Shattering and more: Extending the complete object

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Abstract

Let $\mathcal{F} \subseteq 2^{[m]}$ be a family of subsets of $[m] = \{1, 2, \dots, m\}$. For $S \subseteq [m]$, let $\mathcal{F}|_S$ be the *trace* $\mathcal{F}|_S = \{B \cap S : B \in \mathcal{F}\}$, considered as a multiset. We say $\mathcal{F}$ *shatters* a set $S \subseteq [m]$ if $\mathcal{F}|_S$ has all $2^{|S|}$ possible sets (i.e. complete). We say $\mathcal{F}$ has a shattered set of size k if $\mathcal{F}$ shatters some $S \subseteq [m]$ with $|S| = k$. It is well known that if $\mathcal{F}$ has no shattered k-set then $|\mathcal{F}| \leqslant \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}$. We obtain the same exact bound on $|\mathcal{F}|$ (for m large enough) when forbidding less. Namely, given fixed positive integers t and k, for every set $S \subseteq [m]$ with $|S| = k$, set families $\mathcal{F}$ are such that $\mathcal{F}|_S$ does not have both all possible sets 2^S *and* specified additional sets occurring at least t times. Similar results are proven for *double shattering*, namely when $\mathcal{F}|_S$ does not have all sets $2^{|S|}$ appearing twice. The paper is written in matrix notation with trace replaced by configuration.

Keywords: extremal set theory, shattered set, shattering, VC-dimension, forbidden configurations

Mathematics Subject Classifications: 05D05

1 Introduction

Using the notation $[m] = \{1, 2, \dots, m\}$ and $2^{[m]} = \{S : S \subseteq [m]\}$, we are interested in families of subsets of $[m]$ say $\mathcal{A} \subseteq 2^{[m]}$. We say that $\mathcal{A}$ *shatters* a set $S \subseteq [m]$ if all $2^{|S|}$ sets appear in the *trace* $\mathcal{A}|_S = \{B \cap S : B \in \mathcal{A}\}$ which will be interpreted as a multiset. This paper uses matrix notation. There is a natural correspondence between a family $\mathcal{A} \subseteq 2^{[m]}$

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of subsets of $[m]$ and an m-rowed (0,1)-matrix A that have no repeated columns. The correspondence has the ith column of A being the incidence vector of the ith set $A_i \in \mathcal{A}$, with a 1 in row j if $j \in A_i$ and a 0 in row j if $j \notin A_i$. It is convenient to say a matrix is *simple* if it is a (0,1)-matrix with no repeated columns so that a simple m-rowed matrix A corresponds to a set system $\mathcal{A} \subseteq 2^{[m]}$. Define $\|A\|$ to be the number of columns in A so that $\|A\| = |\mathcal{A}|$.

We define F to be a *configuration* in A if there is a submatrix of A that is a row and column permutation of F. Thus a configuration is the combinatorial equivalent of a submatrix. F need not be simple. Define $F \prec A$ if F is a configuration in A. For a set of rows S, define $A|_S$ to be the submatrix of A given by rows S. Let K_k denote the $k \times 2^k$ matrix corresponding to $2^{[k]}$, hence the adjective *complete*. Then $\mathcal{A}$ shatters S if $K_{|S|} \prec A|_S$.

Definition 1. We say $\mathcal{A}$ has *VC-dimension k* if

$$k = \max\{|S| \,:\, \mathcal{A} \text{ shatters } S\}. \quad \square$$

Thus if $\mathcal{A}$ has VC-dimension k and A is the associated simple matrix, then $K_k \prec A$ and $K_{k+1} \not\prec A$. Let $\mathcal{F}$ be a family of forbidden configurations and define

$$\text{Avoid}(m, \mathcal{F}) = \{A \,:\, A \text{ is } m\text{-rowed and simple}, F \not\prec A \text{ for all } F \in \mathcal{F}\}.$$

The extremal problem becomes

$$\text{forb}(m, \mathcal{F}) = \max\{\|A\| \,:\, A \in \text{Avoid}(m, \mathcal{F})\}.$$

The following Theorem has proved remarkably useful in a variety of contexts.

Theorem 2. *Sauer [8], Perles, Shelah [9], Vapnik, Chervonenkis [10].*

$$\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{1} + \binom{m}{0}. \quad \square \tag{1}$$

The goal of this paper is to show that there are many columns you can add to K_k without changing the bound. Initial investigations (Theorem 1.2 in [5]) were hampered by trying to prove base cases. We use stability results to overcome this.

Let A be an $m_1 \times n_1$ matrix and let B be an $m_2 \times n_2$ matrix. Where $m_1 = m_2$, use the notation $[A|B]$ to denote the concatenation of A, B with $n_1 + n_2$ columns, and define $t \cdot A$ to be the concatenation of t copies of A: $[A|A|\cdots|A]$ with tn_1 columns. Use the notation $A \times B$ to denote the $(m_1 + m_2) \times (n_1 n_2)$ matrix with all possible columns formed from one column from A placed on top of one column from B. Let $\mathbf{1}_p\mathbf{0}_q$ be the column of p 1s on top of q 0s. The following are our two main results. The notation K_2^T refers to the 4×2 matrix that is the transpose of K_2. The matrix F_1 appears in Theorem 4. Let

$$K_2^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Theorem 3. *Assume $k \geqslant 4$ and $t \geqslant 1$. There exists an m_k (depending on k and t) so that for $m > m_k$ we have*

$$\text{forb}(m, [K_k | t \cdot (K_2^T \times K_{k-4})]) = \text{forb}(m, K_k).$$

The proof is in Section 2. The case $[K_k | t \cdot (\mathbf{1}_2\mathbf{0}_2 \times K_{k-4})]$ with $t = 1$ is Theorem 1.2 in [5]. We also considered $2 \cdot K_k$ with success.

Theorem 4. *Assume $k \geqslant 3$ and $t \geqslant 1$. There exists an m_k (depending on k and t) so that for $m > m_k$ we have*

$$\text{forb}(m, [2 \cdot K_k | t \cdot (F_1 \times K_{k-3})]) = \text{forb}(m, 2 \cdot K_k) = \text{forb}(m, K_{k+1}).$$

The proof is in Section 3. Theorem 1.7 in [5] proves the case $[2 \cdot K_k | t \cdot (F_1 \times K_{k-3})])$ with $t = 1$. There are a number of results [1], [4] where a *critical substructure* determines the bound and adding a few columns does not alter the bound.

A useful proof technique is called *standard induction* [1]. Assume A is simple. Permute the rows and columns of A so that r becomes the first row. After deleting row r there may be repeated columns which we place in C_r in the following *standard decomposition* of A:

$$A = \begin{array}{c} r \rightarrow \\ \\ \end{array} \left[\begin{array}{cccc} 0 \cdots 0 & & 1 \cdots 1 & \\ B_r & C_r & C_r & D_r \end{array} \right], \tag{2}$$

where B_r are the columns that appear with a 0 in row r but don't appear with a 1, and D_r are the columns that appear with a 1 but not a 0. We note both $[B_rC_rD_r]$ and C_r are simple $(m-1)$-rowed matrices. If we assume $A \in \text{Avoid}(m, \mathcal{F})$, then $[B_rC_rD_r] \in \text{Avoid}(m-1, \mathcal{F})$ and

$$\|A\| = \|[B_rC_rD_r]\| + \|C_r\| \leqslant \text{forb}(m-1, \mathcal{F}) + \|C_r\|. \tag{3}$$

This means any upper bound on $\|C_r\|$ (as a function of m) automatically yields an upper bound on forb$(m, \mathcal{F})$ by induction. Of course $C_r \in \text{Avoid}(m-1, \mathcal{F})$ but more is true. Let $A \in \text{Avoid}(m, \mathcal{F})$. Define the *inductive children* of $\mathcal{F}$ as the minimal set of configurations $\mathcal{F}'$ which must be avoided in C_r. Potential candidates for an inductive child would be configurations F' such that $[0\,1] \times F'$ cannot appear in A, i.e. there is an $F \in \mathcal{F}$ with $F \prec [0\,1] \times F'$. The uniqueness of the minimal set follows from the following requirement: if we have two configurations F', F'' with $F' \prec F''$ then $F'' \notin \mathcal{F}'$. We ask for $\mathcal{F}'$ to be *minimal* to avoid having an unwieldy set. With this definition, $C_r \in \text{Avoid}(m-1, \mathcal{F}')$ and $\|C_r\| \leqslant \text{forb}(m-1, \mathcal{F}')$.

