On the maximum of the weighted binomial sum $2^{-r}\sum_{i=0}^{r}\binom{m}{i}$

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Abstract

The weighted binomial sum $f_m(r) = 2^{-r}\sum_{i=0}^{r}\binom{m}{i}$ arises in coding theory and information theory. We prove that, for $m \notin \{0, 3, 6, 9, 12\}$, the maximum value of $f_m(r)$ with $0 \leqslant r \leqslant m$ occurs when $r = \lfloor m/3 \rfloor + 1$. We also show this maximum value is asymptotic to $\frac{3}{\sqrt{\pi m}}\left(\frac{3}{2}\right)^m$ as $m \to \infty$.

Mathematics Subject Classifications: 05A10, 11B65, 94B65

1 Introduction

Let m be a non-negative integer, and let $f_m(r)$ be the function:

$$f_m(r) = \frac{1}{2^r}\sum_{i=0}^{r}\binom{m}{i}.$$

This function arises in coding theory and information theory e.g. [2, Theorem 4.5.3]. It is desirable for a linear code to have large rate (to communicate a lot of information) and large minimal distance (to correct many errors). So for a linear code with parameters

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$[n, k, d]$, one wants both k/n and d/n to be large. The case that kd/n is large is studied in [1]. A Reed-Muller code $\mathrm{RM}(r, m)$ has $n = 2^m$, $k = \sum_{i=0}^{r} \binom{m}{i}$ and $d = 2^{m-r}$ by [4, §6.2], and hence kd/n equals $f_m(r)$. It is natural to ask which value of r maximizes $f_m(r)$, and what is the size of the maximum value.

Theorem 1. *Suppose that m, r are integers where $0 \leqslant r \leqslant m$. The maximum value of $f_m(r) = 2^{-r} \sum_{i=0}^{r} \binom{m}{i}$ occurs when $r = \lfloor \frac{m}{3} \rfloor + 1$ provided $m \notin \{0, 3, 6, 9, 12\}$.*

We give an optimal asymptotic bound for the maximum value of $f_m(r)$.

Theorem 2. *Suppose that $m \notin \{0, 1, 3, 6, 9, 12\}$ and $r_0 = \lfloor \frac{m}{3} \rfloor + 1$. Then*

$$\frac{1}{2^{\lfloor \frac{m}{3} \rfloor}} \left(1 - \frac{k+2}{2(r_0+1)}\right) \binom{m}{r_0} < f_m(r_0) < \frac{1}{2^{\lfloor \frac{m}{3} \rfloor}} \binom{m}{r_0} \tag{1}$$

where $k := 3r_0 - m \in \{1, 2, 3\}$. Furthermore,

$$f_m(r_0) < \frac{3}{\sqrt{\pi m}} \left(\frac{3}{2}\right)^m \quad \text{and} \quad \lim_{m \to \infty} f_m(r_0) \sqrt{m} \left(\frac{2}{3}\right)^m = \frac{3}{\sqrt{\pi}}. \tag{2}$$

We prove that $f_m(r)$ increases strictly if $0 \leqslant r \leqslant r_0 := \lfloor \frac{m}{3} \rfloor + 1$ and $m > 12$ (see Theorem 6), and it decreases strictly for $r_0 \leqslant r \leqslant m$ (see Theorem 8). Elementary arguments in Lemma 4(c) show that $f_m(0) < f_m(1) < \cdots < f_m(r_0 - 1)$. More work is required to prove that $f_m(r_0 - 1) < f_m(r_0)$. Determining when $f_m(r)$ decreases involves a delicate inductive proof requiring a growing amount of precision, and inequalities with rational functions such as $X_i = \frac{r-i+1}{m-r+i}$, see Lemma 5. In Section 5 we establish bounds (and asymptotic behavior) for $f_m(r_0)$ using standard methods.

Brendan McKay [3] showed, using approximations for sufficiently large m, that the maximum value of $f_m(r)$ is near $m/3$. His method may well extend to a proof of Theorem 1. If so, it would involve very different techniques from ours.

2 Data, comparisons and strategies

The values of $f_m(0), f_m(1), f_m(2), \ldots, f_m(m-2), f_m(m-1), f_m(m)$ appear to increase to a maximum and then decrease. For 'large' m we see that

$$1 < \frac{m+1}{2} < \frac{m^2+m+2}{8} < \cdots ? \cdots > 8 - \frac{m^2+m+2}{2^{m-2}} > 4 - \frac{m+1}{2^{m-2}} > 2 - \frac{1}{2^{m-1}} > 1.$$

Computer calculations for 'large' m suggest that a maximum value for $f_m(r)$ occurs at $r_0 = \lfloor \frac{m}{3} \rfloor + 1$, see Table 1 which lists the *integer part* $\lfloor f_m(r) \rfloor$. Computing $f_m(r)$ exactly shows that for $m \in \{0, 3, 6, 9, 12\}$ the maximum occurs at $r_0 - 1$ and not r_0, see Table 2. The maximum happens to occur for a unique r, except for $m = 1$.

