The excluded minors for lattice path polymatroids

Joseph E. Bonin
Department of Mathematics
The George Washington University
Washington, D.C. 20052, U.S.A.
jbonin@gwu.edu

Carolyn Chun*
Mathematics Department
United States Naval Academy
Annapolis, MD 21402, U.S.A.
chun@usna.edu

Tara Fife
School of Mathematical Sciences
Queen Mary University of London
Mile End Road, London E1 4NS, United Kingdom
fi.tara@gmail.com

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Abstract

We find the excluded minors for the minor-closed class of lattice path polymatroids as a subclass of the minor-closed class of Boolean polymatroids. Like lattice path matroids and Boolean polymatroids, there are infinitely many excluded minors, but they fall into a small number of easily-described types.

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1 Introduction

We consider only polymatroids where the rank \( \rho(X) \) of each set \( X \) is a nonnegative integer, so a polymatroid on a finite set \( E \) is a function \( \rho : 2^E \to \mathbb{Z} \) that is

1. normalized, that is, \( \rho(\emptyset) = 0 \),
2. non-decreasing, that is, if \( A \subseteq B \subseteq E \), then \( \rho(A) \leq \rho(B) \), and
3. submodular, that is, \( \rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B) \) for all \( A, B \subseteq E \).

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Herzog and Hibi [3] treat some equivalent formulations of polymatroids, which are also called *integer polymatroids* or *discrete polymatroids*. We often write the ground set $E$ of $ho$ as $E(\rho)$. For a positive integer $t$, a *$t$-polymatroid* is a polymatroid $\rho$ on $E$ for which $\rho(e) \leq t$ for all $e \in E$, or, equivalently, $\rho(A) \leq t|A|$ for all $A \subseteq E$. Thus, matroids are $1$-polymatroids. The definitions of deletion and contraction, when cast for matroids using the rank function, generalize directly to polymatroids (see Section 2), and the notion of a minor carries over directly.

Let $[k]$ denote the set $\{1, 2, \ldots, k\}$. Let $E$ be a finite set and let $A_i$, for $i \in [k]$, be (not necessarily distinct) subsets of $E$. We get a polymatroid $\rho$ on $E$ by, for $X \subseteq E$, setting

$$\rho(X) = |\{i : X \cap A_i \neq \emptyset\}|.$$

Such polymatroids are *Boolean polymatroids* or *transversal polymatroids*. The class of Boolean polymatroids is minor-closed, that is, every minor of a polymatroid in this class is also in this class.

Lattice path polymatroids, introduced by Schweig [6], are constructed as follows. (See Figure 1.) Take two lattice paths $P$ and $Q$ from $(1, 0)$ to $(n, k)$, where $P$ never rises above $Q$. These paths bound a region of the plane. We label each north step (a segment from $(i, j)$ to $(i, j + 1)$ where $i$ and $j$ are integers) in this region by its first coordinate. For each $i \in [k]$, let $A_i$ be the set of labels on the north steps in row $i$ of this diagram, with $i = 1$ indexing the lowest row. The polymatroid $\rho$ on $E = [n]$ given by equation (1.1) is the *lattice path polymatroid determined by the paths $P$ and $Q$*. A lattice path polymatroid is any polymatroid that is isomorphic to such a polymatroid from paths. (Schweig [6, 7] treats lattice path polymatroids via their bases, which can be represented by monomials as follows: replace the labels $1, 2, \ldots, n$ by the distinct variables $x_1, x_2, \ldots, x_n$, respectively, and record a lattice path from $(1, 0)$ to $(n, k)$ as the product of the variables on its north steps. Thus, the path in Figure 1 would be recorded as $x_3^2x_6x_7$.)

Lattice path polymatroids have many interesting properties [4, 6, 7]. Most lattice path matroids [1] are not lattice path polymatroids, but the two structures have much in common. Lattice path polymatroids form a minor-closed class of Boolean polymatroids. In this paper, we find the Boolean polymatroids that are excluded minors for the class of lattice path polymatroids. That is, we find the set $\mathcal{E}$ of Boolean polymatroids for which a Boolean polymatroid $\rho$ is a lattice path polymatroid if and only if no member of $\mathcal{E}$ is a minor of $\rho$. The set $\mathcal{E}$, combined with the set of excluded minors for Boolean
polymatroids, which was found by Matúš [5], gives a complete characterization of lattice path polymatroids. (Note however that, as we explain in Section 2, each member of $E$ is a proper minor of infinitely many of the excluded minors that Matúš identified.) Lattice path matroids [2] and Boolean polymatroids [5] have infinitely many excluded minors, and the same is true of lattice path polymatroids: the set $E$ is infinite. There are nine types of excluded minors in $E$, and each type includes infinitely many polymatroids: four of the types are made up of polymatroids on three elements, four other types are made up of polymatroids on four elements, and one type is made up of polymatroids on any number of elements greater than two.

In Section 2, we review the relevant background on polymatroids, minors, Boolean polymatroids, and lattice path polymatroids; in particular, we state the excluded-minor characterization of Boolean polymatroids by Matúš [5] and explain how it relates to our results. In Section 3, we identify the type of excluded minors that can have any number of elements three or greater. In Section 4, we identify the types of excluded minors on three or four elements. In Section 5, specifically Theorem 13, we prove that the collection of excluded minors identified in Sections 3 and 4 is complete.

2 Background

Let $\rho$ be a polymatroid on $E$. For $A \subseteq E$, the deletion $\rho|_A$ and contraction $\rho/A$, which are polymatroids on $E - A$, are given by $\rho|_A(X) = \rho(X)$ and $\rho/A(X) = \rho(X \cup A) - \rho(A)$ for all $X \subseteq E - A$. The minors of $\rho$ are the polymatroids of the form $(\rho|_A)/B$ (equivalently, $(\rho/B)|_A$) for disjoint subsets $A$ and $B$ of $E$.

Let $r_M$ denote the rank function of a matroid $M$. If $M_1, M_2, \ldots, M_k$ are matroids on $E$, then the function $\rho : 2^E \to \mathbb{Z}$ given by

$$\rho(X) = r_{M_1}(X) + r_{M_2}(X) + \cdots + r_{M_k}(X),$$

for $X \subseteq E$, is a polymatroid on $E$. We write $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$ to express this more briefly. (Not all polymatroids have such decompositions into matroid rank functions.) It is easy to check that if $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$, then

$$\rho|_A = r_{M_1\setminus A} + r_{M_2\setminus A} + \cdots + r_{M_k\setminus A} \quad \text{and} \quad \rho/A = r_{M_1/A} + r_{M_2/A} + \cdots + r_{M_k/A}.$$

It is easy to see that the Boolean polymatroid $\rho$ on $E$ given by equation (1.1), using the nonempty sets $A_i$ for $i \in [k]$, can be written as the sum of $k$ rank-1 matroids on $E$, namely,

$$\rho = ru_{1,A_1 \cup U_{0,E-A_1}} + ru_{1,A_2 \cup U_{0,E-A_2}} + \cdots + ru_{1,A_k \cup U_{0,E-A_k}},$$

(2.1)

where, adapting the usual notation $U_{r,n}$ for uniform matroids, $U_{r,A}$ is the rank-$r$ uniform matroid on the set $A$.

For completeness, we next prove a result that, while not new, is crucial to our work: the Boolean polymatroid $\rho$ determines the nonempty sets $A_i$ (but, of course, not how these sets are indexed).
Lemma 1. Let $A_1, A_2, \ldots, A_k$ be (not necessarily distinct) nonempty subsets of a set $E$ and let $\rho$ be the Boolean polymatroid on $E$ given by equation (2.1). For each nonempty subset $Z$ of $E$, the multiplicity of $Z$ in the list $A_1, A_2, \ldots, A_k$ can be computed from $\rho$.

