On the Trivial T-Module of a Graph

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Abstract

Let Γ denote a finite, simple and connected graph. Fix a vertex x of Γ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. In this paper we study the unique irreducible T-module with endpoint 0. We assume that this T-module is thin. The main result of the paper is a combinatorial characterization of this property.

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1 Introduction

Terwilliger algebras of association schemes were defined by Terwilliger in [19], where they were called subconstituent algebras. These noncommutative algebras are generated by the Bose-Mesner algebra of the scheme, together with matrices containing local information about the structure with respect to a fixed vertex. Since then, numerous papers appear in which the Terwilliger algebra was successfully used for studying commutative association schemes and distance-regular graphs, see [7, 8, 10, 11, 12, 13, 14, 15, 16, 17] for the most recent research on the subject. However, the notion of a Terwilliger algebra could be easily generalized to an arbitrary finite, simple and connected graph. This article is a contribution to the growing literature on studies involving Terwilliger algebras of non-distance-regular graphs, see for example [1, 2, 3, 9, 20, 21, 22].

Let us first recall the definition of a Terwilliger algebra (see Section 2 for formal definitions). Let Γ denote a finite, simple, connected graph with vertex set X. Let

$\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Pick a vertex x of Γ and let $\epsilon(x)$ denote its eccentricity. Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ and let E_i^* $(0 \leqslant i \leqslant \epsilon(x))$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ whose (y, y)-entry is equal to 1 if the distance between x and y is i, and 0 otherwise $(y \in X)$. We refer to matrices E_i^* $(0 \leqslant i \leqslant \epsilon(x))$ as *dual idempotents* of Γ with respect to x. The *Terwilliger algebra* $T = T(x)$ is a matrix subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by the adjacency matrix of Γ and the dual idempotents of Γ with respect to x. Algebra T acts on the space of all column vectors with coordinates indexed by X. Observe that T is closed under the conjugate-transpose map, and so T is semi-simple. It follows that each T-module is a direct sum of irreducible T-modules. Therefore, in many instances the algebra T can best be studied via its irreducible modules. We now recall an important parameter which is assigned to every irreducible T-module. Let W denote an irreducible T-module. By the *endpoint* of W we mean $\min\{\ i \mid 0 \leqslant i \leqslant \epsilon(x),\ E_i^* W \neq 0\}$. We say that W is *thin* if $\dim E_i^* W \leqslant 1$ for every $0 \leqslant i \leqslant \epsilon(x)$. It turns out that there exists a unique irreducible T-module with endpoint 0. It was already proved in [18] that this irreducible T-module is thin if Γ is distance-regular around x. The converse, however, is not true. Fiol and Garriga [5] later introduced the concept of *pseudo-distance-regularity* around vertex x, which is based on giving to the vertices of the graph some weights which correspond to the entries of the (normalized) positive eigenvector. They showed that the unique irreducible T-module with endpoint 0 is thin if and only if Γ is pseudo-distance-regular around x (see also [4, Theorem 3.1]).

The main result of this paper is a purely combinatorial characterization of the property, that the irreducible T-module with endpoint 0 is thin (see Theorem 6). This characterization involves the number of walks between vertex x and vertices at some fixed distance from x, which are of a certain shape. Our paper is organized as follows. In Sections 2 and 3 we recall basic definitions and results about Terwilliger algebras and local distance-regularity and pseudo-distance-regularity. In Section 4 we present our main result, and we prove it in Section 5. We conclude the paper with a couple of examples in Section 6.

2 Preliminaries

In this section we review some definitions and basic concepts. Throughout this paper, $\Gamma = (X, \mathcal{R})$ will denote a finite, undirected, connected graph, without loops and multiple edges, with vertex set X and edge set $\mathcal{R}$.

Let $x, y \in X$. The *distance* between x and y, denoted by $\partial(x, y)$, is the length of a shortest xy-path. The *eccentricity of* x, denoted by $\epsilon(x)$, is the maximum distance between x and any other vertex of Γ: $\epsilon(x) = \max\{\partial(x, z) \mid z \in X\}$. Let D denote the maximum eccentricity of any vertex in Γ. We call D the *diameter of* Γ. For an integer i we define $\Gamma_i(x)$ by

$$\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}\,.$$

We will abbreviate $\Gamma(x) = \Gamma_1(x)$. Note that $\Gamma(x)$ is the set of neighbours of x. Observe that $\Gamma_i(x)$ is empty if and only if $i < 0$ or $i > \epsilon(x)$.

We now recall some definitions and basic results concerning a Terwilliger algebra of Γ. Let $\mathbb{C}$ denote the complex number field. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let V denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module*. We endow V with the Hermitian inner product $\langle\, ,\, \rangle$ that satisfies $\langle u, v\rangle = u^\top \overline{v}$ for $u, v \in V$, where $\top$ denotes transpose and $^-$ denotes complex conjugation. For $y \in X$, let $\widehat{y}$ denote the element of V with a 1 in the y-coordinate and 0 in all other coordinates. We observe $\{\widehat{y} | y \in X\}$ is an orthonormal basis for V.

Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ:

$$(A)_{xy} = \begin{cases} 1 & \text{if} \quad \partial(x,y) = 1, \\ 0 & \text{if} \quad \partial(x,y) \neq 1, \end{cases} \qquad (x, y \in X).$$

The *adjacency algebra of* Γ is a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$ generated by the adjacency matrix A of Γ.

