Turán Number of Disjoint Triangles in 4-Partite Graphs

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Abstract

Let $k \geqslant 2$ and $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4$ be integers such that n_4 is sufficiently larger than k. We determine the maximum number of edges of a 4-partite graph with parts of sizes $n_1, \dots, n_4$ that does not contain k vertex-disjoint triangles. For any $r > t \geqslant 3$, we give a conjecture on the maximum number of edges of an r-partite graph that does not contain k vertex-disjoint cliques K_t.

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1 Introduction

Given two graphs G and F, we say that G is F*-free* if G does not contain F as a subgraph. Let K_t denote a complete graph on t vertices, and $T_{n,t}$ denote a balanced complete t-partite graph on n vertices (now known as the *Turán graph*). In 1941, Turán [9] proved that $T_{n,t}$ has the maximum number of edges among all K_{t+1}-free graphs (the case $t = 2$ was previously solved by Mantel [7]). Turán's result initiates the study of Extremal Graph Theory, an important area of research in modern Combinatorics (see the monograph of Bollobás [2]). Let kK_t denote k disjoint copies of K_t. Simonovits [8] studied the Turán problem for kK_t and showed that when n is sufficiently large, the (unique) extremal graph on n vertices is the join of K_{k-1} and the Turán graph $T_{n-k+1,t-1}$.

In this paper we consider Turán problems in multi-partite graphs. Let $K_{n_1,n_2,\dots,n_r}$ denote the complete r-partite graph on parts of sizes $n_1, n_2, \dots, n_r$. This variant of the

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Turán problem was first considered by Zarankiewicz [11], who was interested in the case of forbidding $K_{s,t}$ in (subgraphs of) $K_{a,b}$. Formally, given graphs H and F, we define $\mathrm{ex}(H,F)$ as the maximum number of edges in an F-free subgraph of H. Bollobás, Erdős, and Straus [3] (see also [2, Page 544]) proved the following result. For any subset $I \subseteq [r]$, write $n_I := \sum_{i\in I} n_i$.

Theorem 1. *[3] The extremal number* $\mathrm{ex}(K_{n_1,\dots,n_r}, K_t)$ *is equal to*

$$\max_{\mathcal{P}} \sum_{I\neq I'\in\mathcal{P}} n_I \cdot n_{I'},$$

where the maximum is taken over all partitions $\mathcal{P}$ *of* $[r]$ *into* $t-1$ *parts.*

The problem of forbidding disjoint copies of cliques in multi-partite graphs has been studied recently. Chen, Li and Tu [4] determined $\mathrm{ex}(K_{n_1,n_2}, kK_2)$ and De Silva, Heysse and Young [6] claimed that $\mathrm{ex}(K_{n_1,\dots,n_r}, kK_2) = (k-1)(n_1+\dots+n_{r-1})$ for $n_1 \geqslant \dots \geqslant n_r$. De Silva, Heysse, Kapilow, Schenfisch and Young [5] determined $\mathrm{ex}(K_{n_1,\dots,n_r}, kK_r)$ and raised the question of determining $\mathrm{ex}(K_{n_1,\dots,n_r}, kK_t)$ when $r > t$. After giving another proof of Theorem 1, Bennett, English and Talanda-Fisher [1] reiterated this question.

Problem 2. [5] Determine $\mathrm{ex}(K_{n_1,\dots,n_r}, kK_t)$ when $r > t$.

In this paper we solve Problem 2 for $r = 4$ and $t = 3$ when all n_i's are sufficiently large. To state our result, for $k \geqslant 1$, we define a function of positive integers $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4$:

$$\begin{aligned} g_k(n_1,n_2,n_3,n_4) &:= \max\{(n_1+n_4)(n_2+n_3)+(k-1)n_1,\, n_1(n_2+n_3+n_4)+(k-1)(n_2+n_3)\} \\ &= \begin{cases} (n_1+n_4)(n_2+n_3)+(k-1)n_1 & \text{if } n_1 \leqslant n_2+n_3, \\ n_1(n_2+n_3+n_4)+(k-1)(n_2+n_3), & \text{if } n_1 > n_2+n_3. \end{cases} \end{aligned}$$

When G is a 4-partite graph with parts of sizes $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4$, we define $g_k(G) := g_k(n_1,n_2,n_3,n_4)$. For arbitrary positive integers a,b,c,d, we define that $g_k(a,b,c,d) = g_k(a_1,a_2,a_3,a_4)$, where $a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4$ is a reordering of a,b,c,d.

Theorem 3. *Given* $k \geqslant 1$*, there exists* $N_0(k)$ *such that if* G *is a* kK_3*-free 4-partite graph with parts of sizes* $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4 \geqslant 6k^2$ *and* $n_1+n_2+n_3+n_4 \geqslant N_0(k)$*, then* $e(G) \leqslant g_k(n_1,n_2,n_3,n_4)$*. In other words,* $\mathrm{ex}(K_{n_1,n_2,n_3,n_4}, kK_3) \leqslant g_k(n_1,n_2,n_3,n_4)$*.*

Theorem 3 is tight due to two constructions G_1 and G_2 below. In fact, a subgraph of G_2 was given by De Silva et al. [5] as a potential extremal construction; later Wagner [10] realized that G_1 was a better construction for the $n_1 = n_2 = n_3 = n_4$ case. Let $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4 \geqslant k$. We define two 4-partite graphs with parts $V_1,\dots,V_4$ such that $|V_i| = n_i$. Fix a set Z of $k-1$ vertices in V_4. Let

$$G_1 := K_{V_1\cup V_4,\, V_2\cup V_3} \cup K_{Z,\, V_1} \text{ and } G_2 := K_{V_1,\, V_2\cup V_3\cup V_4} \cup K_{Z,\, V_2\cup V_3},$$

where $K_{V_1,\dots,V_r}$ denotes the complete r-partite graph with parts $V_1,\dots,V_r$. Note that each triangle must intersect Z and thus both G_1 and G_2 are kK_3-free. Moreover, $e(G_1) =$

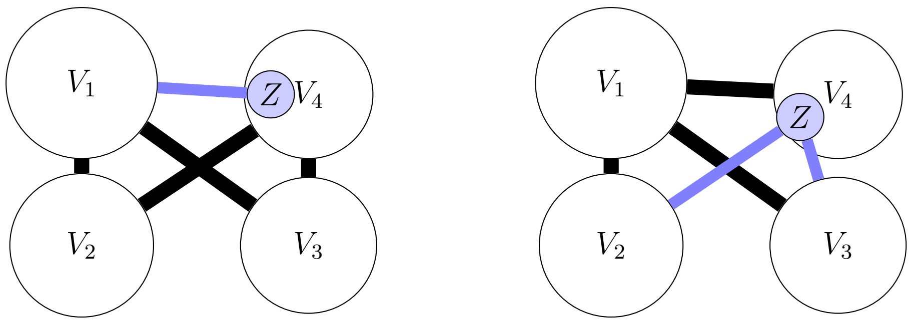

Figure 1: The extremal graphs G_1 and G_2

$(n_1 + n_4)(n_2 + n_3) + (k-1)n_1$ and $e(G_2) = n_1(n_2 + n_3 + n_4) + (k-1)(n_2 + n_3)$. Thus $e(G_2) \leqslant e(G_1)$ if and only if $n_1 \leqslant n_2 + n_3$ and equality holds when $n_1 = n_2 + n_3$.