Determining the relative sizes of $f_m(r)$ and $f_m(r + 1)$ is reduced in Lemma 3 to determining the relative sizes of $\sum_{i=0}^{r} \binom{m}{i}$ and $\binom{m}{r+1}$.

Table 1: Maximum values of $\lfloor f_m(r) \rfloor$ for $0 \leqslant r \leqslant m$ and $m \in \{6, 7, \ldots, 15\}$.

6	1	3	5	5	3	1	1									
7	1	4	7	8	6	3	1	1								
8	1	4	9	11	10	6	3	1	1							
9	1	5	11	16	16	11	7	3	1	1						
10	1	5	14	22	24	19	13	7	3	1	1					
11	1	6	16	29	35	32	23	14	7	3	1	1				
12	1	6	19	37	49	49	39	25	14	7	3	1	1			
13	1	7	23	47	68	74	64	45	27	15	7	3	1	1		
14	1	7	26	58	91	108	101	77	50	29	15	7	3	1	1	
15	1	8	30	72	121	154	155	128	89	54	30	15	7	3	1	1

Lemma 3. *Suppose that* $0 \leqslant r < m$. *Then*

(a) *the inequality* $f_m(r) < f_m(r+1)$ *is equivalent to* $\sum_{i=0}^{r} \binom{m}{i} < \binom{m}{r+1}$,

(b) *if* $\sum_{i=0}^{r} \binom{m}{i} \leqslant \binom{m}{r+1}$, *then* $\sum_{i=0}^{r} \binom{m+1}{i} < \binom{m+1}{r+1}$,

(c) *the inequality* $f_m(r) > f_m(r+1)$ *is equivalent to* $\sum_{i=0}^{r} \binom{m}{i} > \binom{m}{r+1}$, *and*

(d) *if* $\sum_{i=0}^{r} \binom{m}{i} \geqslant \binom{m}{r+1}$, *then* $\sum_{i=0}^{r} \binom{m-1}{i} > \binom{m-1}{r+1}$.

Proof. (a,b) Clearly $f_m(r) < f_m(r+1)$ is equivalent to $2\sum_{i=0}^{r} \binom{m}{i} < \sum_{i=0}^{r+1} \binom{m}{i}$ which is equivalent to $\sum_{i=0}^{r} \binom{m}{i} < \binom{m}{r+1}$. If $r < m$ and $\sum_{i=0}^{r} \binom{m}{i} \leqslant \binom{m}{r+1}$, then

$$\begin{aligned}\sum_{i=0}^{r} \binom{m+1}{i} &= \sum_{i=0}^{r} \frac{m+1}{m-i+1} \binom{m}{i} \leqslant \frac{m+1}{m-r+1} \sum_{i=0}^{r} \binom{m}{i} \\ &< \frac{m+1}{m-r} \binom{m}{r+1} = \binom{m+1}{r+1}.\end{aligned}$$

(c,d) Clearly $f_m(r) > f_m(r+1)$ is equivalent to $2\sum_{i=0}^{r} \binom{m}{i} > \sum_{i=0}^{r+1} \binom{m}{i}$ which, in turn, is equivalent to $\sum_{i=0}^{r} \binom{m}{i} > \binom{m}{r+1}$. If $\sum_{i=0}^{r} \binom{m}{i} \geqslant \binom{m}{r+1}$, then as $m > r \geqslant 0$,

$$\begin{aligned}\sum_{i=0}^{r} \binom{m-1}{i} &= \sum_{i=0}^{r} \frac{m-i}{m} \binom{m}{i} \geqslant \frac{m-r}{m} \sum_{i=0}^{r} \binom{m}{i} \\ &> \frac{m-r-1}{m} \binom{m}{r+1} = \binom{m-1}{r+1}.\end{aligned}$$

□

The following easy lemma elucidates which $r \in \{0, \ldots, m\}$ maximize $f_m(r)$.

Lemma 4. *Let $s_m(m+1) = 2^m$, and for $0 \leqslant r \leqslant m$ define*

$$s_m(r) = \sum_{i=0}^{r} \binom{m}{i}, \qquad t_m(r) = \frac{s_m(r+1)}{s_m(r)}, \quad \text{and} \quad c_m(r) = \frac{\binom{m}{r+1}}{\binom{m}{r}} = \frac{m-r}{r+1}.$$

(a) *If $0 \leqslant r \leqslant m$, then $c_m(r) < t_m(r)$, and if $0 \leqslant r < m$, then $t_m(r+1) < t_m(r)$.*
(b) *If $m \geqslant 2$, then for some r^*, $f_m(0) < \cdots < f_m(r^*)$ and $f_m(r^*+1) > \cdots > f_m(m)$.*
(c) $\max\{f_m(0), \ldots, f_m(m)\} = \max\{f_m(r^*), f_m(r^*{+}1)\}$ *and* $f_m(0) < \cdots < f_m(r_0{-}1)$.

