A Categorification for the Signed Chromatic Polynomial

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Abstract

By coloring a signed graph by signed colors, one obtains the signed chromatic polynomial of the signed graph. For each signed graph we construct graded cohomology groups whose graded Euler characteristic yields the signed chromatic polynomial of the signed graph. We show that the cohomology groups satisfy a long exact sequence which categorifies the signed deletion-contraction rule. This work is motivated by Helme-Guizon and Rong's construction of the categorification for the chromatic polynomial of unsigned graphs.

Mathematics Subject Classifications: 05C15, 05C22

1 Introduction

The chromatic polynomial, which encodes the number of distinct ways to color the vertices of a graph, was introduced by Birkhoff in attempt to attack the four-color problem [1, 2]. Birkhoff's original definition is limited to planar graphs, later Whitney extended this notion to nonplanar graphs. Although this polynomial did not lead to a solution to the

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four-color problem, it is one of the most important polynomials in graph theory. The reader is referred to [18] for a nice introduction to the chromatic polynomial and [9] for a recent breakthrough.

In recent years, a lot of work has been done on the chromatic polynomial and its categorification. Motivated by Khovanov's seminal work on the categorification of the Jones polynomial [10], Helme-Guizon and Rong introduced a categorification for the chromatic polynomial by constructing graded cohomology groups whose graded Euler characteristic is equal to the chromatic polynomial of the graph [6], see also [22]. Later, Jasso-Hernandez and Rong introduced a categorification for the Tutte polynomial [8]. A homology theory for fatgraphs was constructed by Loebl and Moffatt in [11], from which the chromatic polynomial can be recovered as the Euler characteristic. By using a similar idea, Luse and Rong proposed a categorification for the Penrose polynomial, and put forward some relations with other categorifications [13]. Recently, Sazdanovic and Yip constructed a categorification of the chromatic symmetric function [19], which can be considered as a generalization of the chromatic polynomial.

The chromatic cohomology was well studied during the past fifteen years. For example, in [4] M. Chmutov, S. Chmutov and Y. Rong proved the knight move theorem for chromatic cohomology. It follows that the ranks of the cohomology groups are completely determined by the chromatic polynomial. We remark that the original knight move conjecture is false for Khovanov homology [15]. The reader is referred to [7, 12] for some investigations on the torsion part of the chromatic cohomology, which cannot be obtained from the chromatic polynomial.

A signed graph is a graph in which each edge is labeled with a positive sign or a negative sign. The signed graph coloring was first studied by Cartwright and Harary in [5]. In the early 1980's, Zaslavsky tried to use signed colors to color signed graphs [23]. The main principle of how to color a signed graph is that equivalent signed graphs have the same chromatic number. Here two signed graphs are said to be *equivalent* if they are related by finitely many vertex switchings. Zalavsky found some properties of signed graphs and introduced two kinds of chromatic polynomial, namely the chromatic polynomial and the balanced chromatic polynomial. Recently, a good survey on this topic was written by Steffen and Vogel [21].

It's natural to ask whether we can define a categorification for the chromatic polynomial and the balanced chromatic polynomial of signed graphs, following the categorification for the chromatic polynomial of unsigned graphs. The main aim of this paper is to construct two such categorifications. Furthermore, we can also put forward the so-called signed deletion-contraction rule. We show that the cohomology groups satisfy a long exact sequence corresponding to it, which is based on the corresponding exact sequence in Helme-Guizon and Rong's work [6].

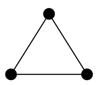
The rest of this paper is arranged as follows. Section 2 is devoted to give a brief introduction to the chromatic polynomial. In Section 3, we give a quick review of the basics of signed graphs and signed graph colorings. Then we recall the notion of signed chromatic polynomials, which combines the chromatic polynomial and the balanced chromatic polynomial of signed graphs. In the beginning of Section 4, we recall the notion of graded dimension of graded $\mathbb{Z}$-modules, then construct the categorifications for the chromatic polynomial and the balanced chromatic polynomial. Several concrete examples are also given. Some basic properties of the cohomology groups in these two categorifications are discussed in Section 5.

2 The chromatic polynomial

We begin our discussion with a brief introduction to the chromatic polynomial. We shall consistently use G to denote a graph, and use $V(G), E(G)$ to denote its vertex set and edge set respectively. A *proper coloring* of G is an assignment of elements from a color set C to $V(G)$, such that adjacent vertices receive different colors. In other words, a proper coloring of G is a map φ from $V(G)$ to C, which requires that for any $v_1, v_2 \in V(G)$, if there exists an edge $e \in E(G)$ connecting v_1 and v_2, then $\varphi(v_1) \neq \varphi(v_2)$. If the color set $C = \{1, 2, \cdots, \lambda\}$, then we denote the number of all proper colorings of G by $P_G(\lambda)$. It follows immediately that if G contains a loop, i.e. an edge that connects a vertex to itself, then $P_G(\lambda) = 0$. By using the deletion-contraction relation

$$P_G(\lambda) = P_{G-e}(\lambda) - P_{G/e}(\lambda),$$

it is not difficult to find that actually $P_G(\lambda)$ defines a polynomial [18], which is called the *chromatic polynomial* of G. Here $G - e$ denotes the graph obtained from G by removing the edge e, and G/e is the graph obtained from G by contracting the edge e.

It's obvious that $P_{N_k}(\lambda) = \lambda^k$, if N_k is the graph consists of k vertices but zero edges. Together with the deletion-contraction relation, these two relations uniquely determine $P_G(\lambda)$. As an example, the chromatic polynomial of $P_3 =$  is equal to $\lambda(\lambda-1)(\lambda-2) = \lambda^3 - 3\lambda^2 + 2\lambda$.

On the other hand, according to the principle of inclusion-exclusion, there is another formula for $P_G(\lambda)$. For each $s \subseteq E(G)$, denote $[G : s]$ the subgraph whose vertex is $V(G)$ and edge set is s, and let $k(s)$ be the number of connected components of $[G : s]$. Then we have

$$P_G(\lambda) = \sum_{s \subseteq E(G)} (-1)^{|s|} \lambda^{k(s)} = \sum_{i \geqslant 0} (-1)^i \sum_{s \subseteq E(G), |s|=i} \lambda^{k(s)}.$$

Note that $\lambda^{k(s)}$ is nothing but the polynomial which counts the ways of colorings of $[G : s]$ such that adjacent vertices have the same color. The formula above plays an important role in the categorification of the chromatic polynomial, see [6] for more details.

3 Signed graph and signed chromatic polynomials

3.1 Signed graph and signed colorings

First, let's take a quick review of signed graphs. Let $SG = (G, \sigma)$ be a signed graph on an ordinary graph $G = (V(G), E(G))$ with a sign on each edge $\sigma : E(G) \to \{1, -1\}$. A circuit of a signed graph is *negative* if the product of edges' signs is negative. We say a signed graph is *unbalanced* if contains a negative circuit, otherwise we say it is *balanced*. Let us use $b(SG)$ to denote the number of balanced components of SG. The following is an example.

Example 1. For the signed graph in Figure 1, there are only 2 negative circuits, say, $v_1v_3v_4v_1$ and $v_1v_3v_4v_2v_1$. Both of them come from the left component. If follows that the left component is unbalanced and the right one is balanced, hence we have $b(SG) = 1$.

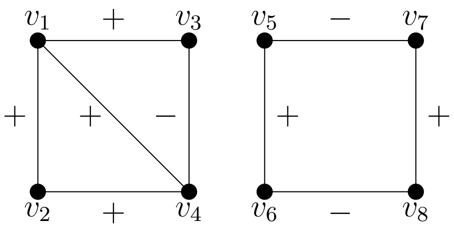


Figure 1: A signed graph with one balanced component and one unbalanced component.

In [23], Zalavsky introduced the signed coloring and chromatic number on signed graphs. Recently, the notion of the chromatic number on signed graphs was modified by E. Máčajová, A. Raspaud and M. Škoviera [14] to make it coincide with the classical chromatic number when all the signs of a signed graph are positive. A signed λ-coloring is a map $k : V(SG) \to K$, where $K = \{-\mu, \ldots, -1, 0, 1, \ldots, \mu\}$ if $\lambda = 2\mu + 1$, and $K = \{-\mu, \ldots, -1, 1, \ldots, \mu\}$ (which is called *zero-free*), if $\lambda = 2\mu$. We call k a *proper λ-coloring* (or just coloring for short) if for each edge $e = v_1v_2$ (it is possible that $v_1 = v_2$), we have $k(v_1) \neq \sigma(e)k(v_2)$. The *chromatic number* $\chi(SG)$ of a signed graph SG is defined to be the smallest λ such that SG admits a proper λ-coloring.