Remark 5. F is the only inductive child of $F \times [0\,1]$. □

Sometimes the interest in Theorem 2 is the exact bound forb(m, K_k) and sometimes the interest is in the asymptotic bound $\Theta(m^{k-1})$. Theorem 3 considers what columns can be added to K_k and still have the same exact bound. The analogous asymptotic question, what columns can be added to K_k and still have the bound $\Theta(m^{k-1})$, is completely settled:

Theorem 6. *[1] Let k be given and let B be an $k \times (k+1)$ matrix with one column of each column sum. Then* $\mathrm{forb}(m, [K_k | t \cdot (K_k \backslash B)])$ *is* $\Theta(m^{k-1})$. *Also if F is a k-rowed configuration and $K_k \prec F$, then* $\mathrm{forb}(m, F)$ *is* $\Theta(m^{k-1})$ *if and only if there is a t and $k \times (k+1)$ matrix B with one column of each column sum where $F \prec [K_k | t \cdot (K_k \backslash B)]$.* □

Using the notation K_4^2 to refer to the 4×6 simple matrix of all columns of sum 2, we obtain from Theorem 6 that $\mathrm{forb}(m, [K_4 | K_4^2])$ is $\Theta(m^4)$. The construction $I_{m/4} \times I_{m/4} \times I_{m/4}^c \times I_{m/4}^c$ has $\Theta(m^4)$ columns and avoids $2 \cdot K_4^2$. Thus adding all six columns of K_4^2 to K_4 cannot preserve the bound. Theorem 3 only answers the question for two columns of sum 2 and we pose remaining questions in Section 5. You may note that increasing t value in Theorem 3 does not increase the bound. The following result is proved in Section 4 showing some elementary cases where increasing t does affect the bound. It is imagined that this is typically the case.

Theorem 7. *Let F be a $k \times \ell$ forbidden configuration and $t \geqslant 1$ be an integer. Then there exists a number M so that* $\mathrm{forb}(m, (t+1) \cdot F) > \mathrm{forb}(m, t \cdot F)$ *when $m > M$.*

2 Extensions of K_k with the same bound

We will need the following fact concerning the bound (1) of Theorem 2 that follow readily from Pascal's identity.

$$\mathrm{forb}(m, K_k) = \mathrm{forb}(m-1, K_k) + \mathrm{forb}(m-1, K_{k-1}). \tag{4}$$

The following is a simple example of what we do.

Theorem 8. $\mathrm{forb}(m, \{[K_2 | \mathbf{1}_1 \mathbf{0}_1], [K_2 | \mathbf{0}_2]\}) = \lceil \frac{3}{2} m \rceil > \mathrm{forb}(m, K_2)$.

Proof. Let $A \in \mathrm{Avoid}(m, \{[K_2 | \mathbf{1}_1 \mathbf{0}_1], [K_2 | \mathbf{0}_2]\})$. Assume $\|A\| > \mathrm{forb}(m, K_2)$. Then there exist rows i, j with $K_2 \prec A|_{\{i,j\}}$. To avoid both $[K_2 | \mathbf{1}_1 \mathbf{0}_1]$ and $[K_2 | \mathbf{0}_2]$, the only column of $A|_{\{i,j\}}$ appearing more than once is $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and so we can delete from A rows i, j and 3 columns to obtain a simple matrix $A' \in \mathrm{Avoid}(m-2, \{[K_2 | \mathbf{1}_1 \mathbf{0}_1], [K_2 | \mathbf{0}_2]\})$.

A construction would be to take $I_{m/2}^c$ and replace each 0 by the 2×3 matrix $\left[\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right]$ and replacing each 1 by the 2×3 block of 1's to obtain an $m \times \frac{3}{2} m$ simple matrix in $\mathrm{Avoid}(m, \{[K_2 | \mathbf{1}_1 \mathbf{0}_1], [K_2 | \mathbf{0}_2]\})$. If m is odd, we note that the outlined construction works for $m+1$ and delete a row. □

Note that if the bound $\mathrm{forb}(m, K_2)$ is exceeded then there is a pair of rows i, j so that that $K_2 \prec A|_{\{i,j\}}$ and hence $[2 \cdot \mathbf{1}_1 \mathbf{0}_1] \not\prec A|_{\{i,j\}}$ and $[2 \cdot \mathbf{0}_2] \not\prec A|_{\{i,j\}}$. The consequence is that we can delete at most 3 columns and two rows from A to obtain a simple matrix A'. This idea is repeated in Lemma 9 and is crucial for Theorem 3.

Consider the following matrices:

$$F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We note that the inductive children of $[K_4 | t \cdot K_2^T]$ are $[K_3 | t \cdot F_2]$, $[K_3 | t \cdot F_3]$ and $[K_3 | t \cdot F_4]$.

Lemma 9. *Let $\mathcal{F} = \{[K_3|t\cdot F_2], [K_3|t\cdot F_3], [K_3|t\cdot F_4]\}$. Let $A \in \text{Avoid}(m, \mathcal{F})$ with a triple of rows $S = \{i, j, k\}$ with $(t+1)\cdot F_2 \not\prec A|_S$, $(t+1)\cdot F_3 \not\prec A|_S$ and $(t+1)\cdot F_4 \not\prec A|_S$. Then we can delete one row and at most $4t$ columns and obtain a simple matrix $A' \in \text{Avoid}(m-1, \mathcal{F})$.*

Proof. Let $A \in \text{Avoid}(m, \mathcal{F})$ with a triple of rows $S = \{i, j, k\}$ with $(t+1)\cdot F_2 \not\prec A|_S$, $(t+1)\cdot F_3 \not\prec A|_S$ and $(t+1)\cdot F_4 \not\prec A|_S$. Using case analysis, we will show that 4 (of the 8 possible) columns of $A|_S$ are restricted to occur at most t times (columns in *short supply*) and the other 4 columns have no restriction (we use the notation that those columns are in *long supply* denoted *l.s.*). Indeed, suppose that $\mathbf{0}_2\mathbf{1}_1$ is in long supply. Since $(t+1)\cdot F_2 \not\prec A_S$, the columns $\mathbf{1}_1\mathbf{0}_2$ and $\mathbf{0}_1\mathbf{1}_1\mathbf{0}_1$ are in short supply. Since $(t+1)\cdot F_3 \not\prec A_S$, the columns $\mathbf{0}_1\mathbf{1}_2$ and $\mathbf{1}_1\mathbf{0}_1\mathbf{1}_1$ are in short supply. We reach analogous conclusions if we assume $\mathbf{1}_2\mathbf{0}_1$ is in long supply. There is only one case up to row permutations:

$$\begin{array}{ccccccccccc} & & \leqslant t & \leqslant t & \leqslant t & \leqslant t & \text{l.s.} & \text{l.s.} & \text{l.s.} & \text{l.s.} \\ A|_{\{i,j,k\}} & \begin{matrix} i \\ j \\ k \end{matrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{array} \tag{5}$$

It is unusual in these investigations that there is only one case as in (5) to consider when forbidding configurations. Deleting row i and the at most $4t$ columns in short supply yields a simple matrix $A' \in \text{Avoid}(m-1, \mathcal{F})$ with $\|A'\| \geqslant \|A\| - 4t$. □

Lemma 10. *Given t, there exists an m_3 so that for $m > m_3$, $\text{forb}(m, \mathcal{F}) = \text{forb}(m, K_3)$.*