Proof. (a) We show $c_m(r) < t_m(r)$ via induction on r. This is true when $r = 0$ as $c_m(0) = m < m+1 = t_m(0)$. Suppose that $0 \leqslant r < m$ and $c_m(r) < t_m(r)$ holds. That is, $\binom{m}{r+1}/\binom{m}{r} < s_m(r+1)/s_m(r)$ holds. Since $c_m(r+1) = \frac{m-r-1}{r+2} < \frac{m-r}{r+1} = c_m(r)$ we have $c_m(r+1) < c_m(r) < t_m(r)$. Using properties of mediants, it follows that

$$c_m(r+1) = \frac{\binom{m}{r+2}}{\binom{m}{r+1}} < \frac{\binom{m}{r+2} + s_m(r+1)}{\binom{m}{r+1} + s_m(r)} < \frac{s_m(r+1)}{s_m(r)} = t_m(r).$$

Hence $c_m(r+1) < t_m(r+1) < t_m(r)$ as $s_m(n+1) = \binom{m}{n+1} + s_m(n)$. This completes the induction, and it also proves that $t_m(r+1) < t_m(r)$, as claimed.

(b) Since $s_m(m+1) = 2^m$, part (a) shows that $1 = t_m(m) < \cdots < t_m(0) = m+1$. Choose an integer r^* such that $t_m(r^*) \leqslant 2 < t_m(r^*-1)$. The following are equivalent: $2 < t_m(r)$; $2s_m(r) < s_m(r+1)$; $f_m(r) < f_m(r+1)$. Thus $2 < t_m(r^*-1) < \cdots < t_m(0)$ implies $f_m(0) < \cdots < f_m(r^*)$. Similarly, $t_m(m-1) < \cdots < t_m(r^*+1) < 2$ and $t_m(r) < 2$ implies $f_m(r+1) < f_m(r)$. Hence $f_m(r^*+1) > \cdots > f_m(m)$.

(c) By part (b), $\max\{f_m(r) \mid 0 \leqslant r \leqslant m\} = \max\{f_m(r^*), f_m(r^*+1)\}$. If $2 \leqslant c_m(r) = \frac{m-r}{r+1}$, then $3r+2 \leqslant m$ and $r \leqslant \lfloor \frac{m-2}{3} \rfloor$. Hence $2 \leqslant c_m(r) < t_m(r)$ by part (a), and $\lfloor \frac{m-2}{3} \rfloor \leqslant r^* - 1$ by the definition of r^*. Thus $r_0 - 1 = \lfloor \frac{m}{3} \rfloor \leqslant r^*$ and it follows from part (b) that $f_m(0) < \cdots < f_m(r_0-1)$. □

Fix m and r where $0 \leqslant r < m$. We shall use the following notation:

$$X_i = \frac{r-i+1}{m-r+i} \qquad \text{for } 0 \leqslant i \leqslant r, \tag{3}$$

$$S_j = 1 + X_{j+1} + X_{j+1}X_{j+2} + \cdots + X_{j+1}X_{j+2}\cdots X_r \qquad \text{for } 0 \leqslant j < r, \tag{4}$$

$$T_j = 1 + X_1 + X_1X_2 + \cdots + X_1X_2\cdots X_j \qquad \text{for } 0 \leqslant j \leqslant r. \tag{5}$$

Our convention in (5) is that $T_0 = 1$ as $T_j = \sum_{i=0}^{j}(\prod_{k=1}^{i} X_k)$ equals 1 when $j = 0$.

Lemma 5. *Fix m, r, j where $0 \leqslant j \leqslant r < m$. Using the above definitions,*

(a) *the inequality $\sum_{i=0}^{j} \binom{m}{r-i} > \binom{m}{r+1}$ is equivalent to $T_j > X_0^{-1}$,*

(b) *the inequality $\sum_{i=0}^{r} \binom{m}{i} < \binom{m}{r+1}$ is equivalent to $S_0 < X_0^{-1}$.*

Proof. For $0 \leqslant i \leqslant r$, we have $\binom{m}{r-i} = X_i \binom{m}{r-i+1}$ so $\binom{m}{r-i} = (\prod_{k=1}^{i} X_k)\binom{m}{r}$ holds. Therefore $\sum_{i=0}^{j} \binom{m}{r-i} = \binom{m}{r} \sum_{i=0}^{j} \left(\prod_{k=1}^{i} X_k\right) = \binom{m}{r} T_j$. Since $\binom{m}{r} = X_0 \binom{m}{r+1}$, the inequality $\sum_{i=0}^{j} \binom{m}{r-i} > \binom{m}{r+1}$ is equivalent to $\binom{m}{r} T_j > X_0^{-1}\binom{m}{r}$ which is equivalent to $T_j > X_0^{-1}$. This proves part (a).

Note that $\sum_{i=0}^{r} \binom{m}{i} = \sum_{i=0}^{r} \binom{m}{r-i} = \binom{m}{r} T_r = \binom{m}{r} S_0$ since $S_0 = T_r$. Since $\binom{m}{r+1} = X_0^{-1}\binom{m}{r}$, the inequality $\sum_{i=0}^{r} \binom{m}{i} < \binom{m}{r+1}$ is equivalent to $\binom{m}{r} S_0 < X_0^{-1}\binom{m}{r}$ which is equivalent to $S_0 < X_0^{-1}$. This proves part (b). □

3 Proof that $f_m(r)$ is increasing for $0 \leqslant r \leqslant r_0$

Recall that $m \geqslant 0$ and $r_0 := \lfloor \frac{m}{3} \rfloor + 1$. We now strengthen Lemma 4(c).

