On the maximum of the weighted binomial sum $2^{-r} \sum_{i=0}^{r} \binom{m}{i}$

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Abstract
The weighted binomial sum $f_m(r) = 2^{-r} \sum_{i=0}^{r} \binom{m}{i}$ arises in coding theory and information theory. We prove that, for $m \not\in \{0, 3, 6, 9, 12\}$, the maximum value of $f_m(r)$ with $0 \leq r \leq m$ occurs when $r = \lfloor m/3 \rfloor + 1$. We also show this maximum value is asymptotic to $\frac{3}{\sqrt{2m}} \left(\frac{3}{2}\right)^m$ as $m \to \infty$.

Mathematics Subject Classifications: 05A10, 11B65, 94B65

1 Introduction
Let $m$ be a non-negative integer, and let $f_m(r)$ be the function:

$$f_m(r) = \frac{1}{2^r} \sum_{i=0}^{r} \binom{m}{i}.$$  

This function arises in coding theory and information theory e.g. [2, Theorem 4.5.3]. It is desirable for a linear code to have large rate (to communicate a lot of information) and large minimal distance (to correct many errors). So for a linear code with parameters

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[n, k, d], one wants both k/n and d/n to be large. The case that kd/n is large is studied in [1]. A Reed-Muller code RM(r, m) has n = 2^m, k = \sum_{i=0}^{r} \binom{m}{i} and d = 2^{m-r} by [4, §6.2], and hence kd/n equals f_m(r). It is natural to ask which value of r maximizes f_m(r), and what is the size of the maximum value.

**Theorem 1.** Suppose that m, r are integers where 0 \leq r \leq m. The maximum value of f_m(r) = 2^{-r} \sum_{i=0}^{r} \binom{m}{i} occurs when r = \left\lfloor \frac{m}{3} \right\rfloor + 1 provided m \not\in \{0, 3, 6, 9, 12\}.

We give an optimal asymptotic bound for the maximum value of f_m(r).

**Theorem 2.** Suppose that m \not\in \{0, 1, 3, 6, 9, 12\} and r_0 = \left\lfloor \frac{m}{3} \right\rfloor + 1. Then

\[
\frac{1}{2^{\left\lfloor m/3 \right\rfloor}} \left(1 - \frac{k + 2}{2(r_0 + 1)}\right) \left(\frac{m}{r_0}\right) < f_m(r_0) < \frac{1}{2^{\left\lfloor m/3 \right\rfloor}} \left(\frac{m}{r_0}\right)
\]

where k := 3r_0 - m \in \{1, 2, 3\}. Furthermore,

\[
f_m(r_0) < \frac{3}{\sqrt{\pi} m} \left(\frac{3}{2}\right)^m \quad \text{and} \quad \lim_{m \to \infty} f_m(r_0) \sqrt{m} \left(\frac{2}{3}\right)^m = \frac{3}{\sqrt{\pi}}.
\]

We prove that f_m(r) increases strictly if 0 \leq r \leq r_0 := \left\lfloor \frac{m}{3} \right\rfloor + 1 and m > 12 (see Theorem 6), and it decreases strictly for r_0 \leq r \leq m (see Theorem 8). Elementary arguments in Lemma 4(c) show that f_m(0) < f_m(1) < \cdots < f_m(r_0 - 1). More work is required to prove that f_m(r_0 - 1) < f_m(r_0). Determining when f_m(r) decreases involves a delicate inductive proof requiring a growing amount of precision, and inequalities with rational functions such as X_i = \frac{r_i}{m - r_i + 1}, see Lemma 5. In Section 5 we establish bounds (and asymptotic behavior) for f_m(r_0) using standard methods.

Brendan McKay [3] showed, using approximations for sufficiently large m, that the maximum value of f_m(r) is near m/3. His method may well extend to a proof of Theorem 1. If so, it would involve very different techniques from ours.

## 2 Data, comparisons and strategies

The values of f_m(0), f_m(1), f_m(2), \ldots, f_m(m-2), f_m(m-1), f_m(m) appear to increase to a maximum and then decrease. For ‘large’ m we see that

\[1 < \frac{m+1}{2} < \frac{m^2+m+2}{8} < \cdots \Rightarrow 8 - \frac{m^2+m+2}{2^{m-2}} > 4 - \frac{m+1}{2^{m-2}} > 2 - \frac{1}{2^{m-1}} > 1.\]

Computer calculations for ‘large’ m suggest that a maximum value for f_m(r) occurs at r_0 = \left\lfloor \frac{m}{3} \right\rfloor + 1, see Table 1 which lists the integer part \lfloor f_m(r) \rfloor. Computing f_m(r) exactly shows that for m \in \{0, 3, 6, 9, 12\} the maximum occurs at r_0 - 1 and not r_0, see Table 2. The maximum happens to occur for a unique r, except for m = 1.