Our proof uses a *progressive induction* (an induction without a base case) on the total number of vertices and a standard induction on k that uses Theorem 1 as the base case.

We conjecture an answer to Problem 2 in general, which includes all aforementioned results [1, 4, 6] and Theorem 3.

Conjecture 4. Given $r > t \geqslant 3$ and $k \geqslant 2$, let $n_1, \dots, n_r$ be sufficiently large. For $I \subseteq [r]$, write $m_I := \min_{i \in I} n_i$. Given a partition $\mathcal{P}$ of $[r]$, let $n_{\mathcal{P}} := \max_{I \in \mathcal{P}} \{n_I - m_I\}$. The Turán number $\mathrm{ex}(K_{n_1,\dots,n_r}, kK_t)$ is equal to

$$\max_{\mathcal{P}} \left\{ (k-1)n_{\mathcal{P}} + \sum_{I \neq I' \in \mathcal{P}} n_I \cdot n_{I'} \right\}, \tag{1.1}$$

where the maximum is taken over all partitions $\mathcal{P}$ of $[r]$ into $t-1$ parts.

The bound (1.1) is achieved by the following graph. Given integers k, t and $n_1, \dots, n_r$ with $r > t$ and $n_i \geqslant k$ for all i, let $\mathcal{P}$ be a partition of $[r]$ into $t-1$ parts that maximizes (1.1). Let G be an r-partite graph whose parts have sizes $n_1, \dots, n_r$. Partition G into $t-1$ parts according to $\mathcal{P}$, namely, let $V_I = \bigcup_{i \in I} V_i$ for every $I \in \mathcal{P}$ and include all edges between V_I and $V_{I'}$ for all $I \neq I' \in \mathcal{P}$. In addition, let $I_0 \in \mathcal{P}$ such that $n_{\mathcal{P}} = n_{I_0} - m_{I_0}$ and let V_{i_0} be the smallest part in V_{I_0}. We choose a set $Z \subseteq V_{i_0}$ of $k-1$ vertices and add all edges between Z and $V_{I_0} \setminus V_{i_0}$.

Verifying Conjecture 4 seems hard due to the complexity of (1.1) – we shall discuss this in the last section.

Notation. Given a graph $G = (V, E)$, let $|G|$ denote the order of G. Suppose A, B are two disjoint subsets of V. Let $e(A) := e(G[A])$ be the number of edges of G in A and $e(A, B)$ be the number of edges of G with one end in A and the other in B. Moreover, let $G \setminus A := G[V \setminus A]$. Denote by

$$e(A; G) := e(G) - e(G \setminus A),$$

the number of edges of G incident to A. Given a vertex x, let $N(x)$ denote the set of neighbors of x. For vertices x, y and z, we often write xyz for $\{x, y, z\}$. We sometimes abuse this notation by using $xy \in A \times B$ to indicate that $x \in A$ and $y \in B$. Given an r-partite graph G, a *crossing set* is a set that contains at most one vertex from each part of G.

2 Proof of Theorem 3

In this section we prove Theorem 3. Define two sequences $N_0(k)$ and $M_0(k)$ recursively by letting $N_0(1) = 1$,

$$M_0(k) = \max\{72(k-1)^3, 96k^2, N_0(k-1)+3\}, \quad \text{and} \quad N_0(k) = M_0(k)^2 \tag{2.1}$$

for $k \geqslant 2$. Given a 4-partite graph G, let $v_4(G)$ denote the size of the smallest part of G. Define $\varphi(G) := e(G) - g_k(G)$. The following theorem is the main step in the proof of Theorem 3.

Theorem 5. *Suppose $k \geqslant 2$ and Theorem 3 holds for $k-1$. Let G be a 4-partite graph of order $|G| > M_0(k)$ and with $v_4(G) \geqslant 6k^2$. If G is kK_3-free and $\varphi(G) > 0$, then we can find a subgraph G' of G such that $|G| - 2 \leqslant |G'| \leqslant |G| - 1$, $v_4(G') \geqslant 6k^2$, and $\varphi(G') > \varphi(G)$.*

Theorem 3 nows follows from Theorem 5 by an induction on k and a progressive induction on $|G|$ (e.g., used in [8]).

Proof of Theorem 3. The base case $k = 1$ follows from Theorem 1 with $N_0(1) = 1$. Let $k \geqslant 2$ and G be a 4-partite graph of order $|G| \geqslant N_0(k)$ and with $v_4(G) \geqslant 6k^2$. Suppose G is kK_3-free and $\varphi(G) > 0$. By Theorem 5, we find a subgraph $G_1 \subset G$ such that $|G| - 2 \leqslant |G_1| \leqslant |G| - 1$, $v_4(G_1) \geqslant 6k^2$, and $\varphi(G_1) > \varphi(G) \geqslant 1$. Repeating this process, we obtain subgraphs $G_1 \supset G_2 \supset G_3 \supset \cdots \supset G_t$ such that $|G| - 2i \leqslant |G_i| \leqslant |G| - i$ and $\varphi(G_i) > i$ for $i = 1, \ldots, t$. We stop at G_t because $|G_t| \leqslant M_0(k)$. Hence,

$$t \geqslant \frac{|G| - |G_t|}{2} \geqslant \frac{N_0(k) - M_0(k)}{2} = \frac{M_0(k)^2 - M_0(k)}{2} = \binom{M_0(k)}{2}.$$

Consequently, $\varphi(G_t) > \binom{M_0(k)}{2}$. However, since $\varphi(G_t) \leqslant e(G_t) \leqslant \binom{M_0(k)}{2}$, this is impossible. □

The rest of this section is devoted to the proof of Theorem 5.

Proof of Theorem 5. Let $k \geqslant 2$ and suppose that

($*$) for any $(k-1)K_3$-free 4-partite graph $\tilde{G}$ with part sizes $n_1' \geqslant n_2' \geqslant n_3' \geqslant n_4' \geqslant 6(k-1)^2$ and $\sum_{i \in [4]} n_i' \geqslant N_0(k-1)$, we have $e(\tilde{G}) \leqslant g_{k-1}(n_1', n_2', n_3', n_4')$.

Let G be a 4-partite graph of order $|G| > M_0(k)$ and with parts of size $n_1 \geqslant n_2 \geqslant n_3 \geqslant n_4 \geqslant 6k^2$. Assume that G is kK_3-free and $\varphi(G) > 0$. Without loss of generality, we assume that G contains $k-1$ disjoint triangles – otherwise we keep adding edges to G until it contains $k-1$ disjoint triangles (as a result, $\varphi(G)$ increases). Our goal is to show that there exists a crossing set $T \subset V(G)$ of size at most 2 such that $\varphi(G) < \varphi(G \setminus T)$ and $v_4(G \setminus T) \geqslant 6k^2$.

