows the claim. □

Define $W' := \{w \in W \mid k_H(w) = |W| - 1\}$ and $W'' = W \setminus W'$. As H is not complete, it follows that W'' is not empty.

Now we establish the following claim.

Claim 2. Let $w_1, w_2 \in W''$ such that $d(w_1, w_2) = 2$. Then the subgraph of G induced on $\{x, y_1, y_2, w_1, w_2\}$ is a 4-claw. As before, for $z_1, z_2 \in \{y_1, y_2, w_1, w_2\}$, let $C(z_1, z_2) := \{z \in V(G) \mid z \sim x, z_1 \sim z \sim z_2\}$ with cardinality $c(z_1, z_2)$. Then we have:

1. $c(y_1, y_2) \in \{24, 25\}$;
2. $c(z_1, z_2) \in \{25, 26\}$ if $\{z_1, z_2\} \in \binom{\{y_1, y_2, w_1, w_2\}}{2}$ and $\{z_1, z_2\} \neq \{y_1, y_2\}$;
3. $$\beta := \sum_{\{z_1,z_2\} \in \binom{\{y_1,y_2,w_1,w_2\}}{2}} c(z_1, z_2) \in \{154, 155\},$$

   and $m_3 = 0$ if $\beta = 154$ (and, $m_3 = 1$ if $\beta = 155$).

Proof. By Equation (8), we see that $0 \leqslant 270 - 4 \times 106 + \binom{4}{2} \times 26 - p = 2 - p$, and hence $p \in \{1, 2\}$, as x, y_1, x', y_2 is an induced quadrangle. Equation (8) also shows that $m_3 + p = 2$. Combining Equations (6) and (7), we obtain $\beta = 6 \times 26 - p = 156 - p$. So we have shown Item (iii). As $c(y_1, y_2) \leqslant 25$ and $c(z_1, z_2) \leqslant 26$ for all $\{z_1, z_2\} \in \binom{\{y_1,y_2,w_1,w_2\}}{2}$, Items (i) and (ii) follow. This shows the claim. □

Now we are going to look at H more closely. For $w_1, w_2 \in W''$ with $d(w_1, w_2) = 2$, define $C'(w_1, w_2) = \{w \in W \mid w_1 \sim w \sim w_2\}$ with cardinality $c'(w_1, w_2)$.

Claim 3. The following hold:

1. If $c(y_1, y_2) = 24$, then $|W| = 82$. In this case, $k_H(w) \in \{81, 53\}$ for $w \in W$ and for $w_1, w_2 \in W''$ with $d(w_1, w_2) = 2$ we have $c'(w_1, w_2) = 26$.
2. If $c(y_1, y_2) = 25$, then $|W| = 83$. In this case $k_H(w) \in \{82, 53, 54\}$ for $w \in W$ and for $w_1, w_2 \in W''$ with $d(w_1, w_2) = 2$ we have $c'(w_1, w_2) \in \{25, 26\}$. Moreover, the induced subgraph of H on $\{w \in W \mid k_H(w) = 54\}$ forms a clique, if $\{w \in W \mid k_H(w) = 54\} \neq \emptyset$.

Proof. We have $|W| = 270 - 2(105+1) + c(y_1, y_2)$. So we obtain $|W| = 82$, if $c(y_1, y_2) = 24$, and $|W| = 83$, if $c(y_1, y_2) = 25$. We already observed that $W'' \neq \emptyset$. Let $w_1, w_2 \in W''$ with $d(w_1, w_2) = 2$.

If $c(y_1, y_2) = 24$, then $c(z_1, z_2) = 26$ for all $\{z_1, z_2\} \in \binom{\{y_1,y_2,w_1,w_2\}}{2} \setminus \{\{y_1, y_2\}\}$, and $m_3 = 0$, by Claim 2. This shows that $k_H(w_1) = k_H(w_2) = 105 - 2 \times 26 = 53$ and $c'(w_1, w_2) = 26$. This shows Item (i).

