Queen Domination of Even Square Boards

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Abstract

The queen's graph $Q_{m\times n}$ has the squares of the $m \times n$ chessboard as its vertices; we identify the $m \times n$ chessboard with a rectangle of width m and height n in the Cartesian plane, having sides parallel to the coordinate axes and placed so that square centers have integer coordinates. Two squares are adjacent if they are in the same row, column, or diagonal of the board. A set D of squares of $Q_{m\times n}$ is a *dominating set* for $Q_{m\times n}$ if every square of $Q_{m\times n}$ is either in D or adjacent to a square in D. The minimum size of a dominating set of $Q_{m\times n}$ is the *domination number*, denoted by $\gamma(Q_{m\times n})$.

We give a new proof of the bound $\gamma(Q_{m\times n}) \geqslant \min\left\{m, n, \left\lceil \frac{m+n-2}{4} \right\rceil\right\}$, with implications for queen domination problems, and then consider square boards.

Let n be an even integer and assume $Q_{n\times n}$ has a dominating set D of size $n/2$ (which implies $\gamma(Q_{n\times n}) = n/2$). For $p \in \{0, 1\}$, let $D_p = \{(x, y) \in D : x + y \equiv p \pmod 2\}$. Say that D is *monochromatic* if $D = D_p$ for some p; otherwise *bichromatic*. We show that if D is bichromatic then $||D_0| - |D_1|| \leqslant 2$ and conjecture that if $n > 4$ then D is monochromatic.

Assume further that D is monochromatic. If $n \equiv 0 \pmod 4$ then $n \in \{4, 12\}$. If $n \equiv 2 \pmod 4$ then odd integers $k = n/2, e, d$ with $1 \leqslant d, e \leqslant k$ satisfy the equation $d^2 + (k-1)e^2 = k(k^2 + 2)/3$. We analyze six infinite sequences of solutions of this equation arising from Fermat-Pell equations, give monochromatic dominating sets of $Q_{n\times n}$ of size $n/2$ for $n = 2, 4, 6, 10, 12, 18, 30$ (new), and 130, and show there are no others with $n < 238$.

Mathematics Subject Classifications: 05C69, 11D25

1 Introduction

For positive integers m, n, let $Q_{m\times n}$ denote the graph whose vertices are the squares of the $m \times n$ chessboard; two squares are adjacent if a chess queen can move from one to the other. That is, the two squares share a row, column, or diagonal of the board.

We will identify the $m \times n$ chessboard with a rectangle of width m and height n in the Cartesian plane, having sides parallel to the coordinate axes and placed so that square centers have integer coordinates. (We defer the specific choice of board placement.) We refer to board squares by the coordinates of their centers; the square (x,y) is in *column* x and *row* y. Columns and rows will be referred to collectively as *orthogonals*. The *difference diagonal* (respectively *sum diagonal*) through square (x,y) is the set of all board squares with centers on the line of slope $+1$ (respectively -1) through the point (x,y). The value of $y - x$ is the same for each square (x,y) on a difference diagonal, and we will refer to the diagonal by this value. Similarly, the value of $y + x$ is the same for each square on a sum diagonal, and we associate this value to the diagonal. Orthogonals and diagonals are collectively referred to as *lines* of the board.

Let $S = \{(x_i, y_i) : 1 \leqslant i \leqslant h\}$ be a set of squares of $Q_{m\times n}$. For each i, let $d_i = y_i - x_i$ be the index of the difference diagonal and $s_i = y_i + x_i$ the index of the sum diagonal occupied by square (x_i, y_i). From the definitions of these indices, we have two constraints relating the orthogonal and diagonal indices: a linear constraint

$$\sum_{i=1}^{h} d_i = \sum_{i=1}^{h} y_i - \sum_{i=1}^{h} x_i \text{ and } \sum_{i=1}^{h} s_i = \sum_{i=1}^{h} y_i + \sum_{i=1}^{h} x_i \tag{1}$$

and a quadratic constraint

$$2\sum_{i=1}^{h} x_i^2 + 2\sum_{i=1}^{h} y_i^2 = \sum_{i=1}^{h} d_i^2 + \sum_{i=1}^{h} s_i^2. \tag{2}$$

The square (x,y) is *even* if $x + y$ is even, *odd* if $x + y$ is odd. We write $\mathbb{Z}_2$ for the set $\{0, 1\}$, frequently used in discussing parity. The parity opposite to $p \in \mathbb{Z}_2$ is denoted $\overline{p}$. For any set S of squares of $Q_{m\times n}$ and $p \in \mathbb{Z}_2$, let S_p be the set of squares of parity p in S.