Determining the relative sizes of f_m(r) and f_m(r+1) is reduced in Lemma 3 to determining the relative sizes of \sum_{i=0}^{r} \binom{m}{i} and \binom{m}{r+1}.
Table 1: Maximum values of $\lceil f_m(r) \rceil$ for $0 \leq r \leq m$ and $m \in \{6,7,\ldots,15\}$.

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Lemma 3. Suppose that $0 \leq r < m$. Then

(a) the inequality $f_m(r) < f_m(r + 1)$ is equivalent to $\sum_{i=0}^{r} \binom{m}{i} < \binom{m}{r+1}$.

(b) if $\sum_{i=0}^{r} \binom{m}{i} \leq \binom{m}{r+1}$, then $\sum_{i=0}^{r} \binom{m+1}{i} \leq \binom{m+1}{r+1}$.

(c) the inequality $f_m(r) > f_m(r + 1)$ is equivalent to $\sum_{i=0}^{r} \binom{m}{i} > \binom{m}{r+1}$, and

(d) if $\sum_{i=0}^{r} \binom{m}{i} \geq \binom{m}{r+1}$, then $\sum_{i=0}^{r} \binom{m-1}{i} \geq \binom{m-1}{r+1}$.

Proof. (a,b) Clearly $f_m(r) < f_m(r + 1)$ is equivalent to $2\sum_{i=0}^{r} \binom{m}{i} < \sum_{i=0}^{r+1} \binom{m}{i}$ which is equivalent to $\sum_{i=0}^{r} \binom{m}{i} < \binom{m}{r+1}$. If $r < m$ and $\sum_{i=0}^{r} \binom{m}{i} \leq \binom{m}{r+1}$, then

$$\sum_{i=0}^{r} \binom{m+1}{i} = \sum_{i=0}^{r} \frac{m+1}{m-i+1} \binom{m}{i} \leq \frac{m+1}{m-r} \sum_{i=0}^{r} \binom{m}{i} < \frac{m+1}{m-r} \binom{m}{r+1}.$$

(c,d) Clearly $f_m(r) > f_m(r + 1)$ is equivalent to $2\sum_{i=0}^{r} \binom{m}{i} > \sum_{i=0}^{r+1} \binom{m}{i}$ which, in turn, is equivalent to $\sum_{i=0}^{r} \binom{m}{i} > \binom{m}{r+1}$. If $\sum_{i=0}^{r} \binom{m}{i} \geq \binom{m}{r+1}$, then as $m > r \geq 0$,

$$\sum_{i=0}^{r} \binom{m-1}{i} = \sum_{i=0}^{r} \frac{m-i}{m} \binom{m}{i} \geq \frac{m-r}{m} \sum_{i=0}^{r} \binom{m}{i} > \frac{m-r-1}{m} \binom{m}{r+1}.$$

The following easy lemma elucidates which $r \in \{0,\ldots,m\}$ maximize $f_m(r)$.
Lemma 4. Let \( s_m(m + 1) = 2^m \), and for \( 0 \leq r \leq m \) define

\[
  s_m(r) = \sum_{i=0}^{r} \binom{m}{i}, \quad t_m(r) = \frac{s_m(r + 1)}{s_m(r)}, \quad c_m(r) = \binom{m}{r+1} = \frac{m-r}{r+1}.
\]

(a) If \( 0 \leq r \leq m \), then \( c_m(r) < t_m(r) \), and if \( 0 \leq r < m \), then \( t_m(r+1) < t_m(r) \).

(b) If \( m \geq 2 \), then for some \( r^* \), \( f_m(0) < \cdots < f_m(r^*) \) and \( f_m(r^* + 1) > \cdots > f_m(m) \).

(c) max\{\( f_m(0), \ldots, f_m(m) \}\} = max\{f_m(r^*), f_m(r^*+1)\} and \( f_m(0) < \cdots < f_m(r_0-1) \).

Proof. (a) We show \( c_m(r) < t_m(r) \) via induction on \( r \). This is true when \( r = 0 \) as \( c_m(0) = m < m + 1 = t_m(0) \). Suppose that \( 0 \leq r \leq m \) and \( c_m(r) < t_m(r) \) holds. That is, \( \binom{m}{r+1}/\binom{m}{r} < s_m(r+1)/s_m(r) \) holds. Since \( c_m(r+1) = \frac{m-r-1}{r+1} < \frac{m-r}{r+1} = c_m(r) \) we have \( c_m(r+1) < c_m(r) < t_m(r) \). Using properties of mediants, it follows that

\[
  c_m(r+1) = \frac{\binom{m}{r+2}}{\binom{m}{r+1}} \leq \frac{\binom{m}{r+2} + s_m(r+1)}{\binom{m}{r+1} + s_m(r)} < \frac{s_m(r+1)}{s_m(r)} = t_m(r).
\]

Hence \( c_m(r+1) < t_m(r+1) < t_m(r) \) as \( s_m(n+1) = \binom{m}{n+1} + s_m(n) \). This completes the induction, and it also proves that \( t_m(r+1) < t_m(r) \), as claimed.