Proof. Let $A \in \text{Avoid}(m, \mathcal{F})$ with $\|A\| > \text{forb}(m, K_3)$ so that $K_3 \prec A|_S$ for some triple of rows S. Hence $(t+1)\cdot F_2 \not\prec A|_S$, $(t+1)\cdot F_3 \not\prec A|_S$ and $(t+1)\cdot F_4 \not\prec A|_S$. Using Lemma 9, we may delete one row and $4t$ columns to obtain a matrix $A^{'} \in \text{Avoid}(m-1, \mathcal{F})$ with $\|A'\| \geqslant \|A\| - 4t$. To continue, we need $\|A'\| > \text{forb}(m-1, K_3)$. Using (1), (4) we have $\text{forb}(p, K_3) - \text{forb}(p-1, K_3) = p$ for all p and so we need $m - 1 \geqslant 4t$ to obtain $\|A'\| > \text{forb}(m-1, K_3)$. We wish to repeat this process $m - m^{1/2}$ times deleting a row and at most $4t$ columns at each step, to obtain at the final step a simple matrix $A''' \in \text{Avoid}(m^{1/2}, \mathcal{F})$ with $\|A'''\| = \|A\| - 4t(m - m^{1/2})$. For this to work we need $m^{1/2} \geqslant 4t$ and so we choose $m_3 \geqslant \lceil (4t)^2 \rceil$ so that at each intermediate simple matrix A'' on m'' rows has $m'' \geqslant 4t$. Similarly, with this requirement on m_3, we can ensure $m'' > m^{1/2} \geqslant 4t$ which yields the number of columns $\|A''\| \geqslant \|A\| - 4t(m - m'') > \text{forb}(m'', K_3)$ using (1), (4). Hence each intermediate matrix A'' has $K_3 \prec A''$ and so the process can proceed to A'''. There is a further restriction on m_3 given below but for this part of the argument it suffices to have $m_3 \geqslant \lceil (4t)^2 \rceil$. Now $\|A'''\| = \|A\| - 4t(m - m^{1/2}) \geqslant \binom{m}{2} - 4t(m - m^{1/2})$ which, for fixed t, is a quadratic in m. The bound for $(t+1)\cdot K_3$ on $m^{1/2}$ rows is $\frac{t+3}{3}\binom{m^{1/2}}{3} + \binom{m^{1/2}}{2} + \binom{m^{1/2}}{1} + \binom{m^{1/2}}{0}$ [3] which is $\Theta(m^{3/2})$ while $\|A'''\|$ is quadratic. So m_3 can be chosen so that $\|A'''\| > \text{forb}(m^{1/2}, (t+1)\cdot K_3)$. Also choose m_3 to satisfy $m_3 \geqslant \lceil (4t)^2 \rceil$. Thus we will obtain $(t+1)\cdot K_3 \prec A'''$. Then $[K_3|t\cdot F_1] \prec (t+1)\cdot K_3 \prec A''' \prec A$, contradicting $A \in \text{Avoid}(m, \mathcal{F})$. □

This doesn't directly solve $[K_4|t\cdot K_2^T]$ by standard induction and (4) since that would require a base case for $\text{forb}(m,[K_4|t\cdot K_2^T])$. The following stability lemma is crucial to our bounds. It says that for $A\in\text{Avoid}(m,\mathcal{F})$ where $K_3\prec A$, we have that $\|A\|$ is less that the expected bound $\text{forb}(m,K_3)$ by a large amount.

Lemma 11. *Let $\mathcal{F}=\{[K_3|t\cdot F_2],[K_3|t\cdot F_3],[K_3|t\cdot F_4]\}$. Let $A\in\text{Avoid}(m,\mathcal{F})$. Assume there is a set S of 3 rows with $(t+1)\cdot F_2\not\prec A|_S$, $(t+1)\cdot F_3\not\prec A|_S$ and $(t+1)\cdot F_4\not\prec A|_S$. Assume $m>m_3+1$. Then $\|A\|\leqslant\text{forb}(m,K_3)-m+4t$.*

Proof. Assume $A\in\text{Avoid}(m,\mathcal{F})$ and $\|A\|>\text{forb}(m,K_3)-m+4t$. Assume $(t+1)\cdot F_2\not\prec A|_S$, $(t+1)\cdot F_3\not\prec A|_S$ and $(t+1)\cdot F_4\not\prec A|_S$. Then, using Lemma 9 ($m>m_3+1$) we can delete one row and at most $4t$ columns to obtain a simple matrix $A'\in\text{Avoid}(m-1,\mathcal{F})$. Then using (4) and assuming $m>4t$, $\|A'\|>\text{forb}(m,K_3)-m+4t-4t=\text{forb}(m-1,K_3)$. By Lemma 10 there is then an $F\in\mathcal{F}$ with $F\prec A'\prec A$, a contradiction. □

We can now prove our main result that gives a surprising number of columns we can append to K_k and still have the same bound. No column α can be added to K_3 and still have $\text{forb}(m,[K_3|\alpha])=\text{forb}(m,K_3)$ since, without loss of generality, α has 2 1's and then $3\cdot\mathbf{1}_2\prec[K_3|\alpha]$ but $\text{forb}(m,3\cdot\mathbf{1}_2)=\frac{4}{3}\binom{m}{2}+\binom{m}{1}+\binom{m}{0}>\text{forb}(m,K_3)$ (e.g. Theorem 17). This idea works for larger k so that for $k=4$ we can only add columns of 2 1's and 2 0's if we want the bound $\text{forb}(m,K_4)$. It also indicates why we need to have the hypothesis $m>m_k$ for some m_k. We offer no insight on the exact value of m_k. Theorem 3 is somewhat anticipated in [5] for the case $t=1$ (Theorems 1.3,1.4 in [5]), but the more general results were handicapped searching for base cases which the proof here has avoided. The columns we add to K_k are t copies of the 2^{k-3} columns $K_2^T\times K_{k-4}$.

Lemma 12. *Assume $k\geqslant 5$. The inductive child of $[K_k|t\cdot(K_2^T\times K_{k-4})]$ is $[K_{k-1}|t\cdot(K_2^T\times K_{k-5})]$.*

Proof. Apply Remark 5 and $[K_k|t\cdot(K_2^T\times K_{k-4})]=[K_{k-1}|t\cdot(K_2^T\times K_{k-5})]\times[0\,1]$. □

Proof of Theorem 3. We use induction on k and m but with a stronger induction hypotheses and prove three facts in turn. We will establish constants c_k,m_k with $m_k>m_{k-1}$ for $k\geqslant 4$. The constant m_3 already exists in Lemma 10.

The overall induction is on k. The base cases for Claim 1(3) and Claim 2(3) are Lemma 10 and the base case for Claim 3(3) is Lemma 11. Claim 1(k) will follow from Claim 2($k-1$) and induction on m with base case m_{k-1}. Claim 2(k) will follow from Claim 1(k) and Claim 3($k-1$) and induction on m with base case m_k. Claim 3(k) will follow from Claim 2(k) and Claim 3($k-1$). The case $k=4$ needs some special treatment.

Claim 1(k). There exists c_k so that for $m\geqslant m_{k-1}$, $\text{forb}(m,[K_k|t\cdot(K_2^T\times K_{k-4})])\leqslant\text{forb}(m,K_k)+c_k$.

Claim 2(k). There exists an m_k so that for $m>m_k$, $\text{forb}(m,[K_k|t\cdot(K_2^T\times K_{k-4})])=\text{forb}(m,K_k)$.

Claim 3(k). Assume $A \in \text{Avoid}(m, [K_k|t \cdot (K_2^T \times K_{k-4})])$. Assume there is some k-tuple S of rows so that $(t+1) \cdot (K_2^T \times K_{k-4}) \nprec A$. Then there exist a constant m_k so that $\|A\| \leqslant \text{forb}(m, K_k) - m + 4t$ for $m > m_k$.