Proof. For each $e \in E$, we define a property $p_e$ that each integer $i \in [k]$ may or may not have: $i$ has property $p_e$ if $e \not\in A_i$. Fix $X \subseteq E$. Now $i \in [k]$ has the properties $p_e$ with $e \in X$ and no other properties if and only if $A_i = E - X$. The number of $i \in [k]$ that have all the properties $p_e$ with $e \in X$, and maybe more, is $k - \rho(X)$. By inclusion-exclusion, it follows that the number of $i \in [k]$ that have the properties $p_e$ with $e \in X$ and no others is

$$\sum_{Y : X \subseteq Y \subseteq E} (-1)^{|Y - X|}(k - \rho(Y)).$$

Since this is the number of $i$ for which $A_i = E - X$, the lemma follows. $\Box$

This argument is closely related to the characterization of Boolean polymatroids by inequalities involving the rank function in Matúš [5, Lemma 2] (see Theorem 4 below).

Lemma 1 leads to a well-defined (up to relabeling the indices) notion of support: given $E$ and $A_i$, for $i \in [k]$, and $\rho$ given by equation (2.1), the support $s(e)$ of $e \in E$ is

$$s(e) = \{i : i \in [k] \text{ and } e \in A_i\}.$$

The support of $\rho$ is $\cup_{e \in E}s(e)$, which is $[k]$ if no set $A_i$ is empty. For example, for the lattice path polymatroid in Figure 1, we have $s(1) = s(2) = \{1\}$, $s(3) = s(4) = \{1, 2\}$, $s(5) = \{1, 2, 3\}$, $s(6) = \{2, 3, 4\}$, $s(7) = s(8) = \{3, 4\}$, and $s(9) = \{4\}$. In the lattice path polymatroid determined by two paths, the support $s(e)$ shows which rows in the diagram intersect the line $x = e$. Note that for any Boolean polymatroid $\rho$ on $E$ and all $X \subseteq E$,

$$\rho(X) = \left| \bigcup_{e \in X} s(e) \right|. \tag{2.2}$$

There is a natural linear order on $[k]$, namely, $1 < 2 < \cdots < k$, as well as on the ground set $E$ of a lattice path polymatroid determined by two paths, but this is not so for the excluded minors. All orders (or orderings) that we consider are linear orders (also known as total orders). They can be denoted by listing the elements from least to greatest, as in $x_1, x_2, \ldots, x_t$, or via the conventional symbol for order, as in $x_1 < x_2 < \cdots < x_t$.

Since lattice path polymatroids in general are just isomorphic to those determined by two paths, it follows that $\rho$ is a lattice path polymatroid if and only if there is some ordering of $[k]$ and some ordering $e_1, e_2, \ldots, e_n$ of $E$ so that,

(S1) for each $i \in [n]$, the support $s(e_i)$ is an interval $[a_i, b_i]$ in the order on $[k]$, and,  

(S2) in that order on $[k]$, we have $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$.

Such an ordering of $E$ is a lattice path ordering of $\rho$, and such an ordering of $[k]$ is a lattice path ordering of the support of $\rho$. Given sets $Z_1, Z_2, \ldots, Z_t$ that partition $[k]$ such that a lattice path ordering of the support of $\rho$ can be obtained by taking any ordering
of the elements of $Z_1$ followed by any ordering of $Z_2$ and so on, finally ending with any ordering of the elements of $Z_\ell$, we also refer to $Z_1, Z_2, \ldots, Z_\ell$ as a lattice path ordering of the support of $\rho$. For ease of notation, we also consider any sequence obtained from $Z_1, Z_2, \ldots, Z_\ell$ by adding copies of the empty set to be a lattice path ordering of the support of $\rho$.

By equation (2.2), knowing the support of each element $e_i$ in a Boolean polymatroid $\rho$ determines $\rho$. Different elements in a Boolean polymatroid may have the same support, so the map taking each element to its support need not be injective. Also, the support $s(e_i)$ depends on the order in which we write the sets $A_1, A_2, \ldots, A_\ell$, and changing the order by a permutation $\pi$ just replaces each support by its image under $\pi$. To determine whether $\rho$ is a lattice path polymatroid, we need to determine whether a lattice path ordering of $\rho$ exists, and its existence goes hand in hand with the existence of a compatible lattice path ordering of the support.

From equation (2.1) and the discussion of minors before it, it follows that, in terms of supports, deleting $e$ from a Boolean polymatroid corresponds to deleting its support set $s(e)$, while contracting $e$ from a Boolean polymatroid corresponds to deleting $s(e)$ and, for each element $f \in E - e$, replacing $s(f)$ by $s(f) - s(e)$ since contracting a nonloop in a rank-1 matroid yields a rank-0 matroid. It now easily follows that minors of lattice path polymatroids are lattice path polymatroids, but we get much more: if $\rho$ is a Boolean polymatroid and $\rho_{/e}$ is a lattice path polymatroid, then so is $\rho_{/e}$ since, given a linear order on $[k]$ in which the support sets for $\rho_{/e}$ are intervals that satisfy (S2), for the induced linear order on $[k] - s(e)$, the support sets for $\rho_{/e}$ are intervals that satisfy (S2). Thus, we have the following lemma, which is highly atypical for minor-closed classes of polymatroids.

**Lemma 2.** If $\rho$ is a Boolean polymatroid that is not a lattice path polymatroid, then $\rho$ is an excluded minor for the class of lattice path polymatroids if and only if each single-element deletion is a lattice path polymatroid.

As noted above, the class of lattice path polymatroids is a minor-closed subclass of the class of Boolean polymatroids. The excluded-minors for the class of Boolean polymatroids were found by Matuš. The next theorem recasts [5, Theorem 3].

**Theorem 3.** The excluded minors for Boolean polymatroids are the polymatroids of the form

$$-c_E r_{U_1,E} + \sum_{X: \emptyset \neq X \subseteq E} c_X r_{U_1,X \oplus U_0,E-X}$$

(2.3)

where $|E| \geq 3$, $c_E$ is a positive integer, all coefficients $c_X$ are nonnegative integers, and $c_X \geq c_E$ if $|X| = |E| - 1$.

Thus, there are infinitely many excluded minors for Boolean polymatroids. To translate this result into an infinite list of the excluded minors, one must take isomorphism into account since acting on $2^E$ with a permutation of $E$ yields isomorphic excluded minors that all have the form in (2.3). It follows from Theorem 3 that each Boolean polymatroid $\rho'$ on $E'$ is a minor of infinitely many of these excluded minors: take any superset
$E$ of $E'$ with $|E - E'| \geq 2$ and let $\rho$ be as in (2.3) where, for $X \subseteq E'$, the coefficient $c_X$ is the number of copies of $t_{U, X \subseteq U, E' - X}$ when $\rho'$ is written as in equation (2.1); then

$$\rho_{|E - E'} = \rho'.$$

Similarly, one can construct excluded minors for Boolean polymatroids that have $\rho'$ as a deletion. Given a characterization of a proper minor-closed class of Boolean polymatroids by its excluded minors relative to the class of Boolean polymatroids, there is an excluded-minor characterization of the class relative to the class of all polymatroids; however, the excluded minors relative to the class of Boolean polymatroids may, in some cases, be more illuminating. Of course, each Boolean polymatroid that is an excluded minor relative to the class of Boolean polymatroids is also an excluded minor relative to the class of all polymatroids.