(b) Since \( s_m(m + 1) = 2^m \), part (a) shows that \( 1 = t_m(m) < \cdots < t_m(0) = m + 1 \). Choose an integer \( r^* \) such that \( t_m(r^*) \leq 2 < t_m(r^* - 1) \). The following are equivalent: \( 2 < t_m(r); 2s_m(r) < s_m(r+1); f_m(r) < f_m(r+1) \). Thus \( 2 < t_m(r^* - 1) < \cdots < t_m(0) \) implies \( f_m(0) < \cdots < f_m(r^*) \). Similarly, \( t_m(m - 1) < \cdots < t_m(r^* - 1) < 2 \) and \( t_m(r) < 2 \) implies \( f_m(r^* + 1) > \cdots > f_m(m) \). Hence \( f_m(r^* + 1) > \cdots > f_m(m) \).

(c) By part (b), \( \text{max}\{f_m(r) | 0 \leq r \leq m\} = \text{max}\{f_m(r^*), f_m(r^* + 1)\} \). If \( 2 < c_m(r) = \frac{m-r}{r+1} \), then \( 3r + 2 \leq m \) and \( r \leq \lfloor \frac{m-2}{3} \rfloor \). Hence \( 2 \leq c_m(r) < t_m(r) \) by part (a), and \( \lfloor \frac{m-2}{3} \rfloor \leq r^* - 1 \) by the definition of \( r^* \). Thus \( r_0 - 1 = \lfloor \frac{m}{3} \rfloor \leq r^* \) and it follows from part (b) that \( f_m(0) < \cdots < f_m(r_0 - 1) \).

Fix \( m \) and \( r \) where \( 0 \leq r < m \). We shall use the following notation:

\[
  (3) \quad X_i = \frac{r - i + 1}{m - r + i} \quad \text{for } 0 \leq i \leq r,
\]

\[
  (4) \quad S_j = 1 + X_{j+1} + X_{j+1}X_{j+2} + \cdots + X_{j+1}X_{j+2} \cdots X_r \quad \text{for } 0 \leq j < r,
\]

\[
  (5) \quad T_j = 1 + X_1 + X_1X_2 + \cdots + X_1X_2 \cdots X_j \quad \text{for } 0 \leq j \leq r.
\]

Our convention in (5) is that \( T_0 = 1 \) as \( T_j = \sum_{i=0}^{j}(\prod_{k=1}^{i} X_k) \) equals 1 when \( j = 0 \).

Lemma 5. Fix \( m, r, j \) where \( 0 \leq j \leq r < m \). Using the above definitions,

(a) the inequality \( \sum_{i=0}^{j} \binom{m}{r-i} > \binom{m}{r+1} \) is equivalent to \( T_j > X_0^{-1} \),

(b) the inequality \( \sum_{i=0}^{r} \binom{m}{i} < \binom{m}{r+1} \) is equivalent to \( S_0 < X_0^{-1} \).

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Proof. For $0 \leq i \leq r$, we have $(m \choose r) = X_i (r-i+1 \choose r)$ so $(m \choose r) = (\prod_{k=1}^{i} X_k) (m \choose r)$ holds. Therefore $\sum_{i=0}^{r} (m \choose r) = (m \choose r) \sum_{i=0}^{r} (\prod_{k=1}^{i} X_k) = (m \choose r) T_j$. Since $(m \choose r) = X_0 (m \choose r)$, the inequality $\sum_{i=0}^{r} (m \choose r) > (r+1) \choose r$ is equivalent to $(m \choose r) T_j > X_0^{-1} (m \choose r)$ which is equivalent to $T_j > X_0^{-1}$. This proves part (a).

Note that $\sum_{i=0}^{r} \binom{m}{i} = \sum_{i=0}^{r} (m \choose r-i) = (m \choose r) T_r = (m \choose r) S_0$ since $S_0 = T_r$. Since $(m \choose r+1) = X_0^{-1} (m \choose r)$, the inequality $\sum_{i=0}^{r} \binom{m}{i} < (r+1) \choose r+1$ is equivalent to $(m \choose r) S_0 < X_0^{-1} (m \choose r)$ which is equivalent to $S_0 < X_0^{-1}$. This proves part (b). □.