We now recall the dual idempotents of Γ. To do this fix a vertex $x \in X$ and let $d = \epsilon(x)$. We view x as a *base vertex*. For $0 \leqslant i \leqslant d$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry as follows:

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if} \quad \partial(x,y) = i, \\ 0 & \text{if} \quad \partial(x,y) \neq i \end{cases} \qquad (y \in X).$$

We call E_i^* the *i-th dual idempotent of* Γ *with respect to* x [19, p. 378]. We also observe (ei) $\sum_{i=0}^d E_i^* = I$; (eii) $\overline{E_i^*} = E_i^*$ $(0 \leqslant i \leqslant d)$; (eiii) $E_i^{*\top} = E_i^*$ $(0 \leqslant i \leqslant d)$; (eiv) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \leqslant i, j \leqslant d)$ where I denotes the identity matrix in $\mathrm{Mat}_X(\mathbb{C})$. By these facts, matrices $E_0^*, E_1^*, \ldots, E_d^*$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. Note that for $0 \leqslant i \leqslant d$ we have

$$E_i^* V = \mathrm{Span}\{\widehat{y} \mid y \in \Gamma_i(x)\},$$

and that

$$V = E_0^* V + E_1^* V + \cdots + E_d^* V \qquad \text{(orthogonal direct sum)}.$$

We call $E_i^* V$ the *i-th subconstituent of* Γ *with respect to* x. Moreover E_i^* is the projection from V onto $E_i^* V$ for $0 \leqslant i \leqslant d$. For convenience we define E_{-1}^* and E_{d+1}^* to be the zero matrix of $\mathrm{Mat}_X(\mathbb{C})$.

We recall the definition of a Terwilliger algebra of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. We call T the *Terwilliger algebra of* Γ *with respect to* x. Recall M is generated by A so T is generated by A and the dual idempotents. We observe T has finite dimension. In addition, by construction T is closed under the conjugate-transpose map and so T is semi-simple. For a vector subspace $W \subseteq V$, we denote by TW the subspace $\{Bw \mid B \in T, w \in W\}$. We now recall the lowering, the flat and the raising matrix of T.

Definition 1. Let $\Gamma = (X, \mathcal{R})$ denote a simple, connected, finite graph. Pick $x \in X$. Let $d = \epsilon(x)$ and let $T = T(x)$ be the Terwilliger algebra of Γ with respect to x. Define $L = L(x)$, $F = F(x)$ and $R = R(x)$ in $\mathrm{Mat}_X(\mathbb{C})$ by

$$L = \sum_{i=1}^{d} E^*_{i-1} A E^*_i, \qquad F = \sum_{i=0}^{d} E^*_i A E^*_i, \qquad R = \sum_{i=0}^{d-1} E^*_{i+1} A E^*_i.$$

We refer to L, F and R as the *lowering*, the *flat* and the *raising matrix with respect to* x, respectively. Note that $L, F, R \in T$. Moreover, $F = F^\top$, $R = L^\top$ and $A = L + F + R$.

Observe that for $y, z \in X$ we have the (z, y)-entry of L equals 1 if $\partial(z, y) = 1$ and $\partial(x, z) = \partial(x, y) - 1$, and 0 otherwise. The (z, y)-entry of F is equal to 1 if $\partial(z, y) = 1$ and $\partial(x, z) = \partial(x, y)$, and 0 otherwise. Similarly, the (z, y)-entry of R equals 1 if $\partial(z, y) = 1$ and $\partial(x, z) = \partial(x, y) + 1$, and 0 otherwise. Consequently, for $v \in E^*_i V$ $(0 \leqslant i \leqslant d)$ we have

$$Lv \in E^*_{i-1} V, \qquad Fv \in E^*_i V, \qquad Rv \in E^*_{i+1} V. \tag{1}$$

By a *T-module* we mean a subspace W of V, such that $TW \subseteq W$. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no T-modules other than 0 and W. Since the algebra T is semi-simple, it turns out that any T-module is an orthogonal direct sum of irreducible T-modules.

Let W be an irreducible T-module. We observe that W is an orthogonal direct sum of the nonvanishing subspaces $E^*_i W$ for $0 \leqslant i \leqslant d$. By the *endpoint* of W we mean $\min\{i \mid 0 \leqslant i \leqslant d,\ E^*_i W \neq 0\}$. We say W is *thin* whenever the dimension of $E^*_i W$ is at most 1 for $0 \leqslant i \leqslant d$.

Observe that the subspace $T\widehat{x} = \{B\widehat{x} \mid B \in T\}$ is a T-module. Suppose that W is an irreducible T-module with endpoint 0. Then, $\widehat{x} \in W$, which implies that $T\widehat{x} \subseteq W$. Since W is irreducible, we therefore have $T\widehat{x} = W$. Hence, $T\widehat{x}$ is the unique irreducible T-module with endpoint 0. We refer to $T\widehat{x}$ as the *trivial T-module*.

3 Distance-regularity and pseudo-distance-regularity

Recall graph $\Gamma = (X, \mathcal{R})$ from Section 2. In this section we recall the notions of (local) distance-regularity and (local) pseudo-distance-regularity of Γ. To do this, fix $x \in X$ and let d denote the eccentricity of x.