Besides using the excluded minors, there is another way to determine whether a polymatroid is Boolean. The next result is part of [5, Lemma 1].

**Theorem 4.** A polymatroid $\rho$ on $E$ is Boolean if and only if, for all $X \subseteq E$,

$$\sum_{Y : X \subseteq Y \subseteq E} (-1)^{|Y - X|} (\rho(E) - \rho(Y)) \geq 0.$$

Given these factors, it is reasonable to seek characterizations of proper minor-closed classes of Boolean polymatroids by their excluded minors relative to the class of Boolean polymatroids. This is exactly what we do for lattice path polymatroids.

### 3 Boolean cycles and a property of lattice path orderings

We start with a necessary condition for an ordering on the ground set $E$ of a Boolean polymatroid $\rho$ to be a lattice path ordering of $E$, and so for $\rho$ to be lattice path.

**Lemma 5.** Let $\rho$ be a Boolean polymatroid on $E$. If $\rho$ is a lattice path polymatroid on $E$ and $e_1, e_2, \ldots, e_n$ is a lattice path ordering of $\rho$, then the following property holds:

**(S3)** If $S \neq \emptyset$ and $S \subseteq s(e_h)$ for at least one $h \in [n]$, then the elements whose supports contain $S$ are $e_i, e_{i+1}, \ldots, e_{j-1}$, and $e_j$ for some $i$ and $j$ with $1 \leq i \leq j \leq n$.

**Proof.** By relabeling if needed, we can take the usual order $1, 2, \ldots, k$ on the support of $\rho$ and let $s(e_i)$ be the interval $[a_i, b_i]$ in $[k]$. Let $i \in [k]$ be least with $S \subseteq s(e_i)$, and let $j \in [k]$ be greatest with $S \subseteq s(e_j)$. The result holds if $i = j$, so assume that $i < j$. Now $S \subseteq s(e_i) \cap s(e_j) = [a_j, b_i]$. Since $a_i \leq a_{i+1} \leq \cdots \leq a_j$ and $b_i \leq b_{i+1} \leq \cdots \leq b_j$, we have $[a_j, b_i] \subseteq s(e_h)$ for all $h$ with $i \leq h \leq j$, as needed.

Recall that, by equation (2.2), a Boolean polymatroid can be given by its supports. That is how we will define each of the excluded minors for lattice path polymatroids.

We first treat a family of excluded minors that, for each $n \geq 3$, contains infinitely many polymatroids with $n$ elements. We call a polymatroid in this family a *Boolean cycle*, or a *Boolean $n$-cycle*, where $n$ is the number of elements. Let $n \geq 3$. The supports of the elements $e_1, e_2, \ldots, e_n$ of a *Boolean $n$-cycle* have the form $s(e_i) = Z_{2i-1} \cup Z_{2i} \cup Z_{2i+1}$, where the indices are taken modulo $2n$ (thus, $s(e_n) = Z_{2n-1} \cup Z_{2n} \cup Z_1$), and
• \( Z_1, Z_2, \ldots, Z_{2n} \) are pairwise disjoint, and
• for all \( i \in [n] \), the set \( Z_{2i-1} \) is nonempty.

The sets \( Z_{2i} \) may be empty. The example of smallest rank and size is three coplanar lines, that is, \( e_1, e_2, e_3 \) each have rank 2 and any set of two or three of them has rank 3. Also, three Boolean cycles have rank 4 and so can be seen as made up of faces of the four-vertex simplex: taking all edges other than one pair of non-coplanar edges gives a 4-cycle; we get Boolean 3-cycles from (a) two planes and the edge that neither contains and (b) a plane and two edges not contained in it. As with the description of the excluded minors for Boolean polymatroids in Theorem 3, the definition of Boolean \( n \)-circuits on \( E \) yields many isomorphic polymatroids; we are, of course, interested in these polymatroids up to isomorphism.

We now prove that each Boolean \( n \)-cycle is an excluded minor.

**Lemma 6.** Let \( \rho \) be a Boolean \( n \)-cycle for some integer \( n \geq 3 \). Then \( \rho \) is an excluded minor for the class of lattice path polymatroids.

**Proof.** Suppose \( \rho \) is lattice path. Since the supports of only two elements contain \( Z_3 \), namely \( e_1 \) and \( e_2 \), these elements are consecutive in the lattice path ordering of \( \rho \). Likewise, since \( Z_{2i-1} \) is only contained in \( s(e_{i-1}) \) and \( s(e_i) \) for all \( i \in \{3, 4, \ldots, n\} \), the ordering on the elements of \( \rho \) must be \( e_1, e_2, \ldots, e_n \). However \( s(e_1) \cap s(e_n) = Z_1 \) and \( Z_1 \not\subseteq s(e_2) \), so (S3) fails, and so \( \rho \) is not lattice path.

Take \( i \in [n] \). Then \( e_{i+1}, e_{i+2}, \ldots, e_n, e_1, e_2, \ldots, e_{i-1} \) is a lattice path ordering of \( \rho \backslash e_i \). Thus, \( \rho \) is an excluded minor for the class of lattice path polymatroids by Lemma 2. \( \square \)

4 Excluded minors with three or four elements

In this section, we define the types of excluded minors that are not Boolean cycles. Each of these types has three or four elements. We also prove that our collection of three-element excluded minors is complete.

We first note that each polymatroid on a set of at most two elements is a lattice path polymatroid; in particular, it is a Boolean polymatroid. This is because, for a polymatroid \( \rho \) on \( E = \{e, f\} \) with \( k = \rho(E) \), we can let \( s(e) \) be the first \( \rho(e) \) elements, and \( s(f) \) the last \( \rho(f) \) elements, of \( [k] \).

We turn to the Boolean excluded minors that have three elements and are not Boolean 3-cycles. They come in types \( T_1, T_2, T_3, \) and \( T_4 \), and there are infinitely many of each type. In order to characterize these four types of Boolean polymatroids, say with elements \( e, f, \) and \( g \), it suffices to identify which of the areas in the Venn diagram of their supports in Figure 2 are empty, and which are nonempty. Note that if \( Z = \emptyset \) and each of \( W, X, \) and \( Y \) is nonempty, then \( \rho \) is a Boolean 3-cycle, independent of whether or not \( T, U, \) or \( V \) are empty.

In the following list, we assume that \( \rho \) is a Boolean polymatroid and \( E(\rho) = \{e, f, g\} \), where the supports of \( e, f, \) and \( g \) are as shown in Figure 2. We view elements of a rank-\( k \)
Figure 2: Venn diagram showing the supports of elements $e$, $f$, and $g$ in a Boolean polymatroid, $\rho$.

Boolean polymatroid in terms of their supports, which are subsets of $[k]$ and so can be identified with faces of a simplex with $k$ vertices, so the examples that we provide are given as sets of faces of a simplex.

$(T_1)$ The polymatroid $\rho$ is type $T_1$ if
- $T$, $U$, $V$, and $Z$ are all nonempty.
  - Example: three lines in rank four that, on the simplex, share a vertex (the vertex does not correspond to an element of the polymatroid).

$(T_2)$ The polymatroid $\rho$ is type $T_2$ if
- $W$, $X$, $Y$, and $Z$ are all nonempty and
- at least one of $T$, $U$, or $V$ is empty.
  - An example: three planes in the four-vertex simplex.