A set D of squares of $Q_{m\times n}$ is a *dominating set* for $Q_{m\times n}$ if every square of $Q_{m\times n}$ is either in D or adjacent to a square in D. The set D is *independent* if no two squares of D share a line.

Let $\gamma(Q_{m\times n})$ denote the minimum size of a dominating set for $Q_{m\times n}$. Let $i(Q_{m\times n})$ denote the minimum size of an independent dominating set of $Q_{m\times n}$.

For all positive integers m, n, set $G(m, n) = 4\gamma(Q_{m\times n}) - m - n + 2$.

Let D be a dominating set of $Q_{m\times n}$ of size $\gamma = \gamma(Q_{m\times n})$.

As D may contain more than one square in an orthogonal, we need to consider multisets of square coordinates.

For $t \in \{\text{row}, \text{column}\}$, let $I(t)$ be the multiset of t-indices of squares in D. Let $L(\text{row}) = I(\text{row}) \setminus \{1, 2, \dots, n\}$ and $L(\text{column}) = I(\text{column}) \setminus \{1, 2, \dots, m\}$. Then for each t the number of type t orthogonals occupied by D is $\gamma - |L(t)|$, and $L(t)$ is the multiset of 'excess' t-indices.

Given a dominating set D of $Q_{m\times n}$, say that a square of $Q_{m\times n}$ is *needy* (meaning it needs diagonal cover by D) if it is not in any orthogonal occupied by D.

The following theorem appeared in [5]; a new and more insightful proof, relying on equation (4), is given here.

Theorem 1. *[5] For all positive integers m, n,*

$$\gamma(Q_{m\times n}) \geqslant \min\left\{m, n, \left\lceil\frac{m+n-2}{4}\right\rceil\right\}. \tag{3}$$

Proof. Without loss of generality, we may assume $m \leqslant n$. It suffices to show that if $\gamma(Q_{m\times n}) \leqslant m-1$ then $\gamma(Q_{m\times n}) \geqslant (m+n-2)/4$. So we assume that $\gamma(Q_{m\times n}) \leqslant m-1$.

Let D be a dominating set of $Q_{m\times n}$ of size $\gamma = \gamma(Q_{m\times n})$. First, suppose $\gamma(Q_{m\times n}) = m-1$. There is then a column not containing a square of D; as each square of D can cover at most three squares of that column, we have $n < 3m-2$, which implies $m-1 > (m+n-2)/4$. Thus we may take $\gamma(Q_{m\times n}) \leqslant m-2$. Then D fails to occupy at least two columns and two rows of $Q_{m\times n}$. Let a be the index of the leftmost empty column, b the index of the rightmost empty column, c the index of the lowest empty row, d the index of the highest empty row. The board has a rectangular sub-board with corner squares $(a, c), (a, d), (b, c),$ and (b, d). The *box border* is the set E of edge squares of this sub-board.

Let $R = R(D)$ be the number of squares of D that have at most one diagonal that meets E. Such a square can diagonally cover at most two squares of E; other squares of D can diagonally cover at most four squares of E. Thus D can diagonally cover at most $2R+4(\gamma-R) = 4\gamma-2R$ squares of E. The *diagonal waste* $W_{\text{diag}}(D)$ of D is the difference between $4\gamma-2R$ and the number of needy squares of E.

The *orthogonal waste* of D is $W_{\text{orth}}(D) = |L(\text{col})| + |L(\text{row})|$, the difference between the maximum number of orthogonals that a square set of size γ could occupy and the number of orthogonals D occupies.

