The excluded minors for lattice path polymatroids

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Abstract

We find the excluded minors for the minor-closed class of lattice path polymatroids as a subclass of the minor-closed class of Boolean polymatroids. Like lattice path matroids and Boolean polymatroids, there are infinitely many excluded minors, but they fall into a small number of easily-described types.

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1 Introduction

We consider only polymatroids where the rank $\rho(X)$ of each set X is a nonnegative integer, so a *polymatroid* on a finite set E is a function $\rho : 2^E \to \mathbb{Z}$ that is

1. *normalized*, that is, $\rho(\emptyset) = 0$,
2. *non-decreasing*, that is, if $A \subseteq B \subseteq E$, then $\rho(A) \leqslant \rho(B)$, and
3. *submodular*, that is, $\rho(A \cup B) + \rho(A \cap B) \leqslant \rho(A) + \rho(B)$ for all $A, B \subseteq E$.

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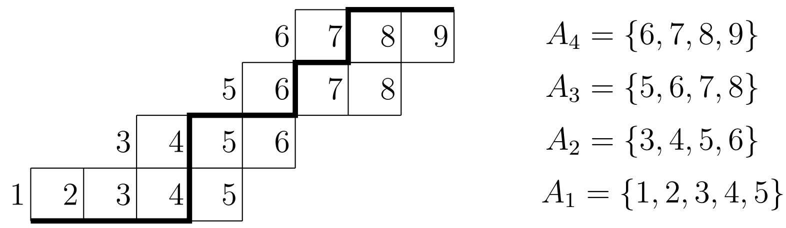

Figure 1: An example of the labeling of north steps and the sets of interest in the construction of a lattice path polymatroid.

Herzog and Hibi [3] treat some equivalent formulations of polymatroids, which are also called *integer polymatroids* or *discrete polymatroids*. We often write the ground set E of ρ as $E(\rho)$. For a positive integer t, a *t-polymatroid* is a polymatroid ρ on E for which $\rho(e) \leqslant t$ for all $e \in E$, or, equivalently, $\rho(A) \leqslant t|A|$ for all $A \subseteq E$. Thus, matroids are 1-polymatroids. The definitions of deletion and contraction, when cast for matroids using the rank function, generalize directly to polymatroids (see Section 2), and the notion of a minor carries over directly.

Let $[k]$ denote the set $\{1, 2, \ldots, k\}$. Let E be a finite set and let A_i, for $i \in [k]$, be (not necessarily distinct) subsets of E. We get a polymatroid ρ on E by, for $X \subseteq E$, setting

$$\rho(X) = \big|\{i \,:\, X \cap A_i \neq \emptyset\}\big|. \tag{1.1}$$

Such polymatroids are *Boolean polymatroids* or *transversal polymatroids*. The class of Boolean polymatroids is minor-closed, that is, every minor of a polymatroid in this class is also in this class.

Lattice path polymatroids, introduced by Schweig [6], are constructed as follows. (See Figure 1.) Take two lattice paths P and Q from $(1, 0)$ to (n, k), where P never rises above Q. These paths bound a region of the plane. We label each north step (a segment from (i, j) to $(i, j + 1)$ where i and j are integers) in this region by its first coordinate. For each $i \in [k]$, let A_i be the set of labels on the north steps in row i of this diagram, with $i = 1$ indexing the lowest row. The polymatroid ρ on $E = [n]$ given by equation (1.1) is the *lattice path polymatroid determined by the paths P and Q*. A *lattice path polymatroid* is any polymatroid that is isomorphic to such a polymatroid from paths. (Schweig [6, 7] treats lattice path polymatroids via their bases, which can be represented by monomials as follows: replace the labels $1, 2, \ldots, n$ by the distinct variables $x_1, x_2, \ldots, x_n$, respectively, and record a lattice path from $(1, 0)$ to (n, k) as the product of the variables on its north steps. Thus, the path in Figure 1 would be recorded as $x_4^2 x_6 x_7$.)

Lattice path polymatroids have many interesting properties [4, 6, 7]. Most lattice path matroids [1] are not lattice path polymatroids, but the two structures have much in common. Lattice path polymatroids form a minor-closed class of Boolean polymatroids. In this paper, we find the Boolean polymatroids that are excluded minors for the class of lattice path polymatroids. That is, we find the set $\mathcal{E}$ of Boolean polymatroids for which a Boolean polymatroid ρ is a lattice path polymatroid if and only if no member of $\mathcal{E}$ is a minor of ρ. The set $\mathcal{E}$, combined with the set of excluded minors for Boolean polymatroids, which was found by Matúš [5], gives a complete characterization of lattice path polymatroids. (Note however that, as we explain in Section 2, each member of $\mathcal{E}$ is a proper minor of infinitely many of the excluded minors that Matúš identified.) Lattice path matroids [2] and Boolean polymatroids [5] have infinitely many excluded minors, and the same is true of lattice path polymatroids: the set $\mathcal{E}$ is infinite. There are nine types of excluded minors in $\mathcal{E}$, and each type includes infinitely many polymatroids: four of the types are made up of polymatroids on three elements, four other types are made up of polymatroids on four elements, and one type is made up of polymatroids on any number of elements greater than two.

In Section 2, we review the relevant background on polymatroids, minors, Boolean polymatroids, and lattice path polymatroids; in particular, we state the excluded-minor characterization of Boolean polymatroids by Matúš [5] and explain how it relates to our results. In Section 3, we identify the type of excluded minors that can have any number of elements three or greater. In Section 4, we identify the types of excluded minors on three or four elements. In Section 5, specifically Theorem 13, we prove that the collection of excluded minors identified in Sections 3 and 4 is complete.

2 Background

Let ρ be a polymatroid on E. For $A \subseteq E$, the *deletion* $\rho_{\backslash A}$ and *contraction* $\rho_{/A}$, which are polymatroids on $E - A$, are given by $\rho_{\backslash A}(X) = \rho(X)$ and $\rho_{/A}(X) = \rho(X \cup A) - \rho(A)$ for all $X \subseteq E - A$. The *minors* of ρ are the polymatroids of the form $(\rho_{\backslash A})_{/B}$ (equivalently, $(\rho_{/B})_{\backslash A}$) for disjoint subsets A and B of E.

Let r_M denote the rank function of a matroid M. If $M_1, M_2, \ldots, M_k$ are matroids on E, then the function $\rho : 2^E \to \mathbb{Z}$ given by

$$\rho(X) = r_{M_1}(X) + r_{M_2}(X) + \cdots + r_{M_k}(X),$$

for $X \subseteq E$, is a polymatroid on E. We write $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$ to express this more briefly. (Not all polymatroids have such decompositions into matroid rank functions.) It is easy to check that if $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$, then

$$\rho_{\backslash A} = r_{M_1 \backslash A} + r_{M_2 \backslash A} + \cdots + r_{M_k \backslash A} \quad \text{and} \quad \rho_{/A} = r_{M_1/A} + r_{M_2/A} + \cdots + r_{M_k/A}.$$

It is easy to see that the Boolean polymatroid ρ on E given by equation (1.1), using the nonempty sets A_i for $i \in [k]$, can be written as the sum of k rank-1 matroids on E, namely,

$$\rho = r_{U_{1,A_1} \oplus U_{0,E-A_1}} + r_{U_{1,A_2} \oplus U_{0,E-A_2}} + \cdots + r_{U_{1,A_k} \oplus U_{0,E-A_k}}, \tag{2.1}$$

where, adapting the usual notation $U_{r,n}$ for uniform matroids, $U_{r,A}$ is the rank-r uniform matroid on the set A.

For completeness, we next prove a result that, while not new, is crucial to our work: the Boolean polymatroid ρ determines the nonempty sets A_i (but, of course, not how these sets are indexed).

