Simplicial Dollar Game

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Abstract

The dollar game is a chip-firing game introduced by Baker as a context in which to formulate and prove the Riemann-Roch theorem for graphs. A divisor on a graph is a formal integer sum of vertices. Each determines a dollar game, the goal of which is to transform the given divisor into one that is effective (nonnegative) using chip-firing moves. We use Duval, Klivans, and Martin's theory of chip-firing on simplicial complexes to generalize the dollar game and results related to the Riemann-Roch theorem for graphs to higher dimensions. In particular, we extend the notion of the degree of a divisor on a graph to a (multi)degree of a chain on a simplicial complex and use it to establish two main results. The first of these generalizes the fact that if a divisor on a graph has large enough degree (at least as large as the genus of the graph), it is winnable; and the second generalizes the fact that trees (graphs of genus 0) are exactly the graphs on which every divisor of degree 0, interpreted as an instance of the dollar game, is winnable.

Mathematics Subject Classifications: 05E45

1 Introduction

Let $G = (V, E)$ be a finite, connected, undirected graph with vertex set V and edge set E. To play the dollar game on G, assign an integer number of dollars to each vertex. Negative integers are interpreted as debt. A *lending move* consists of a vertex giving one of its dollars to each of its neighboring vertices, and a *borrowing move* is the opposite, in which a vertex takes a dollar from each neighbor. Vertices may lend or borrow, regardless of the number of dollars they possess. The goal of the game is to bring all vertices out of debt through a sequence of such moves.

The dollar game was introduced in *Riemann-Roch and Abel-Jacobi theory on a finite graph*, by Baker and Norine ([2]) as a variant of an earlier version due to Biggs ([4]). Baker and Norine's work develops the *divisor theory of graphs*, which views a graph as a discrete version of an algebraic curve or Riemann surface. The assignment of a_v dollars to each vertex v is formally a *divisor* $D = \sum_{v\in V} a_v v$ in the free abelian group $\mathrm{Div}(G) := \mathbb{Z}V$. The net amount of money on the graph is $\deg(D) := \sum_{v\in V} a_v$, the *degree* of D. Divisors D and D' are *linearly equivalent*, denoted $D \sim D'$, if one may be obtained from the other via lending and borrowing moves. The group of divisors modulo linear equivalence is the *Picard group* $\mathrm{Pic}(G)$. Since lending and borrowing moves conserve net wealth, $\mathrm{Pic}(G)$ is graded by degree. Its degree zero component is the *Jacobian group* $\mathrm{Jac}(G)$, which is a finite group with size equal to the number of spanning trees of G. A choice of a vertex v gives an isomorphism

$$\begin{aligned} \mathrm{Pic}(G) &\xrightarrow{\sim} \mathrm{Jac}(G) \oplus \mathbb{Z} \\ [D] &\mapsto ([D - \deg(D)v], \deg(D)). \end{aligned} \tag{1}$$

A divisor is *effective* if its coefficients are nonnegative. Thus, in the language of algebraic geometry, an instance of the dollar game is a divisor $D \in \mathrm{Div}(G)$, and the game is won by finding a linearly equivalent effective divisor.

A fundamental concept introduced in [2] is the notion of the *rank* of a divisor. If there is no effective divisor linearly equivalent to D, then the rank of D is $r(D) = -1$. Otherwise, the rank is the maximum integer k such that $D - E$ is linearly equivalent to an effective divisor for all effective divisors E of degree k. In terms of the dollar game, the rank is a measure of robustness of winnability: the dollar game D is winnable if and only if $r(D) \geqslant 0$, and if $r(D) = k > 0$, it is winnable even after removing k dollars arbitrarily.

The Riemann-Roch theorem for graphs ([2, Theorem 1.12]) has a form nearly identical to that for algebraic curves. It says that for all $D \in \mathrm{Div}(G)$,

$$r(D) - r(K - D) = \deg(D) + 1 - g.$$

Here, $g = |E| - |V| + 1$ and $K = \sum_{v\in V} (\deg_G(v)v - 2)\, v$ where $\deg_G(v)$ is the number of edges incident on v. These play the role of the genus and the canonical divisor, respectively, for an algebraic curve.

Since the rank is at least -1,

$$r(D) = \deg(D) + 1 - g + r(K - D) \geqslant \deg(D) - g.$$

A consequence is that if $\deg(D) \geqslant g$, then the dollar game D is winnable. This result is sharp, too: there are always unwinnable divisors of degree $g - 1$ ([2, Theorem 1.9]). It follows that all divisors of degree 0 are winnable if and only if $g = 0$, i.e., G is a tree. In summary, the dollar game has a minimal "winning degree" g, and that minimal degree is 0 exactly when the game is played on a tree. Our main goal is to generalize these results to a dollar game played on a simplicial complex of any dimension.

Lending moves are sometimes called *vertex-firings* or *chip-firings* (and borrowing moves are *reverse firings*). They arise naturally as an encoding of the discrete Laplacian operator for the graph. Duval, Klivans, and Martin ([8], [9], [10]) use a version of a combinatorial Laplacian to generalize the divisor theory of graphs to higher-dimensional simplicial (and cellular) complexes. In this theory, an i-chain—a formal integer sum of i-dimensional faces—of a complex Δ may be thought of as an assignment of an integer "flow" to each i-face. Firing an i-face f then diverts flow around the $(i+1)$-faces incident on f. The group of i-cycles modulo these firing moves is the *i-th critical group* of the complex, $\mathcal{K}_i(\Delta)$, generalizing the Jacobian group of a graph. By [8, Corollary 4.2], under certain restrictions on Δ, the size of the torsion part of $\mathcal{K}_i(\Delta)$ is the number of torsion-weighted $(i+1)$-dimensional spanning trees of Δ.

In this paper, we interpret Duval, Klivans, and Martin's theory as a higher-dimensional dollar game. A chain on a simplicial complex is thought of as a distribution of wealth among the faces. The goal of the game is to use face-firings to redistribute wealth, leaving no face in debt. For this purpose, the naive version of degree as the net wealth of the system is not appropriate: using that notion of degree, there would be simplicial complexes with chains of arbitrarily negative degree that are winnable and arbitrarily positive degree that are unwinnable. The root of the problem is that, unlike for graphs, lending and borrowing moves on simplicial complexes are not necessarily conservative. Instead, in Definition 4 we introduce a natural generalization of the degree of a divisor on a graph to one that is invariant under firing moves on the chains of a complex. Our main results generalize the properties of divisors on graphs discussed in connection with the Riemann-Roch theorem, above: Theorem 18 shows that if the degree of a chain is sufficiently large, then it is winnable, and Corollary 34 shows that for each i, all $(i-1)$-chains of degree 0 are winnable if and only if the i-skeleton of the complex is a spanning forest, torsion-free in codimension one.

Section 2 sets notation and presents required background on (abstract) simplicial complexes and polyhedral cones. In particular, Δ always denotes a d-dimensional simplicial complex. In Section 3, we recall the definition of the i-dimensional Laplacian L_i and critical group $\mathcal{K}_i(\Delta)$ for Δ and use these to carefully define the dollar game determined by each i-chain. Two i-chains are *linearly equivalent* if their difference is in the image of L_i.

Section 4 defines the degree of each i-chain σ of Δ and relates it the winnability of the dollar game, generalizing results from graphs (the special case $d = 1$) to higher dimensions. Let $\mathcal{H}$ be the minimal additive basis, i.e., the Hilbert basis, for the monoid of nonnegative integer points in the kernel of L_i. Using $\mathcal{H}$, we define the degree of σ as an integer vector $\deg(\sigma) \in \mathbb{Z}^{|\mathcal{H}|}$. By Proposition 6, the degree of a chain is invariant under linear equivalence, with the immediate consequence (Corollary 7) that if the dollar game determined by the chain σ is winnable, then $\deg(\sigma) \geqslant 0$. Lemma 10 is a key technical result showing there is a strictly positive element in the kernel of L_i. By Theorem 13, the group of degree zero i-chains modulo linear equivalence is isomorphic to the torsion part of the i-th critical group. In the special case where $d = 1$, this result generalizes the fact that the Jacobian group of a connected graph is the torsion part of the Picard group (in accordance with isomorphism (1)). Theorem 18 achieves one of our main goals: it says that if the degree of a chain is sufficiently large, its corresponding dollar game is winnable.

Section 5 considers the case where Δ is a pseudomanifold. We compute the critical group of an oriented pseudomanifold (Proposition 21), generalizing [8, Theorem 4.7 and subsequent remarks]. Our main result on pseudomanifolds is a combinatorial description of the Hilbert basis $\mathcal{H}$, described above, in codimension one (Theorem 22). The section ends with an example of calculating minimal degrees δ such that every chain of degree at least as large as δ is winnable.

Section 6 builds on the work of Duval, Klivans, and Martin ([8], [9], [10]) on higher-dimensional forests and critical groups. Our main result is Corollary 34, which shows that all $(i-1)$-chains of degree zero are winnable if and only if the i-skeleton is an i-dimensional spanning forest, torsion-free in codimension one. We also generalize Theorem 3.4 of [8], which for each dimension gives an isomorphism between the critical group and the cokernel of the *reduced Laplacian*—a submatrix of the Laplacian determined by a spanning forest. In Section 6.1, we consider an alternative generalization of the set of divisors of nonnegative degree on a graph due to Corry and Keenan ([6]). We use it to characterize higher-dimensional spanning trees that are acyclic in codimension one in terms of winnability of the dollar game.

Section 7 poses some open questions. Finally, the proofs of Proposition 21 and Theorem 30 are relegated to an appendix to avoid distraction from our main line of argument.

Readers interested in learning more about chip-firing on graphs and its relation to a diverse range of mathematics may wish to consult the textbooks [7] and [15].

2 Preliminaries

2.1 Simplicial complexes

Throughout this paper, Δ is a d-dimensional simplicial complex on the set $V = [n] := \{1, \ldots, n\}$ for some integer n. A subset of V of cardinality $i + 1$ that is an element of Δ is an *i-dimensional face* or *i-face* of Δ, and the collection of all i-faces is denoted Δ_i. Let $f_i = f_i(\Delta) := |\Delta_i|$ be the number of faces of dimension i. The empty set is the single face of dimension -1. The elements of V are called *vertices*. The set of all faces forms a poset under inclusion, graded by dimension, and its maximal elements are the *facets* of Δ. To say that Δ has dimension d means that its highest-dimensional facet has dimension d. The complex Δ is *pure* if all of its facets have dimension d, which we do not assume. If R is a commutative ring, the module of *i-chains*, $C_i(\Delta, R)$, is the free R-module with basis Δ_i. In particular, let $C_i(\Delta)$ denote the integral i-chains, $C_i(\Delta, \mathbb{Z})$. Take $C_i(\Delta, R) = 0$ for $i > d$ and $i < -1$, whereas $C_{-1}(\Delta, R) \approx R$. Given an i-chain $\sigma = \sum_{f\in\Delta_i} a_f f$, we write $\sigma(f) := a_f$ and define the *support* of σ to be $\mathrm{supp}(\sigma) := \{f \in \Delta_i : \sigma(f) \neq 0\}$.

In general, our results will depend on the choice of an orientation of Δ (cf. Example 2). In order for the dollar game to be sensible, this orientation must be acyclic, i.e., for all i, every positive sum of i-faces has nonzero boundary. Since any such orientation induces an acyclic orientation on the 1-skeleton of Δ, every acyclic orientation is the standard orientation up to renumbering of the vertices, so we fix the standard orientation on Δ induced by the natural ordering on the vertex set $V = [n]$. Thus, each i-face is represented by the list of its vertices $\overline{v_0 \cdots v_i}$ with $v_0 < \cdots < v_i$. We fix the lexicographic total ordering on each Δ_i and the corresponding induced isomorphism $C_i(\Delta) \simeq \mathbb{Z}^{f_i}$. If π is a permutation, we write

$$\overline{v_{\pi(0)} \cdots v_{\pi(i)}} = \operatorname{sgn}(\pi)\, \overline{v_0 \cdots v_i}$$

as chains.

For each i, there is a boundary mapping

$$\partial_i \colon C_i(\Delta, R) \to C_{i-1}(\Delta, R)$$

defined by

$$\partial_i(\overline{v_0 \cdots v_i}) := \sum_{j=0}^{i} (-1)^j \overline{v_0 \cdots \widehat{v_j} \cdots v_i},$$

where $\widehat{v_j}$ indicates that v_j is omitted. We have $\partial_i \circ \partial_{i+1} = 0$. The elements of $\ker \partial_i$ are the *i-cycles* and elements of $\operatorname{im} \partial_i$ are *i-boundaries.* The i-th *reduced homology* group is

$$\widetilde{H}_i(\Delta, R) := \ker \partial_i / \operatorname{im} \partial_{i+1}.$$

The ordinary homology groups $H_i(\Delta, R)$ use the same definition, with one change: ∂_0 is taken to be the zero mapping, or equivalently, $C_{-1}(\Delta)$ is defined to be the trivial group. We write simply $\widetilde{H}_i(\Delta)$ and $H_i(\Delta)$ in the case $R = \mathbb{Z}$. The i-th *reduced Betti number* is

$$\tilde{\beta}_i(\Delta) = \operatorname{rank}_{\mathbb{Z}} \widetilde{H}_i(\Delta) = \dim_{\mathbb{Q}} \widetilde{H}_i(\Delta, \mathbb{Q}).$$

Applying the functor $\operatorname{Hom}(\cdot, R)$, we get the dual mapping

$$\partial_{i+1}^t \colon C_i(\Delta, R) \to C_{i+1}(\Delta, R)$$

identifying chain modules with their duals using our fixed orderings of the faces of Δ.