Let $SG = (G, \sigma)$ be a signed graph, each function $u : V(SG) \to \{-1, 1\}$ leads to a switching function on SG: for any $e \in E(SG)$ connecting $v_1, v_2 \in V(SG)$, $\sigma(e)$ will be changed into $u(v_1)\sigma(e)u(v_2)$. In particular, if the edge e is a loop, then the sign $\sigma(e)$ is preserved. If a switching function sends a vertex v to -1 and all other vertices to 1, we call this operation a *vertex switching* on v and denote the new signed graph obtained by $v(SG)$. Two signed graphs are called *equivalent* if they are related by finitely many vertex switchings. Note that vertex switching preserves the sign of any circuit, hence the number of balanced components is invariant under vertex switching. On the other hand, it is obvious that if we are given a proper coloring of SG, after applying the vertex switching on some vertex $v \in V(SG)$, the coloring obtained from the original coloring by reversing the sign of $k(v)$ is a proper coloring for $v(SG)$. It follows that for any given λ, equivalent signed graphs have the same number of proper colorings. In particular, if we are given a balanced signed graph, since a balanced signed graph is equivalent to the corresponding positive graph, it suffices to consider the proper colorings on the unsigned graph.

3.2 Signed chromatic polynomials

As with the ordinary graph, the number of proper signed colorings with λ (odd or even) colors is a polynomial with respect to λ, we call it the *signed chromatic polynomial* and use $P_{SG}(\lambda)$ to denote it[1]. In particular, if $\lambda = 2\mu + 1\,(\mu \in \mathbb{Z})$, we call it the *chromatic polynomial*, otherwise we call it the *balanced chromatic polynomial*, as suggested in [23].

As we mentioned before, a switching function does not change the number of proper colorings, hence it also preserves the signed chromatic polynomial. As with the classical chromatic polynomial, we have the following deletion-contraction relation.

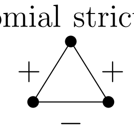
Proposition 2 ([23]). *Let SG be a signed graph and e a positive edge. We use $SG - e$ to denote the signed graph obtained from SG by deleting e, and use SG/e to denote the signed graph obtained from SG by contracting e. The deletion-contraction relation of the signed graph SG with respect to e reads*

$$P_{SG}(\lambda) = P_{SG-e}(\lambda) - P_{SG/e}(\lambda).$$

Notice that the positive edge e above could be a loop, in which case $P_{SG}(\lambda) = 0$. If e is a negative edge but not a loop, we can take vertex switching on one of its endpoint first and then apply the deletion-contraction relation.

Let us use SN_m^n $(m \geqslant n)$ to denote the graph with m vertices and n negative loops, where each negative loop joins one vertex to itself. The signed chromatic polynomial of SN_m^n can be calculated directly

$$P_{SN_m^n}(\lambda) = \begin{cases} \lambda^{m-n}(\lambda-1)^n, & \lambda \text{ is odd;} \\ \lambda^m, & \lambda \text{ is even.} \end{cases}$$

By using the deletion-contraction relation and the signed chromatic polynomial of SN_m^n, we can calculate the signed chromatic polynomial of any signed graph, and verify that both the chromatic polynomial and the balanced chromatic polynomial are well defined as polynomials. We need to point out that the signed chromatic polynomial, as a joint name for both, is not always a polynomial strictly. As an example, let us calculate the signed chromatic polynomial of $SP_3 =$ 

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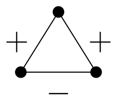
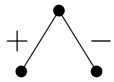
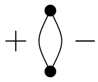
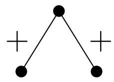
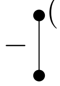
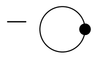
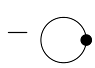
[1]As pointed out by Zaslavsky, signed chromatic polynomial is not a good name since actually $P_{SG}(\lambda)$ is not a polynomial. A better name for $P_{SG}(\lambda)$ should be *chromatic quasipolynomial*.

Example 3.

P  $(\lambda) = P$  $(\lambda) - P$  (λ)

$= P$  $(\lambda) - P$  $(\lambda) + P$  (λ)

$= \lambda(\lambda-1)^2 - \lambda(\lambda-1) + P$  (λ)

$$= \begin{cases} \lambda^3 - 3\lambda^2 + 3\lambda - 1, & \lambda \text{ is odd;} \\ \lambda^3 - 3\lambda^2 + 3\lambda, & \lambda \text{ is even.} \end{cases}$$

As with the unsigned graph, there is another formula for the signed chromatic polynomial deduced from the principle of inclusion-exclusion.

Proposition 4. *Let SG be a signed graph on G, we define $Q_{SG}(\lambda)$ to be the polynomial which calculates the ways of coloring SG such that for any $e \in E(G)$ connecting v_1, $v_2 \in V(G)$, it satisfies $k(v_1) = \sigma(e)k(v_2)$. Then we have*

$$Q_{SG}(\lambda) = \begin{cases} \lambda^{b(SG)}, & \lambda \textit{ is odd or } SG \textit{ is balanced;} \\ 0, & \textit{else.} \end{cases}$$

Proof. For any coloring that satisfies the condition above, the coloring of one component is completely determined by the color on one vertex. If the component is balanced then there are λ colors to choose. However, if the component is unbalanced, we can only assign 0 to it. In other words, we cannot color it when λ is even and there is only one way to color it when λ is odd. Then we obtain

$$Q_{SG}(\lambda) = \begin{cases} \lambda^{b(SG)}, & \lambda \text{ is odd ;} \\ \lambda^{b(SG)} \cdot 0^{n-b(SG)}, & \lambda \text{is even.} \end{cases}$$

where n is the number of components of SG and as usual we set $0^0 = 1$. The proof is finished. □

Proposition 5. *According to the principle of inclusion-exclusion, we have*

$$P_{SG}(\lambda) = \sum_{i \geqslant 0} (-1)^i \sum_{s \subseteq E(G), |s|=i} Q_{[SG:s]}(\lambda),$$

where $[SG : s]$ is the signed subgraph on $[G : s]$.

Proof. We obtain this directly by applying the principle of inclusion-exclusion together with the definition of $Q_{SG}(\lambda)$. □

4 A categorification for the signed chromatic polynomial

As we mentioned above, the signed chromatic polynomial itself is not always a polynomial. However, the chromatic polynomial and the balanced chromatic polynomial are both well defined as polynomials. So in this section, we give the categorifications for these two polynomials respectively. We first recall the definition and some properties of the graded dimension of a graded $\mathbb{Z}$-module.

4.1 Graded dimension of graded modules

Definition 6. Let $\mathcal{M} = \bigoplus_j M_j$ be a graded $\mathbb{Z}$ -module, where M_j denotes the set of homogeneous elements with degree j. The *graded dimension* (also called *quantum dimension*) of $\mathcal{M}$ is the power series

$$q\text{dim}\mathcal{M} := \sum_j q^j \cdot \text{rank}\left(M_j\right),$$

where $\text{rank}\left(M_j\right) = \dim_{\mathbb{Q}}\left(M_j \otimes \mathbb{Q}\right)$.

We remark that the torsion part of $\mathcal{M}$ cannot be detected by the graded dimension. Let $\mathcal{M}$ and $\mathcal{N}$ be two graded $\mathbb{Z}$-modules. Then $\mathcal{M} \oplus \mathcal{N}$ and $\mathcal{M} \otimes \mathcal{N}$ are both graded $\mathbb{Z}$ -modules and the graded dimensions can be obtained as below

$$q\text{dim}(\mathcal{M} \oplus \mathcal{N}) = q\text{dim}(\mathcal{M}) + q\text{dim}(\mathcal{N}), q\text{dim}(\mathcal{M} \otimes \mathcal{N}) = q\text{dim}(\mathcal{M}) \cdot q\text{dim}(\mathcal{N}).$$

The following example of a graded $\mathbb{Z}$-module, which is taken from [6], will be frequently used throughout this paper. It was originally used in the construction of Khovanov homology [10].

Example 7. Let M be the graded free $\mathbb{Z}$-module with two basis elements 1 and x, whose degrees are 0 and 1 respectively. According to the definition of the graded dimension, we have

$$q\text{dim}(M) = q^0 \cdot \text{rank}(\mathbb{Z}) + q^1 \cdot \text{rank}(\mathbb{Z}x) = 1 + q.$$

On the other hand, by using the left identity above, as $M = \mathbb{Z} \oplus \mathbb{Z}x$, we obtain the same result

$$q\text{dim}(M) = q\text{dim}(\mathbb{Z}) + q\text{dim}(\mathbb{Z}x) = 1 + q.$$

Additionally, the tensor product formula above tells us that $q\text{dim}M^{\otimes k} = (1+q)^k$.

Definition 8. Let $\{\ell\}$ be the "degree shift" operation on graded $\mathbb{Z}$-modules. That is, if $\mathcal{M} = \oplus_j M_j$ is a graded $\mathbb{Z}$ -module where M_j denotes the set of elements of $\mathcal{M}$ of degree j, we set $\mathcal{M}\{\ell\}_j := M_{j-\ell}$ so that $q\text{dim}\mathcal{M}\{\ell\} = q^\ell \cdot q\text{dim}\mathcal{M}$. In other words, all the degrees are increased by ℓ.

For example, since $\text{deg}x = 1$, then $\mathbb{Z}\{1\} = \mathbb{Z}x$. And for each $\ell \in \mathbb{N}$, it's easy to check that $\mathcal{M} \otimes \mathbb{Z}\{\ell\} \cong \mathcal{M}\{\ell\}$, the $\mathbb{Z}$-module isomorphic to $\mathcal{M}$ with degree of every homogeneous element raised up by ℓ.