$(T_3)$ The polymatroid $\rho$ is type $T_3$ if
- $T$, $U$, $W$, and $Z$ are all nonempty and
- $V = X = Y = \emptyset$.
  - Example: two planes of the four-vertex simplex, along with one of the two points on the line that the planes share on the simplex.

$(T_4)$ The polymatroid $\rho$ is type $T_4$ if
- $T$, $W$, $Y$, and $Z$ are all nonempty and
- $X = V = \emptyset$.
  - The set $U$ may or may not be empty. Example with $U = \emptyset$: an element of rank four and two lines on the corresponding simplex that share one vertex (the vertex is not in the polymatroid). Example with $U \neq \emptyset$: start with a similar configuration in a four-vertex simplex, but embedded in a five-vertex simplex, with one of the lines extended to a plane that includes the new vertex.
Using equation (2.2), it is easy to translate these descriptions into the values of \( \rho \). For instance, a polymatroid \( \rho \) on \( E = \{e, f, g\} \) is type \( T_1 \) if and only if there are positive integers \( z, t, u, \) and \( v \), and nonnegative integers \( w, x, \) and \( y \), for which

\[
\begin{align*}
\rho(e) &= z + t + w + y, \\
\rho(f) &= z + u + w + x, \\
\rho(g) &= z + v + x + y, \\
\rho(e, f) &= z + t + u + w + x + y, \\
\rho(e, g) &= z + t + v + w + x + y, \\
\rho(f, g) &= z + u + v + w + x + y, \\
\rho(E) &= z + t + u + v + w + x + y.
\end{align*}
\]

We next show that the polymatroids of these four types, along with Boolean 3-cycles, are the 3-element Boolean excluded minors for lattice path polymatroids.

**Lemma 7.** A Boolean polymatroid \( \rho \) is an excluded minor for the class of lattice path polymatroids, where \( |E(\rho)| = 3 \), if and only if \( \rho \) is type \( T_1, T_2, T_3, \) or \( T_4 \) or \( \rho \) is a Boolean 3-cycle.

**Proof.** It is straightforward to check that if \( \rho \) is type \( T_1, T_2, T_3, \) or \( T_4 \) or a Boolean 3-cycle, then \( \rho \) is not lattice path. Since every two-element polymatroid is lattice path, \( \rho \) is therefore an excluded minor for the class of lattice path polymatroids.

Assume then that \( \rho \) is a three-element excluded minor for the class of lattice path polymatroids. Let \( E = \{e, f, g\} \) where the supports of these elements are shown in Figure 2. Suppose first that \( Z = \emptyset \). If \( W = \emptyset \), then \( T, Y, V, X, U \) is a lattice path ordering
of the support of ρ, which is a contradiction. By symmetry, none of W, X, Y is empty, and ρ is a Boolean 3-cycle.

Now assume that Z ≠ ∅. If none of T, U, V is empty, then ρ is type T₁, so we assume not; say V = ∅. If each of W, X, Y is nonempty, then ρ is type T₂, so we assume not. If W = ∅, then T, Y, Z, X, U is a lattice path ordering of the support of ρ, which is a contradiction. Therefore W ≠ ∅ and, without loss of generality, X = ∅. If T = ∅, then Y, Z, W, U is a lattice path ordering of the support of ρ, which is a contradiction. Hence T ≠ ∅. If Y is nonempty, then ρ is type T₄, so we assume not. If U = ∅, then T, W, Z is a lattice path ordering of the support of ρ, which is a contradiction. Therefore U ≠ ∅ and ρ is type T₃.

We now present all four-element Boolean polymatroids, other than Boolean 4-cycles, that are excluded minors for the class of lattice path polymatroids. There are four types, and infinitely many of each type. Let ρ be a Boolean polymatroid on four elements.

(F₁) The polymatroid ρ is type F₁ if the supports have the form A ∪ B, A ∪ B ∪ C, B ∪ C ∪ D, and B ∪ D where

- A, B, C, and D are pairwise disjoint and
- ∅ ∉ {A, B, C}.

○ Example with D ≠ ∅: two planes in the four-vertex simplex along with a line in exactly one of those planes and the line in just the other plane that is coplanar with the first line. Example with D = ∅: a vertex in a triangle, the two lines containing that vertex, and the triangle.

(F₂) The polymatroid ρ is type F₂ if the supports have the form A, B, C, and D where

- A, B, and C are pairwise disjoint, and
- A ∩ D, B ∩ D, and C ∩ D are all nonempty.

○ Example: three noncolinear points along with the plane that contains them.

(F₃) The polymatroid ρ is type F₃ if the supports have the form A, A', B, and C where

- A' ⊆ A and A' ≠ A, and
- B ∩ C = ∅, and
- both B and C are disjoint from A − A', and neither is disjoint from A'.

○ Example: two points, the line they span, and a plane that contains that line.

(F₄) The polymatroid ρ is type F₄ if the supports have the form A ∪ B ∪ C, B ∪ C ∪ D, C ∪ D ∪ E, and A ∪ C ∪ E where

- A, B, C, D, and E are all pairwise disjoint and nonempty.
Figure 4: Venn diagrams showing the supports of elements in a Boolean polymatroid. Areas shaded gray indicate that those sets are empty. A point in an area indicates that that set is nonempty. Areas that are not gray and contain no point may be empty or nonempty.

- Example: take four of the six lines in a four-vertex simplex, with the pair omitted being skew; embed this simplex in a five-vertex simplex and replace each line by the plane that is spanned by it and the new vertex; these four planes are the elements of the polymatroid.

The Venn diagrams of the supports of the element for types $F_2$ and $F_3$ are shown in Figure 4, where a point appears in an area for which the set is known to be nonempty, and a darkened area indicates that the set is empty. It is straightforward to check that if $\rho$ is type $F_1$, $F_2$, $F_3$, or $F_4$, then $\rho$ is not lattice path, but all of its proper minors are.

5 Proof of the main result

In this section we prove our main result: when considered as a subclass of the class of Boolean polymatroids, the set of excluded minors given in the previous two sections for the class of lattice path polymatroids is complete.

We start with an easy result that is worth noting.

**Lemma 8.** If $\rho$ is a Boolean polymatroid that is an excluded minor for the class of lattice path polymatroids, and $e$ and $f$ are distinct elements of $E(\rho)$, then $s(e) \neq s(f)$.

*Proof.* If $s(e) = s(f)$, then any lattice path ordering of $\rho_{\{\epsilon\}}$ can be extended to a lattice path ordering of $\rho$ by adding $\epsilon$ directly following $f$. \hfill $\Box$

Lemmas 9, 10, and 11 treat the case in which, in a Boolean polymatroid that is an excluded minor, the support of some element is contained in another.

**Lemma 9.** Let $\rho$ be a Boolean polymatroid that is an excluded minor for the class of lattice path polymatroids. If $e, f, g \in E(\rho)$ and $s(e) \subset s(f) \subset s(g)$, then $\rho$ is type $F_1$ or $F_3$.