If Σ is a subcomplex of Δ, we assume it has the orientation inherited from Δ (induced by the natural ordering on V) and may write $\partial_{\Sigma,i}$ for its i-th boundary mapping. The *i-skeleton* of Δ, denoted $\operatorname{Skel}_i(\Delta)$, is the subcomplex consisting of all faces of Δ of dimension i or less.

Relative homology is mentioned in Section 5. The *relative chain complex* (with $\mathbb{Z}$-coefficients) for a nonempty subcomplex Σ of Δ is the complex

$$\cdots \to C_i(\Delta)/C_i(\Sigma) \xrightarrow{\bar{\partial}_i} C_{i-1}(\Delta)/C_{i-1}(\Sigma) \to \cdots,$$

where $\bar{\partial}_i$ is induced by ∂_i. The *i-th relative homology group* is

$$H_i(\Delta, \Sigma) := \ker \bar{\partial}_i / \operatorname{im} \bar{\partial}_{i+1}.$$

If $\Sigma = \varnothing$, we take $H_i(\Delta, \Sigma) := H_i(\Delta)$.

2.2 Polyhedral cones

We recall some facts about polyhedral cones, using [11], [13], and [18] as references. Let Q be a *cone* in $\mathbb{R}^n$. For us, this means Q is a subset of $\mathbb{R}^n$ closed under nonnegative linear combinations: if $x, y \in Q$ and $\alpha, \beta \in \mathbb{R}_{\geqslant 0}$, then $\alpha x + \beta y \in Q$. The cone Q is *pointed* if $Q \setminus \{0\}$ is contained in an open half-space in $\mathbb{R}^n$, i.e., there exists $z \in \mathbb{R}^n$ such that $x \cdot z > 0$ for all $x \in Q \setminus \{0\}$ (using the ordinary dot product on $\mathbb{R}^n$). We say Q is *polyhedral* if it is finitely generated, i.e., if there exist $x_1, \ldots, x_\ell \in \mathbb{R}^n$ such that

$$Q = \operatorname{Span}_{\mathbb{R}_{\geqslant 0}} \{x_1, \ldots, x_\ell\} := \left\{ \sum_{i=1}^{\ell} \alpha_i x_i : \alpha_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant \ell \right\}.$$

If the generators $x_1, \ldots, x_\ell$ can be taken to be integral, then Q is a *rational* polyhedral cone.

Let Q be a rational polyhedral cone. Then the semigroup of its integral points, $Q_{\mathbb{Z}} := Q \cap \mathbb{Z}^n$, has a *Hilbert basis* $\mathcal{H}$, defined to be a set of minimal cardinality such that every point of $Q_{\mathbb{Z}}$ is a nonnegative integral combination of elements of $\mathcal{H}$. If Q is pointed, then $\mathcal{H}$ is unique, determined by the property that $x \in \mathcal{H}$ if and only if $x \in Q_{\mathbb{Z}} \setminus \{0\}$ and there do not exist $y, z \in Q_{\mathbb{Z}} \setminus \{0\}$ such that $x = y + z$. If Q is integrally generated by $x_1, \ldots, x_\ell$, let

$$\Pi := \Pi(x_1, \ldots, x_\ell) := \left\{ \sum_{i=1}^{\ell} \alpha_i x_i : 0 \leqslant \alpha_i < 1 \text{ for } 1 \leqslant i \leqslant \ell \right\} \subset \mathbb{R}^n$$

be the corresponding *fundamental parallelepiped*. Then

$$\mathcal{H} \subset \{x_1, \ldots, x_\ell\} \cup \Pi.$$

The *dual* of Q is the rational polyhedral cone

$$Q^* := \{x \in \mathbb{R}^n : x \cdot q \geqslant 0 \text{ for all } q \in Q\},$$

and we have $(Q^*)^* = Q$. The *Minkowski sum* of two rational polyhedral cones Q_1 and Q_2 is the rational polyhedral cone $Q_1 + Q_2 := \{x + y : x \in Q_1, y \in Q_2\}$. We will need the following well-known fact:

$$(Q_1 \cap Q_2)^* = Q_1^* + Q_2^*.$$

2.3 Partial order

Throughout this paper, fix the following "component-wise" partial order on the i-chains of Δ: write $\sigma \geqslant \tau$ if $\sigma(f) \geqslant \tau(f)$ for all faces $f \in \Delta_i$. We say σ is *nonnegative* if $\sigma \geqslant 0$, where 0 denotes the zero i-chain. Fix a similar partial order on $\mathbb{R}^k$: write $v \geqslant w$ if $v_i \geqslant w_i$ for all i; and v is nonnegative if $v \geqslant 0$, where 0 denotes the zero vector.

3 The dollar game

The *i-th Laplacian* of Δ, also know as the i-th *up-down combinatorial Laplacian*, is the mapping

$$L_i := \partial_{i+1} \circ \partial_{i+1}^t : C_i(\Delta) \to C_i(\Delta).$$

The isomorphism $C_i(\Delta) \simeq \mathbb{R}^{f_i}$ identifies L_i with an $f_i \times f_i$ matrix whose rows and columns are indexed by the i-faces.

Think of $\sigma = \sum_{f\in\Delta_i} \sigma(f)f \in C_i(\Delta)$ as a distribution of wealth to the i-faces of Δ: face f has $\sigma(f)$ dollars, interpreted as debt if $\sigma(f)$ is negative. A *borrowing move* at an i-face f redistributes wealth by replacing σ by the i-chain

$$\sigma + L_i f.$$

A *lending move* at f replaces σ by

$$\sigma - L_i f.$$

The goal of the *dollar game* for σ is to bring all faces out of debt through a sequence of lending and borrowing moves. In detail, say σ is *linearly equivalent* to the i-chain σ' and write $\sigma \sim \sigma'$ if there exists $v \in \mathbb{Z}^{f_i}$ such that

$$\sigma' = \sigma + L_i v. \tag{2}$$

Call σ' *effective* if $\sigma' \geqslant 0$. Then σ is *winnable* if there exists an effective σ' linearly equivalent to σ, and *winning* the dollar game determined by σ means finding such a σ'.

The *i-chain class group* is

$$\mathcal{J}_i(\Delta) := C_i(\Delta)/\!\sim\; = \; C_i(\Delta)/\operatorname{im} L_i.$$

So an i-chain σ is winnable if and only if there is an effective chain in its class $[\sigma] \in \mathcal{J}_i(\Delta)$.

The image of the i-th Laplacian is contained in the kernel of the i-th boundary mapping, which allows us to define the *i-th critical group* of Δ introduced by Duval, Klivans, and Martin in [8]:

$$\mathcal{K}_i(\Delta) := \ker \partial_i / \operatorname{im} L_i.$$

Choosing a splitting $\rho\colon \operatorname{im}\partial_i \to C_i(\Delta)$ of the exact sequence of free abelian groups

$$0 \to \ker \partial_i \to C_i(\Delta) \to \operatorname{im}\partial_i \to 0$$

gives a corresponding isomorphism

$$\begin{aligned} \mathcal{J}_i(\Delta) &\to \mathcal{K}_i(\Delta) \oplus \operatorname{im}\partial_i \\ [\sigma] &\mapsto ([\sigma - \rho(\sigma)], \partial_i(\sigma)). \end{aligned} \tag{3}$$

The torsion part of $\mathcal{J}_i(\Delta)$ is thus the torsion part of the critical group, $\mathbf{T}(\mathcal{K}_i(\Delta))$, (which, itself, is sometimes called the critical group of Δ (e.g., in [9])). There is a natural surjection $\mathcal{K}_i(\Delta) \to \widetilde{H}_i(\Delta)$ which is an isomorphism when restricted to the free parts of each group (Corollary 14).

Example 1. Figure 1 illustrates an instance of the dollar game determined by a 1-chain σ on the simplicial complex with two facets: $\overline{123}$ and $\overline{234}$. Calling the winning chain on the right σ', Equation (2) in this case takes the form

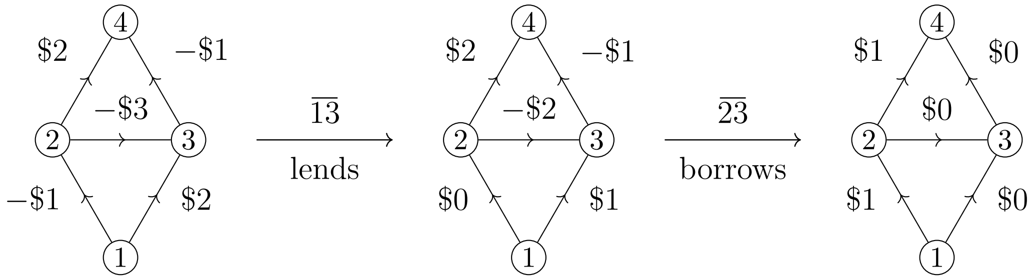

Figure 1: Winning the dollar game $\sigma = -\overline{12} + 2\cdot\overline{13} - 3\cdot\overline{23} + 2\cdot\overline{24} - \overline{34}$ on the 2-dimensional simplicial complex with facets $\overline{123}$ and $\overline{234}$.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -3 \\ 2 \\ -1 \end{bmatrix} + \begin{array}{c} \begin{array}{ccccc} \overline{12} & \overline{13} & \overline{23} & \overline{24} & \overline{34} \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma' = \sigma + L_1 v.$$

Note that in moving from σ to σ', money has been introduced from nowhere: the net amount in σ is $-\$1$, while in σ' it is \$2. While the simplicial dollar game does not conserve the net amount of money, other quantities are conserved, and we will discuss this at length starting in the next section. For now, as an example, it is easy to check that the sum of the amount of money on just the edges $\overline{12}$ and $\overline{13}$ is conserved under lending and borrowing moves. Thus, for instance, if we change the amount of money on $\overline{12}$ in σ from $-\$1$ to $-\$3$, the resulting game could never be won. And that statement would continue to hold no matter how much money we added to the edges $\overline{23}$, $\overline{24}$, and $\overline{34}$.

Example 2. Here we show that winnability depends on the orientation of the simplicial complex. Figure 2 depicts two dollar games on the 2-simplex (the simplicial complex with the single facet $\overline{123}$). The first can be won by lending at the edge $\overline{13}$. The second is not winnable. To see this, note that the sum of the $\overline{13}$ and $\overline{23}$ components of a 1-chain on this complex—which is $-\$2$ for the second game—is invariant under lending and borrowing moves. So one of these games is winnable and the other is not, yet they are the same up to a relabeling of the vertices (which amounts to a change in orientation).

Figure 2: Two dollar games on the edges of a 2-simplex. Only the first is winnable.

Example 3 (Graphs)**.** Let $\Delta = G$ be a connected, undirected graph as in the introduction. In that case, the dollar game for 0-chains on Δ we just defined is the same as the dollar game for graphs from [2]. If the vertices of G are $v_i = i$ for $i = 1, \ldots, n$, then the 0-th Laplacian is the usual discrete Laplacian for a graph:

$$L_0 = \operatorname{diag}(\deg_G(v_1), \ldots, \deg_G(v_n)) - A,$$

the difference of the diagonal matrix of vertex degrees and the adjacency matrix of G. The 0-chain class group and 0-th critical group are the Picard group and Jacobian group, respectively, described in the introduction: $\mathcal{J}_0(\Delta) = \operatorname{Pic}(G)$ and $\mathcal{K}_0(\Delta) = \operatorname{Jac}(G)$. Isomorphism (3) specializes to the usual isomorphism (1) for graphs.

4 Degree

The naive way of generalizing the degree of a divisor on a graph to the degree of an i-chain on a simplicial complex Δ, by simply summing up the coefficients of the i-faces, fails to retain many of the useful properties of the graph-theoretic degree. Under this naive definition of degree, as shown in Example 1, linearly equivalent i-chains can fail to have the same degree, i-chains with negative degree can be winnable, and for a fixed complex, there can exist i-chains of arbitrarily large degree that are unwinnable. This section will introduce a better generalization of degree, avoiding these problems. To summarize the rest of this section: Theorem 13 shows that the group of i-chains of degree zero modulo firing rules is exactly the torsion part of the i-th critical group, as it is in the usual case of connected graphs. Our main result is Theorem 18, which states that i-chains of large enough degree are winnable. Unlike for graphs, it turns out that all i-chains of a given degree may be winnable even though there exists an i-chain of larger degree that is not (cf. Example 37). Corollary 20 says this will not occur if the Hilbert basis $\mathcal{H}_i$ consists of 0-1 vectors.

For divisors on a graph, the degree function, $\deg\colon \mathbb{Z}V \to \mathbb{Z}$, is a linear function with the following two properties:

$$\begin{aligned} \text{invariance under linear equivalence:} &\quad D \sim D' \Rightarrow \deg(D) = \deg(D'),\\ \text{nonnegativity on effective divisors:} &\quad E \geqslant 0 \Rightarrow \deg(E) \geqslant 0. \end{aligned}$$

To generalize the notion of degree to higher dimensions, for each i, we look for a linear function $\deg\colon C_i(\Delta) \to \mathbb{Z}$ with the above two properties. Any such linear function can be represented by $\sigma \mapsto \langle \sigma, \sigma' \rangle$ for a fixed $\sigma' \in C_i(\Delta)$, where $\langle \sigma, \sigma' \rangle := \sum_{f \in \Delta_i} \sigma(f)\sigma'(f)$. To have invariance under linear equivalence, σ' must lie in the kernel of L_i. For the function to be nonnegative on effective chains, σ' must itself be effective. Thus, an integer-valued linear function has our two desired properties if and only if it is expressible as the inner product with an effective i-chain in $\ker L_i$. But no particular one of these functions stands out as a preferred choice. Instead, we will take our generalization to contain the information of the output of all such functions, as we now describe.