Assume for a moment that $y \in \Gamma_i(x)$ $(0 \leqslant i \leqslant d)$ and let z be a neighbour of y. Then, by the triangle inequality,

$$\partial(x, z) \in \{i - 1, i, i + 1\},$$

and so $z \in \Gamma_{i-1}(x) \cup \Gamma_i(x) \cup \Gamma_{i+1}(x)$. For $y \in \Gamma_i(x)$ we therefore define the following numbers:

$$a_i(x, y) = |\Gamma_i(x) \cap \Gamma(y)|, \quad b_i(x, y) = |\Gamma_{i+1}(x) \cap \Gamma(y)|, \quad c_i(x, y) = |\Gamma_{i-1}(x) \cap \Gamma(y)|.$$

We say that $x \in X$ is *distance-regularized* (or that Γ is *distance-regular around* x) if the numbers $a_i(x,y), b_i(x,y)$ and $c_i(x,y)$ do not depend on the choice of $y \in \Gamma_i(x)$ $(0 \leqslant i \leqslant d)$. In this case, the numbers $a_i(x) = a_i(x,y), b_i(x) = b_i(x,y)$ and $c_i(x) = c_i(x,y)$ are called the *intersection numbers of* x.

The concept of pseudo-distance-regularity around a vertex of a graph was introduced in [6] by Fiol, Garriga and Yebra as a natural generalization of distance regularity around a vertex. We now recall this definition.

Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. Let $\rho(A)$ denote the spectral radius of A and let $\upsilon \in V$ denote a Perron-Frobenius vector of A. For $z \in X$ let υ_z denote the z-coordinate of υ. For $y \in \Gamma_i(x)$ $(0 \leqslant i \leqslant d)$ we define numbers $a_i^*(x,y)$, $b_i^*(x,y)$ and $c_i^*(x,y)$ as follows:

$$a_i^*(x,y) = \sum_{z \in \Gamma(y) \cap \Gamma_i(x)} \frac{\upsilon_z}{\upsilon_y}, \quad b_i^*(x,y) = \sum_{z \in \Gamma(y) \cap \Gamma_{i+1}(x)} \frac{\upsilon_z}{\upsilon_y}, \quad c_i^*(x,y) = \sum_{z \in \Gamma(y) \cap \Gamma_{i-1}(x)} \frac{\upsilon_z}{\upsilon_y}.$$

Observe that $a_i^*(x,y) + b_i^*(x,y) + c_i^*(x,y) = \rho(A)$.

We say that vertex $x \in X$ is *pseudo-distance-regularized* (or that Γ is *pseudo-distance-regular around* x) if the numbers $a_i^*(x,y)$, $b_i^*(x,y)$ and $c_i^*(x,y)$ do not depend on the choice of y. In this case, they are denoted by $a_i^*(x)$, $b_i^*(x)$ and $c_i^*(x)$ and they are called the *pseudo-intersection numbers of* Γ *with respect to* x. Moreover, the array

$$\left\{ \begin{matrix} 0 & c_1^*(x) & \cdots & c_{d-1}^*(x) & c_d^*(x) \\ 0 & a_1^*(x) & \cdots & a_{d-1}^*(x) & a_d^*(x) \\ b_0^*(x) & b_1^*(x) & \cdots & b_{d-1}^*(x) & 0 \end{matrix} \right\}$$

is called the *pseudo-intersection array of* Γ *with respect to* x.

Assume now that Γ is distance-regular around x. By [6, Proposition 3.2], Γ is also pseudo-distance-regular around x. However, the converse of this result is not true. In particular, it was shown in [6] that the cartesian product $P_3 \square \cdots \square P_3$ of r paths of length 3 has pseudo-distance-regularized vertices which are not distance-regularized. For the convenience of the reader we would also like to present another example.

Example 2. Let Γ be the connected graph with vertex set $X = \{1,2,3,4,5,6\}$ and edge set $\mathcal{R} = \{\{1,2\},\{1,3\},\{2,4\},\{2,5\},\{3,5\},\{3,6\}\}$. See Figure 1. Let A denote the adjacency matrix of Γ. It is easy to see that $\rho(A) = \sqrt{5}$ and $\upsilon = (2\ \sqrt{5}\ \sqrt{5}\ 1\ 2\ 1)^\top$ is a Perron-Frobenius vector of A. Consider vertex $1 \in X$ and note that $\epsilon(1) = 2$. It is straightforward to check that Γ is pseudo-distance-regular around 1 with the following pseudo-intersection array:

$$\left\{ \begin{matrix} 0 & \frac{2}{\sqrt{5}} & \sqrt{5} \\ 0 & 0 & 0 \\ \sqrt{5} & \frac{3}{\sqrt{5}} & 0 \end{matrix} \right\}$$

However, Γ is not distance-regular around 1. Namely, vertex $4 \in \Gamma_2(1)$ has only one neighbour in $\Gamma(1)$, while vertex $5 \in \Gamma_2(1)$ has two neighbours in $\Gamma(1)$.

As already mentioned in the Introduction, it was proved in [18] that if Γ is distance-regular around x, then the trivial T-module is thin. Fiol and Garriga [5] later proved the following result (see also [4, Theorem 3.1]).

Theorem 3. *Let $\Gamma = (X, \mathcal{R})$ be as in Section 2. Fix $x \in X$ and let $T = T(x)$ denote the corresponding Terwilliger algebra. Then, the trivial T-module is thin if and only if Γ is pseudo-distance-regular around x.*

4 The main result and some products in T

Recall graph $\Gamma = (X, \mathcal{R})$ from Section 2. In this section we state our main result. To do this we need the following definition.