Below is an implication scheme for the inductive argument with $k = 3, 4, 5, \ldots$. The base case $k = 4$ uses Lemmas 10 and 11, rendering these the de facto base cases.

$$\begin{array}{ccccc}
\textbf{Claim 1}(3) & \rightarrow & \textbf{Claim 2}(3) & \rightarrow & \textbf{Claim 3}(3) \\
 & & (m > m_3) & & (m > m_3) \\
 & \swarrow & & \swarrow & \downarrow \\
\textbf{Claim 1}(4) & \rightarrow & \textbf{Claim 2}(4) & \rightarrow & \textbf{Claim 3}(4) \\
(m \geqslant m_3) & & (m > m_4) & & (m > m_4) \\
 & \swarrow & & \swarrow & \downarrow \\
\textbf{Claim 1}(5) & \rightarrow & \textbf{Claim 2}(5) & \rightarrow & \textbf{Claim 3}(5) \\
(m \geqslant m_4) & & (m > m_5) & & (m > m_5) \\
\vdots & & \vdots & & \vdots
\end{array}$$

Proof of Claim 1(k). There exists a $c_k \geqslant 0$ with $\text{forb}(m_{k-1}, [K_k|t \cdot (K_2^T \times K_{k-4})]) = \text{forb}(m_k, K_k) + c_k$ establishing a base case. We use m_{k-1} to determine c_k where m_3 is determined in Lemma 10. We determine values for m_k that also must satisfy $m_k > m_{k-1}$ for $k \geqslant 4$. We use induction on m for $m \geqslant m_{k-1}$. The base case for $m = m_{k-1}$ is given. Assume $m \geqslant m_{k-1}$. Assume $A \in \text{Avoid}(m, [K_k|t \cdot (K_2^T \times K_{k-4})])$.

First assume $k = 4$. We use induction on m assuming $m \geqslant m_3$. The inductive children of $[K_4|t \cdot K_2^T]$ are $\mathcal{F} = \{[K_3|t \cdot F_2], [K_3|t \cdot F_3], [K_3|t \cdot F_4]\}$. Then applying standard induction to A we have $C_r \in \text{Avoid}(m-1, \mathcal{F})$. By Lemma 10, $\|C_r\| \leqslant \text{forb}(m, K_3)$. Note that this will require $m_4 > m_3$. Now $[B_r C_r D_r] \in \text{Avoid}(m-1, [K_4|t \cdot K_2^T])$ and so by induction on m for Claim 1(4),

$$\|A\| = \|B_r C_r D_r\| + \|C_r\| \leqslant \text{forb}(m-1, K_4) + c_4 + \text{forb}(m-1, K_3) = \text{forb}(m, K_4) + c_4, \quad (6)$$

(using (4)) which yields Claim 1(4).

Next, assume $k \geqslant 5$. We use induction on m assuming $m \geqslant m_{k-1}$. Apply standard induction so that $C_r \in \text{Avoid}(m-1, [K_{k-1}|t \cdot (K_2^T \times K_{k-5})])$. By using Claim $2(k-1)$, we deduce that $\|C_r\| \leqslant \text{forb}(m-1, K_{k-1})$. Now $[B_r C_r D_r] \in \text{Avoid}(m-1, [K_k|t \cdot (K_2^T \times K_{k-4})])$ and so by induction on m for Claim $1(k)$, $\|[B_r C_r D_r]\| \leqslant \text{forb}(m-1, K_k) + c_k$. Hence $\|A\| = \|C_r\| + \|[B_r C_r D_r]\| \leqslant \text{forb}(m, K_k) + c_k$ (using (4)) establishing Claim $1(k)$.

Proof of Claim 2(k). Assume $A \in \text{Avoid}(m, [K_k|t \cdot (K_2^T \times K_{k-4})])$. If $K_k \nprec A$, then $\|A\| \leqslant \text{forb}(m, K_k)$ as desired. So assume there is a k-tuple of rows S so that $K_k \prec A|_S$. Then $(t+1) \cdot (K_2^T \times K_{k-4}) \nprec A|_S$. We are able to deduce that $C_r|_{S \backslash r}$ has special behaviour. Note we are not trying to prove that $K_{k-1} \prec C_r|_{S \backslash r}$.

For $k = 4$, then if $(t+1)\cdot F_2 \prec C_r|_{S\setminus r}$ then $(t+1)\cdot K_2^T \prec A|_S$. Since $K_4 \prec A|_S$, we obtain $[K_4|t\cdot K_2^T] \prec A|_S$, a contradiction. Thus $(t+1)\cdot F_2 \not\prec C_r|_{S\setminus r}$. Similarly $(t+1)\cdot F_3 \not\prec C_r|_{S\setminus r}$ and $(t+1)\cdot F_4 \not\prec C_r|_{S\setminus r}$. Then Lemma 11 yields $\|C_r\| \leqslant \text{forb}(m, K_3) - m + 4t$. We have $\|[B_rC_rD_r]\| \leqslant \text{forb}(m-1, K_k) + c_4$ by Claim 1(4). Thus we have

$$\|A\| \leqslant \|[B_rC_rD_r]\| + \|C_r\|$$

$$\leqslant \text{forb}(m-1, K_4) + c_4 + \text{forb}(m-1, K_3) - m + 4t \leqslant \text{forb}(m, K_4),$$

which establishes Claim 2(4) for $m > c_4 + 4t$. So we take $m_4 \geqslant c_4 + 4t$. Here is where we need that c_4 does not depend on m_4 but on m_3.

For $k \geqslant 5$, we have $(t+1)\cdot(K_2^T \times K_{k-5}) \not\prec C_r|_{S\setminus r}$ else $(t+1)\cdot(K_2^T \times K_{k-4}) \prec A|_S$, a contradiction. Apply Claim 3$(k-1)$ to C_r to obtain $\|C_r\| \leqslant \text{forb}(m, K_{k-1}) - m + 4t$. We have $\|[B_rC_rD_r]\| \leqslant \text{forb}(m-1, K_k)$ by induction on m. Thus for $k \geqslant 4$, we have

$$\|A\| \leqslant \|[B_rC_rD_r]\| + \|C_r\|$$

$$\leqslant \text{forb}(m-1, K_k) + c_k + \text{forb}(m-1, K_{k-1}) - m + 4t \leqslant \text{forb}(m, K_k),$$

which establishes Claim 2(k) for $m > c_k + 4t$. So we take $m_k \geqslant c_k + 4t$. Again we recall c_k depends on m_{k-1} and not m_k.

Proof of Claim 3(k). Let $A \in \text{Avoid}(m, [K_k|t\cdot(K_2^T \times K_{k-4})])$ with some k-tuple S of rows so that $(t+1)\cdot(K_2^T \times K_{k-4}) \not\prec A$. Apply standard induction. We will first show $\|C_r\| \leqslant \text{forb}(m-1, K_{k-1}) - m + 4t$.

For $k = 4$, we note that the inductive children of $(t+1)\cdot K_2^T$ are $\{(t+1)\cdot F_2,$ $(t+1)\cdot F_3, (t+1)\cdot F_4\}$. Thus when applying standard induction, we deduce that if S are the k rows with $(t+1)\cdot(K_2^T) \not\prec A$ then $(t+1)\cdot F_2 \not\prec C_r|_{S\setminus r}$ and $(t+1)\cdot F_3 \not\prec C_r|_{S\setminus r}$ and $(t+1)\cdot F_4 \not\prec C_r|_{S\setminus r}$. Assume $m_4 > m_3$. By Lemma 11, we deduce $\|C_r\| \leqslant \text{forb}(m-1, K_3) - m + 4t$.

For $k \geqslant 5$, we have $(t+1)\cdot(K_2^T \times K_{k-4}) \not\prec A|_S$ and so $(t+1)\cdot(K_2^T \times K_{k-5}) \not\prec C_r|_{S\setminus r}$. Assume $m_k > m_{k-1}$. By Claim 3$(k-1)$, $\|C_r\| \leqslant \text{forb}(m-1, K_{k-1}) - m + 4t$, for $k \geqslant 5$.

By Claim 2(k), $\|[B_rC_rD_r]\| \leqslant \text{forb}(m, K_k)$. For $k \geqslant 4$ we have shown $\|C_r\| \leqslant \text{forb}(m-1, K_{k-1}) - m + 4t$. Then using (4), we obtain Claim 3(k).