The set $C := \left\{v \in \mathbb{R}^{f_i} : L_i v \geqslant 0 \text{ and } v \geqslant 0\right\}$ is a pointed, rational, polyhedral cone. Therefore, its set of integer points, $C \cap \mathbb{Z}^{f_i}$, has a unique Hilbert basis $\mathcal{H}$ ([14], [18]). This means that $C \cap \mathbb{Z}^{f_i}$ is exactly the set of nonnegative integer linear combinations of $\mathcal{H}$, and $\mathcal{H}$ is the smallest subset of $C \cap \mathbb{Z}^{f_i}$ with this property. We can now give our definition of degree:

Definition 4. Let $i \in \mathbb{Z}$. The *i-th nonnegative kernel* for Δ is the monoid

$$\ker^+ L_i := \{\sigma \in \ker L_i : \sigma(f) \geqslant 0 \text{ for all } f \in \Delta_i\}.$$

Fix an ordering

$$\mathcal{H}_i = \mathcal{H}_i(\Delta) = (h_1, \ldots, h_{\ell_i})$$

for the elements of the Hilbert basis for $\ker^+ L_i$. The *degree* of $\sigma \in C_i(\Delta)$ is

$$\deg(\sigma) := \deg_i(\sigma) := (\sigma \cdot h_1, \ldots, \sigma \cdot h_{\ell_i})$$

where $\sigma \cdot h_j := \sum_{f \in \Delta_i} \sigma(f) h_j(f)$.

Remark 5. Another possible definition for the degree function is to replace $\mathcal{H}_i$ in the definition with a list of only those elements of the Hilbert basis that are rays of the cone $L_i^+ \otimes \mathbb{R}$. Denoting this variant of the definition of degree by rdeg, we have

$$\deg(\sigma) \geqslant \deg(\sigma') \iff \operatorname{rdeg}(\sigma) \geqslant \operatorname{rdeg}(\sigma')$$

for $\sigma, \sigma' \in C_i(\Delta)$. This means that all our results relating winnability of the dollar game to the degree of a chain will hold using either definition. One advantage of rdeg over deg is that it is easier to compute.

For each i, our definition of degree is a linear function into $\mathbb{Z}^{\ell_i}$, where ℓ_i is the number of elements in $\mathcal{H}_i$, and satisfies the two essential properties described earlier: invariance under linear equivalence is shown below, and nonnegativity on effective chains is obvious. It also specializes to the usual definition of degree in the case of a connected graph, as the Hilbert basis in that case is the sum of all of the vertices of the graph.

Proposition 6. *The degree of an i-chain depends only on its linear equivalence class.*

Proof. It suffices to show that every element of $\operatorname{im} L_i$ has degree zero. If $\tau \in \ker L_i$ and $\sigma \in C_i(\Delta)$, then

$$\langle \tau, L_i \sigma \rangle = \langle L_i^t \tau, \sigma \rangle = \langle L_i \tau, \sigma \rangle = 0,$$

since L_i is symmetric. In particular, $\langle \tau, L_i \sigma \rangle = 0$ for all $\tau \in \ker^+ L_i$. □

Corollary 7. *If an i-chain σ is winnable, then* $\deg(\sigma) \geqslant 0$.

Proof. If σ is winnable, then $\sigma \sim \tau$ for some $\tau \geqslant 0$. Then $\deg(\sigma) = \deg(\tau)$, and since each element of the Hilbert basis $\mathcal{H}_i(\Delta)$ has nonnegative coefficients, $\deg(\tau) \geqslant 0$. □

Remark 8. Using (4), below, the proof of Proposition 6 is easily modified to show that every element of $\operatorname{im}\partial_{i+1}$ has degree zero. Thus, we get the stronger result that degree is a homology invariant.

Definition 9. A vector $\delta \in \mathbb{Z}^{|\mathcal{H}_i|}$ is a *realizable i-degree* if there exists an i-chain σ such that $\deg(\sigma) = \delta$.

It is typically the case that not all degrees are realizable. For instance, consider the 3-simplex with single facet $\overline{1234}$. In this case, the Hilbert basis for $\ker^+ L_2$, computed by Sage ([20]), is

$$\{\overline{123} + \overline{124}, \overline{123} + \overline{234}, \overline{134} + \overline{124}, \overline{134} + \overline{234}\}.$$

Ordering these elements as listed, it is easy to check that there are no 2-chains of degree $(0, 0, 0, 1)$.

In general, the set of realizable i-degrees forms an additive monoid $\mathcal{M}_i(\Delta)$, and Proposition 6 says that the i-class group $\mathcal{J}_i(\Delta)$ is graded by $\mathcal{M}_i$. Given $\delta \in \mathcal{M}_i(\Delta)$, let $\mathcal{J}_i^\delta(\Delta)$ denote the δ-th graded part of $\mathcal{J}_i(\Delta)$. Then there is a faithful action of the group $\mathcal{J}_i^0(\Delta)$ on $\mathcal{J}_i^\delta(\Delta)$ given by addition of i-chains.

4.1 The group of chain classes of degree zero

Our next goal is Theorem 13, identifying the group of degree zero i-chains modulo firing rules with the torsion part of the critical group $\mathcal{K}_i(\Delta)$, and thus generalizing a well-known result from the divisor theory of graphs (cf. Example 16). Letting $K = \mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$, we use the standard notation $X^\perp = \{y \in K : x \cdot y = 0 \text{ for all } x \in X\}$ for the perpendicular space for a subset $X \subseteq K^n$.

By standard linear algebra,

$$\ker L_i = \ker \partial_{i+1}\partial_{i+1}^t = \ker \partial_{i+1}^t. \tag{4}$$

Using the chain property of boundary maps, we identify a useful subset of the kernel:

$$\operatorname{im}\partial_i^t \subseteq \ker \partial_{i+1}^t = \ker L_i.$$

If f is an $(i-1)$-face of Δ, the element $\partial_i^t(f)$ is called the *star* of f; it is a signed sum of the faces radiating from f. If $f = \overline{v_0 \cdots v_{i-1}}$, then each element in the support of its star has the form $\overline{v_0 \cdots v_k v v_{k+1} \cdots v_{i-1}}$ for some vertex v. The set of stars generates $\operatorname{im}\partial_i^t$.

Lemma 10. *For each i, there exists a strictly positive element $\tau \in \ker L_i$, i.e., such that $\tau(f) > 0$ for all $f \in \Delta_i$.*

Proof. For the sake of contradiction, assume no such element τ exists. Then for every $\sigma \in \ker L_i$, let m_σ denote the least (in lexicographic ordering) i-face such that $\sigma(m) \leqslant 0$. Choose a $\sigma \in \ker L_i$ with maximal m_σ. Say $m := m_\sigma = \overline{v_0 \cdots v_i}$, and consider the star $S := \partial_i^t(\overline{v_1 \cdots v_i})$. The coefficient of m in S is 1, and if m_0 is an i-face such that $m_0 < m$, then m_0 begins with a vertex v smaller than v_1, meaning one of two cases occurs: either $m_0 = \overline{v v_1 \cdots v_i}$, in which case the coefficient of m_0 in S is 1, or m_0 does not contain

$\overline{v_1 \cdots v_i}$ as a subface, and the coefficient of m_0 in S is 0. Either way, if $m_0 < m$, then the coefficient of m_0 in S is nonnegative. Now consider $\sigma' := \sigma + (1-\sigma(m))S$. Then $\sigma' \in \ker L_i$, and $\sigma'(f) > 0$ for all faces $f \leqslant m$, contradicting the maximality of m. So our assumption must be false. $\square$

The following is an immediate consequence:

Corollary 11. *If σ is an effective i-chain and $\deg(\sigma) = 0$, then $\sigma = 0$.*

Corollary 12. *For each i, the $\mathbb{Z}$-span of $\ker^+ L_i$ is $\ker L_i$. Hence,*

$$(\ker^+ L_i)^\perp = (\ker L_i)^\perp = (\ker \partial_{i+1}^t)^\perp.$$

Proof. Take a strictly positive element $\tau \in \ker L_i$ that is primitive, i.e., it is not an integer multiple of any other element. We can then complete $\{\tau\}$ to a basis $\{\tau, \sigma_1, \ldots, \sigma_k\}$ for $\ker L_i$. (To see this, consider the exact sequence

$$0 \to \mathbb{Z}\tau \to \mathbb{Z}^n \to \mathbb{Z}^n/\mathbb{Z}\tau \to 0.$$

Since $\mathbb{Z}^n/\mathbb{Z}\tau$ is torsion-free, the sequence splits.) Then, for each nonzero $N \in \mathbb{Z}$, the set

$$\{\tau, \sigma_1 + N\tau, \ldots, \sigma_k + N\tau\}$$

is still a basis for $\ker L_i$. By taking $N \gg 0$, this basis will consist solely of elements $\ker_i^+ L_i$. $\square$

Theorem 13. *For each i, the group of i-chains of degree zero modulo firing rules is isomorphic to the torsion part of the i-th critical group of Δ:*

$$(\ker L_i)^\perp / \operatorname{im}(L_i) = \mathbf{T}(\mathcal{K}_i(\Delta)).$$

Proof. To see that $\operatorname{im} L_i \subseteq (\ker L_i)^\perp$, let $\sigma \in \mathbb{Z}\Delta_i$ and $\tau \in \ker L_i = \ker \partial_{i+1}^t$. Then

$$\langle \tau, L_i\sigma \rangle = \langle \tau, \partial_{i+1}\partial_{i+1}^t \sigma \rangle = \langle \partial_{i+1}^t \tau, \partial_{i+1}^t \sigma \rangle = \langle 0, \partial_{i+1}^t \sigma \rangle = 0.$$

We also have $(\operatorname{im} \partial_i^t)^\perp \subseteq \ker \partial_i$. To see this, take $\sigma \in (\operatorname{im} \partial_i^t)^\perp$ and $\tau \in \mathbb{Z}\Delta_{i-1}$. Then

$$0 = \langle \sigma, \partial_i^t \tau \rangle = \langle \partial_i \sigma, \tau \rangle.$$

Since τ is arbitrary, $\partial_i \sigma = 0$.

Next,

$$\operatorname{im} \partial_i^t \subseteq \ker \partial_{i+1}^t \Rightarrow (\ker L_i)^\perp = (\ker \partial_{i+1}^t)^\perp \subseteq (\operatorname{im} \partial_i^t)^\perp \subseteq \ker \partial_i.$$

Hence,

$$(\ker L_i)^\perp / \operatorname{im} L_i \subseteq \ker \partial_i / \operatorname{im} L_i =: \mathcal{K}_i(\Delta).$$

Since $\dim_{\mathbb{Q}} (\ker L_i)^\perp = \dim_{\mathbb{Q}} (\operatorname{im} L_i)$, the group $(\ker L_i)^\perp / \operatorname{im} L_i$ is finite, and hence torsion. So it is a subset of $\mathbf{T}(\mathcal{K}_i(\Delta))$. To show the opposite inclusion, let $\sigma \in \ker \partial_i$, and suppose there exists a positive integer k such that $k\sigma \in \operatorname{im} L_i$. Say $k\sigma = L_i\tau$, and let $\nu \in \ker L_i = \ker \partial_{i+1}^t$. Then

$$k\langle \nu, \sigma \rangle = \langle \nu, k\sigma \rangle = \langle \nu, L_i\tau \rangle = \langle \partial_{i+1}^t \nu, \partial_{i+1}^t \tau \rangle = 0.$$

Therefore, $\langle \nu, \sigma \rangle = 0$. So each torsion element of $\mathcal{K}_i(\Delta)$ is an element of $(\ker L_i)^\perp / \operatorname{im}(L_i)$. $\square$

Corollary 14. *The natural surjection $\mathcal{K}_i(\Delta) \to \widetilde{H}_i(\Delta)$ is an isomorphism when restricted to the free parts of $\mathcal{K}_i(\Delta)$ and $\widetilde{H}_i(\Delta)$ and a surjection when restricted to the torsion parts.*

Proof. Consider the exact sequence

$$0 \to \operatorname{im} \partial_{i+1} / \operatorname{im} L_i \to \mathcal{K}_i(\Delta) \to \widetilde{H}_i(\Delta) \to 0.$$

We have

$$\operatorname{im} L_i \subseteq \operatorname{im} \partial_{i+1} \subseteq (\ker L_i)^{\perp},$$

where the second inclusion follows by an argument similar to that given for $\operatorname{im} L_i$ at the beginning of the proof of Theorem 13. From Theorem 13, it follows that $\operatorname{im} \partial_{i+1} / \operatorname{im} L_i$ is finite. Tensoring the sequence by $\mathbb{Q}$ then gives the result about the free parts, and since the torsion functor $\mathbf{T}(\,\cdot\,)$ is left-exact, there is a surjection for the torsion parts. $\square$

Remark 15. Let δ be a realizable i-degree, and fix any $\sigma \in C_i(\Delta)$ such that $\deg(\sigma) = \delta$. Then there is a bijection of chain class groups $\mathcal{J}_i^0(\Delta) \to \mathcal{J}_i^\delta(\Delta)$ given by $\omega \mapsto \omega + \sigma$ for each $\omega \in \mathcal{J}_i^0(\Delta)$. By Theorem 13, the group $\mathcal{J}_i^0(\Delta)$ is the torsion part of the (finitely-generated abelian group) $\mathcal{K}_0(\Delta)$ and hence is finite. Thus, there are only finitely many chains to check to determine whether all chains of a given degree are winnable.