Definition 4. Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple and connected graph. Pick $x, y, z \in X$ and let $P = [y = x_0, x_1, \ldots, x_j = z]$ denote a yz-walk. The *shape of P with respect to* x is a sequence of symbols $t_1 t_2 \ldots t_j$, where $t_i \in \{f, \ell, r\}$, and such that $t_i = r$ if $\partial(x, x_i) = \partial(x, x_{i-1}) + 1$, $t_i = f$ if $\partial(x, x_i) = \partial(x, x_{i-1})$ and $t_i = \ell$ if $\partial(x, x_i) = \partial(x, x_{i-1}) - 1$ $(1 \leqslant i \leqslant j)$. We will be using exponential notation for the shapes containing several consecutive identical symbols. For instance, instead of $rrrrfff\ell\ell r$ we simply write $r^4 f^3 \ell^2 r$. For a positive integer i, let $r^i\ell(y)$, $r^i f(y)$ and $r^i(y)$ respectively denote the number of xy-walks of the shape $r^i\ell$, $r^i f$ and r^i with respect to x. We also define $r^0\ell(y) = r^0 f(y) = 0$ for every $y \in X$, and $r^0(y) = 1$ if $y = x$ and $r^0(y) = 0$ otherwise.

For the rest of the paper we adopt the following notation.

Notation 5. Let $\Gamma = (X, \mathcal{R})$ denote a finite, simple, connected graph with vertex set X, edge set $\mathcal{R}$ and diameter D. Let $A \in \mathrm{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ. Fix a vertex $x \in X$ and let d denote the eccentricity of x. Let $E_i^* \in \mathrm{Mat}_X(\mathbb{C})$ $(0 \leqslant i \leqslant d)$ denote the dual idempotents of Γ with respect to x. Let V denote the standard module of Γ and let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x. Let $T\widehat{x}$ denote the unique irreducible T-module with endpoint 0. Let $L = L(x)$, $F = F(x)$ and $R = R(x)$ denote the lowering, the flat and the raising matrix of T, respectively. For $y \in X$, let the numbers $r^i\ell(y)$, $r^i f(y)$ and $r^i(y)$ be as defined in Definition 4.

We are now ready to state our main result.

Theorem 6. *With reference to Notation 5, the following* (i)–(iii) *are equivalent:*

(i) *$T\widehat{x}$ is thin.*

(ii) *Γ is pseudo-distance-regular around x.*

(iii) *For every integer i $(0 \leqslant i \leqslant d)$ there exist scalars α_i, β_i, such that for every $y \in \Gamma_i(x)$ the following hold:*

$$r^{i+1}\ell(y) = \alpha_i \, r^i(y), \qquad r^i f(y) = \beta_i \, r^i(y).$$

Recall that the equivalency of (i) and (ii) of the above theorem was already proved (see Theorem 3). Therefore, we will focus on the equivalency of (i) and (iii) in the rest of this paper. We first evaluate several products in the Terwilliger algebra T that we will need later. The next result is straightforward to prove (using elementary matrix multiplication, comment below Definition 1, and (1)) and is therefore left to the reader.

Lemma 7. *With reference to Notation 5, pick $y \in X$. Then the following* (i)–(iii) *hold for an integer $i \geqslant 0$.*

(i) *The y-entry of $R^i\widehat{x}$ is equal to the number $r^i(y)$.*

(ii) *The y-entry of $LR^i\widehat{x}$ is equal to the number $r^i\ell(y)$.*

(iii) *The y-entry of $FR^i\widehat{x}$ is equal to the number $r^if(y)$.*

Proposition 8. *With reference to Notation 5, the vector $R^i\widehat{x}$ is nonzero for $0 \leqslant i \leqslant d$.*

Proof. Pick $0 \leqslant i \leqslant d$ and $y \in \Gamma_i(x)$ (note that $\Gamma_i(x)$ is nonempty). By Lemma 7(i), the y-entry of $R^i\widehat{x}$ is equal to the number $r^i(y)$. Note that by the definition of $r^i(y)$ and by the choice of y, we have that $r^i(y) > 0$. The result follows. □

5 Proof of the main theorem

With reference to Notation 5, in this section we prove our main theorem. We also display a basis of $T\widehat{x}$ and the matrix representing the action of the adjacency matrix on this basis in the case when $T\widehat{x}$ is thin.

Lemma 9. *With reference to Notation 5, the following* (i), (ii) *are equivalent:*

(i) *$T\widehat{x}$ is thin.*

(ii) *The set $\{R^i\widehat{x} : 0 \leqslant i \leqslant d\}$ is a basis of $T\widehat{x}$.*

In particular, if the above equivalent conditions (i), (ii) *hold, then $E^*_i(T\widehat{x})$ is spanned by $R^i\widehat{x}$ and* $\dim(E^*_i(T\widehat{x})) = 1$ *for $0 \leqslant i \leqslant d$.*

Proof. As $R^i \in T$ for $0 \leqslant i \leqslant d$, we have that $R^i\widehat{x} \in T\widehat{x}$ for $0 \leqslant i \leqslant d$. Furthermore, by Proposition 8 and (1), the vectors $R^i\widehat{x}$ are nonzero, pairwise orthogonal and $R^i\widehat{x} \in E^*_i(T\widehat{x})$ for $0 \leqslant i \leqslant d$. Assume first that $T\widehat{x}$ is thin. Then $E^*_i(T\widehat{x})$ is spanned by $R^i\widehat{x}$ for $0 \leqslant i \leqslant d$. This proves that the set $\{R^i\widehat{x} : 0 \leqslant i \leqslant d\}$ is a basis of $T\widehat{x}$. Conversely, assume that $\{R^i\widehat{x} : 0 \leqslant i \leqslant d\}$ is a basis of $T\widehat{x}$. Then the subspace $E^*_i(T\widehat{x})$ is spanned by $R^i\widehat{x}$, and so $\dim(E^*_i(T\widehat{x})) = 1$ for $0 \leqslant i \leqslant d$. This implies that $T\widehat{x}$ is thin. The result follows. □

Proof of Theorem 6. As already mentioned, the equivalency of Theorem 6(i) and Theorem 6(ii) follows from Theorem 3. We proceed by showing the equivalency of Theorem 6(i) and Theorem 6(iii).