Claim 2(k) yields the desired result. □

As noted in Section 5, it is open for what B is $\text{forb}(m, [K_4|B]) = \text{forb}(m, K_4)$ except it is known that the columns of B must have column sum 2 and we cannot have all 6 columns of sum 2 in B.

3 Extensions of $2\cdot K_k$ with the same bound

In much the same spirit as in Section 2, we will show that many columns can be concatenated with $2\cdot K_k$ without changing the bound. There are new values for m_k determined in this section and independent of Section 2.

Note $\text{forb}(m, 2\cdot K_k) = \text{forb}(m, K_{k+1})$ [6] and so

$$\text{forb}(m, 2\cdot K_k) = \text{forb}(m-1, 2\cdot K_k) + \text{forb}(m-1, 2\cdot K_{k-1}). \tag{7}$$

When we replace K_k by $2\cdot K_k$ we are able to add many more columns than in Theorem 3 without changing the bound for $\text{forb}(m, 2\cdot K_k)$.

Let

$$\mathcal{F}' = \left\{ \left[2\cdot K_2 | t\cdot \begin{bmatrix} 1\,0\,0 \\ 0\,1\,0 \end{bmatrix}\right], \left[2\cdot K_2 | t\cdot \begin{bmatrix} 1\,1\,0 \\ 1\,0\,1 \end{bmatrix}\right], \left[2\cdot K_2 | t\cdot \begin{bmatrix} 1\,1\,0 \\ 1\,0\,0 \end{bmatrix}\right] \right\}. \tag{8}$$

The inductive children of $[2\cdot K_3 | t\cdot F_1]$ are $\mathcal{F}'$. Note that $\text{forb}(m, t\cdot F_1)$ is $\Theta(m^2)$ which is much less that $\text{forb}(m, 2\cdot K_3)$ which is $\Theta(m^3)$. The following Lemma relates to Lemma 9.

Lemma 13. *Let $A \in \text{Avoid}(m, \mathcal{F}')$ and $m \geqslant 2$. Assume there is a pair of rows $S = \{i, j\}$ with $2\cdot K_2 \prec A|_S$. Then we can delete one row and at most $2t$ columns from A to get a matrix $A' \in \text{Avoid}(m-1, \mathcal{F}')$.*

Proof. If A has $\begin{smallmatrix} i \\ j \end{smallmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{smallmatrix} i \\ j \end{smallmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in long supply, then $\begin{smallmatrix} i \\ j \end{smallmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{smallmatrix} i \\ j \end{smallmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ are in short supply. In the same way, if A has $\begin{smallmatrix} i \\ j \end{smallmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{smallmatrix} i \\ j \end{smallmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in long supply, then $\begin{smallmatrix} i \\ j \end{smallmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{smallmatrix} i \\ j \end{smallmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are in short supply. When we note that $(\mathcal{F}')^c = \mathcal{F}'$ (the notation refers to the (0,1)-complements of the matrices), we find that only two column types can be in long supply on rows i, j. In any of these cases, we can delete row i and at most $2t$ columns (columns of short supply) to find a submatrix A' of A with $A' \in \text{Avoid}(m-1, \mathcal{F}')$. ☐

The following result has proof analogous to Lemma 10 but now for $2\cdot K_k$.

Lemma 14. *Let $\mathcal{F}'$ be as in (8). There exists an m_2 so that for $m > m_2$, we have* $\text{forb}(m, \mathcal{F}') = \text{forb}(m, 2\cdot K_2)$.

Proof. Suppose, for the sake of contradiction, that $\text{forb}(m, \mathcal{F}') > \text{forb}(m, 2\cdot K_2)$ with $m > m_2$. Then we can find a simple matrix $A \in \text{Avoid}(m, \mathcal{F}')$ which has $\|A\| > \text{forb}(m, 2\cdot K_2)$ so that on some choice of two rows i, j, we have $2\cdot K_2 \prec A|_{\{i,j\}}$. By Lemma 13, we can delete one row and up to $2t$ columns to find a submatrix $A' \in \text{Avoid}(m-1, \mathcal{F}')$ which has $\|A'\| > \text{forb}(m, 2\cdot K_2) - 2t$ (whenever $m > m_2 > 2t$). We are using $\text{forb}(p, 2\cdot K_2) = \text{forb}(p, K_3)$ which yields $\text{forb}(p, 2\cdot K_2) - \text{forb}(m-1, 2\cdot K_2) = p$, in analogy with Lemma 10. At each of the $m - m^{1/2}$ steps of deletion, we will require that the number of rows is greater than $2t$. Ensure $m_2 > \lceil (2t)^2 \rceil$ to guarantee this. Repeat the process of deletion $m - m^{1/2}$ times to obtain a matrix A'' on $m^{1/2}$ rows and $\binom{m}{2} - 2t(m - m^{1/2})$ columns. Taking $m_2 > \lceil (2t)^2 \rceil$ guarantees we can do these deletions. Now $\text{forb}(m^{1/2}, (t+2)\cdot K_2) = \frac{t+3}{2}\binom{m^{1/2}}{2} + \binom{m^{1/2}}{1} + \binom{m^{1/2}}{0}$ [1], and so for m large enough, we have $\|A''\| > \text{forb}(m^{1/2}, (t+2)\cdot K_2)$. Hence, we choose m_2 large enough so this is true as well as satisfying $m_2 > \lceil (2t)^2 \rceil$. Then for some (and in fact any) $F \in \mathcal{F}'$, we have $F \prec (t+2)\cdot K_2 \prec A'' \prec A$, a contradiction. ☐

Lemma 15. *Let $\mathcal{F}'$ be as in Lemma 13. Let $A \in \text{Avoid}(m, \mathcal{F}')$ and $m > m_2$, where m_2 is as in Lemma 14. Assume that there is a set S of two rows so that $t\cdot \begin{bmatrix} 1\,0\,0 \\ 0\,1\,0 \end{bmatrix} \nprec A|_S$ and $t\cdot \begin{bmatrix} 1\,1\,0 \\ 1\,0\,1 \end{bmatrix} \nprec A|_S$ and $t\cdot \begin{bmatrix} 1\,1\,0 \\ 1\,0\,0 \end{bmatrix} \nprec A|_S$. Then $\|A\| \leqslant \text{forb}(m, 2\cdot K_2) - m + 2t$.*

Proof. Suppose, on the contrary that $\|A\| > \text{forb}(m, 2 \cdot K_2) - m + 2t$. By Lemma 13, we may delete one row and up to $2t$ columns to obtain an $(m-1)$-rowed submatrix A' of A with $A' \in \text{Avoid}(m-1, \mathcal{F}')$ and $\|A'\| > \text{forb}(m, 2\cdot K_2) - m + 2t - 2t = \text{forb}(m-1, 2 \cdot K_2)$. Since $m > m_2$, this contradicts Lemma 14. $\square$

Remark 16. Assume $k \geqslant 4$. The inductive child of $[2 \cdot K_k | t \cdot (F_1 \times K_{k-3})]$ is $[2 \cdot K_{k-1} | t \cdot (F_1 \times K_{k-4})]$.

Theorem 4 is anticipated in Theorem 1.7 in [5] for the case $t = 1$ but progress in [5] was stopped by difficulty with base cases.

Proof of Theorem 4: The proof is a variation on the proof of Theorem 3 with the same induction arguments on m and k. We will be determining constants m_k, c_k where we require $m_k > m_{k-1}$ for $k = 3, 4, \ldots$ and m_2 is determined in Lemma 14. The constant c_k will be determined from m_{k-1}. We use three inductive claims where the basic induction is on k. The base cases for Claim 1(2) and Claim 2(2) are Lemma 14 and the base case for Claim 3(2) is Lemma 15. Claim 1(k) will follow from Claim 2($k-1$) and induction on m with base case m_{k-1}. Claim 2(k) will follow from Claim 1(k) and Claim 3($k-1$) and induction on m with base case m_k.