4.2 A categorification for the chromatic polynomial

In this section, we give a categorification for the chromatic polynomial, i.e. the signed chromatic polynomial with odd λ. In order to do this, we need to introduce a sequence of graded $\mathbb{Z}$-modules and graded differentials.

4.2.1 Cochain groups on signed graphs

Let SG be a signed graph on G, for each signed spanning graph $[SG:s]$ led by $s \subseteq E(SG)$, we assign a graded $\mathbb{Z}$-module $M_s(SG)$ as follows: we assign a copy of M to each balanced component and a copy of $\mathbb{Z}$ to the unbalanced component and then take the tensor product. In this way, it is guaranteed that $q\dim M_s(SG) = Q_{[G:s]}(1+q)$ if $q = \lambda - 1 \in 2\mathbb{N}$, which further ensures the following result.

Proposition 9. *Let SG be a signed graph. For each signed spanning graph $[SG:s]$ led by $s \subseteq E(SG)$, we define the cochain group $C^i(SG) = \bigoplus_{s \subseteq E(SG), |s|=i} M_s(SG)$, then $P_{SG}(1+q) = \sum_{i \geqslant 0} (-1)^i q dim C^i(SG)$, provided that $q \in 2\mathbb{N}$.*

Now we have a cochain group $C^\bullet(SG)$ whose graded Euler characteristic is equal to the chromatic polynomial with $\lambda = 1+q$, where λ is odd and q is even. The next step is to introduce a differential d_s for the chain complex which satisfies $d_s^2 = 0$.

4.2.2 The differential

We first recall the definition of enhanced state of an unsigned graph [6]. Let $G = (V(G), E(G))$ be a graph with an ordering on $E(G)$. An *enhanced state* $S = \{s, c\}$ consists of a subset $s \subseteq E(G)$ and an assignment c which assigns 1 or x to each component of $[G:s]$. For each enhanced state S, we set $i(S) = |s|$ and $j(S)$ to be the number of x in c.

We define a multiplication $m : M \otimes M \to M$ by $m(1 \otimes 1) = 1$, $m(1 \otimes x) = m(x \otimes 1) = x$ and $m(x \otimes x) = 0$. We remark that actually there is a Frobenius algebra structure on M [10]. However, here the only operation we need is the multiplication, since adding an edge never increases the component number.

We set $C^{i,j}(G) = \text{span}\langle S | S \text{ is an enhanced state of } G \text{ with } i(S) = i, j(S) = j\rangle$, where the span is taken over $\mathbb{Z}$. The differential $d : C^{i,j}(G) \to C^{i+1,j}(G)$ is defined as

$$d : S = (s, c) \to \sum_{e \in E(G) - s} (-1)^{n(e)} S_e,$$

where $n(e)$ is the number of edges in s that are ordered before e. Here S_e denotes either an enhanced state or 0, which is defined as follows

- if e connects a component E_i to itself, then the components of $[G : s \cup \{e\}]$ are $E_1, \ldots, E_i \cup \{e\}, \ldots, E_{k(s)}$. We define

$$c_e(E_1) = c(E_1), \ldots, c_e(E_i \cup \{e\}) = c(E_i), \ldots, c_e(E_{k(s)}) = c(E_{k(s)}).$$

- If e connects two components E_i and E_j $(i < j)$, then the components of $[G : s \cup \{e\}]$ are $E_1, \dots, E_{i-1}, E_i \cup E_j \cup \{e\}, E_{i+1}, \dots, E_{j-1}, E_{j+1}, \dots, E_{k(s)}$. We define $c_e(E_i \cup E_j \cup \{e\}) = m(c(E_i) \otimes c(E_j))$ and $c_e(E_l) = c(E_l)$ if $l \neq i, j$.

In particular, if $c(E_i) = c(E_j) = x$, then $c_e(E_i \cup E_j \cup \{e\}) = m(x \otimes x) = 0$. In this case, we set $S_e = 0$. Otherwise, c_e is a coloring and we define S_e to be the enhanced state $(s_e = s \cup \{e\}, c_e)$. It was proven in [6] that $d^2 = 0$. Therefore $C^\bullet(G) = \{\bigoplus_{j \geqslant 0} C^{i,j}(G), d\}$ is a graded cochain complex with graded Euler characteristic the chromatic polynomial $P_G(\lambda)$ evaluated at $\lambda = 1 + q$. The corresponding cohomology groups $H^\bullet(G)$ are called the *chromatic cohomology groups* of G.

Now we turn to the enhanced states for the chromatic polynomial of signed graphs. As before, for a given signed graph SG, we use G to denote the corresponding unsigned graph.

Definition 10. An enhanced state $S = (s, c)$ of G is an *enhanced state* of SG with respect to chromatic polynomial if c assigns 1 for all unbalanced components of $[G : s]$. Now we can define $C^{i,j}(SG)$ as follows

$$C^{i,j}(SG) = \text{span}\langle S | S \text{ is an enhanced state of } SG \text{ with } i(S) = i, j(S) = j \rangle,$$

where $i(S) = |s|$ and $j(S) =$ the number of components that x is assigned to.

It follows immediately that $C^i(SG) = \bigoplus_{j \geqslant 0} C^{i,j}(SG)$. In order to define the differential d_s, we need to define a map f from $C^{i,j}(G)$ to $C^{i,j}(SG)$.

Definition 11. Let SG be a signed graph on G, for each enhanced state $S = (s, c)$ in $C^{i,j}(G)$, we define

$$f(S) = \begin{cases} S, & \text{if } c \text{ assigns 1 to all unbalanced components;} \\ 0, & \text{otherwise.} \end{cases}$$

It's obvious that f extends to a linear map from $C^{i,j}(G)$ onto $C^{i,j}(SG)$, as well as $C^i(G)$ onto $C^i(SG)$. We will use f to denote both of them, if there is no confusion. By using f, we can define a map $d_s = f \circ d \circ f^{-1}$, according to the following commutative diagram.

$$\begin{array}{ccc} C^n(G) & \xrightarrow{\quad d \quad} & C^{n+1}(G) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ C^n(SG) & \xrightarrow{\quad d_s \quad} & C^{n+1}(SG) \end{array}$$

Note that in general f is not injective, for f^{-1}, one just needs to choose an element from the pre-image. The next proposition shows that d_s does not depend on the choice of the pre-image.

Proposition 12. *The map $d_s : C^n(SG) \to C^{n+1}(SG)$, defined by $d_s = f \circ d \circ f^{-1}$ is well defined.*

Proof. Because both f, d are linear functions, it suffices to prove that for each $z \in C^n(G)$ with $f(z) = 0$, we have $f \circ d(z) = 0$. Recall that $C^n(G) = \bigoplus_{s \subseteq E(G), |s|=n} M_s(G)$, where $M_s(G) = M^{\otimes k(s)}$, we suppose $z = \sum_{i=1}^{m} a_i S_i$ $(a_i \neq 0)$, where $S_i = (s_i, c_i)$ $(1 \leqslant i \leqslant m)$ are distinct enhanced states with $|s_i| = n$. Since $f(z) = 0$, it follows that for each $1 \leqslant i \leqslant m$, there exists at least one unbalanced component of $[G : s_i]$ which is assigned an x. For S_1, let us choose an unbalanced component E_1 with $c_1(E_1) = x$. We break down into three cases:

1. If an edge $e \in E(G) - s_1$ is not adjacent to E_1, then the corresponding assignment of $(S_1)_e$ also assigns an x to E_1. Hence $(S_1)_e$ is also sent to 0 according to the definition of f.

2. If an edge $e \in E(G) - s_1$ connects E_1 with another component E_2, then the new component $E_1 \cup E_2 \cup \{e\}$ in $[G : s_1 \cup \{e\}]$ is also unbalanced. If $c_1(E_2) = 1$, then $(c_1)_e(E_1 \cup E_2 \cup \{e\}) = m(x \otimes 1) = x$, hence $f((S_1)_e) = 0$. If $c_1(E_2) = x$, then $(c_1)_e(E_1 \cup E_2 \cup \{e\}) = m(x \otimes x) = 0$, which is also mapped to 0 under f.

3. If an edge $e \in E(G) - s_1$ joins E_1 to itself, then the new component $E_1 \cup \{e\}$ in $[G : s_1 \cup \{e\}]$ is also unbalanced and $(c_1)_e(E_1 \cup \{e\}) = c_1(E_1) = x$, it follows that $f((S_1)_e) = 0$.

For other S_i, the proof is analogous to that of S_1. In summary, we have $f \circ d(z) = 0$, which means that $d_s = f \circ d \circ f^{-1}$ is well defined. ☐

As $d^2 = 0$ and $d_s = f \circ d \circ f^{-1}$, we can directly obtain $d_s^2 = f \circ d^2 \circ f^{-1} = 0$, then we obtain the following cochain complex

$$C^0(SG) \xrightarrow{d_s} C^1(SG) \xrightarrow{d_s} C^2(SG) \xrightarrow{d_s} \cdots \xrightarrow{d_s} C^i(SG) \xrightarrow{d_s} \cdots.$$

Definition 13. For a given signed graph SG, we call the cohomology groups of the cochain complex above the *chromatic cohomology groups* of SG and use $H^i(SG)$ to denote the i-th chromatic cohomology group of SG.