Theorem 6. *If $m \notin \{0, 1, 3, 6, 9, 12\}$, then $f_m(0) < f_m(1) < \cdots < f_m(r_0)$.*

Proof. The statement is easy to check for $m \in \{2, 4, 5\}$. The statement follows from Tables 1 and 2 for $m \in \{7, 8, 10, 11, 13, 14\}$. Suppose now that $m \geqslant 15$.

Table 2: $\square = \max\{f_m(r_0 - 1), f_m(r_0)\}$ for $0 \leqslant m \leqslant 12$, $r_0 = \lfloor m/3 \rfloor + 1$.

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$f_m(r_0-1)$	$\boxed{1}$	$\boxed{1}$	1	$\boxed{2}$	$\frac{5}{2}$	3	$\boxed{\frac{11}{2}}$	$\frac{29}{4}$	$\frac{37}{4}$	$\boxed{\frac{65}{4}}$	22	29	$\boxed{\frac{397}{8}}$
$f_m(r_0)$	$\frac{1}{2}$*	$\boxed{1}$	$\boxed{\frac{3}{2}}$	$\frac{7}{4}$	$\boxed{\frac{11}{4}}$	$\boxed{4}$	$\frac{21}{4}$	$\boxed{8}$	$\boxed{\frac{93}{8}}$	16	$\boxed{\frac{193}{8}}$	$\boxed{\frac{281}{8}}$	$\frac{793}{16}$

Recall that $r_0 = \lfloor \frac{m}{3} \rfloor + 1$ and $m \in \{3r_0 - 3, 3r_0 - 2, 3r_0 - 1\}$. By Lemma 4(c) it suffices to show that $f_m(r_0 - 1) < f_m(r_0)$. If we prove this for $m = 3r_0 - 3$, Lemma 3(b,a) gives it for $m = 3r_0 - 2$ and $m = 3r_0 - 1$ as well, so for $r_0 \geqslant 6$ we want to show $f_{3r_0-3}(r_0 - 1) < f_{3r_0-3}(r_0)$. This is true for $r_0 = 6$ by Table 1. We set $t := r_0 - 1$, $m := 3t$ and we prove, using induction on t, that $f_{3t}(t) < f_{3t}(t + 1)$ holds for all $t \geqslant 6$.

Note that $f_{3t}(t) < f_{3t}(t + 1)$ is equivalent by Lemma 3(a) to $\sum_{i=0}^{t} \binom{3t}{i} < \binom{3t}{t+1}$, and this is equivalent to $S_0 < X_0^{-1}$ by Lemma 5(b). Putting $m = 3t$ and $r = t$ in (3), gives $X_i = \frac{t-i+1}{2t+i}$ and $S_0 = 1 + X_1 + X_1X_2 + \cdots + X_1X_2 \cdots X_t$ by (4).

It follows from $0 < X_t < \cdots < X_5 < X_4$ and $X_4 = \frac{t-3}{2t+4} < \frac{1}{2}$ that

$$S_3 = 1 + X_4 + X_4X_5 + \cdots + X_4X_5 \cdots X_t < 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{t-3}} < \sum_{i=0}^{\infty} \frac{1}{2^i} = 2.$$

The recurrence relation $S_j = 1 + X_{j+1}S_{j+1}$ for $0 \leqslant j < t$ implies that

$$\begin{aligned} S_0 &= 1 + X_1 \left(1 + X_2 \left(1 + X_3S_3\right)\right) < 1 + X_1 \left(1 + X_2 \left(1 + 2X_3\right)\right) \\ &= 1 + \frac{t}{2t+1} \left(1 + \frac{t-1}{2t+2} \left(1 + \frac{2(t-2)}{2t+3}\right)\right). \end{aligned}$$

*Observe that $f_0(1) = 2^{-1}\left(\binom{0}{0}+\binom{0}{1}\right) = 2^{-1}(1+0) = \frac{1}{2}$.

We aim to show that $S_0 < X_0^{-1}$. It suffices to prove $1 + X_1(1 + X_2(1 + 2X_3)) \leqslant X_0^{-1}$ where $X_0 = \frac{t+1}{2t}$. This amounts to proving that

$$1 + \frac{t}{2t+1}\left(1 + \frac{t-1}{2t+2}\left(1 + \frac{2(t-2)}{2t+3}\right)\right) \leqslant \frac{2t}{t+1}.$$

Rearranging, and using the denominator $(2t+1)(2t+2)(2t+3)$, gives

$$0 \leqslant \frac{3t^2 - 17t - 6}{(2t+1)(2t+2)(2t+3)} = \frac{(3t+1)(t-6)}{(2t+1)(2t+2)(2t+3)}$$

This inequality is valid for all $t \geqslant 6$. This completes the proof. □

How might one prove a nice formula such as $\lim_{s\to\infty} \sum_{i=0}^{s} \binom{3s}{i} / \binom{3s}{s} = 2$?