Example 16 (Graphs)**.** Consider again how our structures generalize those on graphs. In the case $d = 1$, the simplicial complex Δ is determined by its 1-skeleton, a graph G. We have two notions of degree for an element $\sigma \in C_i(\Delta)$: as a 0-chain on Δ, there is the degree determined by dot products with elements of the Hilbert basis $\mathcal{H}_0$; and as a divisor on a graph, there is the usual degree given by $\partial_0(\sigma) = \sum_{v \in V} \sigma(v)$. Call the former the Δ-degree, $\deg(\Delta, \sigma)$, of σ, and call the latter the G-degree, $\deg(G, \sigma)$.

By definition, the Picard group $\operatorname{Pic}(G)$ is the set of 0-chains modulo the image of L_0, and hence, coincides with the 0-th class group $\mathcal{J}_0$. Now, $\operatorname{Pic}(G)$ is graded by G-degree, and its G-degree zero part is by definition the Jacobian group $\operatorname{Jac}(G)$. Hence,

$$\operatorname{Jac}(G) = \mathcal{K}_0(\Delta) = \ker \partial_0 / \operatorname{im} L_0.$$

On the other hand, $\mathcal{J}_0(\Delta)$ is graded by Δ-degree. While $\operatorname{Pic}(G) = \mathcal{J}_0(\Delta)$ as groups, in the case where G is not connected, their gradings differ.

If G is connected or, equivalently, $\tilde{\beta}_0(\Delta) = 0$, the Hilbert basis $\mathcal{H}_0$ consists of the all-ones vector $\vec{1}$, and $\deg(\Delta, \sigma) = \sigma \cdot \vec{1} = \partial_0(\sigma) = \deg(G, \sigma)$. Thus, $\operatorname{Pic}(G) = \mathcal{J}_0$ as graded groups, and $\operatorname{Jac}(G)$ is the collection of Δ-degree zero 1-chains. As is well-known, the matrix-tree theorem implies that $|\operatorname{Jac}(G)|$ is the number of spanning trees of G. So $\operatorname{Jac}(G)$ is finite, hence torsion, in agreement with Theorem 13.

Now consider the case where G is not connected. To fix ideas, say G is the graph consisting of the disjoint union of two triangles, one with vertices $1, 2, 3$ and the other with vertices $4, 5, 6$. In this case,

$$\operatorname{Jac}(G) = \mathcal{K}_0(\Delta) \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}.$$

The Hilbert basis $\mathcal{H}_0$ consists of two elements $h_1 = (1,1,1,0,0,0)$ and $h_2 = (0,0,0,1,1,1)$. So if $\sigma \in C_0(\Delta)$, then

$$\deg(G,\sigma) = \textstyle\sum_{i=1}^{6} \sigma_i \quad \text{and} \quad \deg(\Delta,\sigma) = (\textstyle\sum_{i=1}^{3} \sigma_i, \sum_{i=4}^{6} \sigma_i).$$

For instance, if $\sigma = \bar{1} - \bar{4} = (1,0,0,-1,0,0)$, then $\deg(G,\sigma) = 0$ while $\deg(\Delta,\sigma) = (1,-1) \neq (0,0)$. The Δ-degree zero part of $\mathcal{J}_0$ is isomorphic to the direct sum of two copies of the Jacobian group of a triangle, i.e., to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

4.2 Degree/winnability condition

We now show that if the degree of an i-chain is sufficiently large, it is winnable. The proof requires the following lemma:

Lemma 17. *For each integer i, there exists a finite set of i-chains $\mathcal{P}_i$ such that any $\sigma \in C_i(\Delta)$ with $\deg(\sigma) \geqslant 0$ can be written as $\sigma = \zeta + \tau + \phi$ where $\deg(\zeta) = 0$, τ is effective, and $\phi \in \mathcal{P}_i$.*

Proof. Having ordered Δ_i lexicographically, we make the identification $C_i(\Delta,\mathbb{R}) \simeq \mathbb{R}^{f_i}$ where $f_i := |C_i(\Delta)|$. Let $L_i^{\mathbb{R}} := L_i \otimes \mathbb{R} \colon \mathbb{R}^{f_i} \to \mathbb{R}^{f_i}$, and let $\mathcal{O}^+$ be the nonnegative orthant of $\mathbb{R}^{f_i}$. Using dual cones, the fact that σ has degree at least 0 can be expressed as follows:

$$\sigma \in ((\ker L_i^{\mathbb{R}}) \cap \mathcal{O}^+)^* \cap \mathbb{Z}^{f_i} = ((\ker L_i^{\mathbb{R}})^* + (\mathcal{O}^+)^*) \cap \mathbb{Z}^{f_i} = ((\ker L_i^{\mathbb{R}})^* + \mathcal{O}^+) \cap \mathbb{Z}^{f_i}.$$

We can split both $(\ker L_i^{\mathbb{R}})^*$ and $\mathcal{O}^+$ into the Minkowski sum of the integer points they contain and their respective fundamental parallelepipeds P_1 and P_2 (with respect to any choice of integral generators), to get

$$\begin{aligned}((\ker L_i^{\mathbb{R}})^* + \mathcal{O}^+) \cap \mathbb{Z}^{f_i} &= (((\ker L_i^{\mathbb{R}})^* \cap \mathbb{Z}^{f_i} + P_1) + (\mathcal{O}^+ \cap \mathbb{Z}^{f_i} + P_2)) \cap \mathbb{Z}^{f_i} \\ &= (\ker L_i^{\mathbb{R}})^* \cap \mathbb{Z}^{f_i} + \mathcal{O}^+ \cap \mathbb{Z}^{f_i} + (P_1 + P_2) \cap \mathbb{Z}^{f_i}.\end{aligned}$$

Since $\ker L_i^{\mathbb{R}}$ is a linear space, $(\ker L_i^{\mathbb{R}})^* = (\ker L_i^{\mathbb{R}})^{\perp}$. Hence, $(\ker L_i^{\mathbb{R}})^* \cap \mathbb{Z}^{f_i}$ is the set of all i-chains of degree 0, and $\mathcal{O}^+ \cap \mathbb{Z}^{f_i}$ is the set of effective i-chains. So letting $\mathcal{P}_i = (P_1 + P_2) \cap \mathbb{Z}^{f_i}$, which is a finite set since P_1 and P_2 are bounded, completes the proof. $\square$

Theorem 18. *If the degree of a chain is sufficiently large, then it is winnable: for each integer i there exists a realizable i-degree $\delta \in \mathbb{Z}^{|\mathcal{H}_i|}$ such that for all $\sigma \in C_i(\Delta)$, if $\deg(\sigma) \geqslant \delta$, then σ is winnable.*

Proof. Let $\mathcal{S}$ be a set of representatives for $\mathbf{T}(\mathcal{K}_i(\Delta))$, and let $\mathcal{P}_i$ be as in Lemma 17. By finiteness of $\mathcal{S}$ and $\mathcal{P}_i$, there exists an i-chain ω such that the chain $\omega + \gamma + \phi$ is effective for all $\gamma \in \mathcal{S}$ and $\phi \in \mathcal{P}_i$. Set $\delta = \deg(\omega)$, and let σ be an i-chain such that $\deg(\sigma) \geqslant \delta$. Then $\deg(\sigma - \omega) \geqslant 0$, so by Lemma 17 we can write

$$\sigma - \omega = \zeta + \tau + \phi$$

where $\deg(\zeta) = 0$, τ is effective, and $\phi \in \mathcal{P}_i$. Since $\deg(\zeta) = 0$, we have $\zeta \in (\ker^+ L_i)^{\perp} = (\ker L_i)^{\perp}$ by Corollary 12. So by Theorem 13, there exists $\gamma \in \mathcal{S}$ such that $\zeta \sim \gamma$. It follows that σ is winnable:

$$\sigma \sim (\omega + \gamma + \phi) + \tau \geqslant 0.$$

$\square$

Let $\mathcal{W}_i$ be the set of all δ satisfying the conditions in Theorem 18. Then $\mathcal{W}_i$ is partially ordered (§2.3) and bounded below by $0 \in \mathbb{Z}^{|\mathcal{H}_i|}$. So it is natural to consider its set of minimal elements, $\min(\mathcal{W}_i)$. To see that $\min(\mathcal{W}_i)$ is finite, consider the polynomial ideal generated by the monomials $x^\delta := \prod_i x_i^{\delta_i}$ as δ varies over $\mathcal{W}_i$. By the Hilbert basis theorem, this ideal is finitely generated, and its minimal set of generators corresponds with $\min(\mathcal{W}_i)$. See Example 27 for the computation of $\min(\mathcal{W}_1)$ for a hollow tetrahedron.

Intuition coming from the dollar game on graphs may not apply to $\mathcal{W}_i$ on a general simplicial complex. For instance, as in Example 27, there are typically infinitely many nonnegative realizable degrees that are not in $\mathcal{W}_i$. Further, as will be demonstrated in Example 37, it may be the case that all i-chains of a particular realizable degree δ are winnable even though there exists an unwinnable i-chain σ with $\deg(\sigma) \geqslant \delta$.

To finish this section, we describe conditions under which $\delta \in \mathcal{W}_i$ if and only if δ is realizable and all i-chains of degree exactly δ are winnable.

Proposition 19. *Suppose the i-th Hilbert basis $\mathcal{H}_i$ of Δ consists of* 0-1 *vectors, and let σ be an i-chain such that* $\deg(\sigma) \geqslant 0$. *Then there exists an effective i-chain τ (not necessarily linearly equivalent to σ) such that* $\deg(\tau) = \deg(\sigma)$.

Proof. Suppose the result is false, and let σ be a counterexample of minimal degree $\deg(\sigma) \geqslant 0$ (using the component-wise partial order defined in Section 2.3). Note that $\deg(\sigma) \neq 0$. Using notation for dual cones from the proof of Lemma 17, we have

$$\sigma \in (\ker L_i^{\mathbb{R}} \cap \mathcal{O}^+)^* = (\ker L_i^{\mathbb{R}})^* + \mathcal{O}^+ = (\ker L_i^{\mathbb{R}})^\perp + \mathcal{O}^+.$$

The last equality follows because $\ker L_i^{\mathbb{R}}$ is a linear space. Therefore, over $\mathbb{R}$, we have $\sigma = \nu + \tau$ where $\nu \in (\ker L_i^{\mathbb{R}})^\perp$ and $\tau = \sum_{f \in \Delta_i} \tau(f) f$ with $\tau(f) \geqslant 0$ for all $f \in \Delta_i$. So $\tau \cdot h = \sigma \cdot h$ for all $h \in \mathcal{H}_i$, and since $\deg(\sigma) \neq 0$, there exists a face f' such that $\tau(f') > 0$. To compute the degree of the integral chain $\sigma - f'$, let $h = \sum_{f \in \Delta_i} h(f) f$ be an arbitrary element of $\mathcal{H}_i$. Since $h(f') \in \{0, 1\}$, taking dot products,

$$(\sigma - f') \cdot h = (\tau - f') \cdot h = \sum_{f \in \Delta_i} \tau(f) h(f) - h(f') = \sum_{f \neq f'} \tau(f) h(f) + (\tau(f') - 1) h(f') > -1.$$

Since $(\sigma - f') \cdot h \in \mathbb{Z}$ for all $h \in \mathcal{H}_i$, it follows that $\deg(\sigma - f') \geqslant 0$. On the other hand, by Lemma 10, there exists some $h \in \mathcal{H}_i$ such that $h(f') > 0$, and therefore $\deg(\sigma - f')$ is strictly smaller than $\deg(\sigma)$. By minimality, there exists an effective integral i-chain ρ with $\deg(\rho) = \deg(\sigma - f')$. But then $\rho + f'$ is an effective divisor of degree $\deg(\sigma)$, contradicting the fact that σ is a counterexample. □

Corollary 20. *Suppose $\mathcal{H}_i$ consists of* 0-1 *vectors and that there exists a realizable i-degree δ such that every i-chain of degree δ is winnable. Then every i-chain with degree at least δ is winnable.*

Proof. Let $\sigma \in C_i(\Delta)$ with $\deg(\sigma) \geqslant \delta$. By Corollary 19, there exists an effective chain $\tau \in C_i(\Delta)$ of degree $\deg(\sigma) - \delta$. Since $\sigma - \tau$ has degree δ, by hypothesis it is linearly equivalent to an effective chain ρ. Therefore, $\sigma \sim \tau + \rho \geqslant 0$, and σ is winnable. □

5 Pseudomanifolds

In this section we take Δ to be a d-dimensional orientable pseudomanifold. References for pseudomanifolds include [16] and [19]. To say that Δ is a *pseudomanifold* means that it is

1. *pure*: each facet has dimension d;
2. *non-branching*: each $(d-1)$-face is a face of at most two facets; and
3. *strongly connected*: if σ and σ' are facets, there exists a sequence of facets $\sigma_0, \dots, \sigma_k$ with $\sigma_0 = \sigma$ and $\sigma_k = \sigma'$ such that each pair of consecutive facets σ_i and σ_{i+1} share a $(d-1)$-face.

The *boundary* $\partial\Delta$ of Δ is the collection of $(d-1)$-faces of Δ that are faces of exactly one facet. Since Δ is a pseudomanifold, it is a standard result that exactly one of the following must hold in relative homology:

(i) $H_d(\Delta, \partial\Delta) \approx \mathbb{Z}$ and $H_{d-1}(\Delta, \partial\Delta)$ is torsion-free.

(ii) $H_d(\Delta, \partial\Delta) = 0$ and $H_{d-1}(\Delta, \partial\Delta)$ has torsion subgroup $\mathbf{T}(H_{d-1}(\Delta, \partial\Delta)) \approx \mathbb{Z}/2\mathbb{Z}$.