3 Proof that $f_m(r)$ is increasing for $0 \leq r \leq r_0$

Recall that $m \geq 0$ and $r_0 := \lceil m/3 \rceil + 1$. We now strengthen Lemma 4(c).

Theorem 6. If $m \notin \{0, 1, 3, 6, 9, 12\}$, then $f_m(0) < f_m(1) < \cdots < f_m(r_0)$.

Proof. The statement is easy to check for $m \in \{2, 4, 5\}$. The statement follows from Tables 1 and 2 for $m \in \{7, 8, 10, 11, 13, 14\}$. Suppose now that $m \geq 15$.

Table 2: $\square = \max \{f_m(r_0-1), f_m(r_0)\}$ for $0 \leq m \leq 12$, $r_0 = \lceil m/3 \rceil + 1$.

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Recall that $r_0 = \lceil m/3 \rceil + 1$ and $m \in \{3r_0 - 3, 3r_0 - 2, 3r_0 - 1\}$. By Lemma 4(c) it suffices to show that $f_m(r_0-1) < f_m(r_0)$. If we prove this for $m = 3r_0 - 3$, Lemma 3(b,a) gives it for $m = 3r_0 - 2$ and $m = 3r_0 - 1$ as well, so for $r_0 \geq 6$ we want to show $f_{3r_0-3}(r_0-1) < f_{3r_0-3}(r_0)$. This is true for $r_0 = 6$ by Table 1. We set $t := r_0 - 1$, $m := 3t$ and we prove, using induction on $t$, that $f_{3t}(t) < f_{3t}(t+1)$ holds for all $t \geq 6$.

Note that $f_{3t}(t) < f_{3t}(t+1)$ is equivalent by Lemma 3(a) to $\sum_{i=0}^{t} \binom{m}{i} < (3t+1)$, and this is equivalent to $S_0 < X_0^{-1}$ by Lemma 5(b). Putting $m = 3t$ and $r = t$ in (3), gives $X_i = \frac{t+1}{2t+1}$ and $S_0 = 1 + X_1 + X_1 X_2 + \cdots + X_1 X_2 \cdots X_t$ by (4).

It follows from $0 < X_t < \cdots < X_5 < X_4$ and $X_4 = \frac{t-3}{2t+4} < \frac{1}{2}$ that

$$S_3 = 1 + X_4 + X_4 X_5 + \cdots + X_4 X_5 \cdots X_t < 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2t-3} < \sum_{i=0}^{\infty} \frac{1}{2^i} = 2.$$

The recurrence relation $S_j = 1 + X_{j+1} S_{j+1}$ for $0 \leq j < t$ implies that

$$S_0 = 1 + X_1 (1 + X_2 (1 + X_3 S_3)) < 1 + X_1 (1 + X_2 (1 + 2X_3)) = 1 + \frac{t}{2t+1} \left( 1 + \frac{t-1}{2t+2} \left( 1 + \frac{2(t-2)}{2t+3} \right) \right).$$

*Observe that $f_0(1) = 2^{-1} \left( \binom{0}{0} + \binom{0}{1} \right) = 2^{-1}(1+0) = \frac{1}{2}$.
We aim to show that \( S_0 < X_0^{-1} \). It suffices to prove \( 1 + X_1 (1 + X_2 (1 + 2X_3)) \leq X_0^{-1} \) where \( X_0 = \frac{t+1}{2t} \). This amounts to proving that
\[
1 + \frac{t}{2t+1} \left( 1 + \frac{t-1}{2t+2} \left( 1 + \frac{2(t-2)}{2t+3} \right) \right) \leq \frac{2t}{t+1}.
\]
Rearranging, and using the denominator \((2t+1)(2t+2)(2t+3)\), gives
\[
0 \leq \frac{3t^2 - 17t - 6}{(2t+1)(2t+2)(2t+3)} = \frac{(3t+1)(t-6)}{(2t+1)(2t+2)(2t+3)}
\]
This inequality is valid for all \( t \geq 6 \). This completes the proof. \( \square \)

Remark 7. For \( s > 4 \) set \( m = 3s \) and \( r_0 = s + 1 \). Then \( f_m(r_0 - 1) < f_m(r_0) \) by Theorem 6. Hence \( \sum_{i=0}^{s} \binom{3s}{i} < \binom{3s}{s+1} = \frac{2s}{s+1} \binom{3s}{s} \) and so \( \lim_{s \to \infty} \sum_{i=0}^{s} \binom{3s}{i}/\binom{3s}{s} \leq 2 \). We show \( f_m(r_0) > f_m(r_0 + 1) \) in Section 4, and therefore \( \lim_{s \to \infty} \sum_{i=0}^{s} \binom{3s}{i}/\binom{3s}{s} \geq 2 \).