Lemma 1. *Let $A_1, A_2, \ldots, A_k$ be (not necessarily distinct) nonempty subsets of a set E and let ρ be the Boolean polymatroid on E given by equation* (2.1)*. For each nonempty subset Z of E, the multiplicity of Z in the list $A_1, A_2, \ldots, A_k$ can be computed from ρ.*

Proof. For each $e \in E$, we define a property p_e that each integer $i \in [k]$ may or may not have: i has property p_e if $e \notin A_i$. Fix $X \subseteq E$. Now $i \in [k]$ has the properties p_e with $e \in X$ and no other properties if and only if $A_i = E - X$. The number of $i \in [k]$ that have all the properties p_e with $e \in X$, and maybe more, is $k - \rho(X)$. By inclusion-exclusion, it follows that the number of $i \in [k]$ that have the properties p_e with $e \in X$ and no others is

$$\sum_{Y \,:\, X \subseteq Y \subseteq E} (-1)^{|Y-X|}\big(k - \rho(Y)\big).$$

Since this is the number of i for which $A_i = E - X$, the lemma follows. ◻

This argument is closely related to the characterization of Boolean polymatroids by inequalities involving the rank function in Matúš [5, Lemma 2] (see Theorem 4 below).

Lemma 1 leads to a well-defined (up to relabeling the indices) notion of support: given E and A_i, for $i \in [k]$, and ρ given by equation (2.1), the *support* $s(e)$ of $e \in E$ is

$$s(e) = \{i \,:\, i \in [k] \text{ and } e \in A_i\}.$$

The *support of* ρ is $\cup_{e \in E} s(e)$, which is $[k]$ if no set A_i is empty. For example, for the lattice path polymatroid in Figure 1, we have $s(1) = s(2) = \{1\}$, $s(3) = s(4) = \{1,2\}$, $s(5) = \{1,2,3\}$, $s(6) = \{2,3,4\}$, $s(7) = s(8) = \{3,4\}$, and $s(9) = \{4\}$. In the lattice path polymatroid determined by two paths, the support $s(e)$ shows which rows in the diagram intersect the line $x = e$. Note that for any Boolean polymatroid ρ on E and all $X \subseteq E$,

$$\rho(X) = \Big|\bigcup_{e \in X} s(e)\Big|. \tag{2.2}$$

There is a natural linear order on $[k]$, namely, $1 < 2 < \cdots < k$, as well as on the ground set E of a lattice path polymatroid determined by two paths, but this is not so for the excluded minors. All orders (or orderings) that we consider are linear orders (also known as total orders). They can be denoted by listing the elements from least to greatest, as in $x_1, x_2, \ldots, x_t$, or via the conventional symbol for order, as in $x_1 < x_2 < \cdots < x_t$.

Since lattice path polymatroids in general are just isomorphic to those determined by two paths, it follows that ρ is a lattice path polymatroid if and only if there is some ordering of $[k]$ and some ordering $e_1, e_2, \ldots, e_n$ of E so that,

(S1) for each $i \in [n]$, the support $s(e_i)$ is an interval $[a_i, b_i]$ in the order on $[k]$, and,

(S2) in that order on $[k]$, we have $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$.

Such an ordering of E is a *lattice path ordering of* ρ, and such an ordering of $[k]$ is a *lattice path ordering of the support of* ρ. Given sets $Z_1, Z_2, \ldots, Z_\ell$ that partition $[k]$ such that a lattice path ordering of the support of ρ can be obtained by taking any ordering of the elements of Z_1 followed by any ordering of Z_2 and so on, finally ending with any ordering of the elements of Z_ℓ, we also refer to $Z_1, Z_2, \ldots, Z_\ell$ as a *lattice path ordering of the support of* ρ. For ease of notation, we also consider any sequence obtained from $Z_1, Z_2, \ldots, Z_\ell$ by adding copies of the empty set to be a *lattice path ordering of the support of* ρ.

By equation (2.2), knowing the support of each element e_i in a Boolean polymatroid ρ determines ρ. Different elements in a Boolean polymatroid may have the same support, so the map taking each element to its support need not be injective. Also, the support $s(e_i)$ depends on the order in which we write the sets $A_1, A_2, \ldots, A_k$, and changing the order by a permutation π just replaces each support by its image under π. To determine whether ρ is a lattice path polymatroid, we need to determine whether a lattice path ordering of ρ exists, and its existence goes hand in hand with the existence of a compatible lattice path ordering of the support.

From equation (2.1) and the discussion of minors before it, it follows that, in terms of supports, deleting e from a Boolean polymatroid corresponds to deleting its support set $s(e)$, while contracting e from a Boolean polymatroid corresponds to deleting $s(e)$ and, for each element $f \in E - e$, replacing $s(f)$ by $s(f) - s(e)$ since contracting a nonloop in a rank-1 matroid yields a rank-0 matroid. It now easily follows that minors of lattice path polymatroids are lattice path polymatroids, but we get much more: if ρ is a Boolean polymatroid and $\rho_{\backslash e}$ is a lattice path polymatroid, then so is $\rho_{/e}$ since, given a linear order on $[k]$ in which the support sets for $\rho_{\backslash e}$ are intervals that satisfy (S2), for the induced linear order on $[k] - s(e)$, the support sets for $\rho_{/e}$ are intervals that satisfy (S2). Thus, we have the following lemma, which is highly atypical for minor-closed classes of polymatroids.

Lemma 2. *If ρ is a Boolean polymatroid that is a not a lattice path polymatroid, then ρ is an excluded minor for the class of lattice path polymatroids if and only if each single-element deletion is a lattice path polymatroid.*

As noted above, the class of lattice path polymatroids is a minor-closed subclass of the class of Boolean polymatroids. The excluded-minors for the class of Boolean polymatroids were found by Matúš. The next theorem recasts [5, Theorem 3].

Theorem 3. *The excluded minors for Boolean polymatroids are the polymatroids of the form*

$$-c_E\, r_{U_{1,E}} + \sum_{X\,:\,\emptyset \neq X \subsetneq E} c_X\, r_{U_{1,X} \oplus U_{0,E-X}} \tag{2.3}$$

where $|E| \geqslant 3$, c_E is a positive integer, all coefficients c_X are nonnegative integers, and $c_X \geqslant c_E$ if $|X| = |E| - 1$.

Thus, there are infinitely many excluded minors for Boolean polymatroids. To translate this result into an infinite list of the excluded minors, one must take isomorphism into account since acting on 2^E with a permutation of E yields isomorphic excluded minors that all have the form in (2.3). It follows from Theorem 3 that each Boolean polymatroid ρ' on E' is a minor of infinitely many of these excluded minors: take any superset

E of E' with $|E - E'| \geqslant 2$ and let ρ be as in (2.3) where, for $X \subseteq E'$, the coefficient c_X is the number of copies of $r_{U_{1,X} \oplus U_{0,E'-X}}$ when ρ' is written as in equation (2.1); then $\rho_{/E-E'} = \rho'$. Similarly, one can construct excluded minors for Boolean polymatroids that have ρ' as a deletion. Given a characterization of a proper minor-closed class of Boolean polymatroids by its excluded minors relative to the class of Boolean polymatroids, there is an excluded-minor characterization of the class relative to the class of all polymatroids; however, the excluded minors relative to the class of Boolean polymatroids may, in some cases, be more illuminating. Of course, each Boolean polymatroid that is an excluded minor relative to the class of Boolean polymatroids is also an excluded minor relative to the class of all polymatroids.

Besides using the excluded minors, there is another way to determine whether a polymatroid is Boolean. The next result is part of [5, Lemma 1].

Theorem 4. *A polymatroid ρ on E is Boolean if and only if, for all $X \subseteq E$,*

$$\sum_{Y \,:\, X \subseteq Y \subseteq E} (-1)^{|Y-X|}\big(\rho(E) - \rho(Y)\big) \geqslant 0.$$

Given these factors, it is reasonable to seek characterizations of proper minor-closed classes of Boolean polymatroids by their excluded minors relative to the class of Boolean polymatroids. This is exactly what we do for lattice path polymatroids.