We proceed in the following cases. It is easy to see that these cases cover all possibilities. In each case we verify $v_4(G \setminus T) \geqslant 6k^2$ immediately.

Case 0. $\boldsymbol{n_1 > n_2 + n_3}$**.** We will select a one-element set $T \subset V_1$. Since $n_1 > 2n_4$, we have $n_1 - 1 > n_4$ and thus $v_4(G \setminus T) = n_4 \geqslant 6k^2$.

We assume $n_1 \leqslant n_2 + n_3$ in the remaining cases.

Case 1. $\boldsymbol{n_1 > n_3}$ **and** $\boldsymbol{n_2 > n_4}$**.** We will select a crossing set $T \subset V_1 \cup V_2$. Since $n_1 - 1 \geqslant n_2 - 1 \geqslant n_4$, we have $v_4(G \setminus T) = n_4 \geqslant 6k^2$.

Case 2. $\boldsymbol{n_1 = n_2 = n_3 \geqslant n_4 > 6k^2}$**.** We select a one-element set $T \subset V(G)$. Then $v_4(G \setminus T) \geqslant n_4 - 1 \geqslant 6k^2$.

Case 3. $\boldsymbol{n_1 = n_2 = n_3 > n_4 = 6k^2}$**.** We will select a one-element set $T \subset V_1 \cup V_2 \cup V_3$. Since $n_3 - 1 \geqslant n_4$, we have $v_4(G \setminus T) = n_4 = 6k^2$.

Case 4. $\boldsymbol{n_1 > n_2 = n_3 = n_4}$**.** We will select a one-element set $T \subset V_1$. Since $n_1 > n_4$, $v_4(G \setminus T) = n_4 \geqslant 6k^2$.

It remains to show $\varphi(G) < \varphi(G \setminus T)$ in **Cases 0–4**. This is actually easy in **Case 0**.

Case 0. Recall that $\varphi(G) = e(G) - g_k(n_1, n_2, n_3, n_4) > 0$. Since $n_1 > n_2 + n_3$,

$$g_k(n_1, n_2, n_3, n_4) = n_1(n_2 + n_3 + n_4) + (k-1)(n_2 + n_3).$$

First assume that some vertex $v \in V_1$ satisfies $d(v) < n_2 + n_3 + n_4$. Let $T = \{v\}$. Since $n_1 - 1 \geqslant n_2 + n_3$,

$$\begin{aligned} g_k(n_1 - 1, n_2, n_3, n_4) &= (n_1 - 1)(n_2 + n_3 + n_4) + (k-1)(n_2 + n_3) \\ &= g_k(n_1, n_2, n_3, n_4) - (n_2 + n_3 + n_4). \end{aligned}$$

It follows that

$$\varphi(G \setminus \{v\}) = e(G) - d(v) - g_k(n_1 - 1, n_2, n_3, n_4) > e(G) - g_k(n_1, n_2, n_3, n_4) = \varphi(G),$$

as desired. Otherwise, $G[V_1, V_2 \cup V_3 \cup V_4]$ must be complete. Since G is kK_3-free, it follows that $G[V_2 \cup V_3 \cup V_4]$ contains no matching of size k. The result of [6] or a simple induction on k[1] yields that $e(G[V_2 \cup V_3 \cup V_4]) \leqslant (k-1)(n_2 + n_3)$. This shows that $e(G) \leqslant n_1(n_2 + n_3 + n_4) + (k-1)(n_2 + n_3)$, namely, $\varphi(G) = 0$, a contradiction.

In the rest of the proof we assume $n_1 \leqslant n_2 + n_3$ and will resolve **Cases 1–4**.

One difficulty in these cases is that, after we delete a set $T \subseteq V(G)$, the sizes of the four parts of $G \setminus T$ may not follow the order in G. For instance, suppose $n_1 \leqslant n_2 + n_3$ and

[1] If there is a vertex of degree at least $2k-1$, then we can delete it and apply induction; otherwise, as the size of the maximum matching is $k-1$, there are at most $2(k-1)(2k-1) \leqslant (k-1)(n_2 + n_3)$ edges (using $k \ll n_3 \leqslant n_2$).

$T = \{v\} \subseteq V_1$. If $n_1 > n_2$, then the order of the part sizes of $G \setminus T$ is $n_1 - 1 \geqslant n_2 \geqslant n_3 \geqslant n_4$, the same as in G. However, when $n_1 = n_2 > n_3 \geqslant n_4$, the order of the part sizes of $G \setminus T$ is $n_2 \geqslant n_1 - 1 \geqslant n_3 \geqslant n_4$, and the degree estimates we obtain are quite different. Another complication comes from the fact that there are two possible extremal graphs. Even under the assumption that $n_1 \leqslant n_2 + n_3$, we still have to consider the possibility of $n_1' > n_2' + n_3'$ in $G \setminus T$, where n_1', n_2', n_3', n_4' are the part sizes of $G \setminus T$.

Although a case analysis is inevitable, we study the structure of G in Section 2.1 and use it to simplify the presentation of the proofs of **Cases 1–4** in Section 2.2.

2.1 Preparation

We first give several preliminary results. An edge of G is called *rich* if it is contained in at least k triangles whose third vertices are located in the same part of $V(G)$. We show that every triangle in G must contain a rich edge and G contains at most $6(k-1)^2$ rich edges. Let Z be the set of vertices incident to at least one rich edge. Thus, not only is $G \setminus Z$ triangle-free, but also *every edge in $G \setminus Z$ is not contained in any triangle of G* because such a triangle would not contain any rich edge.

We shall use the following simple fact.

Fact 6. *Let G be a 4-partite graph with parts $V_1, \ldots, V_4$ and suppose $x \in V_1$ and $y \in V_2$. Let $n_i := |V_i|$ for $i \in [4]$. Then x and y have at least $d(x) + d(y) - \sum_{i\in[4]} n_i$ common neighbors in G. In particular, if x and y have no common neighbor, then $d(x) + d(y) = \sum_{i\in[4]} n_i$ implies that $xy \in E(G)$, $V_2 \subseteq N(x)$ and $V_1 \subseteq N(y)$. Moreover, if $d(x) + d(y) \geqslant \sum_{i\in[4]} n_i + 2k - 1$, then xy is rich.*

Proof. Note that $|N(x) \cap (V_3 \cup V_4)| = d(x) - |N(x) \cap V_2| \geqslant d(x) - n_2$ and $|N(y) \cap (V_3 \cup V_4)| = d(y) - |N(y) \cap V_1| \geqslant d(y) - n_1$. Let m denote the number of common neighbors of x and y. Then $m \geqslant |N(x) \cap (V_3 \cup V_4)| + |N(y) \cap (V_3 \cup V_4)| - n_3 - n_4 \geqslant d(x) + d(y) - \sum_{i\in[4]} n_i$. So the first part of the fact follows. In particular, if $m = 0$, then $d(x) + d(y) \leqslant \sum_{i\in[4]} n_i$. Moreover, if the equality holds, then the inequalities in previous calculations must be equalities. In particular, $V_2 \subseteq N(x)$ and $V_1 \subseteq N(y)$, which also imply that $xy \in E(G)$.