Remark 7. For $s > 4$ set $m = 3s$ and $r_0 = s + 1$. Then $f_m(r_0 - 1) < f_m(r_0)$ by Theorem 6. Hence $\sum_{i=0}^{s} \binom{3s}{i} < \binom{3s}{s+1} = \frac{2s}{s+1}\binom{3s}{s}$ and so $\lim_{s\to\infty} \sum_{i=0}^{s} \binom{3s}{i} / \binom{3s}{s} \leqslant 2$. We show $f_m(r_0) > f_m(r_0 + 1)$ in Section 4, and therefore $\lim_{s\to\infty} \sum_{i=0}^{s} \binom{3s}{i} / \binom{3s}{s} \geqslant 2$.

4 Proof that $f_m(r)$ is decreasing for $r_0 \leqslant r \leqslant m$

Showing that $f_m(r)$ decreases strictly for $r_0 \leqslant r \leqslant m$ is much harder. Recall that $\binom{r}{i} = 0$ if $i < 0$, and $\binom{r}{i} = \frac{1}{i!}\prod_{j=0}^{i-1}(r - j)$ if $i \geqslant 0$. In this section we prove:

Theorem 8. *If $m \geqslant 2$, then $f_m(\lfloor m/3 \rfloor + 1) > f_m(\lfloor m/3 \rfloor + 2) > \cdots > f_m(m) = 1$.*

Our proof of Theorem 8 depends on two technical lemmas, the first of which proves that the non-leading coefficients of a certain polynomial $A(r)$ are all negative.

First define $B_i(r) = \prod_{\ell=1}^{i}(r-\ell)$. Now $\prod_{\ell=1}^{i}(r-\ell) = r^i + \sum_{k=0}^{i-1} b_{i,k} r^k$ and the coefficients $b_{i,k}$ alternate in sign: for $0 \leqslant k \leqslant i$, we have $b_{i,k} > 0$ if $i - k$ is even and $b_{i,k} < 0$ if $i - k$ is odd. Next define polynomials $A_i(r)$ via:

(6) $\quad A_2(r) = r^2 - 15r - 10 \quad$ and $\quad A_i(r) = (2r + i)A_{i-1}(r) - B_i(r) \quad$ for $i \geqslant 3$.

Clearly $\deg(A_i(r)) = i$ and we may write $A_i(r) = r^i + \sum_{k=0}^{i-1} a_{i,k} r^k$. We use $a_{i,i} = 1$.

Comparing coefficients in this recurrence and $B_i(r) = (r - i)B_{i-1}(r)$, shows that

(Ra) $\quad a_{2,0} = -10, \quad a_{2,1} = -15, \quad a_{i,k} = ia_{i-1,k} + 2a_{i-1,k-1} - b_{i,k} \quad$ for $i \geqslant 3$,

(Rb) $\quad b_{2,0} = 2, \quad b_{2,1} = -3, \quad b_{i,k} = -ib_{i-1,k} + b_{i-1,k-1} \quad$ for $i \geqslant 3$.

Lemma 9. *Let $a_{i,k}, b_{i,k}, A_i(r), B_i(r)$ be as above.*

(a) *If $i \geqslant 2$, then $b_{i,i-1} = -\binom{i+1}{2}$ and $a_{i,i-1} = -\binom{i+4}{2}$.*

(b) *If $i \geqslant 2$ and $0 \leqslant k \leqslant i-1$, then $a_{i,k} \leqslant -2b_{i,k} < 0$ if $i - k$ is even, and $a_{i,k} \leqslant b_{i,k} < 0$ if $i - k$ is odd.*

(c) *If $i \geqslant 2$, then the coefficients $a_{i,k}$ are negative for $0 \leqslant k < i$.*

Proof. (a) Clearly $b_{i,i-1} = -\sum_{j=1}^{i} j = -\binom{i+1}{2}$. The formula for $a_{i,i-1}$ holds for $i = 2$ and by induction using the recurrence (Ra).

(b) We use induction on i. For the base case $i = 2$, either $i - k$ is even and $a_{2,0} = -10 \leqslant -2b_{2,0} = -4$, or $i-k$ is odd and $a_{2,1} = -15 < b_{2,1} = -3$. Thus the claims are true for $i = 2$. Suppose now that $i \geqslant 3$, and the claims are valid for $i - 1$.