3 Boolean cycles and a property of lattice path orderings

We start with a necessary condition for an ordering on the ground set E of a Boolean polymatroid ρ to be a lattice path ordering of E, and so for ρ to be lattice path.

Lemma 5. *Let ρ be a Boolean polymatroid on E. If ρ is a lattice path polymatroid on E and $e_1, e_2, \ldots, e_n$ is a lattice path ordering of ρ, then the following property holds:*

(S3) *if $S \neq \emptyset$ and $S \subseteq s(e_h)$ for at least one $h \in [n]$, then the elements whose supports contain S are $e_i, e_{i+1}, \ldots, e_{j-1}$, and e_j for some i and j with $1 \leqslant i \leqslant j \leqslant n$.*

Proof. By relabeling if needed, we can take the usual order $1, 2, \ldots, k$ on the support of ρ and let $s(e_t)$ be the interval $[a_t, b_t]$ in $[k]$. Let $i \in [k]$ be least with $S \subseteq s(e_i)$, and let $j \in [k]$ be greatest with $S \subseteq s(e_j)$. The result holds if $i = j$, so assume that $i < j$. Now $S \subseteq s(e_i) \cap s(e_j) = [a_j, b_i]$. Since $a_i \leqslant a_{i+1} \leqslant \cdots \leqslant a_j$ and $b_i \leqslant b_{i+1} \leqslant \cdots \leqslant b_j$, we have $[a_j, b_i] \subseteq s(e_h)$ for all h with $i \leqslant h \leqslant j$, as needed. □

Recall that, by equation (2.2), a Boolean polymatroid can be given by its supports. That is how we will define each of the excluded minors for lattice path polymatroids.

We first treat a family of excluded minors that, for each $n \geqslant 3$, contains infinitely many polymatroids with n elements. We call a polymatroid in this family a *Boolean cycle*, or a *Boolean n-cycle*, where n is the number of elements. Let $n \geqslant 3$. The supports of the elements $e_1, e_2, \ldots, e_n$ of a *Boolean n-cycle* have the form $s(e_i) = Z_{2i-1} \cup Z_{2i} \cup Z_{2i+1}$, where the indices are taken modulo $2n$ (thus, $s(e_n) = Z_{2n-1} \cup Z_{2n} \cup Z_1$), and

- $Z_1, Z_2, \ldots, Z_{2n}$ are pairwise disjoint, and
- for all $i \in [n]$, the set Z_{2i-1} is nonempty.

The sets Z_{2i} may be empty. The example of smallest rank and size is three coplanar lines, that is, e_1, e_2, e_3 each have rank 2 and any set of two or three of them has rank 3. Also, three Boolean cycles have rank 4 and so can be seen as made up of faces of the four-vertex simplex: taking all edges other than one pair of non-coplanar edges gives a 4-cycle; we get Boolean 3-cycles from (a) two planes and the edge that neither contains and (b) a plane and two edges not contained in it. As with the description of the excluded minors for Boolean polymatroids in Theorem 3, the definition of Boolean n-circuits on E yields many isomorphic polymatroids; we are, of course, interested in these polymatroids up to isomorphism.

We now prove that each Boolean n-cycle is an excluded minor.

Lemma 6. *Let ρ be a Boolean n-cycle for some integer $n \geqslant 3$. Then ρ is an excluded minor for the class of lattice path polymatroids.*

Proof. Suppose ρ is lattice path. Since the supports of only two elements contain Z_3, namely e_1 and e_2, these elements are consecutive in the lattice path ordering of ρ. Likewise, since Z_{2i-1} is only contained in $s(e_{i-1})$ and $s(e_i)$ for all $i \in \{3, 4, \ldots, n\}$, the ordering on the elements of ρ must be $e_1, e_2, \ldots, e_n$. However $s(e_1) \cap s(e_n) = Z_1$ and $Z_1 \nsubseteq s(e_2)$, so (S3) fails, and so ρ is not lattice path.

Take $i \in [n]$. Then $e_{i+1}, e_{i+2}, \ldots, e_n, e_1, e_2, \ldots, e_{i-1}$ is a lattice path ordering of $\rho_{\backslash e_i}$. Thus, ρ is an excluded minor for the class of lattice path polymatroids by Lemma 2. $\square$

4 Excluded minors with three or four elements

In this section, we define the types of excluded minors that are not Boolean cycles. Each of these types has three or four elements. We also prove that our collection of three-element excluded minors is complete.

We first note that each polymatroid on a set of at most two elements is a lattice path polymatroid; in particular, it is a Boolean polymatroid. This is because, for a polymatroid ρ on $E = \{e, f\}$ with $k = \rho(E)$, we can let $s(e)$ be the first $\rho(e)$ elements, and $s(f)$ the last $\rho(f)$ elements, of $[k]$.

We turn to the Boolean excluded minors that have three elements and are not Boolean 3-cycles. They come in types T_1, T_2, T_3, and T_4, and there are infinitely many of each type. In order to characterize these four types of Boolean polymatroids, say with elements e, f, and g, it suffices to identify which of the areas in the Venn diagram of their supports in Figure 2 are empty, and which are nonempty. Note that if $Z = \emptyset$ and each of W, X, and Y is nonempty, then ρ is a Boolean 3-cycle, independent of whether or not T, U, or V are empty.

In the following list, we assume that ρ is a Boolean polymatroid and $E(\rho) = \{e, f, g\}$, where the supports of e, f, and g are as shown in Figure 2. We view elements of a rank-k

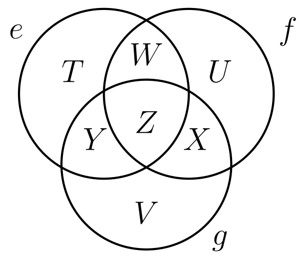

Figure 2: Venn diagram showing the supports of elements e, f, and g in a Boolean polymatroid, ρ.

Boolean polymatroid in terms of their supports, which are subsets of $[k]$ and so can be identified with faces of a simplex with k vertices, so the examples that we provide are given as sets of faces of a simplex.

(T_1) The polymatroid ρ is type T_1 if

- T, U, V, and Z are all nonempty.

◦ Example: three lines in rank four that, on the simplex, share a vertex (the vertex does not correspond to an element of the polymatroid).

(T_2) The polymatroid ρ is type T_2 if

- W, X, Y, and Z are all nonempty and
- at least one of T, U, or V is empty.

◦ An example: three planes in the four-vertex simplex.

(T_3) The polymatroid ρ is type T_3 if

- T, U, W, and Z are all nonempty and
- $V = X = Y = \emptyset$.

◦ Example: two planes of the four-vertex simplex, along with one of the two points on the line that the planes share on the simplex.

(T_4) The polymatroid ρ is type T_4 if

- T, W, Y, and Z are all nonempty and
- $X = V = \emptyset$.

◦ The set U may or may not be empty. Example with $U = \emptyset$: an element of rank four and two lines on the corresponding simplex that share one vertex (the vertex is not in the polymatroid). Example with $U \neq \emptyset$: start with a similar configuration in a four-vertex simplex, but embedded in a five-vertex simplex, with one of the lines extended to a plane that includes the new vertex.