In our case, we are assuming that Δ is an *orientable pseudomanifold*, which by definition means that (i) holds. It is then possible to orient the facets of Δ so that the sum of their boundaries is supported on the boundary of Δ. Letting $f^{(1)}, \dots, f^{(m)} \in C_d(\Delta)$ be the facets of Δ, this means that for each i we can choose $\gamma_i \in \left\{\pm f^{(i)}\right\}$ and define $\gamma = \gamma_1 + \cdots + \gamma_m$ so that $\partial_d(\gamma)$ is supported on $\partial\Delta$. (In particular, if Δ has no boundary, then $\partial_d(\gamma) = 0$.) We call the relative cycle γ a *pseudomanifold orientation* for Δ. Its class $[\gamma] \in H_d(\Delta, \partial\Delta)$ is a choice of generator for the top relative homology group. Recall that the simplicial complexes studied in this paper all come with a fixed underlying orientation as a simplicial complex, upon which the dollar game depends. The orientations of the facets γ_i need not agree with those given by that fixed orientation.

The proof of the following is in the appendix. It was proved in [8] for the case $\widetilde{H}_{d-1}(\Delta) = 0$ and $\partial\Delta = \varnothing$.

Proposition 21. *Suppose Δ is a d-dimensional orientable pseudomanifold. If $\partial\Delta \neq \varnothing$,*

$$\mathcal{K}_{d-1}(\Delta) = \widetilde{H}_{d-1}(\Delta)$$

and otherwise, if Δ has no boundary,

$$\mathcal{K}_{d-1}(\Delta) \simeq (\mathbb{Z}/m\mathbb{Z}) \oplus \widetilde{H}_{d-1}(\Delta)$$

where $m = f_d$ is the number of facets of Δ.

To define the degree of a $(d-1)$-chain on a pseudomanifold Δ, we need to compute the Hilbert basis for $\ker^+ L_{d-1}$. Our main goal for this section is a combinatorial description of this basis. We start by defining the *γ-incidence graph* $\Gamma = \Gamma(\Delta, \gamma)$ as a directed graph whose vertices are the oriented facets $\{\gamma_i\}$. If $\partial\Delta \neq \varnothing$, let $\gamma_0 := 0 \in C_d(\Delta)$, and include it, too, as a vertex of Γ. The edges of Γ are in bijection with the codimension-one faces of Δ. To describe them, let σ be any $(d-1)$-face and write

$$\partial_d^t(\sigma) = \gamma_j - \gamma_i$$

for uniquely determined i and j. (If $\sigma \in \partial\Delta$, then one of i or j will be 0.) Let $\sigma^- := i$ and $\sigma^+ := j$. The directed edge corresponding to σ then starts at γ_{σ^-} and ends at γ_{σ^+}. See Figures 3 and 4 for examples.

Theorem 22 (Hilbert basis for an orientable pseudomanifold)**.** *Let Δ be a pseudomanifold with pseudomanifold orientation γ. Then the Hilbert basis for the nonnegative kernel $\ker^+ L_{d-1}$ is the set of incidence vectors for the simple directed cycles of $\Gamma(\Delta, \gamma)$.*

Proof. Let $\tau = \sum_\sigma a_\sigma \sigma \in C_{d-1}(\Delta) \neq 0$. Then $\tau \in \ker L_{d-1} = \ker \partial_d^t$ if and only if

$$0 = \partial_d^t(\tau) = \sum_\sigma a_\sigma(\gamma_{\sigma^+} - \gamma_{\sigma^-}).$$

Requiring $\tau \in \ker^+ L_{d-1}$ adds the restriction that $a_\sigma \geqslant 0$ for all σ, which is equivalent to saying that τ is a directed cycle in Γ. Then τ is simple if and only if it is not the sum of two other non-trivial directed cycles, which is exactly the requirement that τ belong to the Hilbert basis. ☐

Corollary 23. *Suppose δ is a realizable $(d-1)$-degree on the orientable pseudomanifold Δ of dimension d and that every $(d-1)$-chain of degree δ is winnable. Then every $(d-1)$-chain with degree at least δ is winnable.*

Proof. The result follows immediately from Theorem 22 and Corollary 20. ☐

Example 24. Let Δ be the hollow tetrahedron with facets $\overline{123}$, $\overline{124}$, $\overline{134}$, and $\overline{234}$. A pseudomanifold orientation is given by

$$\gamma = \overline{132} + \overline{124} + \overline{143} + \overline{234} = -\overline{123} + \overline{124} - \overline{134} + \overline{234}.$$

Both Δ and its associated γ-incidence graph $\Gamma(\Delta, \gamma)$ appear in Figure 3. The edges of $\Gamma(\Delta, \gamma)$ are labeled by the corresponding 1-faces of Δ. The incidence vectors for the three simple directed cycles of $\Gamma(\Delta, \gamma)$, and hence the elements of the Hilbert basis for $\ker^+ L_1$, are listed as rows in the table below:

$\overline{12}$	$\overline{13}$	$\overline{14}$	$\overline{23}$	$\overline{24}$	$\overline{34}$
1	1	1	0	0	0
0	0	1	0	1	1
0	1	1	1	1	0

.

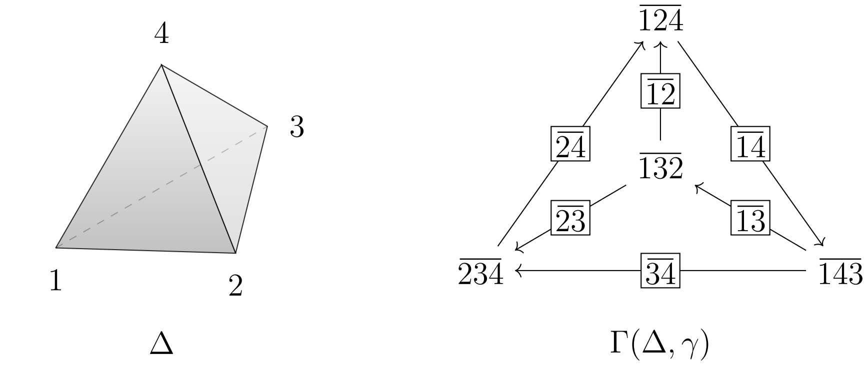

Figure 3: The hollow tetrahedron and its γ-incidence graph (cf. Example 24).

Example 25. Figure 4 shows a triangulated annulus Δ in the plane and its γ-incidence graph for the counter-clockwise orientation,

$$\gamma = \overline{125} + \overline{143} + \overline{154} + \overline{236} + \overline{265} + \overline{346}.$$

The boundary is $\partial\Delta = \left\{\overline{12}, \overline{13}, \overline{23}, \overline{45}, \overline{46}, \overline{56}\right\}$. Since the boundary is nonempty, the γ-incidence graph includes the vertex $*$, representing $0 \in C_d(\Delta)$. The Hilbert basis for $\ker^+ L_1$ has ten elements, two of which are displayed below:

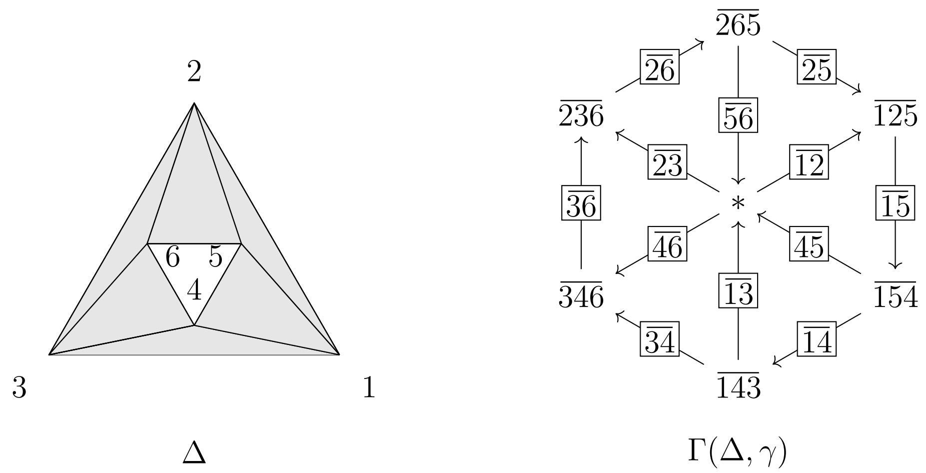

Figure 4: A triangulated annulus and its γ-incidence graph (cf. Example 25).

$\overline{12}$	$\overline{13}$	$\overline{14}$	$\overline{15}$	$\overline{23}$	$\overline{25}$	$\overline{26}$	$\overline{34}$	$\overline{36}$	$\overline{45}$	$\overline{46}$	$\overline{56}$
0	0	1	1	0	1	1	1	1	0	0	0
0	0	0	1	1	1	1	0	0	1	0	0

Two elements in the Hilbert basis for $\ker^+ L_1$.

Example 26. The condition of being orientable as a pseudomanifold is necessary in both Proposition 21 and Theorem 22. The Klein bottle simplicial complex in Figure 5 is a non-orientable pseudomanifold of dimension 2. Computing with Sage ([20]), we find $\mathcal{K}_1(\Delta) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ and that the Hilbert basis for $\ker^+ L_1$ has 14 elements. Three of these basis elements are not 0-1 vectors and, thus, are not incidence vectors of simple cycles in a directed graph.

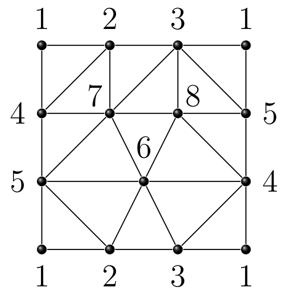

Figure 5: Triangulation of a Klein bottle (cf. Example 26).

Example 27 (Computing minimal winning degrees)**.** Let Δ be the hollow tetrahedron in Example 24, and use lexicographic ordering of the edges of Δ to identify $C_1(\Delta)$ with $\mathbb{Z}^6$, as usual. For the purpose of computing degrees, we can order the elements of the Hilbert basis $\mathcal{H}_1$ for Δ, computed in Example 24, as

$$h_1 = (1,1,1,0,0,0), \quad h_2 = (0,0,1,0,1,1), \quad h_3 = (0,1,1,1,1,0).$$

By Theorem 18, there exists an effective 1-chain $\tau \in \mathbb{Z}^6$ such that every 1-chain of degree at least $\delta := \deg(\tau)$ is winnable. In this example, we compute all minimal such δ (the set $\min(\mathcal{W}_i)$, using earlier notation). We then exhibit an infinite family of nonnegative realizable 1-degrees that are not realizable by winnable 1-chains.

Choose an effective $\tau \in C_1(\Delta) = \mathbb{Z}^6$ with $\deg(\tau) = \delta$, and suppose that every 1-chain of degree at least δ is winnable. Let $\sigma^{(0)}, \sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}$ be representatives for the elements of $\mathcal{K}_1(\Delta) \simeq \mathbb{Z}/4\mathbb{Z}$. Then the equivalence classes of 1-chains of degree δ in $\mathcal{J}_1 := C_1(\Delta)/\operatorname{im} L_1$ are $\tau + \sigma^{(i)}$ for $i = 0, \ldots, 3$ (cf. Remark 15). Each $\sigma^{(i)}$ has degree 0 by Theorem 13. By assumption $\tau + \sigma^{(i)}$ is winnable, so working modulo $\operatorname{im} L_1$, we can choose the $\sigma^{(i)}$ so that each $\tau + \sigma^{(i)}$ is effective. In order to minimize δ, we minimize τ.

First, suppose $\delta_1 = 0$. Since τ is effective and $\tau \cdot h_1 = \tau_1 + \tau_2 + \tau_3 = \delta_1 = 0$, it follows that $\tau_1 = \tau_2 = \tau_3 = 0$. Using this, it similarly follows that $\sigma_1^{(i)} = \sigma_2^{(i)} = \sigma_3^{(i)} = 0$ for $i = 0, 1, 2, 3$. Some linear algebra shows that $\mathcal{K}_1(\Delta)$ is generated by $(0,0,0,1,-1,1)$ and 1-chains in the image of the Laplacian which are 0 in the first three components are exactly those of the form $(0,0,0,4k,-4k,4k)$ for some integer k. So up to re-indexing, $\sigma^{(i)} = (0,0,0,i+4k_i,-i-4k_i,i+4k_i)$ for some integers k_i. Now consider the conditions on τ, besides $\tau \geqslant 0$, required to ensure each $\tau + \sigma^{(i)}$ is effective. These are

$$\begin{pmatrix} \tau_4 \\ \tau_5 \\ \tau_6 \end{pmatrix} \geqslant \begin{pmatrix} -i \\ i \\ -i \end{pmatrix} + k_i \begin{pmatrix} -4 \\ 4 \\ -4 \end{pmatrix}$$

for some integer k_i and for $i = 0, \ldots, 3$. For $i = 0$, we take $k_i = 0$ and see there is no additional condition imposed on τ; for $i = 1$, either $\tau_5 \geqslant 1$ or both τ_4 and τ_6 are at least 3; for $i = 2$, either $\tau_5 \geqslant 2$ or both τ_4 and τ_6 are at least 2; and for $i = 3$, either $\tau_5 \geqslant 3$ or both τ_4 and τ_6 are at least 1. Thus, to minimize τ, there are eight cases to consider. In all of these, $\deg(\tau) \geqslant (0, 3, 3)$.