*Proof.* No 3-element excluded minor has such a chain, so $|E(\rho)| \geq 4$ and $\rho$ has no minors of types $T_1$, $T_2$, $T_3$, or $T_4$. Let $C$ be a subset of $E(\rho)$ of maximum size that can be ordered
so that the support of each element is contained in the support of the next. Suppose that $|C| \geq 3$. By Lemma 8, no two supports are equal. Let $e$ be the element in $C$ with the smallest support. Then $\rho_{\text{e}}$ has a lattice path ordering $e_1, e_2, \ldots, e_n$ where $s(e_i) = [a_i, b_i]$ for all $i$, and $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. By (S3) in Lemma 5, up to reversing the ordering on $\rho_{\text{e}}$, we may assume that $s(e) \subseteq s(e_k) \subseteq s(e_{k+1}) \subseteq \cdots \subseteq s(e_\ell)$ where $1 \leq k < k + |C| - 2 \leq \ell \leq n$. It follows that $a_k = a_{k+1} = \cdots = a_\ell$ and $b_k < b_{k+1} < \cdots < b_\ell$. Let $[a, b]$ be the smallest interval such that $s(e) \subseteq [a, b]$. Thus, $a, b \in s(e)$ and $a_k \leq a \leq b \leq b_k$. Also, $s(e)$ might be a proper subset of $[a, b]$.

We first show that $s(e)$ is an interval in some lattice path ordering on $\rho_{\text{e}}$. To show this, assume that $s(e) \neq [a, b]$. The next two results are the key to being able to reorder $[a, b]$ so that $s(e)$ is an interval and the new ordering of $E(\rho)$ is a lattice path ordering of $\rho_{\text{e}}$.

**Theorem 9.1.** If $p < k$ and $s(e_p) \cap ([a, b] - s(e)) \neq \emptyset$, then $[a, b] \subseteq s(e_p)$.

Note that $a_p \leq a_1 \leq a$ and $b \leq b_p$. If $b_p < b$, then $b \notin s(e_p)$, so $\rho([e, e_p, e_\ell])$ is type $T_4$, but that is impossible. So $b \leq b_p$, and the conclusion of 9.1 holds.

**Theorem 9.2.** If $q > \ell$ and $s(e_q) \cap s(e) \neq \emptyset$, then $[a, b] - s(e) \subseteq s(e_q)$.

If the conclusion failed, then neither a nor the first element in $[a, b] - s(e)$ would be in $s(e_q)$; it would follow that $\rho([e, e_\ell, e_q])$ is type $T_4$, but that is impossible. So 9.2 holds.

With 9.1 and 9.2, we can rearrange $[a, b]$, placing $s(e)$ first and $[a, b] - s(e)$ second, to make $s(e)$ into an interval without changing the lattice path ordering on $\rho_{\text{e}}$. So we may assume that $s(e)$ is the interval $[a, b]$.

If we had $a = a_k$, then $(s(e_{k-1}) - s(e)) \cap s(e_k) \neq \emptyset$ (otherwise inserting $e$ between $e_{k-1}$ and $e_k$ would give a lattice path ordering for $\rho$, which is impossible), so $\rho([e_{k-1}, e, e_\ell])$ would have type $T_3$, contrary to $\rho$ being an excluded minor. So $a_k < a$. Thus, $[a, b]$ cannot be moved to the beginning of $s(e_k)$, so either the support of some element before $e_k$ blocks it, that is,

(i) there is a $p < k$ with $[a_k, a] \cap s(e_p) \neq \emptyset$ and $s(e) \not\subseteq s(e_p)$,

or the support of some element after $e_\ell$ blocks it, and so

(ii) there is a $q > \ell$ with $s(e) \cap s(e_q) \neq \emptyset$ and $a_k \not\subseteq s(e_q)$.

Suppose (i) occurs. If $s(e) \cap s(e_p) \neq \emptyset$, then $\rho([e, e_p, e_\ell])$ is type $T_3$, which is impossible. Therefore $b_p < a$, and $\rho([e, e_p, e_\ell])$ is type $F_3$ (since $s(e)$ and $s(e_p)$ are disjoint, use them as $B$ and $C$). Now assume that (ii) occurs. Since $\rho([e, e_\ell, e_q])$ is not type $T_3$ or $T_4$, we know that $a_q \leq a$ and $b_q = b_\ell$. Since $\rho([e, e_k, e_q])$ is not type $T_3$, we know that $a_q = a$ and $b = b_k$. Then $\rho([e, e_k, e_\ell, e_q])$ is type $F_1$, where $D = \emptyset$. \[ \Box \]

For a Boolean polymatroid $\rho$, we define its support graph $G(\rho)$ to be the graph with vertex set $E(\rho)$ and edge set $\{ef : e, f \in E(\rho), e \neq f, s(e) \cap s(f) \neq \emptyset\}$. Note that, if $G(\rho)$ is an $n$-cycle for some integer $n \geq 4$, then $\rho$ is a Boolean $n$-cycle.
Lemma 10. Let $\rho$ be a Boolean polymatroid that is an excluded minor for the class of lattice path polymatroids. Then $G(\rho)$ is connected. Furthermore, if $s(e) \subsetneq s(f)$ for some elements $e, f \in E(\rho)$, then $G(\rho_{e'})$ is also connected.

Proof. Suppose $G$ has a component with vertex set $A$ where $E(\rho) - A$ is not empty. Then $\rho|A$ and $\rho|(E(\rho) - A)$ each have lattice path orderings, and concatenating these two orderings gives a lattice path ordering of $\rho$, which is a contradiction.

Suppose $s(e) \subsetneq s(f)$, and $G(\rho_{e'})$ has distinct components $X$ and $Y$. Then $s(e)$ has a nonempty intersection with $s(x)$ and $s(y)$ for some $x \in V(X)$ and $y \in V(Y)$. Then $xfy$ is a path connecting $X$ and $Y$ in $G(\rho_{e'})$, which is a contradiction. \qed

We now consider the general case that the support of one element contains another.

Lemma 11. Let the Boolean polymatroid $\rho$ be an excluded minor for the class of lattice path polymatroids. If $s(e) \subsetneq s(f)$ for some $e, f \in E(\rho)$, then $\rho$ is type $T_3$, $T_4$, $F_1$, $F_2$, or $F_3$.

Proof. Let $e_1, e_2, \ldots, e_n$ be a lattice path ordering of $\rho_{e'}$, where $s(e_k) = [a_k, b_k]$ with $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. Say $f$ is $e_i$.

We first show

11.1. if $s(e) \subsetneq s(g)$ for some $g \in E(\rho) - \{e, e_i\}$, then $\rho$ is type $T_3$, $F_1$, or $F_3$.

If $\rho$ is not type $F_1$ or $F_3$, then Lemma 9 gives $s(e_i) \not\subseteq s(g)$ and $s(g) \not\subseteq s(e_i)$ for all such $g$. First assume that all such $g$ satisfy $s(e) = s(e_i) \cap s(g)$. By Lemma 5, it follows that either $e_{i-1}$ or $e_{i+1}$ is such a $g$. If $g = e_{i-1}$, then $s(e) = [a_i, b_{i-1}]$, so $e_1, e_2, \ldots, e_{i-1}, e, e_i, \ldots, e_n$ is a lattice path ordering of $\rho$; if $g = e_{i+1}$, then $s(e) = [a_i, b_{i+1}]$, so $e_1, e_2, \ldots, e_i, e, e_{i+1}, \ldots, e_n$ is a lattice path ordering of $\rho$; both conclusions are impossible, so $s(e) \not\subseteq s(e_i) \cap s(g)$.

Therefore $s\rho\{e, e_i, g\}$, and so $\rho$ is type $T_3$.

Now assume that if $e_j \in E(\rho) - \{e, e_i\}$, then $s(e) \not\subseteq s(e_j)$.

Next we show that

11.2. if $s(e) \cap s(e_j) = \emptyset$ for all $e_j \in E(\rho) - \{e, e_i\}$, then $\rho$ is type $F_2$.