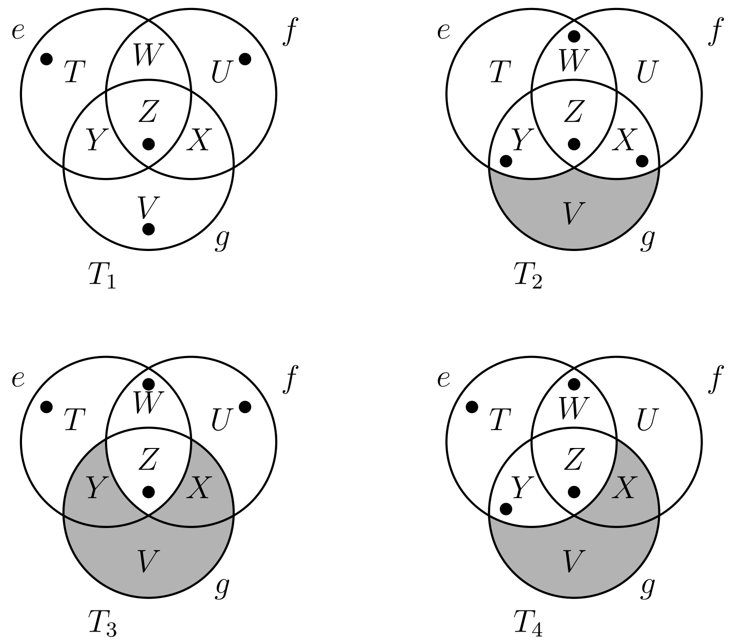

Figure 3: Venn diagrams showing the supports of elements e, f, and g in Boolean polymatroids of types T_1, T_2, T_3, and T_4. Areas shaded gray indicate that those sets are empty. A point in an area indicates that that set is nonempty. The other sets may or may not be empty.

Using equation (2.2), it is easy to translate these descriptions into the values of ρ. For instance, a polymatroid ρ on $E = \{e, f, g\}$ is type T_1 if and only if there are positive integers z, t, u, and v, and nonnegative integers w, x, and y, for which

$$\rho(e) = z + t + w + y, \qquad \rho(f) = z + u + w + x, \qquad \rho(g) = z + v + x + y,$$

$$\rho(e, f) = z + t + u + w + x + y, \qquad \rho(e, g) = z + t + v + w + x + y,$$

$$\rho(f, g) = z + u + v + w + x + y, \qquad \rho(E) = z + t + u + v + w + x + y.$$

We next show that the polymatroids of these four types, along with Boolean 3-cycles, are the 3-element Boolean excluded minors for lattice path polymatroids.

Lemma 7. *A Boolean polymatroid ρ is an excluded minor for the class of lattice path polymatroids, where $|E(\rho)| = 3$, if and only if ρ is type T_1, T_2, T_3, or T_4 or ρ is a Boolean 3-cycle.*

Proof. It is straightforward to check that if ρ is type T_1, T_2, T_3, or T_4 or a Boolean 3-cycle, then ρ is not lattice path. Since every two-element polymatroid is lattice path, ρ is therefore an excluded minor for the class of lattice path polymatroids.

Assume then that ρ is a three-element excluded minor for the class of lattice path polymatroids. Let $E = \{e, f, g\}$ where the supports of these elements are shown in Figure 2. Suppose first that $Z = \emptyset$. If $W = \emptyset$, then T, Y, V, X, U is a lattice path ordering of the support of ρ, which is a contradiction. By symmetry, none of W, X, Y is empty, and ρ is a Boolean 3-cycle.

Now assume that $Z \neq \emptyset$. If none of T, U, V is empty, then ρ is type T_1, so we assume not; say $V = \emptyset$. If each of W, X, Y is nonempty, then ρ is type T_2, so we assume not. If $W = \emptyset$, then T, Y, Z, X, U is a lattice path ordering of the support of ρ, which is a contradiction. Therefore $W \neq \emptyset$ and, without loss of generality, $X = \emptyset$. If $T = \emptyset$, then Y, Z, W, U is a lattice path ordering of the support of ρ, which is a contradiction. Hence $T \neq \emptyset$. If Y is nonempty, then ρ is type T_4, so we assume not. If $U = \emptyset$, then T, W, Z is a lattice path ordering of the support of ρ, which is a contradiction. Therefore $U \neq \emptyset$ and ρ is type T_3. □

We now present all four-element Boolean polymatroids, other than Boolean 4-cycles, that are excluded minors for the class of lattice path polymatroids. There are four types, and infinitely many of each type. Let ρ be a Boolean polymatroid on four elements.

(F_1) The polymatroid ρ is type F_1 if the supports have the form $A \cup B$, $A \cup B \cup C$, $B \cup C \cup D$, and $B \cup D$ where

- A, B, C, and D are pairwise disjoint and
- $\emptyset \notin \{A, B, C\}$.

◦ Example with $D \neq \emptyset$: two planes in the four-vertex simplex along with a line in exactly one of those planes and the line in just the other plane that is coplanar with the first line. Example with $D = \emptyset$: a vertex in a triangle, the two lines containing that vertex, and the triangle.

(F_2) The polymatroid ρ is type F_2 if the supports have the form A, B, C, and D where

- A, B, and C are pairwise disjoint, and
- $A \cap D$, $B \cap D$, and $C \cap D$ are all nonempty.

◦ Example: three noncolinear points along with the plane that contains them.

(F_3) The polymatroid ρ is type F_3 if the supports have the form A, A', B, and C where

- $A' \subseteq A$ and $A' \neq A$, and
- $B \cap C = \emptyset$, and
- both B and C are disjoint from $A - A'$, and neither is disjoint from A'.

◦ Example: two points, the line they span, and a plane that contains that line.

(F_4) The polymatroid ρ is type F_4 if the supports have the form $A \cup B \cup C$, $B \cup C \cup D$, $C \cup D \cup E$, and $A \cup C \cup E$ where

- A, B, C, D, and E are all pairwise disjoint and nonempty.

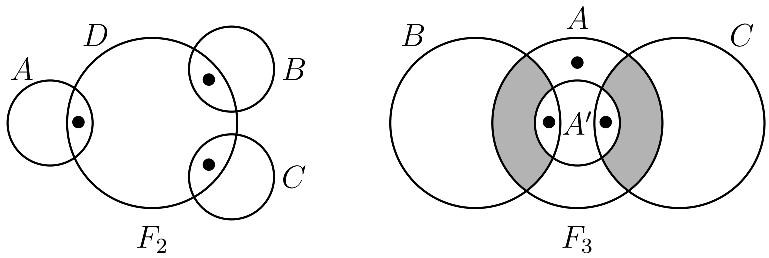

Figure 4: Venn diagrams showing the supports of elements in a Boolean polymatroid. Areas shaded gray indicate that those sets are empty. A point in an area indicates that that set is nonempty. Areas that are not gray and contain no point may be empty or nonempty.

- Example: take four of the six lines in a four-vertex simplex, with the pair omitted being skew; embed this simplex in a five-vertex simplex and replace each line by the plane that is spanned by it and the new vertex; these four planes are the elements of the polymatroid.

The Venn diagrams of the supports of the element for types F_2 and F_3 are shown in Figure 4, where a point appears in an area for which the set is known to be nonempty, and a darkened area indicates that the set is empty. It is straightforward to check that if ρ is type F_1, F_2, F_3, or F_4, then ρ is not lattice path, but all of its proper minors are.

5 Proof of the main result

In this section we prove our main result: when considered as a subclass of the class of Boolean polymatroids, the set of excluded minors given in the previous two sections for the class of lattice path polymatroids is complete.

We start with an easy result that is worth noting.

Lemma 8. *If ρ is a Boolean polymatroid that is an excluded minor for the class of lattice path polymatroids, and e and f are distinct elements of $E(\rho)$, then $s(e) \neq s(f)$.*

Proof. If $s(e) = s(f)$, then any lattice path ordering of $\rho_{\backslash e}$ can be extended to a lattice path ordering of ρ by adding e directly following f. □

Lemmas 9, 10, and 11 treat the case in which, in a Boolean polymatroid that is an excluded minor, the support of some element is contained in another.