For the "moreover" part, note that $d(x) + d(y) \geqslant \sum_{i\in[4]} n_i + 2k - 1$ implies that x and y have at least $2k - 1$ common neighbors and thus at least k common neighbors in one part. Therefore xy is rich. □

Recall that we have assumed that $\varphi(G) > 0$ and $n_1 \leqslant n_2 + n_3$. Thus,

$$e(G) > g_k(n_1, n_2, n_3, n_4) = (n_1 + n_4)(n_2 + n_3) + (k-1)n_1. \tag{2.2}$$

Let R be the subgraph of G induced by the rich edges of G, and let $Z = V(R)$ be the set of the vertices of G that are incident to at least one rich edge.

Claim 7. *Suppose $(*)$, (2.2), and G is kK_3-free. Then the following assertions hold:*

(*i*) *every vertex is contained in at most $k-1$ edges of R whose other ends are located in the same part of G; in particular, the maximum degree of R is at most $3k - 3$;*

(*ii*) $e(R) \leqslant 6(k-1)^2$ *and* $|Z| \leqslant 6(k-1)^2$*;*

(*iii*) *every triangle in* G *contains an edge in* R*.*

Proof. We first show $(i) \Rightarrow (ii)$. Note that if R has a matching of size k, then we can greedily build k vertex-disjoint triangles by extending each rich edge in the matching. This contradicts the assumption that G is kK_3-free. Therefore, the largest matching in R is of size at most $k-1$ and consequently, R has a vertex cover of size at most $2(k-1)$. If the maximum degree of R is at most $3k-3$, then $e(R) \leqslant 2(k-1)(3k-4)+k-1 < 6(k-1)^2$ and $|Z| \leqslant 2(k-1)(3k-4)+2(k-1) = 6(k-1)^2$, confirming (ii).

To see (i), we assume that some vertex v is incident to k rich edges whose other ends are in the same part of G. If there is a copy S of $(k-1)K_3$ in $G \setminus \{v\}$, then we can pick a rich edge in $G \setminus S$ that contains v and then extend this rich edge to a triangle that does not intersect S. This gives a kK_3 in G, a contradiction. Thus, we infer that $G \setminus \{v\}$ is $(k-1)K_3$-free.

Let $n'_1 \geqslant n'_2 \geqslant n'_3 \geqslant n'_4$ be the sizes of four parts of $G \setminus \{v\}$. By $(*)$, we have $e(G \setminus \{v\}) \leqslant g_{k-1}(n'_1, n'_2, n'_3, n'_4)$. To estimate $g_{k-1}(n'_1, n'_2, n'_3, n'_4)$, we first observe that there exists $i_0 \in [4]$ such that $n'_i = n_i$ for all $i \neq i_0$ and $n_{i_0} = n_{i_0} - 1$; and furthermore, $n'_i = |V_i \setminus \{v\}|$ for $i \in [4]$ after relabeling V_1, V_2, V_3, V_4 if necessary (but maintaining $n_i = |V_i|$). This is obvious when $v \in V_{i_0}$ and $n_{i_0} > n_{i_0+1}$. Otherwise, for example, assume that $v \in V_1$ and $n_1 = n_2 > n_3$ (other cases are similar). Then $n'_1 = n_2 = n_1$ and $n'_2 = n_1 - 1 = n_2 - 1$. After relabeling V_1 and V_2, we have $v \in V_2$, and $n'_i = |V_i \setminus \{v\}|$ for $i \in [4]$.

By the definition of g, we consider two cases. When $n'_1 \leqslant n'_2 + n'_3$, we have

$$\begin{aligned} g_{k-1}(n'_1, n'_2, n'_3, n'_4) &= (n'_1 + n'_4)(n'_2 + n'_3) + (k-2)n'_1 \\ &\leqslant \begin{cases} (n_1 + n_4 - 1)(n_2 + n_3) + (k-2)n_1 & \text{if } v \in V_1 \cup V_4, \\ (n_1 + n_4)(n_2 + n_3 - 1) + (k-2)n_1, & \text{if } v \in V_2 \cup V_3. \end{cases} \end{aligned} \tag{2.3}$$

Together with (2.2) and $(*)$, this implies that

$$\begin{aligned} d_G(v) = e(G) - e(G \setminus \{v\}) &> g_k(n_1, n_2, n_3, n_4) - g_{k-1}(n'_1, n'_2, n'_3, n'_4) \\ &\geqslant \begin{cases} n_1 + n_2 + n_3 & \text{if } v \in V_1 \cup V_4, \\ 2n_1 + n_4, & \text{if } v \in V_2 \cup V_3, \end{cases} \end{aligned}$$

which is impossible. When $n'_1 > n'_2 + n'_3$, it must be the case when $n_1 = n_2 + n_3$ and $n'_{i_0} = n_{i_0} - 1$ for $i_0 \in \{2, 3\}$. Thus

$$\begin{aligned} g_{k-1}(n'_1, n'_2, n'_3, n'_4) &= n'_1(n'_2 + n'_3 + n'_4) + (k-2)(n'_2 + n'_3) \\ &= (n_2 + n_3)(n_1 + n_4 - 1) + (k-2)(n_1 - 1). \end{aligned}$$

Together with (2.2) and $(*)$, this implies that $d_G(v) > n_1 + n_2 + n_3$, which is impossible for any $v \in V(G)$.

To see (iii), let S be a triangle in G and consider $G \setminus S$. Since G is kK_3-free, $G \setminus S$ is $(k-1)K_3$-free. By $(*)$, we have $e(G \setminus S) \leqslant g_{k-1}(n'_1, n'_2, n'_3, n'_4)$ where $n'_1 \geqslant n'_2 \geqslant n'_3 \geqslant n'_4$

are the sizes of parts of $G \setminus S$. We observe that there exists $i_0 \in [4]$ such that $n'_i = n_i - 1$ for $i \neq i_0$ and $n'_{i_0} = n_{i_0}$; furthermore, $n'_i = |V_i \setminus S|$ after relabeling V_1, V_2, V_3, V_4 if necessary (while maintaining $n_i = |V_i|$). This is obvious when $S \subset \bigcup_{i \neq i_0} V_i$ and either $i_0 = 1$ or $n_{i_0-1} > n_{i_0}$. Otherwise, for example, assume that $S \subset V_1 \cup V_2 \cup V_3$ and $n_2 > n_3 = n_4$ (other cases are similar). We have $n'_1 = n_1 - 1$, $n'_2 = n_2 - 1$, $n'_3 = n_4 = n_3$ and $n'_4 = n_3 - 1 = n_4 - 1$. After swapping V_3 and V_4, we have $S \subset V_1 \cup V_2 \cup V_4$.