Since d_s is degree preserving, the chromatic cohomology group $H^i(SG)$ can be decomposed as $\bigoplus_{j \geqslant 0} H^{i,j}(SG)$. The chromatic polynomial can be recovered by $P_{SG}(1+q) = \sum_{i \geqslant 0} (-1)^i q \dim C^i(SG) = \sum_{i \geqslant 0} (-1)^i q \dim H^i(SG)$. The following proposition tells us that chromatic cohomology groups are well defined signed graph invariants.

Proposition 14. *The chromatic cohomology groups are independent of the order of the edges.*

Proof. The proof follows the outline given in [6]. Suppose that $E(SG) = \{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\}$, it suffices to prove that $H^i(SG) \cong H^i(SG')$, where

SG' is the same signed graph as SG but the edges are reordered as $\{e_1, \ldots, e_{k+1}, e_k, \ldots, e_n\}$.

Since $C^i(SG) = \bigoplus_{s \subseteq E(SG), |s|=i} M_s(SG)$, where $M_s(SG) = M^{\otimes b([SG:s])}$, it is enough to define an isomorphism g restricted on $M_s(SG)$. Consider the map g_s from $M_s(SG)$ to $M_s(SG')$, which is defined to be

$$g_s = \begin{cases} -id, & \text{if } \{e_k, e_{k+1}\} \subseteq s \\ id, & \text{otherwise} \end{cases}$$

Now we define the map $g : C^i(SG) \to C^i(SG')$ as $g = \bigoplus_{s \subseteq E(SG), |s|=i} g_s$. It is a routine exercise to check that g is a chain map which induces an isomorphism between $H^i(SG)$ and $H^i(SG')$. □

We end this subsection with a concrete example.

Example 15. Let us consider the signed graph SP_3, where $E(SP_3) = \{e_1, e_2, e_3\}$, and $\sigma(e_1) = \sigma(e_2) = +1$, $\sigma(e_3) = -1$, as Figure 2 shows.

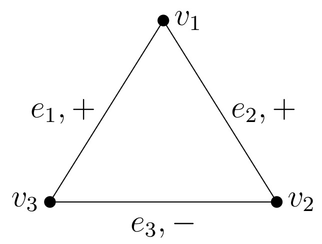


Figure 2: A signed graph SP_3

For each enhanced state $S = (s, c)$ of SP_3, we denote s and c by elements of $\{0, 1\}^3$ and $\{1, x\}^3$, such that the i-th position of s is 1 (0) if $e_i \in s$ ($e_i \notin s$), and all the vertices of the same component have the same color. For example, the enhanced states shown in Figure 3 are $(000, 11x)$, $(100, x1x)$ and $(101, 111)$.

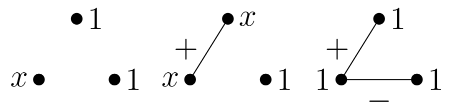


Figure 3: Three enhanced states

One computes

$$\begin{aligned} C^0(SP_3) = \quad & \text{span}\langle (000, xxx), \\ & (000, xx1), (000, x1x), (000, 1xx), \\ & (000, x11), (000, 1x1), (000, 11x), \end{aligned}$$

$$
\begin{aligned}
& \qquad (000,111)\rangle; \\
C^1(SP_3) = \quad & \mathrm{span}\langle (001,xxx),(010,xxx),(100,xxx), \\
& \qquad (001,x11),(001,1xx),(010,xx1), \\
& \qquad (010,11x),(100,x1x),(100,1x1), \\
& \qquad (001,111),(010,111),(100,111)\rangle; \\
C^2(SP_3) = \quad & \mathrm{span}\langle (011,xxx),(101,xxx),(110,xxx), \\
& \qquad (011,111),(101,111),(110,111)\rangle; \\
C^3(SP_3) = \quad & \mathrm{span}\langle (111,111)\rangle. \\
B^0(SP_3) = \quad & \quad 0; \\
B^1(SP_3) = \quad & \mathrm{span}\langle (001,xxx)+(010,xxx), \\
& \qquad (010,xxx)+(100,xxx), \\
& \qquad (100,xxx)+(001,xxx), \\
& \qquad (100,x1x)+(010,xx1)+(001,x11), \\
& \qquad (100,1x1)+(010,xx1)+(001,1xx), \\
& \qquad (100,x1x)+(010,11x)+(001,1xx), \\
& \qquad (001,111)+(010,111)+(100,111)\rangle; \\
B^2(SP_3) = \quad & \mathrm{span}\langle (110,xxx)+(101,xxx),(101,xxx)+(011,xxx) \\
& \qquad (110,111)+(101,111),(101,111)+(011,111)\rangle; \\
B^3(SP_3) = \quad & \mathrm{span}\langle (111,111)\rangle. \\
Z^0(SP_3) = \quad & \mathrm{span}\langle (000,xxx)\rangle; \\
Z^1(SP_3) = \quad & \mathrm{span}\langle (001,xxx),(010,xxx),(100,xxx), \\
& \qquad (001,x11)-(001,1xx), \\
& \qquad (010,xx1)-(010,11x), \\
& \qquad (100,x1x)-(100,1x1), \\
& \qquad (001,x11)+(010,11x)+(100,1x1), \\
& \qquad (001,111)+(010,111)+(100,111)\rangle; \\
Z^2(SP_3) = \quad & \mathrm{span}\langle (011,xxx),(101,xxx),(110,xxx), \\
& \qquad (110,111)+(101,111),(101,111)+(011,111)\rangle; \\
Z^3(SP_3) = \quad & \mathrm{span}\langle (111,111)\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
H^0(SP_3) &= Z^0(SP_3)/B^0(SP_3) \cong \mathbb{Z}\{3\}; \\
H^1(SP_3) &= Z^1(SP_3)/B^1(SP_3) \cong \mathbb{Z}_2\{2\} \oplus \mathbb{Z}\{1\}; \\
H^2(SP_3) &= Z^2(SP_3)/B^2(SP_3) \cong \mathbb{Z}\{1\}; \\
H^3(SP_3) &= Z^3(SP_3)/B^3(SP_3) \cong 0.
\end{aligned}
$$

By using Proposition 9, we can calculate the chromatic polynomial

$$P_{SP_3}(1+q) = \sum_{i\geqslant 0}(-1)^i q\text{dim}C^i(SP_3) = \sum_{i\geqslant 0}(-1)^i q\text{dim}H^i(SP_3) = q^3 - q + q = q^3,$$

where q is even. On the other hand, this result can also be obtained by using the deletion-contraction relation on the signed graph, which is shown in Example 3.

4.2.3 A categorification of the deletion-contraction relation

The aim of this subsection is to introduce a long exact sequence, which recovers the deletion-contraction relation given in Proposition 2 if one takes the Euler characteristic of this long exact sequence. So in some sense, the long exact sequence in Corollary 17 can be considered as a categorification of the deletion-contraction rule in Proposition 2.

Theorem 16. *Let SG be a signed graph and e a positive edge, we have the following short exact sequence*

$$0 \to C^{\bullet-1}(SG/e) \to C^{\bullet}(SG) \to C^{\bullet}(SG-e) \to 0.$$

Proof. Without loss of generality, let us choose an order of $E(SG)$ such that the positive edge e is the first edge. The order of $E(SG)$ induces an order for $E(SG/e)$ and $E(SG-e)$ respectively.

We first introduce a morphism of cochain complexes, say $\widetilde{m} : C^{\bullet}(SG-e) \to C^{\bullet}(SG/e)$. Recall that $C^i(SG) = \bigoplus_{s\subseteq E(SG),|s|=i} M_s(SG)$, where $M_s(SG) = M^{\otimes b([SG:s])}$, it suffices to consider a fixed subset $s \subseteq E(SG-e) = E(SG/e)$. If e joins a component of $[SG-e:s]$ to itself, then we define $\widetilde{m}$ to be $f_{SG/e} \circ id \circ f^{-1}_{SG-e}$, where $f_{SG/e}$ and f_{SG-e} are similarly defined as f in Definition 11. Otherwise, we define $\widetilde{m}$ to be $f_{SG/e} \circ m \circ f^{-1}_{SG-e}$, where m denotes the multiplication $m : M \otimes M \to M$. One can easily check that $\widetilde{m}$ is well defined.

We claim that the complex $C^{\bullet}(SG)$ is the mapping cone of $\widetilde{m} : C^{\bullet}(SG-e) \to C^{\bullet}(SG/e)$.