Lemma 9. *Let ρ be a Boolean polymatroid that is an excluded minor for the class of lattice path polymatroids. If $e, f, g \in E(\rho)$ and $s(e) \subsetneq s(f) \subsetneq s(g)$, then ρ is type F_1 or F_3.*

Proof. No 3-element excluded minor has such a chain, so $|E(\rho)| \geqslant 4$ and ρ has no minors of types T_1, T_2, T_3, or T_4. Let C be a subset of $E(\rho)$ of maximum size that can be ordered so that the support of each element is contained in the support of the next. Suppose that $|C| \geqslant 3$. By Lemma 8, no two supports are equal. Let e be the element in C with the smallest support. Then $\rho_{\backslash e}$ has a lattice path ordering $e_1, e_2, \dots, e_n$ where $s(e_i) = [a_i, b_i]$ for all i, and $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$. By (S3) in Lemma 5, up to reversing the ordering on $\rho_{\backslash e}$, we may assume that $s(e) \subsetneq s(e_k) \subsetneq s(e_{k+1}) \subsetneq \cdots \subsetneq s(e_\ell)$ where $1 \leqslant k < k + |C| - 2 = \ell \leqslant n$. It follows that $a_k = a_{k+1} = \cdots = a_\ell$ and $b_k < b_{k+1} < \cdots < b_\ell$. Let $[a, b]$ be the smallest interval such that $s(e) \subseteq [a, b]$. Thus, $a, b \in s(e)$ and $a_k \leqslant a \leqslant b \leqslant b_k$. Also, $s(e)$ might be a proper subset of $[a, b]$.

We first show that $s(e)$ is an interval in some lattice path ordering on $\rho_{\backslash e}$. To show this, assume that $s(e) \neq [a, b]$. The next two results are the key to being able to reorder $[a, b]$ so that $s(e)$ is an interval and the new ordering of $E(\rho)$ is a lattice path ordering of $\rho_{\backslash e}$.

9.1. *If $p < k$ and $s(e_p) \cap ([a, b] - s(e)) \neq \emptyset$, then $[a, b] \subseteq s(e_p)$.*

Note that $a_p \leqslant a_\ell \leqslant a$ and $b < b_\ell$. If $b_p < b$, then $b \notin s(e_p)$, so $\rho|\{e, e_p, e_\ell\}$ is type T_4, but that is impossible. So $b \leqslant b_p$, and the conclusion of 9.1 holds.

9.2. *If $q > \ell$ and $s(e_q) \cap s(e) \neq \emptyset$, then $[a, b] - s(e) \subseteq s(e_q)$.*

If the conclusion failed, then neither a nor the first element in $[a, b] - s(e)$ would be in $s(e_q)$; it would follow that $\rho|\{e, e_\ell, e_q\}$ is type T_4, but that is impossible. So 9.2 holds.

With 9.1 and 9.2, we can rearrange $[a, b]$, placing $s(e)$ first and $[a, b] - s(e)$ second, to make $s(e)$ into an interval without changing the lattice path ordering on $\rho_{\backslash e}$. So we may assume that $s(e)$ is the interval $[a, b]$.

If we had $a = a_k$, then $(s(e_{k-1}) - s(e)) \cap s(e_k) \neq \emptyset$ (otherwise inserting e between e_{k-1} and e_k would give a lattice path ordering for ρ, which is impossible), so $\rho|\{e_{k-1}, e, e_\ell\}$ would have type T_3, contrary to ρ being an excluded minor. So $a_k < a$. Thus, $[a, b]$ cannot be moved to the beginning of $s(e_k)$, so either the support of some element before e_k blocks it, that is,

(i) there is a $p < k$ with $[a_k, a) \cap s(e_p) \neq \emptyset$ and $s(e) \nsubseteq s(e_p)$,

or the support of some element after e_ℓ blocks it, and so

(ii) there is a $q > \ell$ with $s(e) \cap s(e_q) \neq \emptyset$ and $a_k \notin s(e_q)$.

Suppose (i) occurs. If $s(e) \cap s(e_p) \neq \emptyset$, then $\rho|\{e, e_p, e_\ell\}$ is type T_4, which is impossible. Therefore $b_p < a$, and $\rho|\{e, e_p, e_k, e_\ell\}$ is type F_3 (since $s(e)$ and $s(e_p)$ are disjoint, use them as B and C). Now assume that (ii) occurs. Since $\rho|\{e, e_\ell, e_q\}$ is not type T_3 or T_4, we know that $a_q \leqslant a$ and $b_q = b_\ell$. Since $\rho|\{e, e_k, e_q\}$ is not type T_3, we know that $a_q = a$ and $b = b_k$. Then $\rho|\{e, e_k, e_\ell, e_q\}$ is type F_1, where $D = \emptyset$. □

For a Boolean polymatroid ρ, we define its *support graph* $G(\rho)$ to be the graph with vertex set $E(\rho)$ and edge set $\{ef : e, f \in E(\rho),\ e \neq f,\ s(e) \cap s(f) \neq \emptyset\}$. Note that, if $G(\rho)$ is an n-cycle for some integer $n \geqslant 4$, then ρ is a Boolean n-cycle.

Lemma 10. *Let ρ be a Boolean polymatroid that is an excluded minor for the class of lattice path polymatroids. Then $G(\rho)$ is connected. Furthermore, if $s(e) \subsetneq s(f)$ for some elements $e, f \in E(\rho)$, then $G(\rho_{\backslash e})$ is also connected.*

Proof. Suppose G has a component with vertex set A where $E(\rho) - A$ is not empty. Then $\rho|A$ and $\rho|(E(\rho) - A)$ each have lattice path orderings, and concatenating these two orderings gives a lattice path ordering of ρ, which is a contradiction.

Suppose $s(e) \subsetneq s(f)$, and $G(\rho_{\backslash e})$ has distinct components X and Y. Then $s(e)$ has a nonempty intersection with $s(x)$ and $s(y)$ for some $x \in V(X)$ and $y \in V(Y)$. Then xfy is a path connecting X and Y in $G(\rho_{\backslash e})$, which is a contradiction. □

We now consider the general case that the support of one element contains another.

Lemma 11. *Let the Boolean polymatroid ρ be an excluded minor for the class of lattice path polymatroids. If $s(e) \subsetneq s(f)$ for some $e, f \in E(\rho)$, then ρ is type T_3, T_4, F_1, F_2, or F_3.*

Proof. Let $e_1, e_2, \ldots, e_n$ be a lattice path ordering of $\rho_{\backslash e}$, where $s(e_k) = [a_k, b_k]$ with $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$. Say f is e_i.

We first show

11.1. *if $s(e) \subsetneq s(g)$ for some $g \in E(\rho) - \{e, e_i\}$, then ρ is type T_3, F_1, or F_3.*

If ρ is not type F_1 or F_3, then Lemma 9 gives $s(e_i) \not\subseteq s(g)$ and $s(g) \not\subseteq s(e_i)$ for all such g. First assume that all such g satisfy $s(e) = s(e_i) \cap s(g)$. By Lemma 5, it follows that either e_{i-1} or e_{i+1} is such a g. If $g = e_{i-1}$, then $s(e) = [a_i, b_{i-1}]$, so $e_1, e_2, \ldots, e_{i-1}, e, e_i, \ldots, e_n$ is a lattice path ordering of ρ; if $g = e_{i+1}$, then $s(e) = [a_{i+1}, b_i]$, so $e_1, e_2, \ldots, e_i, e, e_{i+1}, \ldots, e_n$ is a lattice path ordering of ρ; both conclusions are impossible, so $s(e) \subsetneq s(e_i) \cap s(g)$. Therefore $\rho|\{e, e_i, g\}$, and so ρ, is type T_3.

Now assume that if $e_j \in E(\rho) - \{e, e_i\}$, then $s(e) \not\subseteq s(e_j)$.