If $c(y_1, y_2) = 25$, then there is at most one set $\{z_1, z_2\} \in \binom{\{y_1,y_2,w_1,w_2\}}{2} \setminus \{\{y_1, y_2\}\}$ such that $c(z_1, z_2) = 25$ and the others have $c(z_1, z_2) = 26$. If there is one set $\{z_1, z_2\} \in \binom{\{y_1,y_2,w_1,w_2\}}{2} \setminus \{\{y_1, y_2\}\}$ such that $c(z_1, z_2) = 25$, then $m_3 = 0$ and we see that $c'(w_1, w_2) \in \{25, 26\}$. Also we see that in this case we have $|\{z \mid z \sim x, z \sim w_1, z$ is adjacent to at least one of y_1 and $y_2\}| \in \{51, 52\}$. This means that $k_H(w_1) \in \{53, 54\}$. If there is not a set $\{z_1, z_2\} \in \binom{\{y_1,y_2,w_1,w_2\}}{2} \setminus \{\{y_1, y_2\}\}$ such that $c(z_1, z_2) = 25$, then $c(w_1, w_2) = 26$, and hence $c'(w_1, w_2) \in \{25, 26\}$, as $m_3 = 1$ in this case. Again we see that in this case we have $|\{z \mid z \sim x, z \sim w_1, z$ is adjacent to at least one of y_1 and $y_2\}| \in \{51, 52\}$. This means that $k_H(w_1) \in \{53, 54\}$.

In order to show Item (ii), let $w, w' \in W$ be distinct vertices such that $k_H(w) = k_H(w') = 54$. This means that w and w' have both exactly 51 common neighbours with x that are not in W. If $w \not\sim w'$, then this means that $c(w, y_1) + c(w, y_2) + c(w', y_1) + c(w', y_2) - m_3 \leqslant 2 \times 51 = 102$. Now, by Claim 2(iii), we find

$$\begin{aligned} 153 &= 26 + 102 + 25 \\ &\geqslant c(w, w') + c(w, y_1) + c(w, y_2) + c(w', y_1) + c(w', y_2) + c(y_1, y_2) - m_3 \\ &= 154, \end{aligned}$$

which is impossible. This shows the claim. □.

Now we are going to find large cliques in H. For $w \in W''$, define $N_w := \{w_1 \in W \mid d(w, w_1) = 2\}$ with cardinality n_w. The induced subgraph of G on $N_w \cup \{x\}$ is a complete graph, as H does not contain an independent set with order 3, by Claim 1. Let C_w be a maximal clique of G containing $N_w \cup \{x\}$. Note that, by Claim 3, we have $n_w = |W| - k_H(w) - 1 \in \{82 - 53 - 1, 83 - 54 - 1, 83 - 53 - 1\} = \{28, 29\}$. Now let z_1, z_2 be two distinct vertices of G such that for all $v \in N_w$ we have $z_1 \sim v \sim z_2$. Then z_1 and z_2 have at least 28 common neighbours and hence must be adjacent, as $\mu = 27$. This shows that C_w is unique.

Now we will show that G has two distinct maximal cliques C_1 and C_2, each with at least 29 vertices, and intersecting in at least 22 vertices.

Claim 4. Let $w_1, w_2 \in W''$ with $d(w_1, w_2) = 2$. Then there exists a common neighbour $w \in W''$ of w_1 and w_2 such that $|N_w \cap N_{w_1}| \geqslant 21$ or $|N_w \cap N_{w_2}| \geqslant 21$.
Proof. Let $w_1, w_2 \in W''$ with $d(w_1, w_2) = 2$. There are at most three vertices $w \in W$ such that $w \sim z$ for all $z \in N_{w_1}$, as $n_{w_1} \geqslant 28$, $\mu = 27$ and C_{w_1} has at most 32 vertices, by Lemma 13. Similarly, there are at most three vertices $w \in W$ such that $w \sim z$ for all $z \in N_{w_2}$. In particular, the set W' contains at most three vertices.