(i) implies (iii)

Assume that $T\widehat{x}$ is thin. Recall that by Lemma 9 the set $\{R^i\widehat{x} : 0 \leqslant i \leqslant d\}$ is a basis of $T\widehat{x}$, $E_i^*(T\widehat{x})$ is spanned by $R^i\widehat{x}$ and $\dim(E_i^*(T\widehat{x})) = 1$ for $0 \leqslant i \leqslant d$. Consequently, by (1) and since $L, F \in T$, we have that

$$LR^{i+1}\widehat{x} \in E_i^*(T\widehat{x}), \qquad FR^i\widehat{x} \in E_i^*(T\widehat{x})$$

for every $0 \leqslant i \leqslant d$. It follows from the above comments that for every $0 \leqslant i \leqslant d$ there exist scalars α_i, β_i, such that

$$LR^{i+1}\widehat{x} = \alpha_i R^i\widehat{x}, \qquad FR^i\widehat{x} = \beta_i R^i\widehat{x}.$$

The result now follows from Lemma 7.

(iii) implies (i)

Let W denote the vector subspace of V spanned by the vectors $R^i\widehat{x}$ $(0 \leqslant i \leqslant d)$. Since $\widehat{x} \in E_0^*V$, it follows from (1) that $R^i\widehat{x} \in E_i^*V$ for $0 \leqslant i \leqslant d$. By construction and since $R^{d+1}\widehat{x} = 0$, it is clear that W is closed under the action of R. Moreover, by (eiv) from Section 2, the subspace W is invariant under the action of the dual idempotents as well. From Definition 1 and (1) it is easy to see that $L\widehat{x} = F\widehat{x} = 0$.

Recall that by the assumption, for every integer $0 \leqslant i \leqslant d$ there exist scalars α_i, β_i, such that for every $y \in \Gamma_i(x)$ we have

$$r^{i+1}\ell(y) = \alpha_i\, r^i(y), \qquad r^i f(y) = \beta_i\, r^i(y).$$

It follows from Lemma 7 that $LR^{i+1}\widehat{x} = \alpha_i R^i\widehat{x}$ and $FR^i\widehat{x} = \beta_i R^i\widehat{x}$, and so W is invariant under the action of L and F. Since $A = L + F + R$, it follows that W is A-invariant as well. Recall that algebra T is generated by A and the dual idempotents, and so W is a T-module. Note that $R^i\widehat{x} \in T\widehat{x}$ for $0 \leqslant i \leqslant d$, and so $W \subseteq T\widehat{x}$. As W is nonzero and $T\widehat{x}$ is irreducible, we thus have $W = T\widehat{x}$. It is clear that W is thin, since by construction and (1), the subspace E_i^*W is spanned by $R^i\widehat{x}$. This finishes the proof. $\square$

Theorem 10. *With reference to Notation 5, assume that Γ satisfies the equivalent conditions of Theorem 6. Then the set*

$$\mathcal{B} = \left\{R^i\widehat{x} \mid 0 \leqslant i \leqslant d\right\}$$

is a basis of $T\widehat{x}$. Moreover, the matrix representing the action of A on $T\widehat{x}$ with respect to the (ordered) basis $\mathcal{B}$ is given by

$$\begin{pmatrix} 0 & \alpha_0 & & & & \\ 1 & \beta_1 & \alpha_1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & \alpha_{d-2} & \\ & & & 1 & \beta_{d-1} & \alpha_{d-1} \\ & & & & 1 & \beta_d \end{pmatrix}.$$

Proof. By Theorem 6(iii), for every integer $0 \leqslant i \leqslant d$ there exist scalars α_i, β_i, such that for every $y \in \Gamma_i(x)$ we have

$$r^{i+1}\ell(y) = \alpha_i\, r^i(y), \qquad r^i f(y) = \beta_i\, r^i(y).$$

It follows from Lemma 7 that $LR^i\widehat{x} = \alpha_{i-1}R^{i-1}\widehat{x}$ and $FR^i\widehat{x} = \beta_i R^i\widehat{x}$. Recall that $A = L + F + R$, and so the result follows (note also that $\beta_0 = 0$). $\square$

6 Examples

With reference to Notation 5, in this section we present some examples. We first consider graphs which are distance-regular around x.