4 Proof that \( f_m(r) \) is decreasing for \( r_0 \leq r \leq m \)

Showing that \( f_m(r) \) decreases strictly for \( r_0 \leq r \leq m \) is much harder. Recall that \( \binom{r}{i} = 0 \) if \( i < 0 \), and \( \binom{r}{i} = \frac{1}{i} \prod_{j=0}^{i-1} (r-j) \) if \( i \geq 0 \). In this section we prove:

Theorem 8. If \( m \geq 2 \), then \( f_m([m/3]+1) > f_m([m/3]+2) > \cdots > f_m(m) = 1 \).

Our proof of Theorem 8 depends on two technical lemmas, the first of which proves that the non-leading coefficients of a certain polynomial \( A(r) \) are all negative.

First define \( B_i(r) = \prod_{\ell=1}^{i} (r-\ell) \). Now \( \prod_{\ell=1}^{i} (r-\ell) = r^i + \sum_{k=0}^{i-1} b_{i,k} r^k \) and the coefficients \( b_{i,k} \) alternate in sign: for \( 0 \leq k \leq i \), we have \( b_{i,k} > 0 \) if \( i-k \) is even and \( b_{i,k} < 0 \) if \( i-k \) is odd. Next define polynomials \( A_i(r) \) via:

\[
(6) \quad A_2(r) = r^2 - 15r - 10 \quad \text{and} \quad A_i(r) = (2r+i)A_{i-1}(r) - B_i(r) \quad \text{for} \ i \geq 3.
\]

Clearly \( \deg(A_i(r)) = i \) and we may write \( A_i(r) = r^i + \sum_{k=0}^{i-1} a_{i,k} r^k \). We use \( a_{i,i} = 1 \).

Comparing coefficients in this recurrence and \( B_i(r) = (r-i)B_{i-1}(r) \), shows that

(Ra) \( a_{2,0} = -10, \ a_{2,1} = -15, \ a_{i,k} = ia_{i-1,k} + 2a_{i-1,k-1} - b_{i,k} \) for \( i \geq 3 \),

(Rb) \( b_{2,0} = 2, \ b_{2,1} = -3, \ b_{i,k} = -ib_{i-1,k} + b_{i-1,k-1} \) for \( i \geq 3 \).

Lemma 9. Let \( a_{i,k}, b_{i,k}, A_i(r), B_i(r) \) be as above.

(a) If \( i \geq 2 \), then \( b_{i,i-1} = -\binom{i+1}{2} \) and \( a_{i,i-1} = -\binom{i+1}{2} \).

(b) If \( i \geq 2 \) and \( 0 \leq k \leq i-1 \), then \( a_{i,k} \leq 2b_{i,k} < 0 \) if \( i-k \) is even, and \( a_{i,k} \leq b_{i,k} < 0 \) if \( i-k \) is odd.
(c) If \( i \geq 2 \), then the coefficients \( a_{i,k} \) are negative for \( 0 \leq k < i \).

Proof. (a) Clearly \( b_{i-1} = -\sum_{j=1}^{i} j = -(\binom{i+1}{2}) \). The formula for \( a_{i,i-1} \) holds for \( i = 2 \) and by induction using the recurrence \( (Ra) \).

(b) We use induction on \( i \). For the base case \( i = 2 \), either \( i - k \) is even and \( a_{2,0} = -10 \leq -2b_{2,0} = -4 \), or \( i - k \) is odd and \( a_{2,1} = -15 < b_{2,1} = -3 \). Thus the claims are true for \( i = 2 \). Suppose now that \( i \geq 3 \), and the claims are valid for \( i - 1 \).

By part (a), \( a_{i-1,i} = -(\binom{i+1}{2}) < -(\binom{i+2}{2}) = b_{i,i-1} < 0 \) as claimed. It remains to consider \( k \) in the range \( 0 \leq k < i - 1 \). It is useful to set \( a_{i-1,i-1} = b_{i,i-1} = 0 \). Suppose first that \( i - k \) is even. Using the recurrences \( (Ra), (Rb) \) and induction gives

\[
a_{i,k} = i(a_{i-1,k} + b_{i-1,k}) + (2a_{i-1,k-1} - b_{i-1,k-1}) \leq i(b_{i-1,k} + b_{i-1,k}) + (-4b_{i-1,k-1} - b_{i-1,k-1}) \leq -4b_{i-1,k} - 2b_{i-1,k-1} = -2b_{i,k} < 0.
\]

If \( i - k \) is odd, then a similar argument gives

\[
a_{i,k} = i(a_{i-1,k} + b_{i-1,k}) + (2a_{i-1,k-1} - b_{i-1,k-1}) \leq i(-2b_{i-1,k} + b_{i-1,k}) + (2b_{i-1,k-1} - b_{i-1,k-1}) \leq -ib_{i-1,k} + b_{i-1,k-1} = b_{i,k} < 0.
\]

(c) This follows immediately from part (b). \( \square \)

Lemma 10. Suppose that \( j \geq 4 \). Then \( \sum_{i=r-j}^{r} \binom{3r-1}{i} > \binom{3r-1}{r+1} \) holds for all \( r \) in the range \( j \leq r \leq (\binom{j+2}{2}) \).