We will show that if $|D| = \gamma(Q_{m\times n}) \leqslant \min\{m-2, n-2\}$ then

$$R(D) + W_{\text{orth}}(D) + (W_{\text{diag}}(D))/2 = G(m, n). \tag{4}$$

By the definition of $W_{\text{diag}}(D)$, $4\gamma - 2R - W_{\text{diag}}(D)$ is the number of needy squares of E. This number can also be counted as follows. There are $m - (\gamma - |L(\text{col})|)$ columns not occupied by D, necessarily between columns a and b inclusive. These columns meet the rows of E in $2(m-\gamma+|L(\text{col})|)$ needy squares. Similarly the columns of E have $2(n-\gamma+|L(\text{row})|)$ needy squares. As the corners of E are counted twice so far,

$$2(n-\gamma+|L(\text{col})|) + 2(m-\gamma+|L(\text{row})|) - 4 = 4\gamma - 2R - W_{\text{diag}}(D),$$

which simplifies to (4). As all terms on the left side of (4) are nonnegative, (4) implies (3). □

For most minimum dominating sets D of $Q_{m\times n}$, the term $R(D)$ is the least important in (4), but in Figure 1 we show an example on $Q_{9\times 9}$ with $R(D) = 4 = G(9, 9)$.

From this point, we only consider square boards, and adopt the simpler notation Q_n in place of $Q_{n\times n}$.

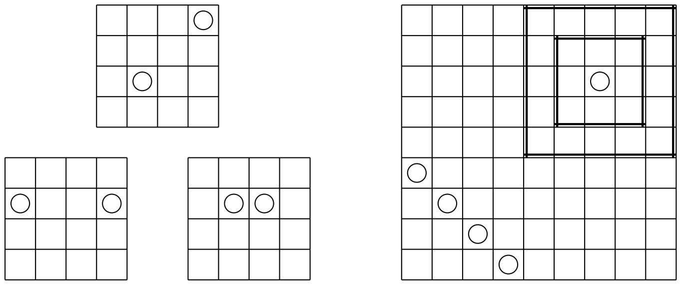

Figure 1: Up to isomorphism there are three minimum dominating sets of Q_4, shown on the left. On the right is a minimum dominating set D of Q_9 from [5] with $R(D) = 4 = G(9,9)$; the box border is outlined.

The task of determining values of $\gamma(Q_n)$ for all positive integers n appears as Problem C18 in the collection [14] of unsolved problems in number theory. Finding these values has interested mathematicians for over 150 years; some of the early references are [2, 11, 19, 20, 21].

V. Raghavan and S. M. Venketesan [18] and P. H. Spencer (see [10] or [22]) independently found very similar proofs for the bound in Theorem 2 below, which also follows immediately from Theorem 1. The last part of Theorem 2 was proved in [23] and [12].

Theorem 2. *For all positive integers n, $\gamma(Q_n) \geqslant (n-1)/2$, and equality holds only for $n = 3, 11$.*

(It is conjectured that for all other odd positive integers n, $\gamma(Q_n) \in \{(n+1)/2, (n+3)/2\}$; this has been confirmed [6, 7, 8, 9, 13, 15, 17] up to $n = 119$. The same workers have jointly confirmed for $n \equiv 1 \pmod 4$ that $\gamma(Q_n) = (n+1)/2$ for $n \leqslant 129$.)

Thus the case $G(n,n) = 0$ is understood. As $G(n,n)$ is even, the next smallest value possible for $G(n,n)$ is 2, which occurs for even n such that $\gamma(Q_n) = n/2$. The rest of this paper begins the process of finding those n for which this occurs.

Definition 3. Say that a set of squares of Q_n is *monochromatic* if all its squares have the same parity; otherwise the set is *bichromatic*.

Up to isomorphism, Q_4 has three dominating sets of size 2, shown in Figure 1. The top set is monochromatic and the other two are bichromatic. Computer search [3, 5] has shown that for each even n, $4 < n \leqslant 24$, if $\gamma(Q_n) = n/2$ then every minimum dominating set of Q_n is monochromatic. This and other indications lead to the following conjecture.

Conjecture 4. For each even integer $n > 4$, if $\gamma(Q_n) = n/2$ then every minimum dominating set of Q_n is monochromatic.