6.1 Distance-regularized vertices

With reference to Notation 5, assume that Γ is distance-regular around x, with the corresponding intersection numbers $a_i(x), b_i(x), c_i(x)$ $(0 \leqslant i \leqslant d)$. Then it is easy to see that for every $y \in \Gamma_i(x)$ $(0 \leqslant i \leqslant d)$ we have

$$r^i(y) = \prod_{j=1}^{i} c_j(x), \qquad r^{i+1}\ell(y) = b_i(x)\prod_{j=1}^{i+1} c_j(x), \qquad r^i f(y) = a_i(x)\prod_{j=1}^{i} c_j(x).$$

Therefore, for every $y \in \Gamma_i(x)$ we have that $r^{i+1}\ell(y) = \alpha_i r^i(y)$ and $r^i f(y) = \beta_i r^i(y)$ with $\alpha_i = b_i(x)c_{i+1}(x)$ and $\beta_i = a_i(x)$. By Theorem 6, the trivial T-module $T\widehat{x}$ is thin.

6.2 Bipartite graphs

With reference to Notation 5, assume that Γ is bipartite. Observe that in this case $r^i f(y) = 0$ for every $0 \leqslant i \leqslant d$ and for every $y \in \Gamma_i(x)$. Therefore, we have the following result.

Corollary 11. *With reference to Notation 5, assume that Γ is bipartite. Then $T\widehat{x}$ is thin if and only if for $0 \leqslant i \leqslant d$ there exist scalars α_i, such that for every $y \in \Gamma_i(x)$ we have $r^{i+1}\ell(y) = \alpha_i r^i(y)$.*

Proof. Immediately from Theorem 6 and the above observation. $\square$

Consider graph Γ from Example 2 (see also Figure 1), and observe that Γ is bipartite. Fix vertex $1 \in X$ and note that $d = 2$. It is easy to see that for every $y \in \Gamma_i(1)$ $(0 \leqslant i \leqslant 2)$ we have $r^{i+1}\ell(y) = \alpha_i r^i(y)$, where $\alpha_0 = 2, \alpha_1 = 3$ and $\alpha_2 = 0$. As Γ is bipartite, it follows from Corollary 11 that $T\widehat{1}$ is thin.

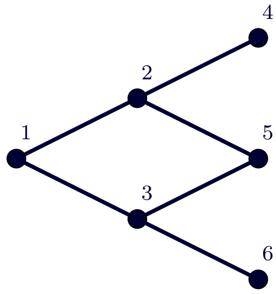

Figure 1: Graph Γ from Example 2.

6.3 Trees

With reference to Notation 5, assume that Γ is a tree. Observe that in this case (as Γ is also bipartite) we have $r^i(y) = 1$ and $r^i f(y) = 0$ for every $0 \leqslant i \leqslant d$ and for every $y \in \Gamma_i(x)$. Therefore, by Theorem 6, $T\widehat{x}$ is thin if and only if for $0 \leqslant i \leqslant d$ there exist scalars α_i, such that for every $y \in \Gamma_i(x)$ we have $r^{i+1}\ell(y) = \alpha_i$. Note however that $r^{i+1}\ell(y) = |\Gamma(y) \cap \Gamma_{i+1}(x)| = b_i(x, y)$. It follows that the trivial module $T\widehat{x}$ is thin if and only if the intersection numbers $b_i(x, y)$ do not depend on the choice of $y \in \Gamma_i(x)$. As $a_i(x, y) = 0$ and $c_i(x, y) = 1$ for every $y \in \Gamma_i(x)$, we have the following corollary of Theorem 6.

Corollary 12. *With reference to Notation 5, assume that* Γ *is a tree. Then* $T\widehat{x}$ *is thin if and only if* Γ *is distance-regular around* x*.*

6.4 Cartesian product $P_3 \square \cdots \square P_3$

Let us first recall the definition of cartesian product of graphs. Let Γ_1 and Γ_2 be finite simple graphs with vertex set X_1 and X_2, respectively. Then the cartesian product of Γ_1 and Γ_2, denoted by $\Gamma_1 \square \Gamma_2$, has vertex set $X_1 \times X_2$. Vertices (x_1, x_2) and (y_1, y_2) are adjacent in $\Gamma_1 \square \Gamma_2$ if and only if either $x_1 = y_1$ and x_2, y_2 are adjacent in Γ_2, or $x_2 = y_2$ and x_1, y_1 are adjacent in Γ_1.

With reference to Notation 5, in this subsection we consider graph $\Gamma = P_3 \square \cdots \square P_3$, the C artesian product of n copies of the path P_3 on 3 vertices (cf. [6, p. 188]). Assume that the vertex set and the edge set of P_3 are $\{0, 1, 2\}$ and $\{\{0, 1\}, \{1, 2\}\}$, respectively. Then the vertex set of Γ is

$$X = \{(y_1, y_2, \ldots, y_n) \mid y_i \in \{0, 1, 2\} \text{ for each } 1 \leqslant i \leqslant n\}.$$

Vertices $y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n)$ are adjacent in Γ if and only if y and z differ in exactly one coordinate (say coordinate i), and $|y_i - z_i| = 1$. Note that Γ is bipartite. We assume that vertex x from Notation 5 is vertex $x = (0, 0, \ldots, 0)$. Observe that $d = 2n$ and that for $0 \leqslant i \leqslant 2n$ we have

$$\Gamma_i(x) = \{(y_1, y_2, \ldots, y_n) \in X \mid y_1 + y_2 + \cdots + y_n = i\}.$$