If $i = 1$, then $s(e) \subseteq [a_1, a_2]$, so we can reorder this interval so that $e, e_1, e_2, \ldots, e_n$ is a lattice path ordering of $\rho$; that is a contradiction, so $i > 1$. Similarly, $i < n$. By Lemma 10, neither $s(e_{i-1})$ nor $s(e_{i+1})$ is disjoint from $s(e_i)$. However, both are disjoint from $s(e)$, and so from each other (since $s(e) \not\subseteq s(e_i)$), so $\rho|\{e, e_{i-1}, e_i, e_{i+1}\}$, and so $\rho$, is type $F_2$ where $s(e_i)$ is $D$.

Now assume that $s(e) \cap s(e_j) \neq \emptyset$ for some $e_j \in E(\rho) - \{e, e_i\}$.

Up to reversing the lattice path ordering of $\rho_{e'}$, we may assume that $s(e) \cap s(e_{i+1}) \neq \emptyset$. Let the supports of $e, e_{i+1}$, and $e$ be represented by the supports of $e, f$, and $g$, respectively in Figure 2. Then $Z \neq \emptyset$ and, since $s(e)$ is properly contained in $s(e_i)$, both $V$ and $X$ are empty, and either $T$ or $W$ is nonempty. Now $Y \neq \emptyset$ since $s(e) \not\subseteq s(e_{i+1})$. If $T$ and $W$ are both nonempty, then $\rho|\{e, e_i, e_{i+1}\}$, and so $\rho$, is type $T_3$. The remaining cases are where the supports of $e, e_{i+1}$, and $e$ are as shown in the Venn diagrams in Figure 5, where $U$ may be empty or nonempty.
Figure 5: Venn diagram showing the supports of $e_i, e_{i+1}$, and $e$.

The case in Figure 5(a) has $Z = [a_{i+1}, b]$. Now $s(e)$ cannot be an interval since otherwise $e_1, e_2, \ldots, e_i, e, e_{i+1}, \ldots, e_n$ would be a lattice path ordering of $\rho$. Since no reordering of $T \cup Y$ having $T$ before $Y$ is a lattice path ordering,

- $i > 1$, and
- there is a $j < i$ with $s(e_j) \cap Y \neq \emptyset$ and $T \not\subseteq s(e_j)$.

Thus, $s(e_j) \cap Z = \emptyset$. If $s(e_j) \cap T \neq \emptyset$, then the proper minor $\rho|\{e, e_j, e_i\}$ is type $T_4$, which is impossible. Thus, $s(e_j) \cap T = \emptyset$, and so $\rho|\{e, e_j, e_i, e_{i+1}\}$, and hence $\rho$ is type $F_3$.

The case in Figure 5(b) has $Y = [a_i, a_{i+1}]$ and $W \cup Z = [a_{i+1}, b]$. Now $s(e)$ cannot be an interval since otherwise $e_1, e_2, \ldots, e_{i-1}, e, e_i, \ldots, e_n$ would be a lattice path ordering of $\rho$. Likewise, we cannot reorder the support of $\rho_e$ to get a lattice path ordering of the support of $\rho$, so either

(i) there is a $j < i$ with $s(e_j) \cap W \neq \emptyset$ and $Z \not\subseteq s(e_j)$, or

(ii) there is a $j > i + 1$ with $s(e_j) \cap Z \neq \emptyset$ and $W \not\subseteq s(e_j)$.

Suppose $j < i$ satisfies (i). Then $Y \subseteq s(e_j)$ and either $s(e_j) \cap Z$ is empty and $\rho|\{e, e_j, e_i\}$ is a Boolean 3-cycle, or $s(e_j) \cap Z$ is not empty and $\rho|\{e, e_j, e_i, e_{i+1}\}$ is type $T_2$. This is impossible since this is a proper minor of $\rho$. Thus, (ii) must occur.

From (ii), we get $s(e_j) \cap Y = \emptyset$. If $s(e_j) \cap W \neq \emptyset$, then the proper minor $\rho|\{e, e_i, e_j\}$ is type $T_4$, which is impossible. Thus, $s(e_j) \cap W = \emptyset$. Then $s(e_j) \cap s(e_i) \subseteq Z$. The proper minor $\rho|\{e, e_{i+1}, e_j\}$ is not type $T_1$, so $s(e_j) = U \cup Z'$ for some $Z' \subseteq Z$. Then $U \neq \emptyset$, since otherwise $s(e_j) \subseteq s(e_{i+1}) \subseteq s(e_i)$, and $\rho$ is type $F_1$ or $F_3$ by Lemma 9, as desired. Let $Z'' = Z\backslash Z'$, which may be empty. Then $s(e) = Y \cup Z' \cup Z''$, and $s(e_i) = Y \cup Z' \cup Z'' \cup W$, and $s(e_{i+1}) = Z' \cup Z'' \cup W \cup U$, and $s(e_j) = Z' \cup U$. The only set in $\{U, W, Y, Z', Z''\}$ that may be empty is $Z''$. If $Z'' \neq \emptyset$, then the proper minor $\rho|\{e, e_{i+1}, e_i\}$ would be type $T_4$, which is impossible. Thus, $Z'' = \emptyset$, and so $\rho|\{e, e_i, e_{i+1}, e_j\}$, and hence $\rho$, is type $F_1$. \hfill \Box

We prove one last lemma before proving the main result.

**Lemma 12.** Let $\rho$ be a Boolean polymatroid that is an excluded minor for the class of lattice path polymatroids. Let $e_1, e_2, \ldots, e_n$ be a lattice path ordering of $\rho_e$ for some $e \in E(\rho)$. If $\emptyset \neq s(e_i) \cap s(e_{i+1}) \subseteq s(e) \subseteq s(e_i) \cup s(e_{i+1})$ for some $i \in \{1, 2, \ldots, n\}$, then $s(e) \not\subseteq s(d)$ for some $e, d \in E(\rho)$.
Proof. Suppose that $s(c) \not\subseteq s(d)$ for all $c, d \in E(\rho)$. Then $a_i < a_{i+1} \leq b_i < b_{i+1}$. Note that if $j \not\in \{i, i+1\}$, then $[a_{i+1}, b_i] \not\subseteq s(e_j) = [a_j, b_j]$, since if $j < i$, then $b_j < b_i$, so $b_i \not\in s(e_j)$, while if $j > i + 1$, then $a_{i+1} < a_j$, so $a_{i+1} \not\in s(e_j)$.

Let $f$ and $g$ represent $e_i$ and $e_{i+1}$, respectively, in Figure 2. From the hypothesis, $T \cup X = \emptyset$ and $Z \not\subseteq \emptyset$. Sets $W$ and $Y$ are both nonempty since $s(e)$ is not contained in either $s(e_i)$ or $s(e_{i+1})$. Sets $U$ and $V$ are also nonempty since $s(e)$ contains neither $s(e_i)$ nor $s(e_{i+1})$.

Now $W \cup Y \cup Z$ is not an interval since otherwise $e_1, e_2, \ldots, e_i, e, e_{i+1}, e_{i+2}, \ldots, e_n$ would be a lattice path ordering on $\rho$, which is false. Note that $[a_i, a_{i+1}] = U \cup W$ and $[a_{i+1}, b_i] = Z$ and $(b_i, b_{i+1}] = V \cup Y$. Since no reordering of the support of $\rho_{\mathcal{C}}$ gives a lattice path ordering of $\rho$, and no support other than $s(e)$, $s(e_i)$, and $s(e_{i+1})$ contains $Z$, there is some element $e_j$ in $E(\rho) - \{e, e_i, e_{i+1}\}$ such that either

(i) $j < i$ and $s(e_j) \cap W \not\subseteq W$ and $U \not\subseteq s(e_j)$, or

(ii) $j > i + 1$ and $s(e_j) \cap Y \not\subseteq Y$ and $V \not\subseteq s(e_j)$.