If $n'_1 \leqslant n'_2 + n'_3$. then $g_{k-1}(n'_1, n'_2, n'_3, n'_4) = (n'_1 + n'_4)(n'_2 + n'_3) + (k-2)n'_1$. By our observation on the values of n'_1, n'_2, n'_3, n'_4, it follows that

$$g_{k-1}(n'_1, n'_2, n'_3, n'_4) \leqslant \max_{j=1,2}\{(n_1 + n_4 - j)(n_2 + n_3 - (3-j))\} + (k-2)n_1.$$

If $n'_1 > n'_2 + n'_3$, then $g_{k-1}(n'_1, n'_2, n'_3, n'_4) = n'_1(n'_2 + n'_3 + n'_4) + (k-2)(n'_2 + n'_3)$. In this case, we must have $n_1 = n_2 + n_3 - t$ for $t = 0, 1$, $n'_2 = n_2 - 1$, and $n'_3 = n_3 - 1$. Thus $n'_i = n_i - 1$ either for $i \in [3]$ or for $i \in \{2, 3, 4\}$, and consequently

$$\begin{aligned} g_{k-1}(n'_1, n'_2, n'_3, n'_4) \leqslant \max\{&(n_1 - 1)(n_2 + n_3 + n_4 - 2) + (k-2)(n_2 + n_3 - 2), \\ &n_1(n_2 + n_3 + n_4 - 3) + (k-2)(n_2 + n_3 - 2). \end{aligned}$$

Since $n_1 = n_2 + n_3 - t$ for $t = 0, 1$, it follows that

$$g_{k-1}(n'_1, n'_2, n'_3, n'_4) \leqslant \max_{j=1,2,3}\{(n_2 + n_3 - (3-j))(n_1 + n_4 - j)\} + (k-2)(n_1 - 1).$$

Putting all cases together with $e(G \setminus S) \leqslant g_{k-1}(n'_1, n'_2, n'_3, n'_4)$, we conclude that

$$e(G \setminus S) \leqslant \max_{j=1,2,3}\{(n_1 + n_4 - j)(n_2 + n_3 - (3-j))\} + (k-2)n_1. \tag{2.4}$$

Recall that $e(S; G) := e(G) - e(G \setminus S)$. We next claim that $e(S; G) \geqslant \frac{3}{2}\sum_{i \in [4]} n_i + 3k$. Indeed, if the maximum in (2.4) is achieved by $j = 1, 2$, then, together with (2.2), it gives

$$e(S; G) > \sum_{i \in [4]} n_i + \min\{n_1 + n_4, n_2 + n_3\} + n_1 - 2 \geqslant \frac{3}{2}\sum_{i \in [4]} n_i + n_4 - 2 \geqslant \frac{3}{2}\sum_{i \in [4]} n_i + 3k,$$

where we used $n_4 \geqslant 6k^2$ in the last inequality. Otherwise, the maximum in (2.4) is achieved by $j = 3$, that is, $e(G \setminus S) \leqslant (n_1 + n_4 - 3)(n_2 + n_3) + (k-2)n_1$. By (2.2), we get

$$\begin{aligned} e(S; G) &> (n_1 + n_4)(n_2 + n_3) + (k-1)n_1 - (n_1 + n_4 - 3)(n_2 + n_3) - (k-2)n_1 \\ &= n_1 + 3n_2 + 3n_3 \geqslant \frac{3}{2}\sum_{i \in [4]} n_i + \frac{n_4}{2} \geqslant \frac{3}{2}\sum_{i \in [4]} n_i + 3k, \end{aligned}$$

where we used the assumption $n_2 + n_3 \geqslant n_1$ and $n_2, n_3 \geqslant n_4$.

Let $S = xyz$ and note that $d(x) + d(y) + d(z) = e(S; G) + 3$. By averaging, without loss of generality, we may assume that

$$d(x) + d(y) \geqslant \frac{2}{3}\left(\frac{3}{2}\sum_{i \in [4]} n_i + 3k\right) = \sum_{i \in [4]} n_i + 2k.$$

By the moreover part of Fact 6, xy is rich and we are done. □

For two disjoint sets $A, B \subseteq V(G)$, let $d(A,B) = e(A,B)/(|A||B|)$ be the density of the bipartite graph with parts A and B. A pair (V_i, V_j) is called *full* if $d(V_i \setminus Z, V_j) = d(V_j \setminus Z, V_i) = 1$; (V_i, V_j) is called *empty* if $e(V_i \setminus Z, V_j) = e(V_i, V_j \setminus Z) = 0$. We have the following observation.

Observation 8. *For distinct $i, j, t \in [4]$, if $d(V_i \setminus Z, V_j) = d(V_i \setminus Z, V_t) = 1$, then (V_j, V_t) must be empty because any edge in (V_j, V_t) but not in $(V_j \cap Z, V_t \cap Z)$ will create a triangle with at most one vertex in Z, contradicting (iii). In particular, if both (V_i, V_j) and (V_i, V_t) are full, then (V_j, V_t) is empty.*

Claim 9. *Fix $i \neq j \in [4]$. If $d(x) + d(y) \geqslant \sum_{i \in [4]} n_i$ for every edge $xy \in V_i \times V_j$, then either*

- *$e(V_i \setminus Z, V_j \setminus Z) = 0$ (this is weaker than (V_i, V_j) being empty) or*
- *$d(V_i \setminus Z, V_j) = d(V_j \setminus Z, V_i) = 1$, and $d(x) + d(y) = \sum_{i \in [4]} n_i$.*

Moreover, if $d(x) + d(y) > \sum_{i \in [4]} n_i$ for every edge $xy \in V_i \times V_j$, then (V_i, V_j) is empty.

Proof. Assume that $\{i, j, t, \ell\} = [4]$. Suppose there is an edge $xy \in (V_i \setminus Z) \times (V_j \setminus Z)$. Note that if x and y have a common neighbor z, then as $x, y \notin Z$, none of the edges of xyz is rich, contradicting (iii). Thus, x and y have no common neighbor. By Fact 6, $d(x) + d(y) \leqslant \sum_{i \in [4]} n_i$. If $d(x) + d(y) \geqslant \sum_{i \in [4]} n_i$, then Fact 6 implies that $V_j \subseteq N(x)$ and $V_i \subseteq N(y)$. In particular, $xy' \in E(G)$ for every $y' \in V_j \setminus Z$. Applying the same argument to the edge xy', we obtain that $V_i \subseteq N(y')$. Similarly, we can derive that $V_j \subseteq N(x')$ for every $x' \in V_i \setminus Z$. Thus, $d(V_i \setminus Z, V_j) = d(V_j \setminus Z, V_i) = 1$.