For $1 \leqslant i \leqslant n$ let us denote by e_i the vertex of Γ, which has i-th coordinate equal to 1, and all other coordinates equal to 0. For vertices $y = (y_1, y_2, \dots, y_n), z = (z_1, z_2, \dots, z_n) \in X$ let $y+z$ denote the n-tuple $(y_1+z_1, y_2+z_2, \dots, y_n+z_n)$. Note that $y+z$ is not necessarily contained in X. Furthermore, let us define $A(y) = \{j \mid 1 \leqslant j \leqslant n, y_j = 0\}$, $B(y) = \{j \mid 1 \leqslant j \leqslant n, y_j = 1\}$ and $C(y) = \{j \mid 1 \leqslant j \leqslant n, y_j = 2\}$. Note that

$$|A(y)| + |B(y)| + |C(y)| = n, \qquad |B(y)| + 2|C(y)| = \partial(x, y). \tag{2}$$

Assume now that $y = (y_1, y_2, \dots, y_n) \in \Gamma_i(x)$. Then $r^i(y)$ equals to the number of walks between x and y in the n-dimensional integer lattice, where for each step of the walk the only possible directions are along one of the "vectors" e_j $(0 \leqslant j \leqslant n)$. This shows that

$$\begin{aligned} r^i(y) &= \binom{i}{y_1}\binom{i-y_1}{y_2}\binom{i-y_1-y_2}{y_3}\cdots\binom{i-y_1-\cdots-y_{n-1}}{y_n} \\ &= \frac{i!(i-y_1)!(i-y_1-y_2)!\cdots(i-y_1-y_2-\cdots-y_{n-1})!}{y_1!(i-y_1)!y_2!(i-y_1-y_2)!\cdots y_{n-1}!(i-y_1-y_2-\cdots-y_{n-1})!y_n!} \\ &= \frac{i!}{y_1!y_2!\cdots y_{n-1}!y_n!} = \frac{i!}{2^{|C(y)|}}. \end{aligned}$$

Observe also that

$$\Gamma(y) \cap \Gamma_{i+1}(x) = \{y + e_j \mid j \in A(y)\} \cup \{y + e_j \mid j \in B(y)\}.$$

Moreover, for $j \in A(y)$ we have $|C(y+e_j)| = |C(y)|$, and for $j \in B(y)$ we have $|C(y+e_j)| = |C(y)| + 1$. It follows that

$$\begin{aligned} r^{i+1}\ell(y) &= \sum_{j \in A(y)} r^{i+1}(y+e_j) + \sum_{j \in B(y)} r^{i+1}(y+e_j) \\ &= \frac{|A(y)|(i+1)!}{2^{|C(y)|}} + \frac{|B(y)|(i+1)!}{2^{|C(y)|+1}} = \frac{(i+1)!}{2^{|C(y)|}}\left(|A(y)| + \frac{|B(y)|}{2}\right). \end{aligned}$$

Finally, it follows from (2) that $|A(y)| + |B(y)|/2 = (2n-i)/2$, and so

$$r^{i+1}\ell(y) = \frac{(i+1)!(2n-i)}{2^{|C(y)|+1}}.$$

This shows that for every $y \in \Gamma_i(x)$ $(0 \leqslant i \leqslant 2n)$ we have $r^{i+1}\ell(y) = \alpha_i r^i(y)$, where $\alpha_i = (i+1)(2n-i)/2$ is independent on the choice of $y \in \Gamma_i(x)$. As Γ is bipartite, it follows from Corollary 11 that $T\widehat{x}$ is thin.

6.5 A construction

In this subsection we show how to construct new graphs, that satisfy the equivalent conditions of Theorem 6 for a certain vertex. To do this, let Γ and Σ denote finite, simple graphs with vertex set X and Y, respectively. Assume that Γ is connected. Fix a vertex

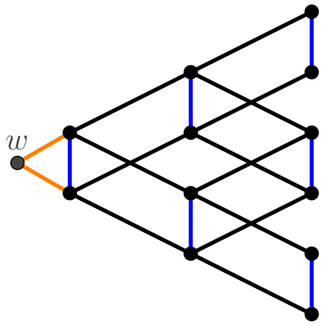

Figure 2: Graph H obtained from the cartesian product $\Gamma \square P_2$ where Γ is the graph from Example 2 and P_2 denotes the path on 2 vertices.

$x \in X$ and consider the Cartesian product $\Gamma \square \Sigma$. Let H denote a graph obtained by adding a new vertex w to the graph $\Gamma \square \Sigma$, and connecting this new vertex w with all vertices (x, y), where y is an arbitrary vertex of Σ. See for example Figure 2.

Note that for an arbitrary vertex (x', y') of H different from w, the distance between w and (x', y') in H is equal to the distance between x and x' in Γ plus one:

$$\partial_H(w, (x', y')) = \partial_\Gamma(x, x') + 1.$$

It follows that $d_H = d + 1$, where d_H is the eccentricity of w in H and d is the eccentricity of x in Γ. Moreover, for $1 \leqslant i \leqslant d_H$ we have

$$H_i(w) = \Gamma_{i-1}(x) \times Y = \{(u, y) \mid u \in \Gamma_{i-1}(x), y \in Y\}.$$