Claim 1(k). There exists c_k so that for $m \geqslant m_{k-1}$, $\text{forb}(m, [2 \cdot K_k | t \cdot (F_1 \times K_{k-3})]) \leqslant \text{forb}(m, 2 \cdot K_k) + c_k$.

Claim 2(k). There is a constant m_k so that $\text{forb}(m, [2 \cdot K_k | t \cdot (F_1 \times K_{k-3})]) = \text{forb}(m, 2 \cdot K_k)$ for $m > m_k$.

Claim 3(k). Assume that $A \in \text{Avoid}(m, [2 \cdot K_k | t \cdot (F_1 \times K_{k-3})])$ has some k-set S of rows so that $t \cdot (F_1 \times K_{k-3}) \not\prec A|_S$. Then there is a constant m_k so that if $m > m_k$, then $\|A\| \leqslant \text{forb}(m, 2 \cdot K_k) - m + 2t$.

Our proof will handle $k = 3$ and $k \geqslant 4$ separately.

Proof of Claim 1(k). We compute c_k from m_{k-1} using

$$\text{forb}(m_{k-1}, [2 \cdot K_k | t \cdot (F_1 \times K_{k-3})]) = \text{forb}(m_{k-1}, 2 \cdot K_k) + c_k, \tag{9}$$

where m_2 is determined in Lemma 14. As in proof of Theorem 3, we prove Claim 1(k) for all $m > m_{k-1}$. Note that in all cases c_k depends on m_{k-1} and moreover (9) provides the base case for induction on m. For proving Claim 1(k) for $k \geqslant 4$, we need Claim 2($k-1$).

We will first prove Claim 1(3). The inductive children of $[2 \cdot K_3 | t \cdot F_1]$ are precisely $\mathcal{F}'$ as in Lemma 13. Using standard induction and by Lemma 14 for $m - 1 > m_2$, we obtain $\|C_r\| \leqslant \text{forb}(m-1, \mathcal{F}') = \text{forb}(m-1, 2 \cdot K_2)$. Now $B_r C_r D_r \in \text{Avoid}(m-1, [2 \cdot K_3 | t \cdot F_1])$ and by induction on m in (9), $\|B_r C_r D_r\| \leqslant \text{forb}(m-1, 2 \cdot K_3) + c_3$. Using (7) and $\|A\| = \|B_r C_r D_r\| + \|C_r\|$ proves Claim 1(3).

To prove Claim 1(k) for $k \geqslant 4$ we note the inductive child of $[2 \cdot K_k | t \cdot (F_1 \times K_{k-3})]$ is $[2 \cdot K_{k-1} | t \cdot (F_1 \times K_{k-4})]$ by Remark 5. By Claim 2($k-1$), there exists a constant m_{k-1} so that $\|C_r\| \leqslant \text{forb}(m-1, 2 \cdot K_{k-1})$ for $m \geqslant m_{k-1}$. By our standard induction

$B_rC_rD_r \in \text{Avoid}(m-1, [2\cdot K_k|t\cdot(F_1\times K_{k-3})])$ and so by induction on m, $B_rC_rD_r \leqslant \text{forb}(m-1, 2\cdot K_k) + c_3$. By (7),

$$\|A\| \leqslant \|B_rC_rD_r\| + \|C_r\| \leqslant \text{forb}(m-1, 2\cdot K_k) + c_k + \text{forb}(m-1, 2\cdot K_{k-1})$$
$$= \text{forb}(m, 2\cdot K_k) + c_k$$

yielding Claim 1(k).

Proof of Claim 2(k). We first handle $k = 3$ and use Claim 1(3). Let $A \in \text{Avoid}(m, [2\cdot K_3|t\cdot F_1])$. If $2\cdot K_3 \not\prec A$, then $\|A\| \leqslant \text{forb}(m, 2\cdot K_3)$ so we may assume that $2\cdot K_3 \prec A$ on some triple $S = \{r, i, j\}$ of rows so that $t\cdot F_1 \not\prec A|_S$. As in the proof of Theorem 3, we show that $C_r|_{S\backslash r}$ has some structure but we do not try to show that $2\cdot K_{k-1} \prec C_r|_{S\backslash r}$. Observe that $t\cdot \left[\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right] \not\prec C_r|_{S\backslash r}$ and $t\cdot \left[\begin{smallmatrix}1&1&0\\1&0&1\end{smallmatrix}\right] \not\prec C_r|_{S\backslash r}$ and $t\cdot \left[\begin{smallmatrix}1&1&0\\1&0&0\end{smallmatrix}\right] \not\prec C_r|_{S\backslash r}$. Apply Lemma 15 with $m > m_2$ to obtain $\|C_r\| \leqslant \text{forb}(m-1, 2\cdot K_3) - m + 2t$. By Claim 1(3), $\|[B_rC_rD_r]\| \leqslant \text{forb}(m-1, 2\cdot K_3) + c_3$. Applying (7) proves Claim 2(3).

For general $k \geqslant 4$, we use Claim 1(k) and Claim 3($k-1$). Let $A \in \text{Avoid}(m, [2\cdot K_k|t\cdot(F_1\times K_{k-3})])$ and assume, as above, that $2\cdot K_k \prec A$ on some k-tuple S of rows. Let $r \notin S$ be a row in A and observe that the row r decomposition of A has $[t\cdot(F_1\times K_{k-4})] \not\prec C_r$ so that Claim 3($k-1$) applies. Hence $\|C_r\| \leqslant \text{forb}(m-1, 2\cdot K_{k-1}) - m + 2t$. By Claim 1($k$), $\|[B_rC_rD_r]\| \leqslant \text{forb}(m-1, 2\cdot K_k) + c_k$. Then $\|A\| = \|[B_rC_rD_r]\| + \|C_r\| \leqslant \text{forb}(m-1, 2\cdot K_k) + c_k + \text{forb}(m-1, 2\cdot K_{k-1}) - m + 2t$. Choosing $m_k \geqslant c_k + 2t$, proves Claim 2(k).

Proof of Claim 3(k). Assume that $A \in \text{Avoid}(m, [2\cdot K_k|t\cdot(F_1\times K_{k-3})])$ has some k-set S of rows so that $t\cdot(F_1\times K_{k-3}) \not\prec A|_S$.

First assume $k = 3$. The proof will use Claim 2(3). Let $S = \{r, i, j\}$. Then $t\cdot \left[\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right] \not\prec A|_{S\backslash r}$ and $t\cdot \left[\begin{smallmatrix}1&1&0\\1&0&1\end{smallmatrix}\right] \not\prec C_r|_{S\backslash r}$ and $t\cdot \left[\begin{smallmatrix}1&1&0\\1&0&0\end{smallmatrix}\right] \not\prec C_r|_{S\backslash r}$ else $t\cdot(F_1\times K_{k-3}) \prec A|_S$, a contradiction. Thus by Lemma 15, $\|C_r\| \leqslant \text{forb}(m-1, 2\cdot K_2) - m + 2t$ for $m > m_2$. Observe that by Claim 2(3) and induction on m, $\|[B_rC_rD_r]\| \leqslant \text{forb}(m-1, 2\cdot K_3)$. Thus $\|A\| = \|[B_rC_rD_r]\| + \|C_r\| \leqslant \text{forb}(m-1, 2\cdot K_3) + \text{forb}(m-1, 2\cdot K_2) - m + 2t$. Applying (7) proves Claim 3(3).

Assume $k \geqslant 4$. The proof will use Claim 2(k) and Claim 3($k-1$). Proceed as above with $t\cdot(F_1\times K_{k-3}) \not\prec A|_S$ and a row r standard decomposition of A with $r \in S$. With $[B_rC_rD_r] \in \text{Avoid}(m-1, [2\cdot K_k|t\cdot(F_1\times K_{k-3})])$ then Claim 2(k) yields $\|[B_rC_rD_r]\| \leqslant \text{forb}(m-1, 2\cdot K_k)$ for $m > m_k$. Since $[2\cdot K_{k-1}|t\cdot(F_1\times K_{k-3})] \not\prec A|_S$, we deduce that $[2\cdot K_{k-1}|t\cdot(F_1\times K_{k-4})] \not\prec C_r|_{S\backslash r}$. Thus by Claim 3($k-1$) we have that $\|C_r\| \leqslant \text{forb}(m-1, 2\cdot K_{k-1}) - m + c_{k-1}$. Applying (7) proves Claim 3(k).