Proof. We apply Lemma 5(a) with \( m = 3r - 1 \). Hence \( X_i = \frac{r-i+1}{2r+i-1} \) by (3). Since \( \sum_{i=r-j}^{r} \binom{m}{i} = \sum_{i=0}^{j} \binom{m}{i} \) it suffices by Lemma 5(a) to prove that

\[
T_j = 1 + X_1 + X_1X_2 + \cdots + X_1X_2\cdots X_j > X_0^{-1}.
\]

We prove that this inequality holds for all \( r \) in the range \( j \leq r \leq (\binom{j+2}{2}) \). This inequality is equivalent to

\[
X_j > X_{j-1}^{-1}(\cdots(X_2^{-1}(X_1^{-1}(X_0^{-1} - 1) - 1) - 1)\cdots) - 1.
\]

The right-side of (7) is a rational function in \( r \), which when \( j = 4 \), equals

\[
\frac{P_4(r)}{Q_4(r)} = \frac{2r + 2}{r - 2}\left(\frac{2r + 1}{r - 1}\left(\frac{2r}{r + 1}\left(\frac{2r}{r + 1} - 1\right) - 1\right) - 1\right) - 1
\]

where the denominator is \( Q_4(r) = (r + 1)r(r - 1)(r - 2) \), and the numerator is \( P_4(r) = (r + 1)r(r - 1)(r - 2) \). Since \( \gcd(P_4(r), Q_4(r)) = (r + 1)r \), the polynomials \( A_2(r) := r^2 - 15r - 10 \) and \( B_2(r) := (r - 1)(r - 2) \) are coprime. The putative inequality (7) when \( j = 4 \) is therefore

\[
\frac{r - 3}{2r + 3} > \frac{P_4(r)}{Q_4(r)} = \frac{A_2(r)}{B_2(r)} = \frac{r^2 - 15r - 10}{(r - 1)(r - 2)}.
\]
Observe that $A_2(r) < r^2 - 15r \leq 0$ for $4 \leq r \leq 15 = \binom{6}{3}$. Thus for $r$ in the range $4 \leq r \leq \binom{6}{3}$, the left side of (7) is positive, and the right side is at most 0. Thus the inequality is valid for $4 \leq r \leq \binom{6}{3}$ and the claim is true for $j = 4$.

Assume now that $j > 4$, and that the claim is true for $j - 1$. Therefore the inequality (7) can be written

$$X_j = \frac{r - j + 1}{2r + j - 1} > \frac{P_j(r)}{Q_j(r)}$$

where

$$\frac{P_j(r)}{Q_j(r)} = (X_{j-1})^{-1} \frac{P_{j-1}(r)}{Q_{j-1}(r)} - 1.$$

Since $(X_{j-1})^{-1} = \frac{2r+j-2}{r-j+2}$, this gives rise to the recurrences:

$$P_j(r) = (2r + j - 2)P_{j-1}(r) - (r - j + 2)Q_{j-1}(r)$$

for $j > 4$,

$$Q_j(r) = (r - j + 2)Q_{j-1}(r)$$

for $j > 4$.

It is clear that $Q_j(r) = (r + 1)r(r - 1) \cdots (r - j + 2) = (r + 1)rB_{j-2}(r)$ holds and $B_{j-2}(r)$ has degree $j - 2$. Furthermore, $(r + 1)r$ divides $\gcd(P_j(r), Q_j(r))$, so the polynomials $A_{j-2}(r)$, which are defined by the similar recurrence (6), satisfy $P_j(r) = (r + 1)rA_{j-2}(r)$ and also have degree $j - 2$.

By Lemma 9, $A_i(r) - r^i$ has negative coefficients and leading coefficient $-(i+4)$. So for $i \geq 2$ and $r \leq \binom{i+4}{2}$, we have $A_i(r) < r^i - \binom{i+4}{2}r^{i-1} \leq 0$. Further, $B_i(r) = \prod_{\ell=1}^i (r - \ell) > 0$ for $r \geq i + 1$. Hence $A_i(r)/B_i(r) < 0$ for $r$ satisfying $i + 2 \leq r \leq \binom{i+4}{2}$. Suppose that $j = i + 2$, then $P_j(r)/Q_j(r) < 0$ for $r$ in the interval $j \leq r \leq \binom{j+2}{2}$. Using the definitions of $P_j(r), Q_j(r)$, the inequality (7) is the same as

$$X_j = \frac{r - j + 1}{2r + j - 1} > \frac{P_j(r)}{Q_j(r)} = \frac{A_{j-2}(r)}{B_{j-2}(r)}.$$

Thus for $r$ satisfying $j \leq r \leq \binom{j+2}{2}$, the left side of (7) is positive, and the right side is negative. Thus the claim is valid for $j \leq r \leq \binom{j+2}{2}$.