By part (a), $a_{i-1,i} = -\binom{i+4}{2} < -\binom{i+2}{2} = b_{i,i-1} < 0$ as claimed. It remains to consider k in the range $0 \leqslant k < i - 1$. It is useful to set $a_{i,-1} = b_{i,-1} = 0$. Suppose first that $i - k$ is even. Using the recurrences (Ra), (Rb) and induction gives

$$\begin{aligned}a_{i,k} &= i(a_{i-1,k} + b_{i-1,k}) + (2a_{i-1,k-1} - b_{i-1,k-1})\\ &\leqslant i(b_{i-1,k} + b_{i-1,k}) + (-4b_{i-1,k-1} - b_{i-1,k-1})\\ &\leqslant 2ib_{i-1,k} - 2b_{i-1,k-1} = -2b_{i,k} < 0.\end{aligned}$$

If $i - k$ is odd, then a similar argument gives

$$\begin{aligned}a_{i,k} &= i(a_{i-1,k} + b_{i-1,k}) + (2a_{i-1,k-1} - b_{i-1,k-1})\\ &\leqslant i(-2b_{i-1,k} + b_{i-1,k}) + (2b_{i-1,k-1} - b_{i-1,k-1})\\ &\leqslant -ib_{i-1,k} + b_{i-1,k-1} = b_{i,k} < 0.\end{aligned}$$

(c) This follows immediately from part (b). □

Lemma 10. *Suppose that $j \geqslant 4$. Then $\sum_{i=r-j}^{r} \binom{3r-1}{i} > \binom{3r-1}{r+1}$ holds for all r in the range $j \leqslant r \leqslant \binom{j+2}{2}$.*

Proof. We apply Lemma 5(a) with $m = 3r - 1$. Hence $X_i = \frac{r-i+1}{2r+i-1}$ by (3). Since $\sum_{i=r-j}^{r} \binom{m}{i} = \sum_{i=0}^{j} \binom{m}{r-i}$ it suffices by Lemma 5(a) to prove that

$$T_j = 1 + X_1 + X_1X_2 + \cdots + X_1X_2 \cdots X_j > X_0^{-1}.$$

We prove that this inequality holds for all r in the range $j \leqslant r \leqslant \binom{j+2}{2}$. This inequality is equivalent to

$$X_j > X_{j-1}^{-1}(\cdots(X_2^{-1}(X_1^{-1}(X_0^{-1} - 1) - 1) - 1)\cdots) - 1. \tag{7}$$

The right-side of (7) is a rational function in r, which when $j = 4$, equals

$$\frac{P_4(r)}{Q_4(r)} = \frac{2r+2}{r-2}\left(\frac{2r+1}{r-1}\left(\frac{2r}{r}\left(\frac{2r-1}{r+1} - 1\right) - 1\right) - 1\right) - 1$$

where the denominator is $Q_4(r) = (r + 1)r(r - 1)(r - 2)$, and the numerator is $P_4(r) = (r + 1)r(r^2 - 15r - 10)$. Since $\gcd(P_4(r), Q_4(r)) = (r + 1)r$, the polynomials $A_2(r) := r^2 - 15r - 10$ and $B_2(r) := (r - 1)(r - 2)$ are coprime. The putative inequality (7) when $j = 4$ is therefore

$$\frac{r-3}{2r+3} > \frac{P_4(r)}{Q_4(r)} = \frac{A_2(r)}{B_2(r)} = \frac{r^2 - 15r - 10}{(r-1)(r-2)}.$$

Observe that $A_2(r) < r^2 - 15r \leqslant 0$ for $4 \leqslant r \leqslant 15 = \binom{6}{2}$. Thus for r in the range $4 \leqslant r \leqslant \binom{6}{2}$, the left side of (7) is positive, and the right side is at most 0. Thus the inequality is valid for $4 \leqslant r \leqslant \binom{6}{2}$ and the claim is true for $j = 4$.

Assume now that $j > 4$, and that the claim is true for $j-1$. Therefore the inequality (7) can be written

$$X_j = \frac{r-j+1}{2r+j-1} > \frac{P_j(r)}{Q_j(r)} \qquad \text{where} \qquad \frac{P_j(r)}{Q_j(r)} = (X_{j-1})^{-1}\frac{P_{j-1}(r)}{Q_{j-1}(r)} - 1.$$

Since $(X_{j-1})^{-1} = \frac{2r+j-2}{r-j+2}$, this gives rise to the recurrences:

$$\begin{aligned} P_j(r) &= (2r+j-2)P_{j-1}(r) - (r-j+2)Q_{j-1}(r) && \text{for } j > 4,\\ Q_j(r) &= (r-j+2)Q_{j-1}(r) && \text{for } j > 4. \end{aligned}$$

It is clear that $Q_j(r) = (r+1)r(r-1)\cdots(r-j+2) = (r+1)rB_{j-2}(r)$ holds and $B_{j-2}(r)$ has degree $j-2$. Furthermore, $(r+1)r$ divides $\gcd(P_j(r), Q_j(r))$, so the polynomials $A_{j-2}(r)$, which are defined by the similar recurrence (6), satisfy $P_j(r) = (r+1)rA_{j-2}(r)$ and also have degree $j-2$.