Now assume $d(x) + d(y) > \sum_{i \in [4]} n_i$ for every edge $xy \in V_i \times V_j$. If $e(V_i \setminus Z, V_j \setminus Z) \neq 0$, then the arguments in the previous paragraph provide a contradiction. Suppose there is an edge $xy \in (V_i \cap Z) \times (V_j \setminus Z)$. As $d(x) + d(y) > \sum_{i \in [4]} n_i$, x and y have some common neighbors in $V_t \cup V_\ell$. But since $y \notin Z$, by (iii), their common neighbors must be in $(V_t \cup V_\ell) \cap Z$. Since $e(V_i \setminus Z, V_j \setminus Z) = 0$, we know that $N(y) \cap V_i \subseteq V_i \cap Z$. Altogether, we obtain that $d(x) + d(y) \leqslant n_j + n_t + n_\ell + |Z| < \sum_{i \in [4]} n_i$, a contradiction. Analogous arguments show that there is no edge in $(V_i \setminus Z) \times (V_j \cap Z)$. Thus, $e(V_i \setminus Z, V_j) = e(V_i, V_j \setminus Z) = 0$, that is, (V_i, V_j) is empty. □

Consider a set $T \subseteq V(G)$ defined in **Cases 1–4** and let n_1', n_2', n_3', n_4' denote the sizes of the parts of $G \setminus T$. Then $\varphi(G) < \varphi(G \setminus T)$ is equivalent to

$$e(G) - g_k(n_1, n_2, n_3, n_4) < e(G \setminus T) - g_k(n_1', n_2', n_3', n_4'),$$

or $e(T; G) < g_k(n_1, n_2, n_3, n_4) - g_k(n_1', n_2', n_3', n_4')$. We will prove by contradiction, assuming that $\varphi(G) \geqslant \varphi(G \setminus T)$, equivalently,

$$e(T; G) \geqslant (n_1 + n_4)(n_2 + n_3) + (k-1)n_1 - g_k(n_1', n_2', n_3', n_4') \tag{2.5}$$

for every $T \subseteq V(G)$ defined in **Cases 1–4**.

The case when $T = \{v\} \subseteq V_1$ occurs in all four cases so we consider it before the cases. Since $n_1 \leqslant n_2 + n_3$, we have three possibilities:

- if $n_1 > n_2$, then $g_k(n_1-1, n_2, n_3, n_4) = (n_1 - 1 + n_4)(n_2 + n_3) + (k-1)(n_1 - 1)$;
- if $n_1 = n_2 > n_4$, then $g_k(n_1-1, n_2, n_3, n_4) = (n_1 + n_4)(n_2 + n_3 - 1) + (k-1)n_1$;
- if $n_1 = n_4$, then $g_k(n_1-1, n_2, n_3, n_4) = (n_1 + n_4 - 1)(n_2 + n_3) + (k-1)n_1$;

Thus (2.5) implies that for every $v \in V_1$,

$$d(v) \geqslant \begin{cases} n_2 + n_3 + k - 1, & \text{if } n_1 > n_2, \\ n_1 + n_4, & \text{if } n_1 = n_2. \end{cases} \tag{2.6}$$

2.2 Proof of Cases 1–4

After these preparations, we return to the proof of **Cases 1–4**. Recall that $n_1 \leqslant n_2 + n_3$ in all these cases. Recall also that $n_i \geqslant 6k^2$ for $i \in [4]$, so we can always assume that $V_i \setminus Z \neq \varnothing$. Moreover, by (2.1), we have $M_0(k) \geqslant N_0(k-1) + 3$, and thus we can apply the induction hypothesis $(*)$ on any $(k-1)K_3$-*free* subgraph $G \setminus S$, whenever $|S| \leqslant 3$ (and thus $v_4(G \setminus S) \geqslant 6k^2 - 3 \geqslant 6(k-1)^2$).

Case 1. $n_1 > n_3$ and $n_2 > n_4$.

In this case (2.5) holds for every crossing set $T = xy \in V_1 \times V_2$. Since the part sizes of $G \setminus \{x, y\}$ are $n_1 - 1 \geqslant \{n_2 - 1, n_3\} \geqslant n_4$. By (2.5), we have

$$\begin{aligned} e(xy; G) &\geqslant (n_1 + n_4)(n_2 + n_3) + (k-1)n_1 - ((n_1 + n_4 - 1)(n_2 + n_3 - 1) + (k-1)(n_1 - 1)) \\ &= \sum_{i \in [4]} n_i + k - 2. \end{aligned}$$

If $xy \in E(G)$, then $d(x) + d(y) = e(xy; G) + 1 \geqslant \sum_{i \in [4]} n_i + k - 1 > \sum_{i \in [4]} n_i$. By Claim 9, (V_1, V_2) is empty. For every $x \in V_1 \setminus Z$, we thus have $d(x) \leqslant n_3 + n_4 < \min\{n_2 + n_3, n_1 + n_4\}$, contradicting (2.6).

Case 2. $n_1 = n_2 = n_3 \geqslant n_4 > 6k^2$.

In this case (2.5) holds for any one-element set $T \subset V(G)$. Write $n_1 = n_2 = n_3 = n$. For any $x \in V_1 \cup V_2 \cup V_3$, by (2.5), we have

$$d(x) = e(\{x\}; G) \geqslant 2n(n + n_4) + (k-1)n - g_k(n, n, n-1, n_4),$$

where $g_k(n, n, n-1, n_4) = (2n-1)(n + n_4) + (k-1)n$ if $n > n_4$ and $g_k(n, n, n-1, n_4) = 2n(n + n_4 - 1) + (k-1)n$ if $n = n_4$. Thus, we have $d(x) \geqslant \min\{2n, n + n_4\} = n + n_4$. Similarly, for $y \in V_4$, by (2.5), we have

$$d(y) = e(\{y\}; G) \geqslant 2n(n + n_4) + (k-1)n - \big(2n(n + n_4 - 1) + (k-1)n\big) = 2n. \tag{2.7}$$

These together imply $d(x) + d(y) \geqslant \sum n_i$ for every edge $xy \in (V_1 \cup V_2 \cup V_3) \times V_4$. For $i = 1, 2, 3$, Claim 9 implies that either (V_i, V_4) is full or $e(V_i \setminus Z, V_4 \setminus Z) = 0$. If $e(V_i \setminus Z, V_4 \setminus Z) = 0$ holds for at least two values of $i \in \{1, 2, 3\}$, then for every $y \in V_4 \setminus Z$, we have $d(y) \leqslant n + |Z| < 2n$ (as $n \geqslant M_0(k)/4 > 6k^2$), contradicting (2.7).