**Proof of Theorem 8.** It follows from $\sum_{i=0}^m \binom{m}{i} = 2^m$ that $f_m(m) = 1$. Since $r_0 := \lceil m/3 \rceil + 1$, we have $m \in \{3r_0 - 3, 3r_0 - 2, 3r_0 - 1\}$. If we can prove that $f_m(r_0) > f_m(r_0 + 1)$ for $m = 3r_0 - 1$, then $f_m(r_0) = f_m(r_0 + 1)$ holds for $m = 3r_0 - 2$ and $3r_0 - 3$ by Lemma 3(d). With the notation in Lemma 4, we have $2 > t_m(r_0)$ and hence $r^* \leq r_0$. Therefore $f_m(r_0 + 1) > \cdots > f_m(m)$ holds by Lemma 4(b).

In summary, it remains to prove $\sum_{i=0}^{r_0} \binom{3r_0-1}{i} \leq \binom{3r_0-1}{r_0+1}$ for $r_0 \geq 1$. This is true for $r_0 = 1, 2, 3$ since $\frac{3}{2} > 1$, $4 > 13/4$, and $93/8 > 163/16$. For each $r_0 \geq 4$ set $j = r_0$. Then $j \geq 4$ and $\sum_{i=0}^{r_0} \binom{m}{i} > \binom{m}{r_0+1}$ follows by Lemma 10. This completes the proof.

**Proof of Theorem 1.** The result follows from Theorems 6 and 8. There are two equal sized maxima if $m = 1$, otherwise the maximum is unique.
5 Estimating $f_m(r_0)$

This section is devoted to proving asymptotically optimal bounds for $f_m(r_0)$.

Proof of Theorem 2. We first prove the upper bound in (1). This is true if $m = 1$. For $m \not\in \{0, 1, 3, 6, 9, 12\}$ and $r_0 = [m/3] + 1$ it follows from Theorem 6 that $f_m(r_0 - 1) < f_m(r_0)$ and by Lemma 3(a) that $\sum_{i=0}^{r_0-1} \binom{m}{i} < \binom{m}{r_0}$. Therefore $\sum_{i=0}^{r_0} \binom{m}{i} < 2\binom{m}{r_0}$ and the upper bound follows. For the lower bound, $f_m(r_0) > f_m(r_0 + 1)$ holds by Theorem 8, and so $\sum_{i=0}^{r_0} \binom{m}{i} > \binom{m}{r_0+1}$ by Lemma 3(c). Hence $2^{-r_0} \binom{m}{r_0+1} < f_m(r_0)$, and the lower bound of (1) follows from

$$
\binom{m}{r_0 + 1} = \frac{2r_0 - k}{r_0 + 1} \binom{m}{r_0} = \left(2 - \frac{k + 2}{r_0 + 1}\right) \binom{m}{r_0}.
$$

To prove (2), we use binomial approximations.

Suppose that $0 < p < 1$ and $q := 1 - p$. If $pn$ is an integer, then $qn = n - pn$ is an integer, and Stirling’s approximation $n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + O(\frac{1}{n}))$ gives

$$
\binom{n}{pn} = \frac{c^n}{\sqrt{2\pi pqn}} \left(1 + O\left(\frac{1}{n}\right)\right) \quad \text{where } c = \frac{1}{p^p q^q}.
$$

Paraphrasing [2, Lemma 4.7.1] gives the following upper and lower bounds:

$$
\frac{c^n}{\sqrt{2\pi pqn}} \leq \binom{n}{pn} \leq \frac{c^n}{\sqrt{2\pi pqn}} \quad \text{where } c = \frac{1}{p^p q^q}.
$$

Henceforth set $p = \frac{1}{3}$, so $q = \frac{2}{3}$ and $c = \frac{3}{22/3}$. Therefore $c^3 = \frac{27}{4}$ and

$$
c^3 \binom{3r_0}{2r_0} = \frac{1}{2^{20}} \left(\frac{27}{4}\right)^{r_0} = \left(\frac{27}{8}\right)^{r_0} = \left(\frac{3}{2}\right)^{3r_0} \quad \text{and} \quad \frac{1}{\sqrt{2pq}} = \frac{3}{2}.
$$

We write $m = 3r_0 - k$ where $k \in \{1, 2, 3\}$.