Next we show that

11.2. *if $s(e) \cap s(e_j) = \emptyset$ for all $e_j \in E(\rho) - \{e, e_i\}$, then ρ is type F_2.*

If $i = 1$, then $s(e) \subseteq [a_1, a_2)$, so we can reorder this interval so that $e, e_1, e_2, \ldots, e_n$ is a lattice path ordering of ρ; that is a contradiction, so $i > 1$. Similarly, $i < n$. By Lemma 10, neither $s(e_{i-1})$ nor $s(e_{i+1})$ is disjoint from $s(e_i)$. However, both are disjoint from $s(e)$, and so from each other (since $s(e) \subsetneq s(e_i)$), so $\rho|\{e, e_{i-1}, e_i, e_{i+1}\}$, and so ρ, is type F_2 where $s(e_i)$ is D.

Now assume that $s(e) \cap s(e_j) \neq \emptyset$ for some $e_j \in E(\rho) - \{e, e_i\}$.

Up to reversing the lattice path ordering of $\rho_{\backslash e}$, we may assume that $s(e) \cap s(e_{i+1}) \neq \emptyset$. Let the supports of e_i, e_{i+1}, and e be represented by the supports of e, f, and g, respectively in Figure 2. Then $Z \neq \emptyset$ and, since $s(e)$ is properly contained in $s(e_i)$, both V and X are empty, and either T or W is nonempty. Now $Y \neq \emptyset$ since $s(e) \not\subseteq s(e_{i+1})$. If T and W are both nonempty, then $\rho|\{e, e_i, e_{i+1}\}$, and so ρ, is type T_4. The remaining cases are where the supports of e_i, e_{i+1}, and e are as shown in the Venn diagrams in Figure 5, where U may be empty or nonempty.

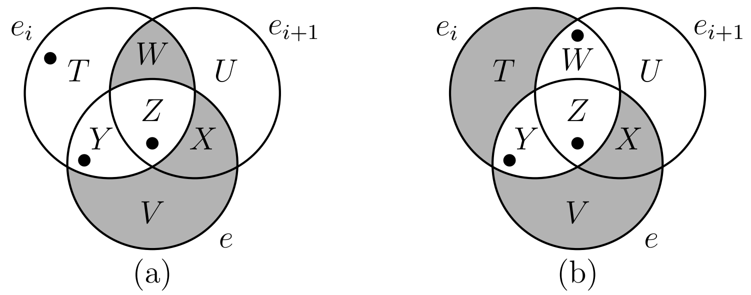

Figure 5: Venn diagram showing the supports of e_i, e_{i+1}, and e.

The case in Figure 5(a) has $Z = [a_{i+1}, b_i]$. Now $s(e)$ cannot be an interval since otherwise $e_1, e_2, \dots, e_i, e, e_{i+1}, \dots, e_n$ would be a lattice path ordering of ρ. Since no reordering of $T \cup Y$ having T before Y is a lattice path ordering,

- $i > 1$, and
- there is a $j < i$ with $s(e_j) \cap Y \neq \emptyset$ and $T \not\subseteq s(e_j)$.

Thus, $s(e_j) \cap Z = \emptyset$. If $s(e_j) \cap T \neq \emptyset$, then the proper minor $\rho|\{e, e_j, e_i\}$ is type T_4, which is impossible. Thus, $s(e_j) \cap T = \emptyset$, and so $\rho|\{e, e_j, e_i, e_{i+1}\}$, and hence ρ, is type F_3.

The case in Figure 5(b) has $Y = [a_i, a_{i+1})$ and $W \cup Z = [a_{i+1}, b_i]$. Now $s(e)$ cannot be an interval since otherwise $e_1, e_2, \dots, e_{i-1}, e, e_i, \dots, e_n$ would be a lattice path ordering of ρ. Likewise, we cannot reorder the support of $\rho_{\backslash e}$ to get a lattice path ordering of the support of ρ, so either

(i) there is a $j < i$ with $s(e_j) \cap W \neq \emptyset$ and $Z \not\subseteq s(e_j)$, or

(ii) there is a $j > i + 1$ with $s(e_j) \cap Z \neq \emptyset$ and $W \not\subseteq s(e_j)$.

Suppose $j < i$ satisfies (i). Then $Y \subseteq s(e_j)$ and either $s(e_j) \cap Z$ is empty and $\rho|\{e, e_j, e_{i+1}\}$ is a Boolean 3-cycle, or $s(e_j) \cap Z$ is not empty and $\rho|\{e, e_j, e_{i+1}\}$ is type T_2. This is impossible since this is a proper minor of ρ. Thus, (ii) must occur.

From (ii), we get $s(e_j) \cap Y = \emptyset$. If $s(e_j) \cap W \neq \emptyset$, then the proper minor $\rho|\{e, e_i, e_j\}$ is type T_4, which is impossible. Thus, $s(e_j) \cap W = \emptyset$. Then $s(e_j) \cap s(e_i) \subseteq Z$. The proper minor $\rho|\{e, e_{i+1}, e_j\}$ is not type T_1, so $s(e_j) = U \cup Z'$ for some $Z' \subseteq Z$. Then $U \neq \emptyset$, since otherwise $s(e_j) \subsetneq s(e_{i+1}) \subsetneq s(e_i)$, and ρ is type F_1 or F_3 by Lemma 9, as desired. Let $Z'' = Z \backslash Z'$, which may be empty. Then $s(e) = Y \cup Z' \cup Z''$, and $s(e_i) = Y \cup Z' \cup Z'' \cup W$, and $s(e_{i+1}) = Z' \cup Z'' \cup W \cup U$, and $s(e_j) = Z' \cup U$. The only set in $\{U, W, Y, Z', Z''\}$ that may be empty is Z''. If $Z'' \neq \emptyset$, then the proper minor $\rho|\{e, e_{i+1}, e_j\}$ would be type T_4, which is impossible. Thus, $Z'' = \emptyset$, and so $\rho|\{e, e_i, e_{i+1}, e_j\}$, and hence ρ, is type F_1. $\square$

We prove one last lemma before proving the main result.

Lemma 12. *Let ρ be a Boolean polymatroid that is an excluded minor for the class of lattice path polymatroids. Let $e_1, e_2, \dots, e_n$ be a lattice path ordering of $\rho_{\backslash e}$ for some $e \in E(\rho)$. If $\emptyset \neq s(e_i) \cap s(e_{i+1}) \subseteq s(e) \subseteq s(e_i) \cup s(e_{i+1})$ for some $i \in \{1, 2, \dots, n\}$, then $s(c) \subsetneq s(d)$ for some $c, d \in E(\rho)$.*

Proof. Suppose that $s(c) \not\subseteq s(d)$ for all $c, d \in E(\rho)$. Then $a_i < a_{i+1} \leqslant b_i < b_{i+1}$. Note that if $j \notin \{i, i+1\}$, then $[a_{i+1}, b_i] \not\subseteq s(e_j) = [a_j, b_j]$, since if $j < i$, then $b_j < b_i$, so $b_i \notin s(e_j)$, while if $j > i+1$, then $a_{i+1} < a_j$, so $a_{i+1} \notin s(e_j)$.

Let f and g represent e_i and e_{i+1}, respectively, in Figure 2. From the hypothesis, $T \cup X = \emptyset$ and $Z \neq \emptyset$. Sets W and Y are both nonempty since $s(e)$ is not contained in either $s(e_i)$ or $s(e_{i+1})$. Sets U and V are also nonempty since $s(e)$ contains neither $s(e_i)$ nor $s(e_{i+1})$.

Now $W \cup Y \cup Z$ is not an interval since otherwise $e_1, e_2, \ldots, e_i, e, e_{i+1}, e_{i+2}, \ldots, e_n$ would be a lattice path ordering on ρ, which is false. Note that $[a_i, a_{i+1}) = U \cup W$ and $[a_{i+1}, b_i] = Z$ and $(b_i, b_{i+1}] = V \cup Y$. Since no reordering of the support of $\rho_{\backslash e}$ gives a lattice path ordering of ρ, and no support other than $s(e)$, $s(e_i)$, and $s(e_{i+1})$ contains Z, there is some element e_j in $E(\rho) - \{e, e_i, e_{i+1}\}$ such that either

(i) $j < i$ and $s(e_j) \cap W \neq \emptyset$ and $U \not\subseteq s(e_j)$, or

(ii) $j > i+1$ and $s(e_j) \cap Y \neq \emptyset$ and $V \not\subseteq s(e_j)$.