Notice that

$$C^i(SG) = \bigoplus_{s\subseteq E(SG),|s|=i} M_s(SG) = \bigoplus_{e\notin s\subseteq E(SG),|s|=i} M_s(SG) \oplus \bigoplus_{e\in s\subseteq E(SG),|s|=i} M_s(SG),$$

and there is an obvious isomorphism between

$$C^i(SG-e) = \bigoplus_{s\subseteq E(SG-e),|s|=i} M_s(SG-e)$$

and

$$\bigoplus_{e\notin s\subseteq E(SG),|s|=i} M_s(SG).$$

For the second summand $\bigoplus_{e\in s\subseteq E(SG),|s|=i} M_s(SG)$, one observes that for any $s \in E(SG/e)$, there is a one-to-one correspondence between the components of $[SG/e : s]$ and the components of $[SG : s \cup \{e\}]$. Since the sign of e is positive, then a component of $[SG/e : s]$ is balanced if and only if the corresponding component in $[SG : s \cup \{e\}]$ is balanced. It follows that

$$\bigoplus_{e\in s\subseteq E(SG),|s|=i} M_s(SG) \cong \bigoplus_{s\subseteq E(SG/e),|s|=i-1} M_s(SG/e) = C^{i-1}(SG/e),$$

and hence

$$C^i(SG) = C^i(SG-e) \bigoplus C^{i-1}(SG/e).$$

Consider the differential $d'_s : C^i(SG) \to C^{i+1}(SG)$ given by $\begin{pmatrix} d_1 & 0 \\ \widetilde{m} & -d_2 \end{pmatrix}$, where d_1 denotes the differential on $C^\bullet(SG-e)$ and d_2 denotes the differential on $C^\bullet(SG/e)$. Recall that e is the first edge, it is not difficult to find that d'_s coincides with the differential d_s defined in Proposition 12. This finishes the proof of the claim and the result follows directly. □

Corollary 17. *Let SG be a signed graph and e a positive edge, we have the following long exact sequence*

$$\cdots \to H^{i-1}(SG/e) \to H^i(SG) \to H^i(SG-e) \to H^i(SG/e) \to \cdots.$$

Remark 18. It would be helpful to describe the two maps $\alpha^* : H^{i-1}(SG/e) \to H^i(SG)$ and $\beta^* : H^i(SG) \to H^i(SG-e)$ intuitively. In order to understand α^*, it suffices to consider the map $\alpha : C^{i-1}(SG/e) \to C^i(SG)$. With a given enhanced state $S = (s, c)$ of SG/e, notice that $[SG/e : s]$ and $[SG : s \cup \{e\}]$ not only have the same number of components but also the same number of balanced components, since e is positive. Then we can define $\alpha(S) = (s \cup \{e\}, c_e)$, which induces the map $\alpha^* : H^{i-1}(SG/e) \to H^i(SG)$. For β^*, let us choose an enhanced state $S = (s, c)$ of SG. If $e \notin s$, then we define $\beta(S) = S$, which is also an enhanced state of $SG - e$, since removing e from a balanced component yields a balanced component. Otherwise, if $e \in s$ then we set $\beta(S) = 0$. The map $\beta^* : H^i(SG) \to H^i(SG-e)$ can be induced from β.

4.3 A categorification for the balanced chromatic polynomial

In this subsection, we discuss how to categorify the balanced chromatic polynomial, i.e. the signed chromatic polynomial with even λ.

4.3.1 Cochain groups on signed graphs

Let SG be a signed graph on G, for each signed subgraph $[SG : s]$ led by $s \subseteq E(SG)$, we assign a graded $\mathbb{Z}$-module $M^b_s(SG)$ as follows:

- if $[SG : s]$ is balanced, we assign a copy of M to each component and then take the tensor product, i.e. $M^b_s(SG) = M^{\otimes k(s)}$ if $[SG : s]$ is balanced;
- if $[SG : s]$ is unbalanced, we assign 0 to it.

In this case, it is guaranteed that $q\dim M_s^b(SG) = Q_{[SG:s]}(1+q)$ when q is odd. As an analogy of Proposition 9, we have the following result.

Proposition 19. *Let SG be a signed graph, we define $C_b^i(SG) = \bigoplus_{s \subseteq E(G), |s|=i} M_s^b(SG)$, then*

$$P_{SG}(1+q) = \sum_{i \geqslant 0}(-1)^i \sum_{s \subseteq E(G), |s|=i} q dim M_s^b(SG) = \sum_{i \geqslant 0}(-1)^i q dim C_b^i(SG),$$

if q is an odd integer.

4.3.2 The differential

Definition 20. An enhanced state $S = (s, c)$ of G is an *enhanced state* of SG for balanced chromatic polynomial if $[G : s]$ is balanced.

As before, we set

$$C_b^{i,j}(SG) = \text{span}\,\langle S | S \text{ is an enhanced state of } SG \text{ with } i(S) = i, j(S) = j \rangle\,,$$

where $i(S) = |s|$ and $j(S)$ equals the number of components that x is assigned to. It follows immediately that $C_b^i(SG) = \bigoplus_{j \geqslant 0} C_b^{i,j}(SG)$.

Now we introduce a map $f_b : C^{i,j}(G) \to C_b^{i,j}(SG)$ for the balanced chromatic polynomial.

Definition 21. Let SG be a signed graph on G, for each enhanced state $S = (s, c)$ in $C^{i,j}(G)$, we define

$$f_b(S) = \begin{cases} S, & [SG : s] \text{ is balanced;} \\ 0, & \text{otherwise.} \end{cases}$$

We extend f_b to a linear projection from $C^{i,j}(G)$ to $C_b^{i,j}(SG)$, or, from $C^i(G)$ to $C_b^i(SG)$ if one sums over j. Let us still use f_b to denote it.

By using f_b, we define the differential d_b as the below, which is similar to the definition of d_s.

$$\begin{array}{ccc} C^n(G) & \xrightarrow{\quad d \quad} & C^{n+1}(G) \\ \downarrow{\scriptstyle f_b} & & \downarrow{\scriptstyle f_b} \\ C_b^n(SG) & \xrightarrow{\quad d_b \quad} & C_b^{n+1}(G) \end{array}$$

Proposition 22. *The map $d_b : C_b^n(SG) \to C_b^{n+1}(SG)$ defined by $d_b = f_b \circ d \circ f_b^{-1}$ is well defined.*

Proof. Similar to the proof of Proposition 12, it suffices to show that for any $z = \sum_{i=1}^{m} a_i S_i$ ($a_i \neq 0$), if $f_b(z) = 0$ then $f_b \circ d(z) = 0$. Here $S_i = (s_i, c_i)$ for some $s_i \subseteq E(SG)$ with $|s_i| = n$. According to the definition of f_b, each $[SG : s_i]$ contains at least one unbalanced component. Notice that the differential d corresponds to the following three cases

- adding an edge connecting an unbalanced component to itself;
- adding an edge connecting an unbalanced component with another component;
- adding an edge which is not adjacent to unbalanced components.

Notice that each case above gives rise to a new unbalanced component. It follows that $f_b \circ d(S_i) = 0$ and hence $f_b \circ d(z) = 0$. □

Since $d^2 = 0$ and $d_b = f_b \circ d \circ f_b^{-1}$, we obtain $d_b^2 = f_b \circ d^2 \circ f_b^{-1} = 0$ and hence we have the following cochain complex

$$C_b^0(SG) \xrightarrow{d_b} C_b^1(SG) \xrightarrow{d_b} C_b^2(SG) \xrightarrow{d_b} \cdots \xrightarrow{d_b} C_b^i(SG) \xrightarrow{d_b} \cdots .$$

Definition 23. We call the cohomology groups of the cochain complex above the *balanced chromatic cohomology groups* of SG and denote them by $H_b^\bullet(SG)$.

Similar to Proposition 14, the balanced chromatic cohomology groups $H_b^\bullet(SG)$ are independent of the choice of the order of edges. Now we give two examples. The first one, as a supplement of Example 15, calculates the balanced chromatic cohomology groups of the signed graph SP_3. The second one is devoted to calculate the balanced chromatic cohomology groups of SP_2, see Figure 4.

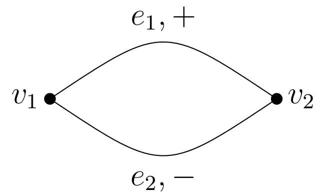


Figure 4: An unbalanced signed graph SP_2

Example 24. We use the same notions as that in Example 15. It is easy to find that for each $i \in \{0, 1, 2\}$, we have $C_b^i(SP_3) = C^i(SP_3)$, $B_b^i(SP_3) = B^i(SP_3)$ and $C_b^3(SP_3) = 0$, $B_b^3(SP_3) = 0$. On the other hand, $Z_b^0(SP_3) = Z^0(SP_3)$, $Z_b^1(SP_3) = Z^1(SP_3)$, $Z_b^2(SP_3) = C^2(SP_3)$, $Z_b^3(SP_3) = 0$. Then we obtain the balanced chromatic cohomology groups as below

$$\begin{aligned}
&H_b^0(SP_3) = Z^0(SP_3)/B^0(SP_3) = H^0(SP_3) \cong \mathbb{Z}\{3\};\\
&H_b^1(SP_3) = Z^1(SP_3)/B^1(SP_3) = H^1(SP_3) \cong \mathbb{Z}_2\{2\} \oplus \mathbb{Z}\{1\};\\
&H_b^2(SP_3) = Z^2(SP_3)/B^2(SP_3) \cong \mathbb{Z}\{1\} \oplus \mathbb{Z};\\
&H_b^3(SP_3) = 0.
\end{aligned}$$

We calculate the balanced chromatic polynomial of SG

$$P_{SP_3}(1+q) = \sum_{i \geqslant 0} (-1)^i q \mathrm{dim} C_b^i(SP_3) = \sum_{i \geqslant 0} (-1)^i q \mathrm{dim} H_b^i(SP_3) = q^3 - q + q + 1 = q^3 + 1,$$

where q is odd. This result coincides with the result obtained in Example 3.