This implies that at least two of (V_1, V_4), (V_2, V_4), and (V_3, V_4) must be full. Without loss of generality, assume (V_1, V_4) and (V_2, V_4) are full. By Observation 8, (V_1, V_2) is empty. Next, we claim that (V_3, V_4) is empty. Indeed, let $x \in V_2 \setminus Z$ and recall that $d(x) \geqslant n+n_4$. Since (V_1, V_2) is empty, we have $d(x) \leqslant n + n_4$. Thus, $d(x) = n + n_4$ and in particular $V_3 \subseteq N(x)$. Since this holds for every $x \in V_2 \setminus Z$, it follows that $d(V_2 \setminus Z, V_3) = 1$. Thus (V_3, V_4) is empty by Observation 8. Together with (ii), we infer

$$e(G) = e(G[Z])+e(V\setminus Z; G) < \binom{|Z|}{2}+(n_1+n_2)(n_3+n_4) \leqslant (n_1+n_2)(n_3+n_4)+(k-1)n_1,$$

contradicting (2.2), The previous inequality follows from $\binom{|Z|}{2} \leqslant 18(k-1)^4 \leqslant (k-1)n_1$, which follows from $n_1 \geqslant M_0(k)/4$ and (2.1).

Case 3. $\boldsymbol{n_1 = n_2 = n_3 > n_4 = 6k^2}$**.**

Write $n_1 = n_2 = n_3 = n$. We assume that

$$n_1 \geqslant 30k^2, \tag{2.8}$$

as otherwise $\sum n_i \leqslant 3 \cdot 30k^2 + 6k^2 \leqslant M_0(k)$ by (2.1), contradicting the assumption $|G| > M_0(k)$. By (2.6) and the similarity of V_1, V_2, and V_3, we have $d(x) \geqslant n + n_4$ for every $x \in V_1 \cup V_2 \cup V_3$. We claim that for $y \in V_4$,

$$d(y) \leqslant 2n + 2k - 1. \tag{2.9}$$

Otherwise, pick k neighbors $x_1, \ldots, x_k$ of y from the same part of G. For each i, since $d(x_i) \geqslant n + n_4$, we have $d(x_i) + d(y) \geqslant \sum n_i + 2k - 1$, yielding that $x_i y$ is rich by Fact 6. However, this contradicts (i).

Claim. The graph $G[V_1 \cup V_2 \cup V_3]$ is K_3-free.

Proof. Suppose instead, there exists a triangle $xyz \in V_1 \times V_2 \times V_3$. Without loss of generality, assume that $d(x) \geqslant d(y) \geqslant d(z)$. We first claim that

$$d(x) + d(y) + d(z) \geqslant 5n + 2n_4 + k. \tag{2.10}$$

Otherwise $d(x)+d(y)+d(z) \leqslant 5n+2n_4+k-1$ and $e(xyz; G) = d(x)+d(y)+d(z)-3 \leqslant 5n + 2n_4 + k - 4$. Then, by (2.2),

$$\begin{aligned} e(G \setminus \{x, y, z\}) &= e(G) - e(xyz; G) > g_k(n,\, n, n, n_4) - (5n + 2n_4 + k - 4) \\ &= 2n(n + n_4) + (k - 1)n - (5n + 2n_4 + k - 4) \\ &= (2n - 2)(n - 1 + n_4) + (k - 2)(n - 1) \\ &= g_{k-1}(n - 1, n - 1, n - 1, n_4). \end{aligned}$$

By induction hypothesis $(*)$, we obtain a copy of $(k-1)K_3$ in $G \setminus \{x, y, z\}$. Together with the triangle xyz, this contradicts the assumption G is kK_3-free.

We next claim that at least two of xy, yz, xz are rich and thus all $x, y, z \in Z$. Indeed, if $d(x) < 2n + n_4 - k$, then by (2.10),

$$d(y) + d(z) > 5n + 2n_4 + k - (2n + n_4 - k) = 3n + n_4 + 2k > \sum n_i + 2k - 1.$$

By Fact 6, yz is rich. Since $d(x)$ is the largest, this argument implies that all three edges of xyz are rich, as desired. Otherwise, $d(x) \geqslant 2n+n_4-k$ and recall that $d(y) \geqslant d(z) \geqslant n+n_4$. Thus

$$d(x) + d(y) \geqslant d(x) + d(z) \geqslant 3n + 2n_4 - k \geqslant \sum n_i + 2k - 1$$

because $n_4 = 6k^2 \geqslant 3k - 1$. By Fact 6, both xy and xz are rich, as desired.

The claim in the previous paragraph applies to all triangles in $V_1 \cup V_2 \cup V_3$. Therefore, all the common neighbors of x and y in $V_1 \cup V_2 \cup V_3$ are in Z and consequently, $|N(x) \cap N(y)| \leqslant |Z| + |V_4| \leqslant 6k^2 + n_4$, and consequently, $d(x) + d(y) \leqslant \sum n_i + 6k^2 + n_4 = 3n + 2n_4 + 6k^2$. On the other hand, (2.10) and the assumption $d(x) \geqslant d(y) \geqslant d(z)$ imply that

$$d(x) + d(y) \geqslant \frac{2}{3}(5n + 2n_4 + k) = \frac{10}{3}n + \frac{4}{3}n_4 + \frac{2}{3}k > 3n + 2n_4 + 6k^2 \tag{2.11}$$

because $n \geqslant 30k^2 = 2n_4 + 18k^2$ by (2.8). This gives a contradiction. □

By the claim, $G[V_1 \cup V_2 \cup V_3]$ is K_3-free, and thus has at most $2n^2$ edges by Theorem 1. Together with (2.9) and (2.8), we obtain that

$$e(G) \leqslant 2n^2 + n_4 \cdot (2n + 2k - 1) = 2n(n + n_4) + (2k - 1)n_4 < 2n(n + n_4) + (k - 1)n,$$

contradicting (2.2).

Case 4. $\boldsymbol{n_1 > n_2 = n_3 = n_4}$**.**

Assume $n_2 = n_3 = n_4 = n$ and recall that $n_1 \leqslant 2n$. We first claim that

$$d(x) \leqslant 3n \text{ for all } x \in V_1, \text{ and } d(y) \leqslant n_1 + n + k - 1 \text{ for all } y \in V_2 \cup V_3 \cup V_4. \tag{2.12}$$

Indeed, the bound $d(x) \leqslant 3n$ for $x \in V_1$ is trivial. Suppose to the contrary, that there is a vertex $y \in V_2 \cup V_3 \cup V_4$ with $d(y) \geqslant n_1 + n + k$. It follows that $|N(y) \cap V_1| \geqslant d(y) - 2n \geqslant k$. Assume that $x_1, \ldots, x_k \in N(y) \cap V_1$. By (2.6), we have $d(x_j) \geqslant 2n + k - 1$. Thus, we infer that $d(x_j) + d(y) \geqslant n_1 + 3n + 2k - 1$. By Fact 6, we have $x_1y, \ldots, x_ky \in E(R)$. However, this contradicts (i).