Up to reversing the order on $\rho_{\mathcal{C}}$, we can assume that (i) holds. Then $s(e_j) \cap (Z \cup Y \cup V) = \emptyset$. Consider $\rho \{e, e_j, e_i\}$. Some element in $W$ is in the supports of all of these, $a_j < a_i$, and some element in $U$ is in $s(e_i)$ but not in $s(e) \cup s(e_j)$ and $Y$ is nonempty and is disjoint from $s(e_i) \cup s(e_j)$. Therefore the proper minor $\rho \{e, e_j, e_i\}$ is type $T_1$, which is a contradiction.

We now prove the main result.

**Theorem 13.** A polymatroid is lattice path if and only if it is Boolean and contains no minor that is a Boolean n-cycle, with $n > 2$, or a type $T_1, T_2, T_3, T_4, F_1, F_2, F_3$, or $F_4$ polymatroid.

**Proof.** The class of lattice path polymatroids is minor-closed and is contained in the class of Boolean polymatroids. Since none of the polymatroids listed above is lattice path, a polymatroid is not lattice path if it has any of these as a minor.

If the list in the theorem is not complete, let $\rho$ be an excluded minor for the class of lattice path polymatroids that is a Boolean polymatroid but not among those identified above. Since $\rho$ is Boolean and is an excluded minor, by Lemmas 1 and 8 each element in $E(\rho)$ can be identified by its support. Since every polymatroid with at most two elements is lattice path, $|E(\rho)| \geq 3$. By Lemma 7, $|E(\rho)| \geq 4$.

Let $G = G(\rho)$. By Lemma 10, $G$ is connected. Let $e$ be a vertex of lowest degree that is not a cut vertex. Let $e_1, e_2, \ldots, e_n$ be a lattice path ordering on $\rho_{\mathcal{C}}$, where $s(e_i) = [a_i, b_i]$ for all $i$ and $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. Since $G \setminus e$ is connected and Lemma 11 implies that no support contains another, $a_{i-1} < a_i \leq b_{i-1} < b_i$ for each $i \in \{2, 3, \ldots, n\}$. Furthermore, $G$ contains $e_1e_2 \ldots e_n$ as a path, although this may not be an induced path. We show that

**13.1. the degree $d(e) > 1$.**

Suppose $d(e) = 1$. Then $s(e) \cap [a_1, b_n]$ is contained one of the following sets:
(i) $[a_1, a_2)$,
(ii) $(b_{n-1}, b_n]$,
(iii) $(b_{i-1}, a_{i+1})$ for some $i \in \{2, 3, \ldots, n-1\}$.

These options yield the contradictions (i) $e, e_1, e_2, \ldots, e_n$ is a lattice path ordering of $\rho$, (ii) $e_1, e_2, \ldots, e_n, e$ is a lattice path ordering of $\rho$, (iii) $\rho|\{e, e_{i-1}, e_i, e_{i+1}\}$ is type $F_2$. Thus 13.1 follows.

Next we show that in the support graph $G$,

13.2. If $e$ is adjacent to non-adjacent vertices $e_i$ and $e_j$, where $i < j$, then $e$ is adjacent to some $e_k$ where $i < k < j$.

Suppose $e$ is adjacent to $e_i$ and $e_j$ but to none of $e_{i+1}, e_{i+2}, \ldots, e_{j-1}$. Take a shortest path $P$ between $e_i$ and $e_j$ in the graph that $G$ induces on $\{e_i, e_{i+1}, \ldots, e_j\}$. Then $P$ together with $e$ forms an induced cycle in $G$ with at least four vertices, and $\rho|V(P) \cup \{e\}$ is a Boolean $m$-cycle for some integer $m \geq 4$. This contradiction proves 13.2.

Next we show that

13.3. $d(e) \geq 3$.

Suppose $d(e) = 2$, and let $e_i$ and $e_j$ be adjacent to $e$, where $i < j$. By 13.2, $e_i, e_j$ is an edge in $G$. Let $f$ represent $e_i$ and $g$ represent $e_j$ in Figure 2. If $Z = \emptyset$, then $W, X, Y$ are all nonempty, so $\rho|\{e, e_i, e_j\}$ is a Boolean 3-cycle, which is a contradiction. Thus, $Z \neq \emptyset$. Then $Z$ is contained in the support of every element in $e_i, e_{i+1}, \ldots, e_j$, so $j = i + 1$. Since $\rho|\{e, e_i, e_{i+1}\}$ is not type $T_1$ or $T_2$, some set in $\{T, U, V\}$ is empty and therefore some set in $\{W, X, Y\}$ is empty. Since no element has its support contained in the support of another element, at least one of the following statements must hold:

(i) $T \cup X = \emptyset$ and $U, V, W, Y$ are all nonempty;
(ii) $U \cup Y = \emptyset$ and $T, V, W, X$ are all nonempty; or
(iii) $V \cup W = \emptyset$ and $T, U, X, Y$ are all nonempty.

If (i) occurs, then we get a contradiction by Lemma 12. Up to reversing the order on $\rho|_e$, we may assume that (ii) occurs. If $i \neq 1$, then $s(e_{i-1}) \cap (e_i) \neq \emptyset$ while $s(e_{i-1}) \cap s(e) = \emptyset$, so $s(e_{i-1}) \cap X \neq \emptyset$. Then $b_{i-1} \geq a_{i+1}$, and $W \subseteq s(e_{i-1})$, contrary to $e$ and $e_{i-1}$ not being adjacent. So $i = 1$ and $e, e_1, e_2, \ldots, e_{n-1}$ is a lattice path ordering of $\rho$, which is a contradiction. This completes the proof of 13.3.

Assume that $e$ is adjacent to $e_i, e_j$, and $e_k$, with $i < j < k$, and to no $e_h$ with $i < h < j$ or $j < h < k$. By 13.2, both $e_ie_j$ and $e_je_k$ are edges in $G$. Let $f$ represent $e_i$ and $g$ represent $e_j$ in Figure 2. If $Z = \emptyset$, then $W, X, Y$ are all nonempty, and $\rho|\{e, e_i, e_j\}$ is a Boolean 3-cycle, which is impossible. Thus, $Z \neq \emptyset$, i.e., $s(e) \cap s(e_i) \cap s(e_j) \neq \emptyset$. Thus, $s(e) \cap s(e_h) \neq \emptyset$ for all $h$ with $i \leq h < j$. The same argument shows that $s(e) \cap s(e_j) \cap s(e_k) \neq \emptyset$, and so $s(e) \cap s(e_h) \neq \emptyset$ for all $h$ with $j < h < k$. Thus, $j = i + 1$ and $k = i + 2$.

We show that
13.4. \(e_ie_{i+2}\) is an edge in \(G\).