Example 25. Let SP_2 be an unbalanced signed graph on P_2, where $E(SP_2) = \{e_1, e_2\}$ and $\sigma(e_1) = +1$, $\sigma(e_2) = -1$, as Figure 4 shows. For each enhanced state $S = (s, c)$ of SP_2, we denote s and c similarly to Example 15, then we can calculate the cochain groups and the cohomology groups as follows.

$$
\begin{aligned}
&C_b^0(SP_2) = \text{span}\langle (00, xx), (00, 1x), (00, x1), (00, 11)\rangle;\\
&C_b^1(SP_2) = \text{span}\langle (01, xx), (10, xx), (01, 11), (10, 11)\rangle;\\
&C_b^2(SP_2) = 0;\\
&B_b^0(SP_2) = 0;\\
&B_b^1(SP_2) = \text{span}\langle (10, xx) + (01, xx), (10, 11) + (01, 11)\rangle;\\
&B_b^2(SP_2) = 0;\\
&Z_b^0(SP_2) = \text{span}\langle (00, xx), (00, 1x) - (00, x1)\rangle;\\
&Z_b^1(SP_2) = \text{span}\langle (01, xx), (10, xx), (01, 11), (10, 11)\rangle;\\
&Z_b^2(SP_2) = 0;\\
&H_b^0(SP_2) = Z_b^0(SP_2)/B_b^0(SP_2) \cong \mathbb{Z}\{2\} \oplus \mathbb{Z}\{1\};\\
&H_b^1(SP_2) = Z_b^1(SP_2)/B_b^1(SP_2) \cong \mathbb{Z}\{1\} \oplus \mathbb{Z};\\
&H_b^2(SP_2) = Z_b^2(SP_2)/B_b^2(SP_2) \cong 0.
\end{aligned}
$$

According to Proposition 19, the balanced chromatic polynomial can be calculated as

$$P_{SG}(1+q) = \sum_{i\geqslant 0}(-1)^i q\text{dim}C_b^i(SG) = \sum_{i\geqslant 0}(-1)^i q\text{dim}H_b^i(SG) = q^2 + q - q - 1 = q^2 - 1,$$

where q is an odd integer. And we can check this result intuitively: suppose there are $1 + q$ colors, if we assign one color to v_1, to which there are $1 + q$ approaches. There are $q - 1$ colors left to be chosen to color v_2. According to the multiplication principle, $P_{SP_2}(1 + q) = (q + 1)(q - 1) = q^2 - 1$, if q is odd.

We remark that it was proved that for unsigned graphs, the knight move conjecture holds for the chromatic cohomology with rational coefficients [4]. More precisely, Chmutov, Chmutov and Rong proved that for a unsigned graph G, the nontrivial cohomology groups come in isomorphic pairs: $H^{i,|V(G)|-i}(G;\mathbb{Q}) \cong H^{i+1,|V(G)|-i-2}(G;\mathbb{Q})$. According to the examples above, we find that this is not true for signed chromatic cohomology. It is worthy to point out that Lowrance and Sazdanovic provided a complete picture of integral chromatic homology for unsigned graphs in [12].

4.3.3 Another long exact sequence

As an analogy to Corollary 17, the deletion-contraction relation for the balanced chromatic polynomial also corresponds to a long exact sequence. This can be derived from the following theorem.

Theorem 26. *Let SG be a signed graph and e a positive edge, we have the following short exact sequence*

$$0 \to C_b^{\bullet-1}(SG/e) \to C_b^{\bullet}(SG) \to C_b^{\bullet}(SG-e) \to 0.$$

Proof. The proof is a mimic of that of Theorem 16. For cochain groups, we still have

$$C_b^i(SG) = \bigoplus_{e\notin s\subseteq E(SG),|s|=i} M_s^b(SG) \oplus \bigoplus_{e\in s\subseteq E(SG),|s|=i} M_s^b(SG) = C_b^i(SG-e) \oplus C_b^{i-1}(SG/e),$$

where e is set as the first edge. The morphism $\widetilde{m_b} : C_b^{\bullet}(SG-e) \to C_b^{\bullet}(SG/e)$ can be similarly defined as

$$\widetilde{m_b} = \begin{cases} f_b \circ id \circ f_b^{-1}, & \text{if } e \text{ connects one component to itself;} \\ f_b \circ m \circ f_b^{-1}, & \text{if } e \text{ connects two different components,} \end{cases}$$

where m denotes the multiplication on M and f_b is the map introduced in Definition 21. Let us use d_b^1 and d_b^2 to denote the differential maps on $C^{\bullet}(SG-e)$ and $C^{\bullet}(SG/e)$ respectively. It is not difficult to check that the differential $d_b' = \begin{pmatrix} d_b^1 & 0 \\ \widetilde{m_b} & -d_b^2 \end{pmatrix}$ defined on $C_b^{\bullet}(SG-e) \bigoplus C_b^{\bullet-1}(SG/e)$ coincides with the differential d_b defined on $C_b^{\bullet}(SG)$. In other words, the complex $C_b^{\bullet}(SG)$ is the mapping cone of $\widetilde{m_b} : C_b^{\bullet}(SG-e) \to C_b^{\bullet}(SG/e)$. The result follows. □

Corollary 27. *Let SG be a signed graph and e a positive edge, we have the following long exact sequence*

$$\cdots \to H_b^{i-1}(SG/e) \to H_b^i(SG) \to H_b^i(SG-e) \to H_b^i(SG/e) \to \cdots .$$

Remark 28. After this paper was finished, Sergei Chmutov told us that Ben O'Connor and Andrew Krieger categorified the signed Tutte polynomial in 2013. But they did not write any paper about this.

Remark 29. It is well known that for a given link diagram, there exists an associated signed planar graph, called the *Tait graph*. Conversely, each signed planar graph also corresponds to a link diagram. An unsigned planar graph, which can be regarded as a positive planar graph, corresponds to an alternating link diagram. From this viewpoint, several investigations on the Khovanov homology based on the study of chromatic cohomology were carried out, see [16, 17] for example. Since each signed planar graph corresponds to a link diagram, it is natural to ask whether we can recover Khovanov homology from the categorification of the signed chromatic polynomial. From this perspective, our categorification is not the "right" approach. The main reason is, as we mentioned before, the main principle of signed coloring is that equivalent signed graphs have the same number of colorings. In other words, vertex switching preserves the signed chromatic polynomial. The corresponding operation of vertex switching in a link diagram is region crossing change [20], which switches all the crossing points on the boundary of a region. It is known that region crossing change is an unknotting operation for proper links [3], which contain in particular all knots. It follows that Khovanov homology is not preserved under region crossing change.

5 Some properties

5.1 Relation to the chromatic cohomology groups of unsigned graphs

In Section 4, we introduced two cochain complexes for signed graphs based on the categorification of the chromatic polynomial of unsigned graphs, which was proposed by Laure Helme-Guizon and Yongwu Rong in [6]. So it is natural to consider the relation among the cohomology groups of these three cochain complexes. Obviously, if all the edges of a signed graph are positive, then all these cohomology groups coincide. This conclusion can be enhanced a little bit as follows.

For a signed graph SG, let us use $n_b(SG)$ to denote the length of the shortest unbalanced circuits. In other words, any $[SG : s]$ is balanced provided that $|s| \leqslant n_b(SG) - 1$.

Proposition 30. *For any $i \leqslant n_b(SG) - 2$, we have $H^i(SG) = H^i_b(SG) = H^i(G)$.*

Proof. For any $i \geqslant n_b$, by replacing all the cochain groups $C^i(G), C^i(SG)$ and $C^i_b(SG)$ with 0, one obtains three new cochain complexes. Since there is no negative circuit now, these three cochain complexes are exactly the same, hence the cohomology groups are isomorphic mutually. □

5.2 Disjoint union of two signed graphs

Let SG_1 and SG_2 be two signed graphs, we denote their disjoint union by $SG_1 \sqcup SG_2$. On the cochain complex level, we have $C^\bullet(SG_1 \sqcup SG_2) = C^\bullet(SG_1) \otimes C^\bullet(SG_2)$ and $C^\bullet_b(SG_1 \sqcup SG_2) = C^\bullet_b(SG_1) \otimes C^\bullet_b(SG_2)$. As a corollary of the Künneth theorem, the cohomology groups of SG_1, SG_2 and their disjoint union satisfy the following relations.

Proposition 31. *For each $i \in \mathbb{N}$, we have*

$$H^i(SG_1 \sqcup SG_2) \cong [\bigoplus_{p+q=i} H^p(SG_1) \otimes H^q(SG_2)] \oplus [\bigoplus_{p+q=i+1} H^p(SG_1) * H^q(SG_2)],$$

$$H^i_b(SG_1 \sqcup SG_2) \cong [\bigoplus_{p+q=i} H^p_b(SG_1) \otimes H^q_b(SG_2)] \oplus [\bigoplus_{p+q=i+1} H^p_b(SG_1) * H^q_b(SG_2)],$$

where $*$ is the torsion product of abelian groups.