If true, this would give a short proof of the main theorem of [23]: that a dominating set of Q_{4k+3} with size $2k+1$ is monochromatic.

The best we can do here is to prove that a bichromatic dominating set of Q_n with size $n/2$ must have nearly equal numbers of squares of each color.

Before doing this, we note it is known for each of the values $n = 3, 11$ (those for which $\gamma(Q_n) = (n-1)/2$) that there is one isomorphism class of minimum dominating sets, and the sets are monochromatic. So Conjecture 4 could be stated: if $G(n, n) \leqslant 2$ then except for $n = 4$, every minimum dominating set of Q_n is monochromatic. However, for those n with $G(n, n) \geqslant 4$ (that is, $\gamma(Q_n) \geqslant (n+1)/2$), bichromatic dominating sets are common. In particular, $\gamma(Q_n) = (n+1)/2$ for $n = 5, 7, 9, 13$; as shown in [5], for $n = 5$, 7 of the 37 isomorphism classes of minimum dominating sets are monochromatic; for $n = 7$, it is 4 of 13 classes; for $n = 9$, it is 6 of 21, and for $n = 13$, it is 14 of 41.

2 Bichromatic dominating sets of size k for Q_{2k}

Definition 5. Distinct orthogonals of the same type are *parallel*. The *distance* between parallel orthogonals is the absolute value of the difference of their indices.

Our next definition requires a lemma.

Lemma 6. *Let n be an even integer, $n \geqslant 4$, and D a bichromatic dominating set of size $n/2$ for Q_n. For $t \in \{row, column\}$ there exist parallel empty t-orthogonals at odd distance.*

Proof. As D is bichromatic, for at least one $t \in \{\text{row}, \text{column}\}$ there are occupied t-orthogonals of opposite parity and thus empty t-orthogonals of opposite parity. Without loss of generality we may assume this is true for columns. Place the $n \times n$ board with lower left corner having center $(1, 1)$. If the desired conclusion does not hold, all the empty rows have indices of the same parity; by flipping the board across its horizontal midline if necessary, we may assume rows $1, 3, \ldots, n-1$ are occupied. Then any empty column with index of parity p contains $k = n/2$ needy squares of parity p. At most two of these squares can be covered by each square of D, so D contains at least $k/2$ squares of each parity. As $|D| = k$, we see that k is even and D contains exactly $k/2$ squares of each parity. This implies that for every square q of parity p in D and every empty column with index of parity p, each diagonal of q covers a needy square in that column.

As k is even, n is a multiple of four. We have already seen that each dominating set of size 2 for Q_4 is either monochromatic or satisfies the conclusion of this lemma, and $\gamma(Q_8) = 5$, so we may assume that $n \geqslant 12$. Thus there are at least six unoccupied columns, implying there are at least three unoccupied columns with indices $c_1 < c_2 < c_3$ of the same parity. Then $s = (c_2, n)$ is the top needy square in column c_2. If s is covered by D along its difference diagonal, that diagonal passes above all needy squares of column c_3, which is not possible by the preceding paragraph. Similarly, if s is covered along its sum diagonal, that diagonal passes above all needy squares of column c_1 and this is not possible. Thus the conclusion holds. □

For the rest of this section, we assume that n is even, $n > 4$, and D is a bichromatic dominating set of Q_n of size $k = n/2$.

Definition 7. Since $n > 4$ and $|D| = n/2$, for each $t \in \{\text{row}, \text{column}\}$ there are at least three empty t-orthogonals, so there are empty t-orthogonals at even distance. Thus we can make the following definition: among all even distances between parallel empty t-orthogonals, let $m(t, 0)$ be the maximum.

As D is bichromatic and $n > 4$, there exist parallel empty orthogonals at odd distance. By Lemma 6, for each $t \in \{\text{row}, \text{column}\}$ we can define $m(t, 1)$ to be the maximum such odd distance. As $m(t, 0)$ and $m(t, 1)$ have different parity they are not equal; given D, we choose $v_t = v_t(D) \in \mathbb{Z}_2$ such that $m(t, v_t) > m(t, \overline{v_t})$.