By Lemma 9, $A_i(r) - r^i$ has negative coefficients and leading coefficient $-\binom{i+4}{2}$. So for $i \geqslant 2$ and $r \leqslant \binom{i+4}{2}$, we have $A_i(r) < r^i - \binom{i+4}{2}r^{i-1} \leqslant 0$. Further, $B_i(r) = \prod_{\ell=1}^{i}(r-\ell) > 0$ for $r \geqslant i+1$. Hence $A_i(r)/B_i(r) < 0$ for r satisfying $i+2 \leqslant r \leqslant \binom{i+4}{2}$. Suppose that $j = i+2$, then $P_j(r)/Q_j(r) < 0$ for r in the interval $j \leqslant r \leqslant \binom{j+2}{2}$. Using the definitions of $P_j(r), Q_j(r)$, the inequality (7) is the same as

$$X_j = \frac{r-j+1}{2r+j-1} > \frac{P_j(r)}{Q_j(r)} = \frac{A_{j-2}(r)}{B_{j-2}(r)}.$$

Thus for r satisfying $j \leqslant r \leqslant \binom{j+2}{2}$, the left side of (7) is positive, and the right side is negative. Thus the claim is valid for $j \leqslant r \leqslant \binom{j+2}{2}$. □

Proof of Theorem 8. It follows from $\sum_{i=0}^{m}\binom{m}{i} = 2^m$ that $f_m(m) = 1$. Since $r_0 := \lfloor m/3 \rfloor + 1$, we have $m \in \{3r_0-3, 3r_0-2, 3r_0-1\}$. If we can prove that $f_m(r_0) > f_m(r_0+1)$ for $m = 3r_0-1$, then $f_m(r_0) > f_m(r_0+1)$ holds for $m = 3r_0-2$ and $3r_0-3$ by Lemma 3(d). With the notation in Lemma 4, we have $2 > t_m(r_0)$ and hence $r^* \leqslant r_0$. Therefore $f_m(r_0+1) > \cdots > f_m(m)$ holds by Lemma 4(b).

In summary, it remains to prove $\sum_{i=0}^{r_0}\binom{3r_0-1}{i} > \binom{3r_0-1}{r_0+1}$ for $r_0 \geqslant 1$. This is true for $r_0 = 1, 2, 3$ since $\frac{3}{2} > 1$, $4 > \frac{13}{4}$ and $\frac{93}{8} > \frac{163}{16}$. For each $r_0 \geqslant 4$ set $j = r_0$. Then $j \geqslant 4$ and $\sum_{i=0}^{r_0}\binom{m}{i} > \binom{m}{r_0+1}$ follows by Lemma 10. This completes the proof. □

Proof of Theorem 1. The result follows from Theorems 6 and 8. There are two equal sized maxima if $m = 1$, otherwise the maximum is unique. □

5 Estimating $f_m(r_0)$

This section is devoted to proving asymptotically optimal bounds for $f_m(r_0)$.

Proof of Theorem 2. We first prove the upper bound in (1). This is true if $m = 1$. For $m \notin \{0, 1, 3, 6, 9, 12\}$ and $r_0 = \lfloor m/3 \rfloor + 1$ it follows from Theorem 6 that $f_m(r_0 - 1) < f_m(r_0)$ and by Lemma 3(a) that $\sum_{i=0}^{r_0-1} \binom{m}{i} < \binom{m}{r_0}$. Therefore $\sum_{i=0}^{r_0} \binom{m}{i} < 2\binom{m}{r_0}$ and the upper bound follows. For the lower bound, $f_m(r_0) > f_m(r_0+1)$ holds by Theorem 8, and so $\sum_{i=0}^{r_0} \binom{m}{i} > \binom{m}{r_0+1}$ by Lemma 3(c). Hence $2^{-r_0}\binom{m}{r_0+1} < f_m(r_0)$, and the lower bound of (1) follows from

$$\binom{m}{r_0+1} = \frac{2r_0 - k}{r_0+1}\binom{m}{r_0} = (2 - \frac{k+2}{r_0+1})\binom{m}{r_0}.$$

To prove (2), we use binomial approximations.

Suppose that $0 < p < 1$ and $q := 1 - p$. If pn is an integer, then $qn = n - pn$ is an integer, and Stirling's approximation $n! = \sqrt{2\pi n}(\frac{n}{e})^n(1 + \mathrm{O}(\frac{1}{n}))$ gives

$$\binom{n}{pn} = \frac{c^n}{\sqrt{2\pi pqn}}\left(1 + \mathrm{O}\left(\frac{1}{n}\right)\right) \qquad \text{where } c = \frac{1}{p^p q^q}. \tag{8}$$

Paraphrasing [2, Lemma 4.7.1] gives the following upper and lower bounds:

$$\frac{c^n}{\sqrt{8pqn}} \leqslant \binom{n}{pn} \leqslant \frac{c^n}{\sqrt{2\pi pqn}} \qquad \text{where } c = \frac{1}{p^p q^q}. \tag{9}$$

Henceforth set $p = \frac{1}{3}$, so $q = \frac{2}{3}$ and $c = \frac{3}{2^{2/3}}$. Therefore $c^3 = \frac{27}{4}$ and

$$\frac{c^{3r_0}}{2^{r_0}} = \frac{1}{2^{r_0}}\left(\frac{27}{4}\right)^{r_0} = \left(\frac{27}{8}\right)^{r_0} = \left(\frac{3}{2}\right)^{3r_0} \qquad \text{and} \qquad \frac{1}{\sqrt{2pq}} = \frac{3}{2}.$$

We write $m = 3r_0 - k$ where $k \in \{1, 2, 3\}$.