Next, suppose $\delta_2 = 0$. By a similar argument (or by symmetry, swapping vertex 1 with 4 and vertex 2 with 3), we find minimal τ have degree at least $(3, 0, 3)$. Finally, suppose $\delta_3 = 0$. In that case, $\tau_2 = \tau_3 = \tau_4 = \tau_5 = 0$ and $\sigma_2^{(i)} = \sigma_3^{(i)} = \sigma_4^{(i)} = \sigma_5^{(i)} = 0$ for all i. However, requiring a chain of the form $(a, 0, 0, 0, 0, b)$ to represent an element in $\mathcal{K}_1(\Delta)$—and hence be in the kernel of ∂_1—forces $a = b = 0$. That is not possible since the $\sigma^{(i)}$ are a full set of representatives for $\mathcal{K}_1(\Delta)$. So we must have $\delta_3 \geqslant 1$.

Combining the above, we conclude δ is greater than or equal to one of $(0, 3, 3)$, $(3, 0, 3)$, or $(1, 1, 1)$. In fact, these three degrees are minimal winning degrees for Δ since there exist four effective 1-chains of each degree that are pairwise not linearly equivalent. We list these chains in the table below:

degree δ	representatives for $\mathcal{J}_1(\Delta)$			
$(0, 3, 3)$	$(0, 0, 0, 3, 0, 3)$,	$(0, 0, 0, 2, 1, 2)$,	$(0, 0, 0, 1, 2, 1)$,	$(0, 0, 0, 0, 3, 0)$
$(3, 0, 3)$	$(3, 0, 0, 3, 0, 0)$,	$(2, 1, 0, 2, 0, 0)$,	$(1, 2, 0, 1, 0, 0)$,	$(0, 3, 0, 0, 0, 0)$
$(1, 1, 1)$	$(1, 0, 0, 1, 0, 1)$,	$(1, 0, 0, 0, 1, 0)$,	$(0, 1, 0, 0, 0, 1)$,	$(0, 0, 1, 0, 0, 0)$

.

On a graph, there are only finitely many nonnegative degrees realizable by unwinnable divisors. That is not usually the case for a general simplicial complex. For instance, on our current Δ, consider the family of 1-chains $\sigma = (a, -b, b, 0, 0, 0)$ where $a \geqslant 0$ and $b > 0$. We have $\deg(\sigma) = (a, b, 0) \geqslant 0 = (0, 0, 0)$. Let τ be any effective 1-chain of degree $(a, b, 0)$. Taking the dot product of τ with each h_i, it follows that $\tau = (a, 0, 0, 0, 0, b)$, and thus $\sigma - \tau = (0, -b, b, 0, 0, -b)$. However, computing the Hermite normal form for L_1, we see that $\operatorname{im} L_1$ is spanned by $(1, 0, -1, 3, -2, 3)$, $(0, 1, -1, 1, -1, 2)$, and $(0, 0, 0, 4, -4, 4)$. It is straightforward to check that $\sigma - \tau \notin \operatorname{im} L_1$, and hence $\sigma \not\sim \tau$. Hence, σ is not winnable.

6 Forests

It is well-known that the dollar game on a graph is winnable for all initial configurations of degree zero if and only if the graph is a tree (e.g., cf. [2]). In this section, that result is extended to higher dimensions. We first recall the basics of trees on simplicial complexes as developed by Duval, Klivans, and Martin in [8] and [9]. In [8], it is shown that under certain circumstances, each critical group is isomorphic to the cokernel of a certain submatrix of the corresponding Laplacian matrix called the *reduced Laplacian*. Theorem 30 generalizes that result by loosening the hypotheses.

Definition 28. A *spanning i-forest of* Δ is an i-dimensional subcomplex $\Upsilon \subseteq \Delta$ with $\mathrm{Skel}_{i-1}(\Upsilon) = \mathrm{Skel}_{i-1}(\Delta)$ and satisfying the three conditions

1. $\widetilde{H}_i(\Upsilon) = 0$;

2. $\tilde{\beta}_{i-1}(\Upsilon) = \tilde{\beta}_{i-1}(\Delta)$;

3. $f_i(\Upsilon) = f_i(\Delta) - \tilde{\beta}_i(\mathrm{Skel}_i(\Delta))$.

In the case where $\tilde{\beta}_{i-1}(\Delta) = 0$, a spanning i-forest is called a *spanning i-tree*. The complex Δ is a *forest* if it is a spanning forest of itself, i.e., if $\widetilde{H}_d(\Delta) = 0$. If, in addition, $\tilde{\beta}_{d-1}(\Delta) = 0$, then Δ is a *tree*.

Remarks. Let Υ be an i-dimensional subcomplex of Δ sharing the same $(i-1)$-skeleton.

1. For a graph G, the above definition says that a (one-dimensional) spanning forest contains all of the vertices of G and: (i) has no cycles, (ii) has the same number of components as G, and (iii) has $m - c$ edges, where m is the number of edges and c is the number of components of G.

2. The condition $\widetilde{H}_i(\Upsilon) = 0$ is equivalent to the elements of the set
$$A := \{\partial_{\Upsilon,i}(f) : f \in \Upsilon_i\}$$
being linearly independent (over $\mathbb{Z}$ or, equivalently, over $\mathbb{Q}$).

3. Since Υ and Δ have the same $(i-1)$-skeleton, $\partial_{\Delta,i-1} = \partial_{\Upsilon,i-1}$, and hence, $\tilde{\beta}_{i-1}(\Upsilon) = \tilde{\beta}_{i-1}(\Delta)$ is equivalent to $\operatorname{rank}\operatorname{im}\partial_{\Upsilon,i} = \operatorname{rank}\operatorname{im}\partial_{\Delta,i}$.

4. It follows from the previous two remarks that Υ is a spanning i-forest if and only if A, defined above, is a basis for $\operatorname{im}\partial_{\Delta,i}$ over $\mathbb{Q}$, i.e, the columns of the matrix $\partial_{\Delta,i}$ corresponding to the i-faces of Υ are a $\mathbb{Q}$-basis for the column space of $\partial_{\Delta,i}$. In particular, spanning i-forests always exist.

5. Since $\partial_{\Delta,j} = \partial_{\mathrm{Skel}_i(\Delta),j}$ for all $j \leqslant i$, it follows the j-th reduced homology groups, Betti numbers, and critical groups for Δ and for $\mathrm{Skel}_i(\Delta)$ are the same for all $j < i$. In particular, this implies that the j-forests (resp., j-trees) of Δ are the same as those for $\mathrm{Skel}_i(\Delta)$ for all $j \leqslant i$.

Proposition 29 ([8, Prop 3.5], [9])**.** *Any two of the three conditions defining a spanning i-forest implies the remaining condition.*

The proof of the following is in the appendix. It generalizes a result in [8], where it is proved with the assumptions that Δ is pure, that $\tilde{\beta}_i(\Delta) = 0$ for all $i < d$, and that $\widetilde{H}_{i-1}(\Upsilon) = 0$.

Theorem 30. *Let Υ be an i-dimensional spanning forest of Δ such $\widetilde{H}_{i-1}(\Upsilon) = \widetilde{H}_{i-1}(\Delta)$. Let $\Theta := \Delta_i \setminus \Upsilon_i$. Define the* reduced Laplacian *$\tilde{L}$ of Δ with respect to Υ to be the square submatrix of L_i consisting of the rows and columns indexed by Θ. Then there is an isomorphism*
$$\mathcal{K}_i(\Delta) \xrightarrow{\sim} \mathbb{Z}\Theta / \operatorname{im}\tilde{L}$$
obtained by setting the faces of Υ_i equal to 0.

Definition 31. Define the *i-complexity* or *i-forest number* of Δ to be

$$\tau := \tau_i(\Delta) := \sum_{\Upsilon \subseteq \Delta} |\mathbf{T}(\widetilde{H}_{i-1}(\Upsilon))|^2$$

where the sum is over all spanning i-forests Υ of Δ.

Proposition 32. *$\tau_i(\Delta) = 1$ if and only if* $\mathrm{Skel}_i(\Delta)$ *is a spanning i-forest of Δ and $\widetilde{H}_{i-1}(\Delta)$ is torsion-free. If* $\mathrm{Skel}_i(\Delta)$ *is a spanning i-forest, regardless of whether $\widetilde{H}_{i-1}(\Delta)$ is torsion-free, then* $\mathrm{Skel}_i(\Delta)$ *is the unique spanning i-forest of Δ.*

Proof. Suppose that $\tau_i(\Delta) = 1$. Then Δ possesses a unique spanning i-forest Υ, and $\widetilde{H}_{i-1}(\Upsilon)$ is torsion-free. Considering ∂_i as a matrix, it follows that its set of columns has a unique maximal linearly independent subset: those columns corresponding to the faces of Υ. Since the columns of ∂_i are all nonzero, it must be that the columns corresponding to Υ are the only columns, i.e., $f_i(\Upsilon) = f_i(\Delta)$, and hence $\Upsilon = \mathrm{Skel}_i(\Delta)$. It follows that $\widetilde{H}_{i-1}(\Delta) = \widetilde{H}_{i-1}(\Upsilon)$ and hence is torsion-free.

Now suppose $\mathrm{Skel}_i(\Delta)$ is a spanning i-forest and let $\Upsilon \subseteq \Delta$ be any spanning i-forest. Since $\widetilde{H}_i(\mathrm{Skel}_i(\Delta)) = 0$, it follows from condition 3 of Definition 28 that

$$f_i(\Upsilon) = f_i(\Delta) - \tilde{\beta}_i(\mathrm{Skel}_i(\Delta)) = f_i(\Delta).$$

Hence, $\Upsilon = \mathrm{Skel}_i(\Delta)$. So $\mathrm{Skel}_i(\Delta)$ is the unique spanning i-forest of Δ. Further, if $\widetilde{H}_{i-1}(\mathrm{Skel}_i(\Delta))$ is torsion free, then $\tau_i(\Delta) = |\mathbf{T}(\widetilde{H}_{i-1}(\Delta))|^2 = 1$. □

Theorem 33 ([9, Theorem 8.1])**.** $|\mathbf{T}(\mathcal{K}_{i-1}(\Delta))| = \tau_i(\Delta)$.[1]

Corollary 34. *All $(i-1)$-chains of degree 0 on Δ are winnable if and only if $\tau_i(\Delta) = 1$.*

Proof. By Proposition 6 and Corollary 11, an $(i-1)$-chain of degree 0 is winnable if and only if it is linearly equivalent to the zero chain. The $(i-1)$-chains of degree 0 are the elements $(\ker^+ L_{i-1})^\perp = (\ker L_{i-1})^\perp$. Hence, by Theorem 13, all $(i-1)$-chains of degree 0 are winnable if and only if $\mathbf{T}(\mathcal{K}_{i-1}(\Delta)) = 0$. The result then follows from Theorem 33. □

Remark 35. As discussed in the introduction, Corollary 34 generalizes the result that all divisors of degree 0 on a graph are winnable if and only if the graph is a tree. However, for graphs, Corollary 34 says that all divisors of degree 0 on a *forest* are winnable. This apparent contradiction is resolved by the fact that for unconnected graphs, our simplicial notion of degree differs from the usual one for graphs. See Example 16.

Example 36. Simply being a spanning tree is not enough to guarantee winnability of all degree 0 divisors. Figure 6 illustrates a two-dimensional complex P which is a triangulation of the real projective plane. We have $\widetilde{H}_0(P) = \widetilde{H}_2(P) = 0$, and $\widetilde{H}_1(P) \approx \mathbb{Z}/2\mathbb{Z}$. Therefore, P is a spanning tree with tree number $\tau_2(P) = 4$. The cycle $\sigma := \overline{12} + \overline{23} - \overline{13}$ is a 1-chain in the image of ∂_2 and hence, by Remark 8, has degree 0. As argued in the first line of the proof of Corollary 34, if σ were winnable, it would be linearly equivalent

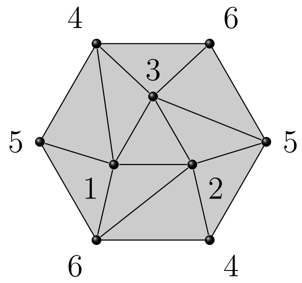

Figure 6: A triangulation of the real projective plane.

to the zero chain. We used Sage ([20]) to find that $\mathcal{K}_1(P) \approx \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\sigma \notin \operatorname{im} L_1$. Hence, 2σ is winnable, but σ is not.

Example 37. This example demonstrates that all i-chains of degree 0 of a complex can be winnable, even though there are unwinnable i-chains of nonnegative degree. Let Δ be the three-dimensional simplicial complex with facets

$$
\begin{gathered}
(1,2,3,4),(1,2,3,6),(1,2,3,7),(1,2,4,6),(1,2,5,7),(1,3,4,7),\\
(1,3,5,7),(1,4,5,6),(1,4,5,7),(1,4,6,7),(2,3,4,7),(2,3,5,6),\\
(2,3,5,7),(2,4,5,6),(3,4,5,7),(3,5,6,7),(4,5,6,7).
\end{gathered}
$$

We have $\tilde{H}_3(\Delta) \cong 0$ and $\tilde{H}_2(\Delta) \cong \mathbb{Z}$; so by Proposition 32, it follows that Δ is a forest with $\tau_3(\Delta) = 1$. Corollary 34 then implies that all 2-chains on Δ of degree 0 are winnable.

The Hilbert basis of $\ker^+ L_2$ for Δ has 445 elements.[2] Let A be the matrix whose rows are these Hilbert basis elements. Each 2-face of Δ may be considered as a chain and, thus, has a degree. These degrees form the 33 columns of A. It follows that the degrees of all effective 2-chains are precisely the nonnegative integer linear combinations of the columns of A. The Hilbert basis for the polyhedral cone generated by the columns of A consists of the columns of A and one other element δ. By the characterization of the Hilbert basis, δ cannot be realized by any effective two-chain, but using linear algebra it is possible to find non-effective two-chains of degree δ, one of which is

$$(1,2,3) - (1,2,7) + (1,3,5) + (1,3,6) + (1,4,6) + (1,6,7) + (2,4,5).$$

Thus, the above 2-chain is unwinnable but has nonnegative degree.