In particular, when SG_2 is a trivial graph, i.e. the graph with exactly one vertex, it is easy to find that $H^i(SG_2) = H^i_b(SG_2) = \mathbb{Z} \oplus \mathbb{Z}x$. It follows that

$$H^i(SG_1 \sqcup SG_2) \cong H^i(SG_1) \otimes (\mathbb{Z} \oplus \mathbb{Z}x),$$

$$H^i_b(SG_1 \sqcup SG_2) \cong H^i_b(SG_1) \otimes (\mathbb{Z} \oplus \mathbb{Z}x).$$

5.3 Vertex switching operation

Recall that two signed graphs are equivalent if they are related by a sequence of vertex switchings. As we mentioned before, equivalent signed graphs have the same signed chromatic polynomial. The following result tells us that vertex switching not only preserves the signed chromatic polynomial, but also the signed chromatic cohomology groups.

Proposition 32. *If two signed graphs SG_1, SG_2 are equivalent, then $H^i(SG_1) \cong H^i(SG_2)$ and $H^i_b(SG_1) \cong H^i_b(SG_2)$.*

Proof. It suffices to consider the case that SG_2 is obtained from SG_1 by applying vertex switching on a vertex $v \in V(SG_1)$. Notice that for any $s \subseteq E(SG_1)$, a component in $[SG_1 : s]$ is balanced if and only if the corresponding component in $[SG_2 : s]$ is also balanced. This induces a cochain map from $C^\bullet(SG_1)$ to $C^\bullet(SG_2)$ and another cochain map from $C^\bullet_b(SG_1)$ to $C^\bullet_b(SG_2)$, both of which induce isomorphisms between the cohomology groups. □

5.4 Contracting a pendant edge

Definition 33. Let SG be a signed graph, suppose $v \in V(SG)$ is a vertex of degree one. We call the edge incident with v a *pendant edge* of SG.

For a given signed graph SG and a positive pendant edge e (one can apply vertex switching on v if necessary), Proposition 2 tells us that

$$P_{SG}(\lambda) = P_{SG-e}(\lambda) - P_{SG/e}(\lambda) = (\lambda - 1)P_{SG/e}(\lambda).$$

The following proposition can be seen as a categorification of this result.

Proposition 34. *Let e be a pendant edge in a signed graph SG. For each i, we have $H^i(SG) \cong H^i(SG/e)\{1\}$ and $H^i_b(SG) \cong H^i_b(SG/e)\{1\}$.*

Proof. We only prove $H^i(SG) \cong H^i(SG/e)\{1\}$, the balanced version can be proved analogously. By switching v if necessary, we assume the sign of e is positive and e is the first edge. The key observation is, for any $s \subseteq E(SG)$ the pendant edge e has no effect on the balanced components of $[SG : s]$. The main idea of the proof is similar to the unsigned case [6]. We sketch the outline here.

According to Corollary 17, we have the following long exact sequence

$$\cdots \to H^{i-1}(SG/e) \to H^i(SG) \to H^i(SG - e) \to H^i(SG/e) \to \cdots.$$

On the other hand, since $SG - e = SG/e \cup \{v\}$, Proposition 31 tells us that

$$H^i(SG - e) \cong H^i(SG/e) \otimes (\mathbb{Z} \oplus \mathbb{Z}x) \cong H^i(SG/e) \oplus H^i(SG/e)\{1\}.$$

By identifying $H^i(SG - e)$ with $H^i(SG/e) \oplus H^i(SG/e)\{1\}$, it suffices to show that the map $\gamma^* : H^i(SG/e) \oplus H^i(SG/e)\{1\} \to H^i(SG/e)$ sends $(x, 0)$ to x.

In fact, for any $x = [\sum_i a_i S_i] \in H^i(SG/e)$, where $S_i = (s_i, c_i)$, we extend each S_i to be an enhanced state in $C^i(SG - e)$ by adding an isolated vertex v with color 1. Then the map γ^* sends each $[(s_i, c_i)]$ to $[(s_i \cup \{e\})/e, (c_i)_e]$. Notice that adding a positive pendant edge preserves the balance of each component. On the other hand, since v is colored by 1 and multiplication by 1 is just the identity map, it follows that $\gamma^*((x, 0)) = x$.

Therefore γ^* is surjective and hence the long exact sequence splits into infinitely many short exact sequences

$$0 \to H^i(SG) \to H^i(SG/e) \oplus H^i(SG/e)\{1\} \xrightarrow{\gamma^*} H^i(SG/e) \to 0.$$

It follows from Lemma 3.10 in [6] that $H^i(SG) \cong H^i(SG/e)\{1\}$. □

5.5 Loops and parallel edges

We first discuss the effect of positive/negative loops on the signed chromatic cohomology. Similarly to the fact that signed graphs with positive loops have zero signed chromatic polynomial, positive loops also kill the signed chromatic cohomology.

Proposition 35. *If a signed graph SG has a positive loop e, then $H^i(SG) = H^i_b(SG) = 0$.*

Proof. The assumption e is a positive loop implies that $SG/e = SG - e$. By investigating the map γ^* in the following long exact sequence

$$\cdots \to H^{i-1}(SG/e) \to H^i(SG) \to H^i(SG - e) \xrightarrow{\gamma^*} H^i(SG/e) \to H^{i+1}(SG) \to \cdots,$$

it is not difficult to find that γ^* is an isomorphism. We conclude that $H^i(SG) = 0$. The unbalanced case can be proved similarly. □

Proposition 36. *If a signed graph SG has a negative loop e, then $H^i_b(SG) = H^i_b(SG-e)$.*

Proof. Since e is negative, Corollary 27 does not work here. However, if we go back to the cochain complex, we have the following decomposition

$$C^i_b(SG) = \bigoplus_{e \notin s \subseteq E(SG), |s|=i} M^b_s(SG) \oplus \bigoplus_{e \in s \subseteq E(SG), |s|=i} M^b_s(SG).$$

If a subset $s \subseteq E(SG)$ includes the negative loop e, then $[SG : s]$ is unbalanced, therefore the associated $M^b_s(SG) = 0$. It follows that the second summand above vanishes and the result follows immediately. □

Now we turn to discuss the effect of parallel edges on signed chromatic cohomology. Recall that two edges join the same pair of vertices, then these two edges are called *parallel edges*. Here we allow the two endpoints coincide with each other.

Proposition 37. *Let SG be a signed graph, and $e, e' \in E(SG)$ are a pair of parallel edges with the same sign. Then we have $H^i(SG) = H^i(SG - e')$ and $H^i_b(SG) = H^i_b(SG - e')$. In other words, the signed chromatic cohomology groups are unchanged if one replaces the parallel edges with the same sign by a single one.*

Proof. If both e and e' are positive, we divide our discussion into two cases.

- The two endpoints of e and e' coincide, i.e. both e and e' are positive loops. In this case, the result follows from Proposition 35.

- The two endpoints of e and e' are distinct. Then e becomes a positive loop in SG/e', hence we obtain $H^i(SG/e') = H^i_b(SG/e') = 0$. It follows from the two long exact sequences that $H^i(SG) = H^i(SG - e')$ and $H^i_b(SG) = H^i_b(SG - e')$.

If both e and e' are negative, there are also two situations.

- Both e and e' are not loops. By switching one of the two endpoints of e and e' we obtain two positive parallel edges, which has been discussed above.

- Both e and e' are loops. The balanced case $H^i_b(SG) = H^i_b(SG - e')$ follows directly from Proposition 36. The rest of the proof is devoted to show that $H^i(SG) = H^i(SG - e')$. Suppose both e and e' connects $v \in V(SG)$ to itself. We define a new signed graph SG' by splitting v into two vertices, say v_1, v_2, and adding a new positive edge e_0 which connects v_1 and v_2. All edges incident to v in SG are now incident to v_1 in SG'. See Figure 5.