Claim 2(k) yields the desired result. □

4 Multiple copies of a configuration

It is interesting to consider $t\cdot F$ for various F and show that increasing t increases the bound. This contrasts with Theorems 3 and 4 where

$\text{forb}(m, [G|t{\cdot}F]) = \text{forb}(m, G)$ for choices of G, F. Note that $\text{forb}(m, t{\cdot}F)$ is asymptotically less than $\text{forb}(m, G)$ in these cases.

There are many examples of forbidden configurations with a parameter t where the upper bound currently known depends on t such as in Theorem 17 but we are lacking constructions in general to show how forb depends on t in many other cases. Note that our proof of Theorem 7 will spend the bulk of the time on F consisting of a single column.

Theorem 17. *[2] Let $p > q$ be given. Then for large enough m and $t > 2$*

$$\text{forb}(m, t \cdot \mathbf{1}_p\mathbf{0}_q) \leqslant \sum_{i=0}^{p-1} \binom{m}{i} + \left(1 + \frac{t-2}{p+1}\right)\binom{m}{p} + \sum_{i=m-q+1}^{m} \binom{m}{i}$$

with equality for m, p, t satisfying $\binom{p+1-i}{p-i}$ divides $\binom{m-i}{p-i}$ for $i = 1, 2, \ldots, p-1$. □

The leading term in $\text{forb}(m, t \cdot \mathbf{1}_p\mathbf{0}_q)$ is $(1 + \frac{t-2}{p+1})\frac{m^p}{p!}$ when the divisibility conditions are satisfied. Note that for a constant ℓ, the expression $\binom{m-\ell}{p}$ is a polynomial in m of degree m^p and the leading term in $\binom{m-\ell}{p}$ is $\frac{m^p}{p!}$ with other terms $O(m^{p-1})$. Constructions rely on Keevash [7].

Theorem 18. *[7] Let p, t be given. There exists a simple matrix A, all of whose columns sums are $p + 1$, with $A \in \text{Avoid}(m, (\lambda + 1) \cdot \mathbf{1}_p)$ and $\|A\| = \frac{\lambda}{p+1}\binom{m}{p}$ for m, p, t satisfying $\binom{p+1-i}{p-i}$ divides $\binom{m-i}{p-i}$ for $i = 1, 2, \ldots, p-1$.* □

The finite nature of the divisibility conditions ensures that there exists a constant $c_p \leqslant \prod_{i=1}^{p-1}\binom{p+1-i}{p-i}$ such that the divisibility conditions are always satisfied for some $m' \in \{m - c_p, m - c_p + 1, \ldots, m\}$. Then the construction $A' \in \text{Avoid}(m', (\lambda + 1) \cdot \mathbf{1}_p)$ with all column sums $p + 1$, can be made into a matrix $A \in \text{Avoid}(m, (\lambda + 1) \cdot \mathbf{1}_p)$ by appending $m - m' < c_p$ rows of 0's and so the leading term (in m) for $\|A\|$ by this construction is still $(\frac{\lambda}{p+1})\frac{m^p}{p!}$.

Lemma 19. *Given t, p, there exist constants c_1, M so that, for $m > M$,*

$$\left(2 + \frac{t-3}{p+1}\right)\frac{m^p}{p!} - c_1 m^{p-1} \leqslant \text{forb}(m, t \cdot \mathbf{1}_p\mathbf{0}_p) \tag{10}$$

Proof. We can construct a matrix $A \in \text{Avoid}(m, t \cdot \mathbf{1}_p\mathbf{0}_p)$ with $\|A\|$ having leading term $(2 + \frac{t-3}{p+1})\frac{m^p}{p!}$ by forming A from all columns of sum $1, 2, \ldots, p$ and $m - p, m - p + 1, \ldots, m$ as well as the above matrix of Theorem 18 with λ replaced by $t - 3$. □

We expect this construction is optimal (when the divisibility conditions are satisfied). The case $p = q$ of Theorem 17 has only been solved for $p = q = 2$ (see ArXiv reference in [2]) .

Definition 20. For $A \in \text{Avoid}(m, t \cdot \mathbf{1}_p\mathbf{0}_p)$, let a_p^+ be the number of columns of sum p or $m-p$, let a_{p+1}^+ be the number of columns of sum $p+1$ or $m-p-1$ and let a_{other} be the number of columns of sum in $\{p+2, p+3, \ldots, m-p-2\}$.

We may ignore columns of at most $p-1$ 1's and columns of at least $m-p+1$ 1's which have no configuration $\mathbf{1}_p\mathbf{0}_p$.

For the following lemma, we use the pigeonhole argument of Lemma 2.2 of [2]. We are counting the occurrences of the configuration $\mathbf{1}_p\mathbf{0}_p$ and note that any set of $2p$ rows can have at most $\binom{2p}{p}(t-1)$ such configurations. The result is more than what is needed for the proof of Theorem 7 but can be viewed as a weak stability result. Namely matrices close to the bounds have the number of columns of each type $a_p^+, a_{p+1}^+, a_{\text{other}}$ close to their expected values arising from the known design based constructions with $a_p^+ = 2\binom{m}{p}$, $a_{p+1}^+ = \frac{t-3}{p+1}\binom{m}{p}$ and $a_{\text{other}} = 0$.

Lemma 21. *Let m, p, t be given and let $A \in \text{Avoid}(m, t \cdot \mathbf{1}_p\mathbf{0}_p)$ with $a_p^+ + a_{p+1}^+ + a_{\text{other}} \geqslant (2 + \frac{t-3}{p+1})\binom{m}{p} - c_1 m^{p-1}$ with c_1 as in Lemma 19. Then, there is some M so that, for $m > M$, we have*

$$\binom{p}{p}\binom{m-p}{p}a_p^+ + \binom{p+1}{p}\binom{m-p-1}{p}a_{p+1}^+ + \binom{p+2}{p}\binom{m-p-2}{p}a_{\text{other}} \leqslant \binom{m}{2p}\binom{2p}{p}(t-1). \tag{11}$$

Further, there exist constants $c_2, c_3, c_4, c_5 > 0$ so that

$$2\binom{m}{p} - c_2 m^{p-1} \leqslant a_p^+ \leqslant 2\binom{m}{p}, \tag{12}$$

$$\frac{t-3}{p+1}\binom{m}{p} - c_3 m^{p-1} \leqslant a_{p+1}^+ \leqslant \frac{t-3}{p+1}\binom{m}{p} + c_4 m^{p-1}, \text{ and} \tag{13}$$

$$a_{\text{other}} \leqslant c_5 m^{p-1} \tag{14}$$

whenever $m > M$.

Proof. The upper bound on a_p^+ (12) follows from the simplicity of A. It yields

$$a_{p+1}^+ + a_{\text{other}} \geqslant \frac{t-3}{p+1}\binom{m}{p} - c_1 m^{p-1}. \tag{15}$$

A column of column sum k has $\binom{k}{p}\binom{m-k}{p}$ configurations $\mathbf{1}_p\mathbf{0}_p$ and for $p+2 \leqslant k \leqslant m-p-2$ we check $\binom{k}{p}\binom{m-k}{p} \geqslant \binom{p+2}{p}\binom{m-p-2}{p}$. The configurations $\mathbf{1}_p\mathbf{0}_p$ can appear on $2p$ rows in up to $\binom{2p}{p}$ orderings but at most $t-1$ times to avoid $t \cdot \mathbf{1}_p\mathbf{0}_p$. The pigeonhole principle then gives (11).