6.1 Spanning trees acyclic in codimension one

Definition 38. For each integer i, let

$$\Lambda_i(\Delta) = \operatorname{Span}_{\mathbb{Z}_{\geqslant 0}} \{\partial_{i+1}(f) : f \in \Delta_{i+1}\} \subset C_i(\Delta) := \mathbb{Z}\Delta_i.$$

[1] In [9], this theorem is stated only for $i = \dim(\Delta)$. The version stated here follows by restricting to $\operatorname{Skel}_i(\Delta)$ (cf. Remark 5).

[2] We used the PyNormaliz package in Sage ([20]) for the Hilbert basis computations in this example.

and

$$X_i(\Delta) := \{\sigma \in C_i(\Delta) : \partial_i(\sigma) \in \Lambda_{i-1}(\Delta)\}.$$

The above definition was introduced by S. Corry and L. Keenan ([6]). Since $\Lambda_{-1}(\Delta) = \mathbb{Z}_{\geqslant 0}$ and, therefore, $X_0(\Delta) = \{\sigma \in C_0(\Delta) : \partial_0(\sigma) \geqslant 0\}$, they regarded the sets $X_i(\Delta)$ as generalizing the notion of divisors of nonnegative degree on a graph and explored their relation to the winnability of the dollar game. They conjectured the equivalence of (1) and (2) in the following proposition and proved it in the case $i = 2$ on a simplicial surface.

Proposition 39. *The following are equivalent for $i \leqslant d$:*

(1) *Every $\sigma \in X_{i-1}(\Delta)$ is winnable.*

(2) $\mathcal{K}_{i-1}(\Delta) = 0$.

(3) $\mathrm{Skel}_i(\Delta)$ *is a spanning i-tree of Δ and $\widetilde{H}_{i-1}(\Delta) = 0$.*

In particular, when $i = d$, the three conditions are equivalent to Δ being a tree, acyclic in codimension one.

Proof. We first note that since Δ has the standard orientation, the only nonnegative element of $\ker \partial_{i-1}$ is 0. To see this, suppose $\sigma = \sum_{f \in \Delta_i} a_f f \neq 0$ with $a_f \geqslant 0$ for all f. Let $\overline{v_0 \cdots v_i}$ be the lexicographically largest element in the support of σ (with $v_0 < \cdots < v_i$). For each $v \in V$ such that $v \leqslant v_0$, let $g_v := \overline{v v_1 \cdots v_i}$. Then the coefficient of $\overline{v_1 \cdots v_i}$ in $\partial_{i-1}(\sigma)$ is $\sum_{v \in V} a_{g_v} > 0$. Hence, $\sigma \notin \ker \partial_{i-1}$. We will need this fact later in the proof.

Letting $\mathcal{E}$ denote the set of effective $(i-1)$-chains, we can write $X_{i-1}(\Delta) = \mathcal{E} + \ker \partial_{i-1}$. Thus, (1) is equivalent to $\mathcal{E} + \ker \partial_{i-1} \subseteq \mathcal{E} + \operatorname{im} L_{i-1}$, which in turn is equivalent to

$$(1)' \qquad \mathcal{E} + \ker \partial_{i-1} = \mathcal{E} + \operatorname{im} L_{i-1}$$

since $\operatorname{im} L_{i-1} \subseteq \ker \partial_{i-1}$. Now, if $\mathcal{K}_{i-1}(\Delta) = 0$, then $\operatorname{im} L_{i-1} = \ker \partial_{i-1}$, and $(1)'$ holds. Conversely, suppose $(1)'$ holds, and let $\sigma \in \ker \partial_{i-1}$. By $(1)'$, there exist $\tau \in \mathcal{E}$ and $\phi \in \operatorname{im} L_{i-1} \subseteq \ker \partial_{i-1}$ such that $\sigma = \tau + \phi$. But then $\sigma - \phi \in \mathcal{E} \cap \ker \partial_{i-1} = \{0\}$, which implies $\sigma = \phi \in \operatorname{im} L_{i-1}$. It follows that $\mathcal{K}_{i-1}(\Delta) = 0$. Therefore, (1) is equivalent to (2).

We now prove the equivalence of (2) and (3) using Proposition 32. If $\mathcal{K}_{i-1}(\Delta) = 0$, then $1 = |\mathbf{T}(\mathcal{K}_{i-1})| = \tau_i(\Delta)$ by Theorem 33. Further, the natural surjection $\mathcal{K}_{i-1}(\Delta) \to \widetilde{H}_{i-1}(\Delta)$ implies $\widetilde{H}_{i-1}(\Delta) = 0$. Hence, $\mathrm{Skel}_i(\Delta)$ is a spanning i-tree of Δ. Conversely, suppose that $\mathrm{Skel}_i(\Delta)$ is a spanning i-tree and $\widetilde{H}_{i-1}(\Delta) = 0$. Then $\tau_i(\mathrm{Skel}_i(\Delta)) = 1$, which implies that $\mathcal{K}_{i-1}(\Delta)$ is free by Theorem 33. However, the free part of $\mathcal{K}_{i-1}(\Delta)$ is the same as the free part of $\widetilde{H}_{i-1}(\Delta)$ by Corollary 14. Therefore, $\mathcal{K}_{i-1}(\Delta) = 0$. □

Example 40. This example shows that condition $\widetilde{H}_{i-1}(\Delta) = 0$ in part (3) of Proposition 39 is necessary. Consider the simplicial complex Δ pictured in Figure 7. By inspection, $\widetilde{H}_2(\Delta) = 0$ and $\widetilde{H}_1(\Delta) \simeq \mathbb{Z} \neq 0$. So the complex is a forest but not a tree.

One may compute directly that $\mathcal{K}_1(\Delta) \simeq \mathbb{Z}$ or argue as follows. By Proposition 32, we have $\tau_2(\Delta) = 1$. By Theorem 33, it follows that $|T(\mathcal{K}_1(\Delta))| = 1$. Then Corollary 14 says $\mathcal{K}_1(\Delta) = \widetilde{H}_1(\Delta) \simeq \mathbb{Z}$.

Now consider a generator for the first homology such as

$$\sigma = (0, 0, 0, 1, -1, 1) = \overline{23} - \overline{24} + \overline{34}.$$

The Hilbert basis $\mathcal{H}_1$ for $\ker^+ L_1$, computed by Sage ([20]), is given by the rows of the table

$\overline{12}$	$\overline{13}$	$\overline{14}$	$\overline{23}$	$\overline{24}$	$\overline{34}$
0	0	0	0	0	1
0	0	1	0	1	0
0	1	0	1	0	0
1	1	1	0	0	0

.

Ordering the elements of $\mathcal{H}_1$ as they appear in the table, top-to-bottom, we have $\deg(\sigma) = (1, -1, 1, 0) \not\geq 0$. So σ is not winnable even though $\partial_1(\sigma) = 0 \in \Lambda_0(\Delta)$.

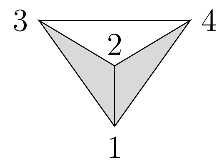

Figure 7: A simplicial complex with facets $\overline{123}$, $\overline{124}$, and $\overline{34}$ (cf. Example 40).

7 Further work

There is still much to be learned about winnability of the dollar game on a simplicial complex. Here, we will present three general open areas of investigation: computation of minimal winning degrees, algorithms for determining winnability, and generalization of the rank function.

Theorem 18 says there exists a realizable degree δ such that all i-chains of degree at least δ are winnable. Call any minimal such δ a *minimal winning degree* for i-chains on Δ. For divisors on connected graphs, there is one minimal winning degree, $g = |E| - |V| + 1$. We know of no such formulas in higher dimensions.

1. Is there a simple combinatorial description of the set of minimal winning degrees for the i-chains of a simplicial complex?

2. It would be nice to compute minimal winning degrees for a class of simplicial complexes. For example, what are the minimal winning degrees for $(d - 2)$-chains on the d-dimensional simplex?

On a graph, there are three standard methods of determining whether the dollar game is winnable, and if it is winnable, finding a sequence of moves leading to a winning position. One of these is a greedy algorithm. It proceeds as follows:

(i) Check if the divisor is effective. If so, the divisor is winnable.

(ii) Modify the divisor by borrowing at any vertex with a negative amount of dollars, prioritizing vertices that have borrowed earlier in the algorithm.

(iii) If all vertices have been forced to borrow, the original divisor is unwinnable. Otherwise, return to step (i).

The proof of the validity of this greedy algorithm (cf. [7, Section 3.1]) relies on two main facts. First, a vertex cannot be brought out of debt by only borrowing at other vertices, and second, the only way to leave a divisor unchanged through a series of borrowing moves is to borrow at every vertex an equal number of times. Neither of these two facts remains true for chains on a simplicial complex, so an immediate translation of the greedy algorithm fails in higher dimensions. The ideas in this paper suggest possible fixes for the second fact. For instance, one might attempt to modify the algorithm to avoid borrowing at any combination of vertices forming an element of the Hilbert basis $\mathcal{H}_i(\Delta)$ of the nonnegative kernel $\ker_i^+ L_i$. Our attempts in this direction have failed due to the first fact. So we propose the question:

3. Can the greedy algorithm for the dollar game on graphs be generalized to one for simplicial complexes?

Another method for determining winnability of the dollar game on a graph is through q-reduction of a divisor ([2], [3]). In this method, given a divisor, one computes a linearly equivalent standard form for the divisor with respect to a chosen vertex q. The game is winnable if and only if q is out of debt in this standard form. Knowing whether q-reduction generalizes to chains on a simplicial complex would be of general interest to the chip-firing community ([1, Problem 17], [12]). Perhaps the methods of [17] could be employed. In that work, q-reduction is interpreted as an instance of Gröbner reduction of the lattice ideal of the graph Laplacian. We formulate the general question in the context of the dollar game:

4. Can one define an efficiently computable standard representative of the equivalence class of a chain on a simplicial complex which is effective if and only if the chain is winnable?

A third way of computing winnability for graphs is to determine whether a certain simplex, defined using the columns of the Laplacian matrix, contains integer points (cf. [7, Section 2.3] or [5]). This method easily extends to the dollar game on a simplicial complex, and it is the one we use in our own computations. However, the general problem of determining whether a simplex has integer points is NP-hard unless the dimension is fixed. Even so, for graphs, q-reduction provides a method of determining winnability of a divisor that is polynomial in the size of the divisor and the size of the graph ([3]).

5. Is there any efficient algorithm for determining winnability of the dollar game on a simplicial complex?

The rank function, discussed in the introduction, is a measure of the robustness of winnability of a divisor on a graph. As noted in [2, Remark 1.13], for a divisor D on an algebraic curve, the same definition for rank would give $r(D) = \ell(D) - 1$, where $\ell(D)$ is the dimension of the vector space of global sections of the line bundle associated with D, appearing in the standard formulation of the Riemann-Roch theorem for curves. The Riemann-Roch theorem for divisors D on an algebraic surface can be thought of as a refinement of a lower bound on $\ell(D)$ in terms of data associated with D and the structure of the surface (by dropping the superabundance term). This motivates the following:

6. Is there a generalization of the rank function to 1-chains on a simplicial complex of dimension 2, measuring robustness of winnability and perhaps related to the Riemann-Roch theorem for algebraic surfaces? If so, can one find a combinatorial lower bound for it?

Appendix

In this appendix, we prove Proposition 21 and Theorem 30. The proof of Proposition 21 requires the following lemma.

Lemma 41. *Let Δ be a d-dimensional orientable pseudomanifold without boundary. Let $\gamma_1, \ldots, \gamma_m$ be the facets of Δ oriented so that $\gamma = \gamma_1 + \cdots + \gamma_m$ is a pseudomanifold orientation for Δ, i.e., such that $\partial_d(\gamma) = 0$. Let σ, τ be two $(d-1)$-chains in the image of ∂_d, and write*

$$\sigma = \sum_{i=1}^{m} s_i \partial_d(\gamma_i), \qquad \tau = \sum_{i=1}^{m} t_i \partial_d(\gamma_i)$$

for some integers $\{s_i\}$ and $\{t_i\}$. Then σ and τ are linearly equivalent if and only if $\sum_{i=1}^{m} s_i = \sum_{i=1}^{m} t_i \bmod m$.