In what follows, we use subscripts to distinguish the number of walks of a particular shape in H and in Γ. For example, for $x' \in \Gamma_i(x)$, we denote the number of walks from x to x' of shape $r^i \ell$ with respect to x by $r^i \ell_\Gamma(x')$. For $(x', y') \in H_i(w)$, we denote the number of walks from w to (x', y') of shape $r^i \ell$ with respect to w by $r^i \ell_H((x', y'))$. It is easy to see that for $(x', y') \in H_i(w)$ $(1 \leqslant i \leqslant d_H)$ we have

$$\begin{aligned} r^i_H((x', y')) = r^{i-1}_\Gamma(x'), \qquad r^{i+1}\ell_H((x', y')) = r^i \ell_\Gamma(x'), \\ r^i f_H((x', y')) = r^{i-1} f_\Gamma(x') + |\Sigma(y')| r^{i-1}_\Gamma(x'), \end{aligned} \tag{3}$$

where $\Sigma(y')$ is the set of neighbours of y' in Σ. Assume now that for vertex x of Γ the equivalent conditions of Theorem 6 are satisfied, and that Σ is regular with valency k. It follows from (3) that for $1 \leqslant i \leqslant d_H$ and for every $(x', y') \in H_i(w)$ we have

$$r^{i+1}\ell_H((x', y')) = r^i \ell_\Gamma(x') = \alpha_{i-1} r^{i-1}_\Gamma(x') = \alpha_{i-1} r^i_H((x', y'))$$

and

$$r^i f_H((x', y')) = r^{i-1} f_\Gamma(x') + |\Sigma(y')| r^{i-1}_\Gamma(x') = (\beta_{i-1} + k) r^{i-1}_\Gamma(x').$$

As we also have $r\ell_H(w) = |Y| = |Y| r^0_H(w)$ and $f_H(w) = 0$, we see that vertex w of H satisfies the condition of Theorem 6(iii). Therefore, by Theorem 6, the trivial $T(w)$-module is thin.

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References

[1] B. Fernández, Certain graphs with exactly one irreducible T-module with endpoint 1, which is thin, submitted.

[2] B. Fernández and Š. Miklavič, On the Terwilliger algebra of distance-biregular graphs, *Linear Algebra Appl.* **597** (2020), 18–32.

[3] B. Fernández and Š. Miklavič, On bipartite graphs with exactly one irreducible T-module with endpoint 1, which is thin, *European J. Combin.* **97** (2021), 103387.

[4] M. A. Fiol, On pseudo-distance-regularity, *Linear Algebra Appl.* **323** (2001), 145–165.

[5] M. A. Fiol and E. Garriga, On the algebraic theory of pseudo-distance-regularity around a set, *Linear Algebra Appl.* **298** (1999), 115–141.

[6] M. A. Fiol, E. Garriga and J. L. A. Yebra, Locally pseudo-distance-regular graphs, *J. Combin. Theory Ser. B* **68(2)** (1996), 79–205.

[7] N. Hamid and M. Oura, Terwilliger algebras of some group association schemes, *Math. J. Okayama Univ.* **61** (2019), 199–204.

[8] A. Hanaki, Modular Terwilliger algebras of association schemes, *Graphs Combin.* **37.5** (2021), 1521–1529.

[9] S. D. Li, Y. Z. Fan, T. Ito, M. Karimi, and J. Xu, The isomorphism problem of trees from the viewpoint of Terwilliger algebras, *J. Combin. Theory Ser. A* **177** (2021).

[10] X. Liang, T. Ito and Y. Watanabe, The Terwilliger algebra of the Grassmann scheme $J_q(N, D)$ revisited from the viewpoint of the quantum affine algebra $U_q\left(\widehat{\mathfrak{sl}}_2\right)$, *Linear Algebra Appl.*, **596** (2020), 117-144.

[11] M. S. MacLean and Š. Miklavič, On a certain class of 1-thin distance-regular graphs, *Ars Math. Contemp.* **18.2** (2020), 187–210.

[12] M. S. MacLean and S. Penjić, A combinatorial basis for Terwilliger algebra modules of a bipartite distance-regular graph, *Discrete Math.* **344.7** (2021), 112393.

[13] J. V. S. Morales, On Lee association schemes over $\mathbb{Z}_4$ and their Terwilliger algebra, *Linear Algebra Appl.* **510** (2016), 311–328.

[14] M. Muzychuk and B. Xu, Terwilliger algebras of wreath products of association schemes, *Linear Algebra Appl.* **493** (2016), 146–163.

[15] S. Penjić, On the Terwilliger algebra of bipartite distance-regular graphs with $\Delta_2 = 0$ and $c_2 = 2$, *Discrete Math.* **340** (2017), 452–466.

[16] Y. Y. Tan, Y. -Z. Fan, T. Ito and X. Liang, The Terwilliger algebra of the Johnson scheme $J(N, D)$ revisited from the viewpoint of group representations, *European J. Combin.* **80** (2019), 157–171.

[17] H. Tanaka and T. Wang, The Terwilliger algebra of the twisted Grassmann graph: the thin case, *Electron. J. Comb.* **27** (4) (2020), #P4.15.

[18] P. Terwilliger, *Algebraic Graph Theory.* Available at https://icu-hsuzuki.github.io/lecturenote/

[19] P. Terwilliger, The subconstituent algebra of an association scheme, (part I), *J. Algebraic Combin.* **1** (1992), 363–388.

[20] P. Terwilliger and A. Žitnik, The quantum adjacency algebra and subconstituent algebra of a graph, *J. Combin. Theory Ser. A* **166** (2019), 297–314.

[21] M. Tomiyama, The Terwilliger algebra of the incidence graph of the Hamming graph, *J. Algebraic Combin.* **48** (2018), 77–118.

[22] J. Xu, T. Ito, and S. Li, Irreducible Representations of the Terwilliger Algebra of a Tree, *Graphs Combin.* **37.5** (2021), 1749–1773.