We now prove the upper bound for $f_m(r_0)$ in (2). It follows from

$$
\binom{m}{r_0} = \binom{3r_0 - k}{r_0} = \frac{2r_0 - k + 1}{3r_0 - k + 1} \binom{3r_0 - k + 1}{r_0} \leq \frac{2}{3} \binom{3r_0 - k + 1}{r_0}
$$

that $\binom{m}{r_0} \leq \left(\frac{2}{3}\right)^{k} \binom{3r_0}{r_0}$. Setting $n = 3r_0$ and $p = \frac{1}{3}$ in (9) and using $m < n$ shows

$$
\frac{2}{2^2} \binom{3r_0}{r_0} = \frac{2}{2^2} \binom{n}{pn} \leq \frac{2}{\sqrt{2\pi pqn}} \binom{3}{2} \binom{3r_0}{r_0} < \frac{3}{\sqrt{\pi m}} \binom{3}{2}.
$$

Using $\binom{m}{r_0} \leq \left(\frac{3}{2}\right)^{-k} \binom{3r_0}{r_0}$ and $m = 3r_0 - k$ gives

$$
f_m(r_0) \leq \frac{2}{2^2} \binom{m}{r_0} \leq \frac{2}{2^2} \binom{3}{2}^{-k} \binom{3r_0}{r_0} < \frac{3}{\sqrt{\pi m}} \binom{3}{2}^m.
$$
We now consider approximate lower bounds for \( f_m(r_0) \). Our argument involves constants depending on \( k \) but not \( r_0 \) whose values are not relevant here. We have

\[
\binom{m}{r_0 + 1} = 2 \left( 1 + O \left( \frac{1}{r_0} \right) \right) \binom{m}{r_0} = 4 \left( 1 + O \left( \frac{1}{r_0} \right) \right) \binom{m}{r_0 - 1}.
\]

Further, if \( k = 1, 2 \) and \( r_0 \geq 1 \) it follows that

\[
\binom{m}{r_0 - 1} = \binom{3r_0 - k - 1}{r_0 - 1} = \frac{3r_0 - k}{2r_0 - k + 1} \binom{3r_0 - k - 1}{r_0 - 1} = \left( 3 + \frac{k - 3}{2(2r_0 - k + 1)} \right) \binom{3r_0 - k - 1}{r_0 - 1} > \frac{3}{2} \binom{3r_0 - k - 1}{r_0 - 1}.
\]

Hence \( \binom{m}{r_0 - 1} \geq \left( \frac{3}{2} \right)^{3-k} \binom{3r_0 - 3}{r_0 - 1} \) holds for \( k \in \{1, 2, 3\} \) and \( r_0 \geq 1 \).

Setting \( n = 3r_0 - 3 \) and \( p = \frac{1}{3} \) in (8) yields

\[
\frac{1}{2^{r_0}} \binom{3r_0 - 3}{r_0 - 1} = \frac{1}{2^{r_0}} \binom{n}{pn} = \frac{1}{2^{r_0} \sqrt{2pq\pi n}} \left( 1 + O \left( \frac{1}{n} \right) \right).
\]

However, \( \frac{1}{2^{r_0}} = \left( \frac{3}{2} \right)^{3r_0} \) and \( \frac{c^{3r_0 - 3}}{2\sqrt{\pi n}} = \frac{3c^{3r_0 - 3}}{2\sqrt{\pi n}} = \frac{2}{9\sqrt{\pi n}} (1 + O \left( \frac{1}{m} \right)) \). Therefore

\[
\frac{1}{2^{r_0}} \binom{3r_0 - 3}{r_0 - 1} = \frac{2}{9\sqrt{\pi m}} \left( \frac{3}{2} \right)^{3r_0} \left( 1 + O \left( \frac{1}{m} \right) \right).
\]

The above bounds give

\[
f_m(r_0) \geq \frac{1}{2^{r_0}} \binom{m}{r_0 + 1} \geq 4 \left( 1 + O \left( \frac{1}{r_0} \right) \right) \left( \frac{3}{2} \right)^{3-k} \frac{1}{2^{r_0}} \binom{3r_0 - 3}{r_0 - 1} = 4 \left( 1 + O \left( \frac{1}{m} \right) \right) \left( \frac{3}{2} \right)^{3-k} \frac{2}{9\sqrt{\pi m}} \left( \frac{3}{2} \right)^{3r_0} = \left( 1 + O \left( \frac{1}{m} \right) \right) \frac{3}{\sqrt{\pi m}} \left( \frac{3}{2} \right)^{m}.
\]

Finally, since \( 1 + O \left( \frac{1}{m} \right) \to 1 \) as \( m \to \infty \), the limit in (2) follows. \( \square \)

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**References**


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