Up to reversing the order on $\rho_{\backslash e}$, we can assume that (i) holds. Then $s(e_j) \cap (Z \cup Y \cup V) = \emptyset$. Consider $\rho|\{e, e_j, e_i\}$. Some element in W is in the supports of all of these, $a_j < a_i$, and some element in U is in $s(e_i)$ but not in $s(e) \cup s(e_j)$ and Y is nonempty and is disjoint from $s(e_i) \cup s(e_j)$. Therefore the proper minor $\rho|\{e_j, e_i, e\}$ is type T_1, which is a contradiction. $\square$

We now prove the main result.

Theorem 13. *A polymatroid is lattice path if and only if it is Boolean and contains no minor that is a Boolean n-cycle, with $n > 2$, or a type T_1, T_2, T_3, T_4, F_1, F_2, F_3, or F_4 polymatroid.*

Proof. The class of lattice path polymatroids is minor-closed and is contained in the class of Boolean polymatroids. Since none of the polymatroids listed above is lattice path, a polymatroid is not lattice path if it has any of these as a minor.

If the list in the theorem is not complete, let ρ be an excluded minor for the class of lattice path polymatroids that is a Boolean polymatroid but not among those identified above. Since ρ is Boolean and is an excluded minor, by Lemmas 1 and 8 each element in $E(\rho)$ can be identified by its support. Since every polymatroid with at most two elements is lattice path, $|E(\rho)| \geqslant 3$. By Lemma 7, $|E(\rho)| \geqslant 4$.

Let $G = G(\rho)$. By Lemma 10, G is connected. Let e be a vertex of lowest degree that is not a cut vertex. Let $e_1, e_2, \ldots, e_n$ be a lattice path ordering on $\rho_{\backslash e}$, where $s(e_i) = [a_i, b_i]$ for all i and $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$. Since $G\backslash e$ is connected and Lemma 11 implies that no support contains another, $a_{i-1} < a_i \leqslant b_{i-1} < b_i$ for each $i \in \{2, 3, \ldots, n\}$. Furthermore, G contains $e_1 e_2 \ldots e_n$ as a path, although this may not be an induced path. We show that

13.1. *the degree $d(e) > 1$.*

Suppose $d(e) = 1$. Then $s(e) \cap [a_1, b_n]$ is contained one of the following sets:

(i) $[a_1, a_2)$,

(ii) $(b_{n-1}, b_n]$, or

(iii) (b_{i-1}, a_{i+1}) for some $i \in \{2, 3, \ldots, n-1\}$.

These options yield the contradictions (i) $e, e_1, e_2, \ldots, e_n$ is a lattice path ordering of ρ, (ii) $e_1, e_2, \ldots, e_n, e$ is a lattice path ordering of ρ, (iii) $\rho|\{e, e_{i-1}, e_i, e_{i+1}\}$ is type F_2. Thus 13.1 follows.

Next we show that in the support graph G,

13.2. *if e is adjacent to non-adjacent vertices e_i and e_j, where $i < j$, then e is adjacent to some e_k where $i < k < j$.*

Suppose e is adjacent to e_i and e_j but to none of $e_{i+1}, e_{i+2}, \ldots, e_{j-1}$. Take a shortest path P between e_i and e_j in the graph that G induces on $\{e_i, e_{i+1}, \ldots, e_j\}$. Then P together with e forms an induced cycle in G with at least four vertices, and $\rho|V(P) \cup \{e\}$ is a Boolean m-cycle for some integer $m \geqslant 4$. This contradiction proves 13.2.

Next we show that

13.3. $d(e) \geqslant 3$.

Suppose $d(e) = 2$, and let e_i and e_j be adjacent to e, where $i < j$. By 13.2, e_ie_j is an edge in G. Let f represent e_i and g represent e_j in Figure 2. If $Z = \emptyset$, then W, X, and Y are all nonempty, so $\rho|\{e, e_i, e_j\}$ is a Boolean 3-cycle, which is a contradiction. Thus, $Z \neq \emptyset$. Then Z is contained in the support of every element in $e_i, e_{i+1}, \ldots, e_j$, so $j = i + 1$. Since $\rho|\{e, e_i, e_{i+1}\}$ is not type T_1 or T_2, some set in $\{T, U, V\}$ is empty and therefore some set in $\{W, X, Y\}$ is empty. Since no element has its support contained in the support of another element, at least one of the following statements must hold:

(i) $T \cup X = \emptyset$ and U, V, W, and Y are all nonempty;

(ii) $U \cup Y = \emptyset$ and T, V, W, and X are all nonempty; or

(iii) $V \cup W = \emptyset$ and T, U, X, and Y are all nonempty.

If (i) occurs, then we get a contradiction by Lemma 12. Up to reversing the order on $\rho_{\backslash e}$, we may assume that (ii) occurs. If $i \neq 1$, then $s(e_{i-1}) \cap (e_i) \neq \emptyset$ while $s(e_{i-1}) \cap s(e) = \emptyset$, so $s(e_{i-1}) \cap X \neq \emptyset$. Then $b_{i-1} \geqslant a_{i+1}$, and $W \subseteq s(e_{i-1})$, contrary to e and e_{i-1} not being adjacent. So $i = 1$ and $e, e_1, e_2, \ldots, e_{n-1}$ is a lattice path ordering of ρ, which is a contradiction. This completes the proof of 13.3.

Assume that e is adjacent to e_i, e_j, and e_k, with $i < j < k$, and to no e_h with $i < h < j$ or $j < h < k$. By 13.2, both e_ie_j and e_je_k are edges in G. Let f represent e_i and g represent e_j in Figure 2. If $Z = \emptyset$, then W, X, and Y are all nonempty, and $\rho|\{e, e_i, e_j\}$ is a Boolean 3-cycle, which is impossible. Thus, $Z \neq \emptyset$, i.e., $s(e) \cap s(e_i) \cap s(e_j) \neq \emptyset$. Thus, $s(e) \cap s(e_h) \neq \emptyset$ for all h with $i \leqslant h \leqslant j$. The same argument shows that $s(e) \cap s(e_j) \cap s(e_k) \neq \emptyset$, and so $s(e) \cap s(e_h) \neq \emptyset$ for all h with $j \leqslant h \leqslant k$. Thus, $j = i + 1$ and $k = i + 2$.

We show that

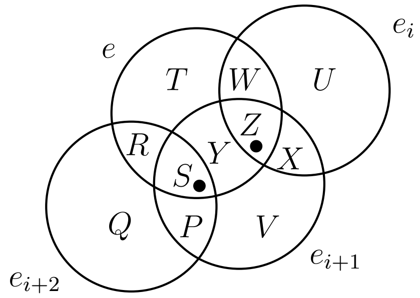

Figure 6: Venn diagram showing the supports of e, e_i, e_{i+1}, and e_{i+2}.

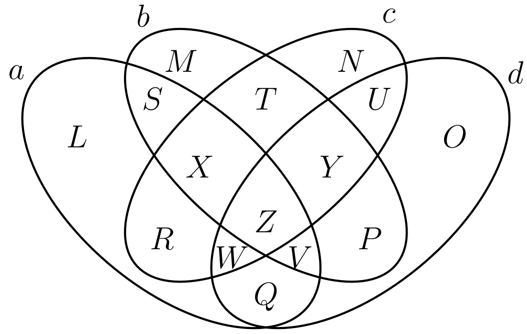

Figure 7: Venn diagram showing the supports of elements a, b, c, and d.