We now prove the upper bound for $f_m(r_0)$ in (2). It follows from

$$\binom{m}{r_0} = \binom{3r_0 - k}{r_0} = \frac{2r_0 - k + 1}{3r_0 - k + 1}\binom{3r_0 - k + 1}{r_0} \leqslant \frac{2}{3}\binom{3r_0 - k + 1}{r_0}$$

that $\binom{m}{r_0} \leqslant (\frac{2}{3})^k\binom{3r_0}{r_0}$. Setting $n = 3r_0$ and $p = \frac{1}{3}$ in (9) and using $m < n$ shows

$$\frac{2}{2^{r_0}}\binom{3r_0}{r_0} = \frac{2}{2^{r_0}}\binom{n}{pn} \leqslant \frac{2}{2^{r_0}}\frac{c^{3r_0}}{\sqrt{2\pi pqn}} = \frac{2}{\sqrt{2\pi pqn}}\left(\frac{3}{2}\right)^{3r_0} < \frac{3}{\sqrt{\pi m}}\left(\frac{3}{2}\right)^{3r_0}.$$

Using $\binom{m}{r_0} \leqslant (\frac{3}{2})^{-k}\binom{3r_0}{r_0}$ and $m = 3r_0 - k$ gives

$$f_m(r_0) \leqslant \frac{2}{2^{r_0}}\binom{m}{r_0} \leqslant \frac{2}{2^{r_0}}\left(\frac{3}{2}\right)^{-k}\binom{3r_0}{r_0} < \frac{3}{\sqrt{\pi m}}\left(\frac{3}{2}\right)^{m}.$$

We now consider approximate lower bounds for $f_m(r_0)$. Our argument involves constants depending on k but not r_0 whose values are not relevant here. We have

$$\binom{m}{r_0+1} = 2\left(1+\mathrm{O}\left(\frac{1}{r_0}\right)\right)\binom{m}{r_0} = 4\left(1+\mathrm{O}\left(\frac{1}{r_0}\right)\right)\binom{m}{r_0-1}.$$

Further, if $k=1,2$ and $r_0 \geqslant 1$ it follows that

$$\begin{aligned}\binom{m}{r_0-1} &= \binom{3r_0-k}{r_0-1} = \frac{3r_0-k}{2r_0-k+1}\binom{3r_0-k-1}{r_0-1} \\ &= \left(\frac{3}{2}+\frac{k-3}{2(2r_0-k+1)}\right)\binom{3r_0-k-1}{r_0-1} > \frac{3}{2}\binom{3r_0-k-1}{r_0-1}.\end{aligned}$$

Hence $\binom{m}{r_0-1} \geqslant (\frac{3}{2})^{3-k}\binom{3r_0-3}{r_0-1}$ holds for $k \in \{1,2,3\}$ and $r_0 \geqslant 1$.

Setting $n = 3r_0-3$ and $p=\frac{1}{3}$ in (8) yields

$$\frac{1}{2^{r_0}}\binom{3r_0-3}{r_0-1} = \frac{1}{2^{r_0}}\binom{n}{pn} = \frac{1}{2^{r_0}}\frac{c^{3r_0-3}}{\sqrt{2pq\pi n}}\left(1+\mathrm{O}\left(\frac{1}{n}\right)\right).$$

However, $\frac{c^{3r_0}}{2^{r_0}} = (\frac{3}{2})^{3r_0}$ and $\frac{c^{-3}}{\sqrt{2pq\pi n}} = \frac{3c^{-3}}{2\sqrt{\pi n}} = \frac{2}{9\sqrt{\pi n}} = \frac{2}{9\sqrt{\pi m}}(1+\mathrm{O}\left(\frac{1}{m}\right))$. Therefore

$$\frac{1}{2^{r_0}}\binom{3r_0-3}{r_0-1} = \frac{2}{9\sqrt{\pi m}}\left(\frac{3}{2}\right)^{3r_0}\left(1+\mathrm{O}\left(\frac{1}{m}\right)\right).$$

The above bounds give

$$\begin{aligned}f_m(r_0) &\geqslant \frac{1}{2^{r_0}}\binom{m}{r_0+1} \geqslant 4\left(1+\mathrm{O}\left(\frac{1}{r_0}\right)\right)\left(\frac{3}{2}\right)^{3-k}\frac{1}{2^{r_0}}\binom{3r_0-3}{r_0-1} \\ &= 4\left(1+\mathrm{O}\left(\frac{1}{m}\right)\right)\left(\frac{3}{2}\right)^{3-k}\frac{2}{9\sqrt{\pi m}}\left(\frac{3}{2}\right)^{3r_0} = \left(1+\mathrm{O}\left(\frac{1}{m}\right)\right)\frac{3}{\sqrt{\pi m}}\left(\frac{3}{2}\right)^{m}.\end{aligned}$$

Finally, since $1+\mathrm{O}\left(\frac{1}{m}\right) \to 1$ as $m \to \infty$, the limit in (2) follows. □

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