Let $Z := \{z \in W \setminus (N_{w_1} \cup N_{w_2}) \mid z$ has at least $n_{w_1} - 7$ neighbours in N_{w_1} and at least $n_{w_2} - 7$ neighbours in $N_{w_2}\}$. If $Z \neq \emptyset$, then the subgraph of H induced on Z is a clique, as any two vertices $z_1, z_2 \in Z$ have at least $14 + 14 = 28$ common neighbours and $\mu = 27$. It follows that $|Z| \leqslant 1 + k_H(z) - (n_{w_1} - 7 + n_{w_2} - 7) \leqslant 1 + 54 - 2 \times 21 = 13$, where z is any vertex in Z. As $c'(w_1, w_2) \geqslant 25 > 3 + 3 + 13 = 19$, there exists a vertex $w \in W \setminus (N_{w_1} \cup N_{w_2})$ with at most $n_{w_1} - 8$ neighbours in N_{w_1} or with at most $n_{w_2} - 8$ neighbours in N_{w_2} and w has a non-neighbour in both N_{w_1} and N_{w_2}. Without loss of generality, we may assume that w has at most $n_{w_1} - 8$ neighbours in N_{w_1}. By Lemma 15, the vertex w has either at least $n_{w_1} - 7$ neighbours in $N_{w_1} \cup \{x\}$ or at most 8 neighbours in $N_{w_1} \cup \{x\}$ (as $n_{w_1} \in \{28, 29\}$). This means that w has at most 8 neighbours in $N_{w_1} \cup \{x\}$, and hence at most 7 neighbours in N_{w_1}. This means that N_w and N_{w_1} intersect in at least 21 vertices. This shows the claim. □

Claim 5. Let $w, w_1 \in W''$ such that $|N_w \cap N_{w_1}| \geqslant 21$. Then C_w and C_{w_1} both have exactly 29 vertices and they intersect in precisely 27 vertices.

Proof. The maximal cliques C_w and C_{w_1} have both at least 29 vertices and they intersect in at least 22 vertices as x is an element of both C_w and C_{w_1}. Hence, by Lemma 16, they have both exactly 29 vertices and intersect in exactly 27 vertices, as both are maximal cliques. □

Now we will show that $W' = \emptyset$ and that $|W| = 82$.

Claim 6.

(i) The set W' is empty.

(ii) The cardinality of W is equal to 82.

Proof. (i) Let z in W' and let $w_1, w_2 \in W''$ be two vertices at distance 2. Then C_{w_i} contains $N_{w_i} \cup \{x, z\}$ for $i = 1, 2$ and hence has at least 30 vertices. Now there exists a common neighbour $w \in W''$ such that without loss of generality $|N_w \cap N_{w_1}| \geqslant 21$ by Claim 4. But this is impossible by Claim 5.

(ii) By Claims 2 and 3, we have $|W| \in \{82, 83\}$. Let us assume that $|W| = 83$. As, by (i), $W' = \emptyset$, we have $k_H(w) \in \{53, 54\}$, by Claim 3. We have seen that the vertices $w \in W$ with $k_H(w) = 54$ form a clique, if there are any. So this means that there are at most 31 of them, as otherwise we would have a clique with 33 vertices, which is impossible by Lemma 13. This means that there are at least 52 vertices w of W with $k_H(w) = 53$. Hence, there must be two distinct non-adjacent vertices $w_1, w_2 \in W$ with $k_H(w_1) = k_H(w_2) = 53$. Therefore we have that $n_{w_1} = n_{w_2} = 29$ and we obtain that both the cliques C_{w_1} and C_{w_2} have at least $1 + 29 = 30$ vertices. Now by Claim 4, there exists a common neighbour $w \in W''$ such that without loss of generality $|N_w \cap N_{w_1}| \geqslant 21$. But this is again impossible, by Claim 5. This finishes the proof of the claim. □

So we are in the case where $|W| = 82$. In this case, by Claim 3, the graph H is a 53-regular graph (as $W' = \emptyset$) such that any two distinct non-adjacent vertices have exactly 26 common neighbours inside H, so this means H is a co-edge-regular graph with parameters $(n', k', \mu') = (82, 53, 26)$. Also, $n_w = 28$ for all $w \in W$.

Now we will show that H contains two cliques C_1 and C_2, each with 28 vertices, and such that they intersect in exactly 2 vertices.

Claim 7. The graph H contains two cliques C_1 and C_2, each with 28 vertices, and that they intersect in exactly 2 vertices.

Proof. Let w_1, w_2 be two distinct non-adjacent vertices of H. Then, by Claims 4 and 5, there exists a common neighbour w of w_1 and w_2 such that, without loss of generality, the (maximal) cliques C_w and C_{w_1} both have exactly 29 vertices and intersect in exactly 27 vertices. Let z be a vertex of C_w but not of C_{w_1} and z_1 a vertex of C_{w_1} but not of C_w. Then $z \not\sim z_1$, as C_w and C_{w_1} are maximal cliques and $\mu = 27$. As C_w and C_{w_1} both contain x as a vertex, we see that $C_w - \{x\}$ and $C_{w_1} - \{x\}$ both are cliques inside H with exactly 28 vertices and they intersect in exactly 26 vertices. Now the vertex w_2 is a vertex of $C_{w_1} - \{x\}$, but not a vertex of $C_w - \{x\}$. Now we consider N_{w_2}. Then the two vertices of C_w which are not in C_{w_1} are both in N_{w_2}. This shows that $C_w - \{x\}$ and $C_{w_2} - \{x\}$

are two cliques with 28 vertices in H such that they intersect in exactly 2 vertices. This shows the claim. $\square$