We next claim that there is no rich edge in $V_1 \times (V_2 \cup V_3 \cup V_4)$. Suppose to the contrary, that $xy \in V_1 \times (V_2 \cup V_3 \cup V_4)$ is a rich edge. By (2.12), we have $e(xy; G) = d(x) + d(y) - 1 \leqslant n_1 + 4n + k - 2$. By (2.2), it follows that

$$\begin{aligned} e(G \setminus \{x, y\}) &= e(G) - e(xy; G) > 2n(n_1 + n) + (k - 1)n_1 - (n_1 + 4n + k - 2) \\ &= 2n(n_1 + n - 2) + (k - 2)(n_1 - 1) \\ &= g_{k-1}(n_1 - 1, n, n, n - 1). \end{aligned}$$

By induction hypothesis $(*)$, $G \setminus \{x, y\}$ contains a copy S of $(k-1)K_3$. Since xy is rich, we can find a triangle in $G \setminus S$ containing xy, contradicting the assumption that G is kK_3-free.

Now we show that there is no triangle intersecting V_1. Suppose to the contrary, there is a triangle xyz with $x \in V_1$. If $d(x) + d(z) \geqslant n_1 + 3n + 2k - 1$, then, by Fact 6, xy is rich, contradicting our earlier claim. We thus assume that $d(x)+d(z) < n_1+3n+2k-1$. Together with (2.12), it gives that $d(x)+d(y)+d(z) < 2n_1+4n+3k-2$, and $e(xyz; G) = d(x) + d(y) + d(z) - 3 < 2n_1 + 4n + 3k - 5$. By (2.2), it follows that

$$\begin{aligned} e(G \setminus \{x, y, z\}) &= e(G) - e(xyz; G) > 2n(n_1 + n) + (k-1)n_1 - (2n_1 + 4n + 3k - 5) \\ &= (n_1 + n - 2)(2n - 1) + (k-2)(n_1 - 1) + n - 2k + 1 \\ &= g_{k-1}(n_1 - 1, n, n-1, n-1) + n - 2k + 1. \end{aligned}$$

By $(*)$, $G \setminus \{x, y, z\}$ contains a copy of $(k-1)K_3$. Together with the triangle xyz, this contradicts the assumption that G is kK_3-free.

We assumed that G contains $k-1$ disjoint triangles. Let T_1 be a triangle of G. By the claim of the previous paragraph, T_1 must be in $V_2 \cup V_3 \cup V_4$. Moreover, by (iii), T_1 must contain a rich edge xy. Below we show that

$$e(G \setminus \{x, y\}) > g_{k-1}(n_1, n, n-1, n-1). \tag{2.13}$$

Then, by $(*)$, $G \setminus \{x, y\}$ contains a copy S of $(k-1)K_3$. Since xy is rich, we can find a triangle in $G \setminus S$ containing xy, contradicting the assumption that G is kK_3-free.

We first assume that $n_1 = 2n$. If $d(x) + d(y) > 6n$, then x and y have a common neighbor in V_1, contradicting the earlier claim that there is no triangle intersecting V_1. We thus assume that $d(x) + d(y) \leqslant 6n$. Thus $e(xy; G) \leqslant 6n - 1$. By (2.2), it follows that

$$\begin{aligned} e(G \setminus \{x, y\}) &> g_k(2n, n, n, n) - (6n - 1) \\ &= 3n \cdot 2n + 2n(k-1) - (6n-1) \\ &= 2n(3n-2) + (k-2)(2n-1) + k - 1 \\ &= g_{k-1}(2n, n, n-1, n-1) + k - 1. \end{aligned}$$

Thus (2.13) holds. Second, assume $n_1 < 2n$. By (2.12), we have $e(xy; G) = d(x) + d(y) - 1 \leqslant 2(n_1 + n + k - 1) - 1$. By (2.2), it follows that

$$\begin{aligned} e(G \setminus \{x, y\}) &> g_k(n_1, n, n, n) - (2n_1 + 2n + 2k - 3) \\ &= (n_1 + n)2n + (k-1)n_1 - (2n_1 + 2n + 2k - 3) \\ &= (n_1 + n - 1)(2n - 1) + (k-2)n_1 + n - 2k + 2 \\ &= g_{k-1}(n_1, n, n-1, n-1) + n - 2k + 2. \end{aligned}$$

Thus (2.13) holds.

The proof of Theorem 5 is now completed. □

3 Concluding remarks

In this paper we solved Problem 2 for $r = 4$ and $t = 3$ when all n_i's are large. The idea in our proof should be helpful for proving Conjecture 4 in general. However, to determine the maximum in (1.1), there are quite a few cases to consider even when $r = 5$ and $t = 3$. Indeed, suppose $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_5$ and $\{I, I'\}$ is the bipartition of [5] that attained the maximum in (1.1). Assume $1 \in I$. Depending on the values of $n_1, \ldots, n_5$, it is possible to have

$$I = \{1\} \text{ or } \{1,2\} \text{ or } \{1,3\} \text{ or } \{1,4\} \text{ or } \{1,5\} \text{ or } \{1,4,5\}.$$

Another open problem is to find the smallest $N_0(k)$ such that Theorem 3 holds. The $N_0(k)$ provided in our proof is a doubly exponential function of k. Indeed, by (2.1) and $N_0(1) = 1$, we have $M_0(2) = 96 \cdot 2^2 = 384$ and $N_0(2) = 384^2$. It is easy to see that $N_0(k) = (N_0(k-1)+3)^2$ for $k \geqslant 3$. Thus $N_0(k-1)^2 \leqslant N_0(k) \leqslant 2N_0(k-1)^2$ for $k \geqslant 3$. It follows that

$$N_0(2)^{2^{k-2}} \leqslant N_0(k) \leqslant \big(2N_0(2)\big)^{2^{k-2}}.$$

It is interesting to know whether one can reduce $N_0(k)$ to a polynomial function (or even a linear function) of k.

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References

[1] P. Bennett, S. English, and M. Talanda-Fisher. Weighted Turán problems with applications. *Discrete Math.*, 342(8):2165–2172, 2019.

[2] B. Bollobás. *Extremal graph theory*, volume 11 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1978.

[3] B. Bollobás, P. Erdős, and E. G. Straus. Complete subgraphs of chromatic graphs and hypergraphs. *Utilitas Math.*, 6:343–347, 1974.

[4] H. Chen, X. Li, and J. Tu. Complete solution for the rainbow numbers of matchings. *Discrete Math.*, 309(10):3370 – 3380, 2009.

[5] J. De Silva, K. Heysse, A. Kapilow, A. Schenfisch, and M. Young. Turán numbers of vertex-disjoint cliques in r-partite graphs. *Discrete Math.*, 341(2):492–496, 2018.

[6] J. De Silva, K. Heysse, and M. Young. Rainbow number for matchings in r-partite graphs, preprint.

[7] W. Mantel. Problem 28. *Wiskundige Opgaven*, 10:60–61, 1907.

[8] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 279–319. Academic Press, New York, 1968.

[9] P. Turán. On an extremal problem in graph theory, (Hungarian). *Mat. Fiz. Lapok*, 48:436–452, 1941.

[10] A. Z. Wagner. Refuting conjectures in extremal combinatorics via linear programming. *J. Combin. Theory Ser. A*, 169:105130, 2020.

[11] K. Zarankiewicz. Problem p 101. *Colloq. Math.*, 3:19–30, 1954.