Suppose not. Then the supports of \(e, e_i, e_{i+1},\) and \(e_{i+2}\) are as shown in Figure 6. Since \(\rho\) has no restriction that is type \(T_1,\) at least one of \(Q, T \cup W,\) and \(V \cup X\) is empty, as is at least one of \(R \cup T, U,\) and \(P \cup V.\) Since no restriction of \(\rho\) is type \(T_2,\) either \(P = \emptyset\) or \(R = \emptyset;\) also, either \(W = \emptyset\) or \(X = \emptyset.\) No support contains another support, so \(Q \neq \emptyset\) and \(U \neq \emptyset.\) Since \(s(e) \not\subseteq s(e_{i+1}),\) at least one of \(T, W,\) and \(R\) is nonempty. Similarly at least one of \(P, V,\) and \(X\) is nonempty. After possibly reversing the order on \(\rho_e,\) it follows that \(R, T, V,\) and \(X\) are empty, and \(P, Q, S, U, W,\) and \(Z\) are nonempty. Now Lemma 12 gives a contradiction since \(\emptyset \neq s(e_i) \cap s(e_{i+1}) \subseteq s(e) \subseteq s(e_i) \cup s(e_{i+1}).\) This proves 13.4.

Let the elements in \(\{e, e_i, e_{i+1}, e_{i+2}\}\) be represented by \(\{a, b, c, d\}\) in Figure 7. We show that

13.5. \(Z = \emptyset.\)

Suppose \(Z \neq \emptyset.\) No restriction of \(\rho\) to any set of three elements in \(\{a, b, c, d\}\) is type \(T_1.\) Hence, for each triple \(\{x, y, z\} \subseteq \{a, b, c, d\},\) at least one set in \(s(x) - (s(y) \cup s(z))\) and \(s(y) - (s(x) \cup s(z))\) and \(s(z) - (s(x) \cup s(y))\) is empty. Without loss of generality, \(s(d) - (s(a) \cup s(b)) = O \cup U = \emptyset,\) and at least one of \(L \cup Q, M \cup P,\) and \(N\) is also empty. Hence, at least one of \(L \cup Q \cup O, M \cup P \cup O,\) and \(N \cup U \cup O\) is empty. Up to changing the labels of \(a, b, c,\) and \(d,\) we may assume that \(N \cup O \cup U = \emptyset.\) Since no element has its support contained in another by Lemma 11, each of the following sets is nonempty:
$P \cup Y$, $Q \cup W$, $T \cup Y$, and $R \cup W$. Since neither $\rho\{a, b, c\}$ nor $\rho\{a, b, d\}$ is type $T_2$, both $S \cup V$ and $S \cup X$ are empty.

Since no support contains another, none of the following sets is empty: $L \cup Q$, $M \cup P$, $L \cup R$, $M \cup T$, $R \cup T$, and $P \cup Q$. Since neither $\rho\{a, c, d\}$ nor $\rho\{b, c, d\}$ is type $T_1$, it follows that at least one of $L, P$, and $T$ is empty and at least one of $M, Q$, and $R$ is empty. Since neither $\rho\{a, c, d\}$ nor $\rho\{b, c, d\}$ is type $T_2$, it follows that at least one of $Q, R$, and $Y$ is empty and at least one of $P, T$, and $W$ is empty. Thus,

(i) if $P = \emptyset$, then $R = \emptyset$ and $L, M, Q, T, W$, and $Y$ are all nonempty;

(ii) if $T = \emptyset$, then $Q = \emptyset$ and $L, M, P, R, W$, and $Y$ are all nonempty;

(iii) if $L = \emptyset$, then $M \cup W \cup Y = \emptyset$ and $P, Q, R$, and $T$ are all nonempty, and $\rho\{a, b, c, d\}$ is type $F_4$, where $A = Q, B = R, C = Z, D = T$, and $E = P$.

Since (iii) gives a contradiction and (ii) is obtained from (i) by switching the labels on $a$ and $b$, we assume that (i) holds. Note that $b, c, d, a$ is a lattice path ordering of $\rho\{a, b, c, d\}$ and $M, T, Y, Z, W, Q, L$ is a lattice path ordering of the support of $\rho\{a, b, c, d\}$. Up to reversing the lattice path ordering on $\rho_e$, where $e_i < e_{i+1} < e_{i+2}$, the fact that $L, M, Q, T, W$, and $Y$ are all nonempty implies that $(b, c, d, a)$ is one of $(e, e_i, e_{i+1}, e_{i+2})$, $(e_i, e, e_{i+1}, e_{i+2})$, $(e_i, e_{i+1}, e, e_{i+2})$, and $(e, e_{i+1}, e_{i+2}, e)$.

By Lemma 12, $e \not\in \{c, d\}$, so $e \in \{a, b\}$. Then $s(e_i) \cap s(e_{i+2}) = [a_{i+2}, b_i]$ is either $Y \cup Z$ or $Z \cup W$, both of which are nonempty. Then

$$s(e_i) \cap s(e_{i+2}) \subseteq s(e_{i+1}) \subseteq s(e_i) \cup s(e_{i+2}).$$

We next show that $e_i$ and $e_{i+2}$ are consecutive elements in a lattice path ordering of $\rho|_{e_{i+1}}$, but that is impossible by Lemma 12, and so that will complete the proof of 13.5. The two options for $(b, c, d, a)$ show that $s(e_i) \cap s(e_{i+2}) \subseteq s(e)$. Now $[a_{i+2}, b_i] \subseteq s(e_{i-1})$ since $b_i > b_{i-1}$. Similarly, $[a_{i+2}, b_i] \subseteq s(e_{i+3})$. Thus, $e_i$ and $e_{i+2}$ are the two elements in $\rho|_{e_{i-1}}$ that contain $[a_{i+2}, b_i]$ in their supports, and so, as claimed, they must be consecutive in the lattice path ordering on $\rho|_{e_{i+1}}$ by Lemma 5. Thus we have shown 13.5.

We show that

13.6. $s(x) \cap s(y) \cap s(z) = \emptyset$ for every triple $\{x, y, z\} \subseteq \{a, b, c, d\}$.

Suppose $s(a) \cap s(b) \cap s(c)$ is nonempty. Then $X \neq \emptyset$. Since $\rho\{a, b, c\}$ is not type $T_1$, at least one of the sets $L \cup Q$, $M \cup P$, and $N \cup U$ is empty. Up to relabeling $a$, $b$, and $c$, we may assume that $N \cup U = \emptyset$. Since $\rho\{a, b, c\}$ is not type $T_2$, at least one of the sets $R \cup W$, $S \cup V$, and $T \cup Y$ is empty. Two of those options yield the contradiction that some support contains another, so $S \cup V = \emptyset$; avoiding other instances of some support containing another implies that $L \cup Q, R \cup W, T \cup Y, M \cup P$ are all nonempty. Then

$$s(a) = L \cup Q \cup R \cup W \cup X,$$

$$s(c) = R \cup W \cup X \cup T \cup Y,$$

$$s(b) = X \cup T \cup Y \cup M \cup P.$$
Thus $L \cup Q, R \cup W, X, T \cup Y, M \cup P$ is a lattice path ordering of the support of $\rho|\{a, b, c\}$. Now $s(d) = Q \cup W \cup Y \cup P \cup O$. If $Q \cup W$ and $Y \cup P$ are both nonempty, then $\rho|\{a, b, d\}$ is a Boolean 3-cycle, which is a contradiction. Therefore either $Q \cup W$ or $Y \cup P$ is empty. Hence either $s(a) \cap s(d) = \emptyset$ or $s(b) \cup s(d) = \emptyset$, which is a contradiction. Then 13.6 follows by symmetry.

Thus, the sets $V, W, X, Y,$ and $Z$ in Figure 7 are empty. Since $s(y) \cap s(z) \neq \emptyset$ for all $\{y, z\} \subseteq \{a, b, c, d\}$, the following sets are nonempty: $P, Q, R, S, T$, and $U$. Therefore $\rho|\{a, b, c\}$ is a Boolean 3-cycle. This contradiction completes our proof. □

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References