13.4. *e_ie_{i+2} is an edge in G.*

Suppose not. Then the supports of e, e_i, e_{i+1}, and e_{i+2} are as shown in Figure 6. Since ρ has no restriction that is type T_1, at least one of Q, $T \cup W$, and $V \cup X$ is empty, as is at least one of $R \cup T$, U, and $P \cup V$. Since no restriction of ρ is type T_2, either $P = \emptyset$ or $R = \emptyset$; also, either $W = \emptyset$ or $X = \emptyset$. No support contains another support, so $Q \neq \emptyset$ and $U \neq \emptyset$. Since $s(e) \not\subseteq s(e_{i+1})$, at least one of T, W, and R is nonempty. Similarly at least one of P, V, and X is nonempty. After possibly reversing the order on $\rho_{\backslash e}$, it follows that R, T, V, and X are empty, and P, Q, S, U, W, and Z are nonempty. Now Lemma 12 gives a contradiction since $\emptyset \neq s(e_i) \cap s(e_{i+1}) \subseteq s(e) \subseteq s(e_i) \cup s(e_{i+1})$. This proves 13.4.

Let the elements in $\{e, e_i, e_{i+1}, e_{i+2}\}$ be represented by $\{a, b, c, d\}$ in Figure 7. We show that

13.5. *$Z = \emptyset$.*

Suppose $Z \neq \emptyset$. No restriction of ρ to any set of three elements in $\{a, b, c, d\}$ is type T_1. Hence, for each triple $\{x, y, z\} \subseteq \{a, b, c, d\}$, at least one set in $s(x) - (s(y) \cup s(z))$ and $s(y) - (s(x) \cup s(z))$ and $s(z) - (s(x) \cup s(y))$ is empty. Without loss of generality, $s(d) - (s(a) \cup s(b)) = O \cup U = \emptyset$, and at least one of $L \cup Q$, $M \cup P$, and N is also empty. Hence, at least one of $L \cup Q \cup O$, $M \cup P \cup O$, and $N \cup U \cup O$ is empty. Up to changing the labels of a, b, c, and d, we may assume that $N \cup O \cup U = \emptyset$. Since no element has its support contained in another by Lemma 11, each of the following sets is nonempty:

$P \cup Y$, $Q \cup W$, $T \cup Y$, and $R \cup W$. Since neither $\rho|\{a,b,c\}$ nor $\rho|\{a,b,d\}$ is type T_2, both $S \cup V$ and $S \cup X$ are empty.

Since no support contains another, none of the following sets is empty: $L \cup Q$, $M \cup P$, $L \cup R$, $M \cup T$, $R \cup T$, and $P \cup Q$. Since neither $\rho|\{a,c,d\}$ nor $\rho|\{b,c,d\}$ is type T_1, it follows that at least one of L, P, and T is empty and at least one of M, Q, and R is empty. Since neither $\rho|\{a,c,d\}$ nor $\rho|\{b,c,d\}$ is type T_2, it follows that at least one of Q, R, and Y is empty and at least one of P, T, and W is empty. Thus,

(i) if $P = \emptyset$, then $R = \emptyset$ and L, M, Q, T ,W, and Y are all nonempty;

(ii) if $T = \emptyset$, then $Q = \emptyset$ and L, M, P, R, W, and Y are all nonempty;

(iii) if $L = \emptyset$, then $M \cup W \cup Y = \emptyset$ and P, Q, R, and T are all nonempty, and $\rho|\{a,b,c,d\}$ is type F_4, where $A = Q$, $B = R$, $C = Z$, $D = T$, and $E = P$.

Since (iii) gives a contradiction and (ii) is obtained from (i) by switching the labels on a and b, we assume that (i) holds. Note that b, c, d, a is a lattice path ordering of $\rho|\{a,b,c,d\}$ and M, T, Y, Z, W, Q, L is a lattice path ordering of the support of $\rho|\{a,b,c,d\}$. Up to reversing the lattice path ordering on $\rho_{\setminus e}$, where $e_i < e_{i+1} < e_{i+2}$, the fact that L, M, Q, T, W, and Y are all nonempty implies that (b,c,d,a) is one of $(e, e_i, e_{i+1}, e_{i+2})$, $(e_i, e, e_{i+1}, e_{i+2})$, $(e_i, e_{i+1}, e, e_{i+2})$, and $(e_i, e_{i+1}, e_{i+2}, e)\}$.

By Lemma 12, $e \notin \{c,d\}$, so $e \in \{a,b\}$. Then $s(e_i) \cap s(e_{i+2}) = [a_{i+2}, b_i]$ is either $Y \cup Z$ or $Z \cup W$, both of which are nonempty. Then

$$s(e_i) \cap s(e_{i+2}) \subseteq s(e_{i+1}) \subseteq s(e_i) \cup s(e_{i+2}).$$

We next show that e_i and e_{i+2} are consecutive elements in a lattice path ordering of $\rho_{\setminus e_{i+1}}$, but that is impossible by Lemma 12, and so that will complete the proof of 13.5. The two options for (b,c,d,a) show that $s(e_i) \cap s(e_{i+2}) \not\subseteq s(e)$. Now $[a_{i+2}, b_i] \not\subseteq s(e_{i-1})$ since $b_i > b_{i-1}$. Similarly, $[a_{i+2}, b_i] \not\subseteq s(e_{i+3})$. Thus, e_i and e_{i+2} are the only elements in $\rho_{\setminus e_{i-1}}$ that contain $[a_{i+2}, b_i]$ in their supports, and so, as claimed, they must be consecutive in the lattice path ordering on $\rho_{\setminus e_{i+1}}$ by Lemma 5. Thus we have shown 13.5.

We show that

13.6. $s(x) \cap s(y) \cap s(z) = \emptyset$ *for every triple* $\{x,y,z\} \subseteq \{a,b,c,d\}$.

Suppose $s(a) \cap s(b) \cap s(c)$ is nonempty. Then $X \neq \emptyset$. Since $\rho|\{a,b,c\}$ is not type T_1, at least one of the sets $L \cup Q$, $M \cup P$, and $N \cup U$ is empty. Up to relabeling a, b, and c, we may assume that $N \cup U = \emptyset$. Since $\rho|\{a,b,c\}$ is not type T_2, at least one of the sets $R \cup W$, $S \cup V$, and $T \cup Y$ is empty. Two of those options yield the contradiction that some support contains another, so $S \cup V = \emptyset$; avoiding other instances of some support containing another implies that $L \cup Q$, $R \cup W$, $T \cup Y$, and $M \cup P$ are all nonempty. Then

$$\begin{aligned} s(a) &= L \cup Q \cup R \cup W \cup X, \\ s(c) &= R \cup W \cup X \cup T \cup Y, \\ s(b) &= X \cup T \cup Y \cup M \cup P. \end{aligned}$$

Thus $L \cup Q, R \cup W, X, T \cup Y, M \cup P$ is a lattice path ordering of the support of $\rho|\{a, b, c\}$. Now $s(d) = Q \cup W \cup Y \cup P \cup O$. If $Q \cup W$ and $Y \cup P$ are both nonempty, then $\rho|\{a, b, d\}$ is a Boolean 3-cycle, which is a contradiction. Therefore either $Q \cup W$ or $Y \cup P$ is empty. Hence either $s(a) \cap s(d) = \emptyset$ or $s(b) \cup s(d) = \emptyset$, which is a contradiction. Then 13.6 follows by symmetry.

Thus, the sets V, W, X, Y, and Z in Figure 7 are empty. Since $s(y) \cap s(z) \neq \emptyset$ for all $\{y, z\} \subseteq \{a, b, c, d\}$, the following sets are nonempty: P, Q, R, S, T, and U. Therefore $\rho|\{a, b, c\}$ is a Boolean 3-cycle. This contradiction completes our proof. $\square$

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