Proof. Let ξ be a $(d-1)$-face of Δ. Then ξ is contained in exactly two facets, say γ_i and γ_j, and $L_{d-1}(\xi) = \pm(\partial_d(\gamma_i) - \partial_d(\gamma_j))$. By strong connectivity, it follows that $\partial_d(\gamma_i) - \partial_d(\gamma_j)$ is in the image of L_{d-1} for *any* pair $1 \leqslant i, j \leqslant m$, and thus,

$$\operatorname{im}(L_{d-1}) = \operatorname{Span}_{\mathbb{Z}} \{\partial_d(\gamma_i) - \partial_d(\gamma_j) : 1 \leqslant i, j \leqslant m\} = \{\textstyle\sum_{i=1}^{m} a_i \partial_d(\gamma_i) : \sum_{i=1}^{m} a_i = 0\}.$$

So linear equivalence of σ and τ is equivalent to being able to write

$$\sum_{i=1}^{m} (s_i - t_i) \partial_d(\gamma_i) = \sum_{i=1}^{m} a_i \partial_d(\gamma_i) \tag{5}$$

for some integers a_i summing to 0. Since the $\partial_d(\gamma_i)$ do not form a basis for the image of ∂_d, we cannot directly conclude something about the relation between the coefficients on both sides of equation (5). However, note that the existence of arbitrary integers a_i (not necessarily summing to 0) such that equation (5) holds is equivalent to

$$\rho := \sum_{i=1}^{m}(s_i - t_i - a_i)\gamma_i \in C_d(\Delta)$$

being in $\ker \partial_d = H_d(\Delta) = \mathbb{Z}\gamma$, and thus to the existence of an integer ℓ such that $\rho = \ell(\gamma_1+\cdots+\gamma_m)$. In this case, since the γ_i form a basis for $C_d(\Delta)$, we conclude $s_i - t_i - a_i = \ell$ for $i = 1, \ldots, m$. Summing, we have

$$\sum_{i=1}^{m} s_i = \sum_{i=1}^{m} t_i + \sum_{i=1}^{m} a_i \bmod m.$$

The result follows: if σ and τ are linearly equivalent, we can take $\sum_{i=1}^{m} a_i = 0$ and conclude that $\sum_{i=1}^{m} s_i = \sum_{i=1}^{m} t_i \bmod m$. Conversely, if $\sum_{i=1}^{m} s_i = \sum_{i=1}^{m} t_i + \ell m$ for some integer ℓ, set $a_i := s_i - t_i - \ell$ for all i. Then (5) holds, and so σ and τ are linearly equivalent. $\square$

Proof of Proposition 21. The projection mapping from the critical group to the relative homology group in codimension one gives the short exact sequence

$$0 \to \operatorname{im}\partial_d/\operatorname{im} L_{d-1} \to \mathcal{K}_{d-1}(\Delta) \to \widetilde{H}_{d-1}(\Delta) \to 0. \tag{6}$$

Let $\gamma = \gamma_1 + \cdots + \gamma_m$ be as in the statement of Lemma 41, and first consider the case where $\partial\Delta \neq \varnothing$. Reasoning as in the beginning of the lemma, we still have

$$X := \operatorname{Span}_{\mathbb{Z}}\{\partial_d(\gamma_i) - \partial(\gamma_j) : 1 \leqslant i, j \leqslant m\} \subseteq \operatorname{im} L_{d-1}.$$

Given any $f \in \partial\Delta$, there exists a unique γ_k whose boundary contains f in its support. Hence, $L_{d-1}(f) = \pm\partial_d(\gamma_k)$. Since $\operatorname{im} L_{d-1}$ contains X and $\partial_d(\gamma_k)$, it contains all of the $\operatorname{im}\partial_d(\gamma_i)$. So $\operatorname{im} L_{d-1} = \operatorname{im}\partial_d$, and hence, $\mathcal{K}_{d-1}(\Delta) = \widetilde{H}_{d-1}(\Delta)$, as claimed.

Now consider the case where $\partial\Delta = \varnothing$. Since Δ is an orientable pseudomanifold, $\widetilde{H}_{d-1}(\Delta)$ is torsion-free, and thus sequence (6) splits. By the lemma, the mapping

$$\begin{aligned} \mathbb{Z}/m\mathbb{Z} &\to \operatorname{im}\partial_d/\operatorname{im} L_{d-1} \\ k &\mapsto k\partial_d(\gamma_1) \end{aligned}$$

is an isomorphism. The result follows. $\square$

Our proof of Theorem 30 follows the general outline of that in [8] with substantial modifications.

Proof of Theorem 30. Considering the commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}\Upsilon_i & \xrightarrow{\partial_{\Upsilon,i}} & \mathbb{Z}\Upsilon_{i-1} & \xrightarrow{\partial_{\Upsilon,i-1}} & \mathbb{Z}\Upsilon_{i-2} \\ \downarrow & & \| & & \| \\ \mathbb{Z}\Delta_i & \xrightarrow{\partial_{\Delta,i}} & \mathbb{Z}\Delta_{i-1} & \xrightarrow{\partial_{\Delta,i-1}} & \mathbb{Z}\Delta_{i-2} \end{array},$$

we see

$$\operatorname{im}\partial_{\Upsilon,i} \subseteq \operatorname{im}\partial_{\Delta,i} \subseteq \ker\partial_{\Delta,i-1} = \ker\partial_{\Upsilon,i-1}.$$

Thus, there is a short exact sequence

$$0 \to \operatorname{im}\partial_{\Delta,i}/\operatorname{im}\partial_{\Upsilon,i} \to \widetilde{H}_{i-1}(\Upsilon) \to \widetilde{H}_{i-1}(\Delta) \to 0.$$

By hypothesis, $\widetilde{H}_{i-1}(\Upsilon) = \widetilde{H}_{i-1}(\Delta)$, and hence

$$\operatorname{im}\partial_{\Upsilon,i} = \operatorname{im}\partial_{\Delta,i}. \tag{7}$$

We now describe a basis for $\ker\partial_{\Delta,i}$. For each $\theta \in \Theta$, since $\operatorname{im}\partial_{\Upsilon,i} = \operatorname{im}\partial_{\Delta,i}$,

$$\partial_{\Delta,i}(\theta) = \sum_{\tau\in\Upsilon_i} a_\theta(\tau)\partial_{\Upsilon,i}(\tau) \tag{8}$$

for some $a_\theta(\tau) \in \mathbb{Z}$. Since $\widetilde{H}_i(\Upsilon) = 0$, the boundary mapping $\partial_{\Upsilon,i}$ is injective, and thus the coefficients $a_\theta(\tau)$ are uniquely determined. Define

$$\alpha(\theta) := \sum_{\tau\in\Upsilon_i} a_\theta(\tau)\tau$$

and extend linearly to get a well-defined mapping $\alpha : \mathbb{Z}\Theta \to \mathbb{Z}\Upsilon_i$. For each $\theta \in \Theta$, let

$$\hat{\theta} := \theta - \alpha(\theta).$$

We claim

$$\ker\partial_{\Delta,i} = \{\hat{\theta} : \theta \in \Theta\}.$$

The $\hat{\theta}$ are linearly independent elements of the kernel. To show they span, suppose $\gamma = \sum_{\sigma\in\Delta_i} b_\sigma\sigma \in \ker\partial_{\Delta,i}$. Consider

$$\gamma' := \gamma - \sum_{\sigma\in\Theta} b_\sigma\hat{\sigma} = \sum_{\sigma\in\Upsilon_i} b_\sigma\sigma + \sum_{\sigma\in\Theta} b_\sigma(\sigma - \hat{\sigma}) = \sum_{\sigma\in\Upsilon_i} b_\sigma\sigma + \sum_{\sigma\in\Theta} b_\sigma\alpha(\sigma).$$

Then since γ and the $\hat{\sigma}$ are in $\ker\partial_{\Delta,i}$, so is γ'. Further, since each $\alpha(\sigma) \in \mathbb{Z}\Upsilon_i$, so is γ'. But $\partial_{\Delta,i}$ restricted to Υ_i is equal to $\partial_{\Upsilon,i}$, which is injective. It follows that

$$\gamma = \sum_{\sigma\in\Delta_i} b_\sigma\sigma = \sum_{\sigma\in\Theta} b_\sigma\hat{\sigma}.$$

We thus have an isomorphism

$$\pi\colon \mathbb{Z}\Theta \xrightarrow{\sim} \ker\partial_{\Delta,i}$$

determined by $\sigma \mapsto \hat{\sigma}$ with inverse given by setting elements of Υ_i equal to 0:

$$\sum_{\sigma \in \Delta_i} b_\sigma \sigma \longmapsto \sum_{\sigma \in \Theta} b_\sigma \sigma.$$

Next, we claim there is a commutative diagram with exact rows

$$\begin{tikzcd} \mathbb{Z}\Theta \arrow[r, "\tilde{L}"] \arrow[d, "\iota"] & \mathbb{Z}\Theta \arrow[r] \arrow[d, "\pi", "\wr"'] & \operatorname{cok} \tilde{L} \arrow[r] \arrow[d, dashed] & 0 \\ \mathbb{Z}\Delta_i \arrow[r, "L_i"] & \ker \partial_{\Delta,i} \arrow[r] & \mathcal{K}_i(\Delta) \arrow[r] & 0 \end{tikzcd}$$

where ι is the natural inclusion. To check commutativity of the square on the left, let $\theta \in \Theta$. Then by definition of $\widetilde{L}$ and the fact that $\iota(\theta)$ is supported on Θ,

$$L_i \iota(\theta) = \rho + \tilde{L}\theta$$

for some $\rho \in \mathbb{Z}\Upsilon_i$. We then have $\pi^{-1}(\rho + \tilde{L}\theta) = \tilde{L}\theta$, as required. Hence, there is a well-defined vertical mapping $\operatorname{cok} \widetilde{L} \to \mathcal{K}_i(\Delta)$ on the right. By the snake lemma, that mapping is an isomorphism if and only if the mapping

$$\mathbb{Z}\Theta \to \mathbb{Z}\Delta_i / \ker L_i$$

given by composing ι with the quotient mapping is surjective. Therefore, to finish the proof, it suffices to show that for all $\gamma \in \Upsilon_i$, there exists $\delta \in \mathbb{Z}\Theta$ such that $\gamma + \delta \in \ker L_i$ (so then $\gamma = -\delta \bmod \ker L_i$).

Now $\ker L_i = \ker \partial_{\Delta,i+1} \partial^t_{\Delta,i+1} = \ker \partial^t_{\Delta,i+1}$. To get a description of $\ker \partial^t_{\Delta,i+1}$, consider the exact sequence

$$\mathbb{Z}\Delta_{i+1} \xrightarrow{\partial_{\Delta,i+1}} \mathbb{Z}\Delta_i \to \operatorname{cok} \partial_{\Delta,i+1} \to 0.$$

Applying the left-exact functor $\operatorname{Hom}(\cdot, \mathbb{Z})$, gives the exact sequence

$$\mathbb{Z}\Delta_{i+1} \xleftarrow{\partial^t_{\Delta,i+1}} \mathbb{Z}\Delta_i \leftarrow (\operatorname{cok} \partial_{\Delta,i+1})^* \leftarrow 0, \tag{9}$$

where we have identified $\mathbb{Z}\Delta_i$ and $\mathbb{Z}\Delta_{i+1}$ with their duals (using the bases Δ_i and Δ_{i+1}, respectively). There is an exact sequence,

$$0 \to \ker \partial_{\Delta,i} / \operatorname{im} \partial_{\Delta,i+1} \to \mathbb{Z}\Delta_i / \operatorname{im} \partial_{\Delta,i+1} \to \mathbb{Z}\Delta_i / \ker \partial_{\Delta,i} \to 0,$$

i.e,

$$0 \to \widetilde{H}_i(\Delta) \to \operatorname{cok} \partial_{\Delta,i+1} \to \mathbb{Z}\Delta_i / \ker \partial_{\Delta,i} \to 0. \tag{10}$$

However,

$$\mathbb{Z}\Delta_i / \ker \partial_{\Delta,i} \xrightarrow{\sim} \operatorname{im} \partial_{\Delta,i} = \operatorname{im} \partial_{\Upsilon,i} \simeq \mathbb{Z}\Upsilon_i$$

using (7) and the fact that $\partial_{\Upsilon,i}$ is injective. Since $\mathbb{Z}\Upsilon_i$ is free, sequence (10) splits:

$$\operatorname{cok} \partial_{\Delta,i+1} \approx \widetilde{H}_i(\Delta) \oplus \mathbb{Z}\Upsilon_i, \tag{11}$$

with each $\gamma \in \Upsilon_i$ identified with its class in $\operatorname{cok}\partial_{\Delta,i+1}$. Given $\gamma \in \Upsilon_i$, let $\gamma^*\colon \mathbb{Z}\Upsilon_i \to \mathbb{Z}$ be the dual function. Then use isomorphism (11), to identify γ^* with an element of $(\operatorname{cok}\partial_{\Delta,i+1})^*$. The image of γ^* in $\mathbb{Z}\Delta_i$ under the mapping in (9) is

$$\gamma + \sum_{\theta\in\Theta} a_\theta(\gamma)\theta,$$

which by exactness of (9) is an element of $\ker \partial^t_{\Delta,i+1}$. Letting $\delta := \sum_{\theta\in\Theta} a_\theta(\gamma)\theta$, we see that $\gamma + \delta \in \ker \partial^t_{\Delta,i+1}$, as required. □

Remark 42. Theorem 30 generalizes Theorem 3.4 of [8]. Remark 3.5 of [8] considers the case where Δ is the 6-vertex simplex, $i = 2$, and Υ is a certain triangulation of the real projective plane (shown in Fig. 3 of [10]). In this case,

$$\widetilde{H}_1(\Delta) = 0 \neq \widetilde{H}_1(\Upsilon) = \mathbb{Z}/2\mathbb{Z},$$

and

$$\mathcal{K}_2(\Delta) = (\mathbb{Z}/6\mathbb{Z})^4 \;\not\simeq\; \mathbb{Z}\Theta/\operatorname{im}\widetilde{L} \simeq (\mathbb{Z}/12\mathbb{Z}) \oplus (\mathbb{Z}/6\mathbb{Z})^3 \oplus (\mathbb{Z}/2\mathbb{Z}).$$

This example is given in [8] to show that the condition $\widetilde{H}_{i-1}(\Delta) = \widetilde{H}_{i-1}(\Upsilon) = 0$ in Theorem 3.4 cannot be dropped. Here, it serves the same purpose for the more relaxed hypothesis $\widetilde{H}_{i-1}(\Delta) = \widetilde{H}_{i-1}(\Upsilon)$ of Theorem 30.

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