Proof of Theorem 3. Let C_1 and C_2 be two cliques of H with 28 vertices intersecting in exactly 2 vertices, say z and z'. Consider the partition $\pi = \{\{z, z'\}, (V(C_1) \setminus V(C_2)) \cup (V(C_2) \setminus V(C_1)), N_z\}$ of the vertex set W of H. The quotient matrix Q of π satisfies:

$$Q = \begin{pmatrix} 1 & 52 & 0 \\ 2 & 37 & 14 \\ 0 & 26 & 27 \end{pmatrix},$$

as the induced subgraph of H on N_z is complete and H is co-edge-regular with $\mu' = 26$. This means that there are exactly $12 \times 26 = 312$ edges between $V(C_1) \setminus V(C_2)$ and $V(C_2) \setminus V(C_1)$.

Next, we will show that there must be many more edges between $V(C_1) \setminus V(C_2)$ and $V(C_2) \setminus V(C_1)$, which gives a contradiction.

There are at most 5 vertices in N_z that are adjacent to all vertices in $V(C_1) \setminus V(C_2)$, as any clique in G has at most 32 vertices. Also there are at most 5 vertices in N_z that are adjacent to all vertices in $V(C_2) \setminus V(C_1)$. So there exists a vertex $u \in N_z$ that is not adjacent to all vertices in $V(C_1) \setminus V(C_2)$ and not adjacent to all vertices in $V(C_2) \setminus V(C_1)$. As u has exactly 26 neighbours in $(V(C_1) \setminus V(C_2)) \cup (V(C_2) \setminus V(C_1))$, we may assume, without loss of generality, that u has at least 13 neighbours in $V(C_1) \setminus V(C_2)$. So u has at least 14 neighbours in the clique $C_1 \cup \{x\}$, a clique with 29 vertices. By Lemma 15, the vertex u has at least 23 neighbours in $C_1 \cup \{x\}$ and hence at least 22 neighbours in C_1. This means that u has at most 4 neighbours in C_2. Now consider the clique C_u. Then C_u and $\{x\} \cup C_2$ intersect in at least 25 vertices. As both have 29 vertices, by Lemma 15, they intersect in exactly 27 vertices. This means that $|N_u \cap V(C_2)| = 26$ and hence $|N_u \cap V(C_1)| = 4$. We find that any vertex of $N_u \cap (V(C_2) \setminus V(C_1))$ has at least 5 neighbours in $C_1 \cup \{x\}$. Now consider the join $K_4 \nabla H$ of K_4 and H, where K_4 is a 4-clique with vertices $\{v_1, v_2, v_3, v_4\}$. As H is an induced subgraph of G and G has smallest eigenvalue -3, we see that $\theta_{\min}(H) \geqslant -3$, by Lemma 7. It follows that $\theta_{\min}(K_4 \nabla H) = -3$, by Lemma 12. Now consider the clique C of $K_4 \nabla H$ on $V(C_1) \cup \{v_1, v_2, v_3, v_4\}$. It has 32 vertices and any vertex w of $N_u \cap (V(C_2) \setminus V(C_1))$ has at least 8 neighbours in C. By Lemma 15, this means that w has at least 27 neighbours in C. It follows that w has at least 23 neighbours in C_1 and hence at least 21 neighbours in $V(C_1) \setminus V(C_2)$. This means that there are at least $|N_u \cap (V(C_2) \setminus V(C_1))| \times 21 = 24 \times 21 = 504$ edges between $V(C_1) \setminus V(C_2)$ and $V(C_2) \setminus V(C_1)$. This is a contradiction with the fact that there are exactly $12 \times 26 = 312$ edges between $V(C_1) \setminus V(C_2)$ and $V(C_2) \setminus V(C_1)$. This contradiction shows that $|W| \neq 82$, and hence there does not exist a strongly regular graph with parameters $(1911, 270, 105, 27)$. This shows Theorem 3. $\square$

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