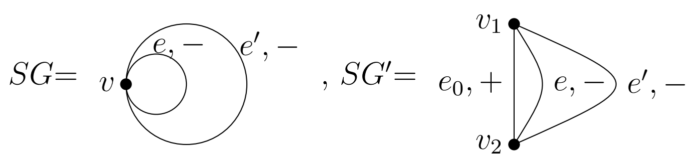


Figure 5: Adding a new vertex

Now we have $SG = SG'/e_0, SG - e' = (SG' - e')/e_0$ and e, e' are negative edges connecting v_1 and v_2 in SG' and $SG' - e_0$. According to our previous discussion, if we switch v_2, delete e' and then switch v_2 again, these operations induce isomorphisms $H^i(SG') \cong H^i(SG' - e')$ and $H^i(SG' - e_0) \cong H^i(SG' - e_0 - e')$. The homomorphism $\delta^* : H^i(SG'/e_0) \to H^i((SG' - e')/e_0)$ in the commutative diagram below can be defined as follows. Given an enhanced state $S = (s, c)$ of SG'/e_0, if $e' \notin s$, then we define $\delta(S) = S$, which is also an enhanced state of $(SG' - e')/e$. Otherwise, we define $\delta(S) = 0$. This map makes the diagram below commutative and induces a homomorphism $\delta^* : H^i(SG'/e_0) \to H^i((SG' - e')/e_0)$.

$$\begin{array}{ccccccccc}
H^i(SG') & \longrightarrow & H^i(SG' - e_0) & \longrightarrow & H^i(SG'/e_0) & \longrightarrow & H^{i+1}(SG') & \longrightarrow & H^{i+1}(SG' - e_0) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \delta^* & & \downarrow \cong & & \downarrow \cong \\
H^i(SG' - e') & \longrightarrow & H^i(SG' - e' - e_0) & \longrightarrow & H^i((SG' - e')/e_0) & \longrightarrow & H^{i+1}(SG' - e') & \longrightarrow & H^{i+1}(SG' - e' - e_0)
\end{array}$$

We conclude that

$$H^i(SG) \cong H^i(SG'/e_0) \cong H^i((SG' - e')/e_0) \cong H^i(SG - e'),$$

the fact that δ^* is an isomorphism is derived from the five lemma. The proof is finished. □

5.6 Trees and polygon graphs

As an application of the properties discussed above, we describe the signed chromatic cohomology groups for some classes of signed graphs.

Example 38. Let SN_m^n be the signed graph with m vertices, on n of which there is a negative loop. When $m = n = 1$, we can calculate the signed chromatic cohomology groups for SN_1^1 as follows.

$$\begin{array}{ll} C^0(SN_1^1) = \mathrm{span}\langle (0,1),(0,x)\rangle & C^1(SN_1^1) = \mathrm{span}\langle (1,1)\rangle \\ B^0(SN_1^1) = 0 & B^1(SN_1^1) = \mathrm{span}\langle (1,1)\rangle \\ Z^0(SN_1^1) = \mathrm{span}\langle (0,x)\rangle & Z^1(SN_1^1) = \mathrm{span}\langle (1,1)\rangle \\ H^0(SN_1^1) = Z^0(SN_1^1)/B^0(SN_1^1) \cong \mathbb{Z}x & H^1(SN_1^1) = Z^1(SN_1^1)/B^1(SN_1^1) \cong 0 \\ C_b^0(SN_1^1) = \mathrm{span}\langle (0,1),(0,x)\rangle & C_b^1(SN_1^1) = 0 \\ B_b^0(SN_1^1) = 0 & B_b^1(SN_1^1) = 0 \\ Z_b^0(SN_1^1) = \mathrm{span}\langle (0,1),(0,x)\rangle & Z_b^1(SN_1^1) = 0 \\ H_b^0(SN_1^1) = Z_b^0(SN_1^1)/B_b^0(SN_1^1) \cong \mathbb{Z}\oplus\mathbb{Z}x & H_b^1(SN_1^1) = Z_b^1(SN_1^1)/B_b^1(SN_1^1) \cong 0 \end{array}$$

Then Proposition 31 implies that

$$H^i(SN_m^n) \cong \begin{cases} (\mathbb{Z}\oplus\mathbb{Z}\{1\})^{\otimes(m-n)} \otimes (\mathbb{Z}\{1\})^{\otimes n}, & i = 0; \\ 0, & i \geqslant 1. \end{cases}$$

$$H_b^i(SN_m^n) \cong \begin{cases} (\mathbb{Z}\oplus\mathbb{Z}\{1\})^{\otimes m}, & i = 0; \\ 0, & i \geqslant 1. \end{cases}$$

We remark that the fact $H_b^i(SN_m^n) \cong H_b^i(SN_m^0)$ also can be deduced from Proposition 36.

Example 39. Let $ST_n = (T_n, \sigma)$, where T_n is a tree with n edges. By repeatedly using Proposition 34 one obtains

$$H^i(ST_n) \cong H_b^i(ST_n) \cong \begin{cases} \mathbb{Z}\{n\}\oplus\mathbb{Z}\{n+1\}, & i = 0; \\ 0, & i \geqslant 1. \end{cases}$$

Example 40. Let $SP_n = (P_n, \sigma)$ be an unbalanced polygon graph with n edges. When $n = 1$, the case $SP_1 = SN_1^1$ has been discussed in Example 38. For this reason, next let us assume $n \geqslant 2$.

We remark that although unbalanced polygon graphs on the unsigned polygon graph P_n are not unique, they are all equivalent. In order to see this, first notice that two unbalanced polygon graphs with only one negative edge are equivalent. If an unbalanced polygon graph has more than three negative edges, choose two of them such that we can find a positive path connecting them. By applying vertex switching on the endpoints of this positive path we obtain a new unbalanced polygon graph with two negative edges less.

We label the vertices of SP_n by $v_1, v_2, \dots, v_n$ monotonically so that each v_i is adjacent to $v_{i\pm1}$ $(1 \leqslant i \leqslant n)$, where $v_0 = v_n$ and $v_{n+1} = v_1$. Let e be a positive edge connecting v_1 and v_n, the $SP_n/e = SP_{n-1}$ and $SP_n - e$ is a tree. Then we have the following long exact sequence

$$\cdots \to H^{i-1}(SP_n - e) \to H^{i-1}(SP_n/e) \to H^i(SP_n) \to H^i(SP_n - e) \to \cdots.$$

As $SP_n - e$ is a tree, we have $H^i(SP_n - e) = 0$ for all $i \geqslant 1$. Thus for any $i \geqslant 2$, we have $H^i(SP_n) \cong H^{i-1}(SP_n/e) \cong H^{i-1}(SP_{n-1})$, and it follows that

$$H^i(SP_n) \cong \begin{cases} H^1(SP_{n-i+1}), & \text{if } 2 \leqslant i \leqslant n; \\ 0, & \text{if } i > n. \end{cases}$$

Then for each $n \geqslant 2$, $H^n(SP_n) \cong H^1(SP_1) \cong 0$, $H^{n-1}(SP_n) \cong H^2(SP_3) \cong \mathbb{Z}\{1\}$, which has been calculated in Example 15.

On the other hand, it has been calculated in [6] that

$$\text{For } i = 0,\ H^0(P_n) \cong \begin{cases} \mathbb{Z}\{n\} \oplus \mathbb{Z}\{n-1\} & \text{if } n \text{ is even and } n \geqslant 2; \\ \mathbb{Z}\{n\} & \text{if } n \text{ is odd and } n \geqslant 2; \\ 0 & \text{if } n = 1. \end{cases}$$

$$\text{For } i > 0,\ H^i(P_n) \cong \begin{cases} \mathbb{Z}_2\{n-i\} \oplus \mathbb{Z}\{n-i-1\} & \text{if } n-i \geqslant 2 \text{ and } n \text{ is even}; \\ \mathbb{Z}\{n-i\} & \text{if } n-i \geqslant 2 \text{ and } n \text{ is odd}; \\ 0 & \text{if } n-i \leqslant 1. \end{cases}$$

And using Proposition 30, we have $H^i(SP_n) = H^i(P_n)$ for all $i \leqslant n-2$, then we obtain all the cohomology groups.

$$\text{For } i = 0,\ H^0(SP_n) \cong \begin{cases} \mathbb{Z}\{n\} \oplus \mathbb{Z}\{n-1\} & \text{if } n \text{ is even and } n \geqslant 2; \\ \mathbb{Z}\{n\} & \text{if } n \text{ is odd and } n \geqslant 2; \\ \mathbb{Z}\{1\} & \text{if } n = 1. \end{cases}$$

$$\text{For } i > 0,\ H^i(SP_n) \cong \begin{cases} \mathbb{Z}_2\{n-i\} \oplus \mathbb{Z}\{n-i-1\} & \text{if } i \leqslant n-2 \text{ and } n \text{ is even}; \\ \mathbb{Z}\{n-i\} & \text{if } i \leqslant n-2 \text{ and } n \text{ is odd}; \\ \mathbb{Z}\{1\} & \text{if } i = n-1; \\ 0 & \text{if } i \geqslant n. \end{cases}$$

We work out the balanced cohomology groups parallelly.

$$\text{For } i = 0,\ H_b^0(SP_n) \cong \begin{cases} \mathbb{Z}\{n\} \oplus \mathbb{Z}\{n-1\} & \text{if } n \text{ is even and } n \geqslant 2; \\ \mathbb{Z}\{n\} & \text{if } n \text{ is odd and } n \geqslant 2; \\ \mathbb{Z}\{1\} \oplus \mathbb{Z} & \text{if } n = 1. \end{cases}$$

$$\text{For } i > 0,\ H_b^i(P_n) \cong \begin{cases} \mathbb{Z}_2\{n-i\} \oplus \mathbb{Z}\{n-i-1\} & \text{if } i \leqslant n-2 \text{ and } n \text{ is even}; \\ \mathbb{Z}\{n-i\} & \text{if } i \leqslant n-2 \text{ and } n \text{ is odd}; \\ \mathbb{Z}\{1\} \oplus \mathbb{Z} & \text{if } i = n-1; \\ 0 & \text{if } i \geqslant n. \end{cases}$$

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