While $m > (p+2)(2p-1)/p$, we have that $\binom{p+1}{p}\binom{m-p-1}{p} < \binom{p+2}{p}\binom{m-p-2}{p}$. Hence

$$\binom{m-p}{p}a_p^+ + \binom{p+1}{p}\binom{m-p-1}{p}(a_{p+1}^+ + a_{\text{other}}) \leqslant \binom{m}{2p}\binom{2p}{p}(t-1).$$

Substitute $a_{p+1}^+ + a_{\text{other}} \geqslant (2+\frac{t-3}{p+1})\binom{m}{p} - a_p^+ - c_1 m^{p-1}$ and rearrange to obtain

$$\binom{m-p-1}{p}\binom{m}{p}(2(p+1)+t-3) - \binom{m}{2p}\binom{2p}{p}(t-1) - c_1 m^{p-1}(p+1)\binom{m-p-1}{p} \leqslant \left(\binom{p+1}{p}\binom{m-p-1}{p} - \binom{m-p}{p}\right) a_p^+.$$

The expressions are polynomials in m when p is viewed as constant. Here, the left side of the inequality has a leading term $\frac{2p}{p!^2}m^{2p}$, whereas the right side sees a_p^+ multiplied by a polynomial with a leading term $\frac{p}{p!}m^p$ (from $p\binom{m}{p}$). Thus $a_p = 2\binom{m}{p} + O(m^{p-1})$ and so there exists a constant c_2 so that the lower bound in (12) holds.

Next, substitute the lower bound for a_p^+ into (11) to obtain

$$\binom{p+1}{p}\binom{m-p-1}{p}a_{p+1}^+ + \binom{p+2}{p}\binom{m-p-2}{p}a_{\text{other}} \leqslant \binom{m}{2p}\binom{2p}{p}(t-1) - \binom{p}{p}\binom{m-p}{p}(2\binom{m}{p} - c_2 m^{p-1})$$

Now (15) can be multiplied by $-\binom{p+1}{p}\binom{m-p-1}{p}$ and added to the above inequality to obtain an inequality for a_{other}. The coefficient of a_{other} on the left side becomes $\binom{p+2}{p}\binom{m-p-2}{p} - \binom{p+1}{p}\binom{m-p-1}{p}$ which has leading term $\frac{p^2+p}{2}m^p$ and the right hand side has leading term that is a multiple of m^{2p-1} (the terms m^{2p} cancel) and so for some constant c_5 we obtain the upper bound on a_{other} in (14). Combining this with (15) yields a constant c_3 for which the lower bound on a_{p+1}^+ in (13) holds.

Combining the hypothesis $a_p^+ + a_{p+1}^+ + a_{\text{other}} \geqslant (2+\frac{t-3}{p+1})\binom{m}{p} - c_1 m^{p-1}$ with lower bounds for a_p^+ and a_{other} yields the upper bound in (13) with a suitable constant c_4. □

Theorem 22. *Let m, p, t be given. Then*

$$\left(2+\frac{t-3}{p+1}\right)\binom{m}{p} - c_1 m^{p-1} \leqslant \text{forb}(m, t\cdot \mathbf{1}_p\mathbf{0}_p) \leqslant \left(2+\frac{t-3}{p+1}\right)\binom{m}{p} + (c_4+c_5)m^{p-1}.$$

Proof. Use Lemma 19 for the lower bound and use the upper bounds on a_p+, $a_{p+1}+$ a_{other} arising in Lemma 21 to obtain the upper bound. □

We now can prove Theorem 7.

Proof of Theorem 7: Assume first $\ell \geqslant 2$. Assume $\text{forb}(m, t\cdot F) = \text{forb}(m, (t+1)\cdot F)$. Take an extremal matrix $A \in \text{Avoid}(m, t\cdot F)$ with $\|A\| = \text{forb}(m, t\cdot F) = \text{forb}(m, (t+1)\cdot F)$ and a $m\times 1$ column α not in A. Consider $A' = [A|\alpha]$. Then $(t+1)\cdot F \prec A'$ on some $((t+1)\ell)$-set of columns of A'. Since $\ell \geqslant 2$, we can take a $t\ell$-subset of these columns which does not include the column α, on which $t\cdot F \prec A$, a contradiction.

Assume $\ell = 1$ and $F = \mathbf{1}_p\mathbf{0}_q$ with $p \geqslant q$. The case where $p > q$ can be verified via the exact bounds in [2] and the construction ideas. So, let $p = q$. By Theorem 22, note that

$$\text{forb}(m, t \cdot F) \leqslant \left(2 + \frac{t-3}{p+1}\right)\binom{m}{p} + (c_4 + c_5)m^{p-1},$$

while

$$\text{forb}(m, (t+1) \cdot F) \geqslant \left(2 + \frac{t-2}{p+1}\right)\binom{m}{p} - c_1 m^{p-1}.$$

For m large enough, we obtain $\text{forb}(m, (t+1)\cdot F) > \text{forb}(m, t\cdot F)$, concluding the proof. □

5 Problems

We were only able to add 2 different columns of sum 2 (but each taken multiple times) to K_4 in Theorem 3. Is this the best we can do?

Problem 23. Let

$$F_5 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_6 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad F_7 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Show that $\text{forb}(m, [K_4|F_5]) > \text{forb}(m, K_4)$ and $\text{forb}(m, [K_4|F_6]) > \text{forb}(m, K_4)$. □

Constructions are hard to come by. Note that $\text{forb}(m, t\cdot F_6)$ is $\Theta(m^2)$ and $\text{forb}(m, t\cdot F_5)$ is conjectured to be $\Theta(m^2)$ ([1]). It is possible that even $\text{forb}(m, [K_4|t\cdot F_7]) = \text{forb}(m, K_4)$. We need some new constructions!

Although in the case of extensions of K_k we could not add columns of column sum 1 or $k-1$, this is not obvious when extending $2 \cdot K_k$. We note that F_1 has two columns of sum 1 that can be added to $2 \cdot K_3$ without changing the bound. Similarly $F_1 \times [0\,1]$ has two columns of sum 1 that can be added to $2 \cdot K_4$ without changing the bound.

Note we do not know the bound for $[2 \cdot K_3|t \cdot I_3]$ although we expect it may be larger than $\text{forb}(m, 2 \cdot K_3)$ while $\text{forb}(m, [t \cdot I_3])$ is known to be quadratic [1].

Problem 24. Show that $\text{forb}(m, [2 \cdot K_3|I_3]) > \text{forb}(m, 2 \cdot K_3)$.

In [4], a number of extensions to $2 \cdot \mathbf{1}_1\mathbf{0}_2$ are shown to have the same bound as $\text{forb}(m, \mathbf{1}_1\mathbf{0}_2)$ (Theorem 3.2 in [4]). In these cases we call $\mathbf{1}_1\mathbf{0}_2$ a critical substructure. You might guess that $\text{forb}(m, [K_3|\mathbf{1}_1\mathbf{0}_2]) = \text{forb}(m, 2 \cdot \mathbf{1}_1\mathbf{0}_2)$ but this is not the case since $3 \cdot \mathbf{0}_2 \prec [K_3|\mathbf{1}_1\mathbf{0}_2]$ and hence $\text{forb}(m, [K_3|\mathbf{1}_1\mathbf{0}_2]) \geqslant \text{forb}(m, 3 \cdot \mathbf{0}_2) = \frac{4}{3}\binom{m}{2} + \binom{m}{1} + \binom{m}{0}$. We have to be careful looking for substructures yielding big bounds.

Problem 25. Show that there exists a constant c so that $\text{forb}(m, \{[K_3|t\cdot\mathbf{0}_3], [K_3|t\cdot\mathbf{1}_3\}) \leqslant \text{forb}(m, K_3) + c$. □

This would, by an easy induction, show that $\text{forb}(m, \{[K_k|t \cdot \mathbf{0}_k], [K_k|t \cdot \mathbf{1}_k]\}) \leqslant \text{forb}(m, K_k) + cm^{k-2}$. Even better would be to show that $\text{forb}(m, \{[K_3|t\cdot\mathbf{0}_3], [K_3|t\cdot\mathbf{1}_3]\}) = \text{forb}(m, K_3)$.

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