For $t \in \{\text{row}, \text{column}\}$, $I(t) \setminus L(t)$ is the set of indices of occupied t-orthogonals. Let $E(t)$ denote the set $\{1, \ldots, n\} \setminus (I(t) \setminus L(t))$ of indices of unoccupied t-orthogonals.

For $t \in \{\text{row}, \text{column}\}$ and any positive integer d, let $\mathcal{P}(t, d)$ be the partition of $E(t)$ into arithmetic progressions with step d and maximal length. Let $o(t, d)$ be the number of progressions in $\mathcal{P}(t, d)$ having odd length. Let $f(t, d)$ be the number of pairs of empty t-orthogonals at distance d.

Theorem 8. *Let n be an even integer, $n > 4$, and let D be a bichromatic dominating set of size $n/2$ for Q_n. For $t \in \{row, column\}$, let d be a positive integer such that $f(t, d) > 0$. Then any two empty t-orthogonals at distance d contain a set $S(t, d)$ of $|E(\bar{t})| + o(\bar{t}, d)$ needy squares, with no three of these squares covered by any one square of D.*

Proof. Let $h, h + d$ be the indices of two empty t-orthogonals. Let S' be the set of $2|E(\bar{t})|$ squares at which empty $\bar{t}$-orthogonals meet t-orthogonals $h, h + d$. All squares of S' are needy; we wish to reduce S' to a subset $S(t, d)$ no three of whose squares are covered by any one square of D. It suffices to arrange that no diagonal contains two squares of $S(t, d)$. To do this, for any square of side length d whose corner squares are in t-orthogonals $h, h + d$ and also in empty $\bar{t}$-orthogonals, we remove two corner squares adjacent along a t-orthogonal, as follows.

For each arithmetic progression $A = \{i, i + d, \ldots, i + (l - 1)d\}$ of length l in $\mathcal{P}(\bar{t}, d)$, we remove the squares in t-orthogonals $h, h+d$ that are in $\bar{t}$-orthogonals $i + jd$ for odd j. Thus we remove $l-1$ squares from progressions of odd length l and l squares from progressions of even length l. Write $\mathcal{P}(\bar{t}, d) = \{A_1, \ldots, A_m\}$ with the length l_i of A_i odd for $1 \leqslant i \leqslant o(\bar{t}, d)$. Then we are removing $\sum_{i=1}^{o(\bar{t},d)}(l_i - 1) + \sum_{i=o(\bar{t},d)+1}^{m} l_i = [\sum_{i=1}^{m} l_i] - o(\bar{t}, d) = |E(\bar{t})| - o(\bar{t}, d)$ squares from S'. Since $|S'| = 2|E(\bar{t})|$, this leaves a set $S(t, d)$ of the desired size. $\square$

Corollary 9. *Let D be a bichromatic dominating set of size $n/2$ for Q_n. Let $t \in \{row, column\}$ and let d be an odd integer such that $f(t, d) > 0$. Then for each $q \in \mathbb{Z}_2$,*

$$|D_q| \geqslant \left\lceil \frac{|E(\bar{t})| + o(\bar{t}, d)}{4} \right\rceil. \tag{5}$$

Proof. Since d is odd, each empty type $\bar{t}$ orthogonal that meets the set $S(t, d)$ of Theorem 8 does so at one even and one odd square. Thus $S(t, d)$ contains equal numbers of squares of each parity. As a square of parity q in D covers at most two squares in $S(t, d)$, each of parity q, $2|D_q| \geqslant |S(t, d)|/2$. By Theorem 8, $|S(t, d)| = |E(\bar{t})| + o(\bar{t}, d)$; the conclusion then follows. $\square$

Lemma 10. *Let n be an even integer, $n > 4$, and let D be a bichromatic dominating set of size $n/2$ for Q_n. Then $|E(t)| = (n/2) + |L(t)|$.*

For each $t \in \{row, column\}$, $f(t, m(t, v_t)) = 1$ and $o(t, m(t, v_t)) = |E(t)| - 2$, and $f(t, m(t, \overline{v_t})) \in \{1, 2\}$ and $o(t, m(t, \overline{v_t})) \in \{|E(t)| - 2, |E(t)| - 4\}$.

Proof. From the definition $E(t) = \{1, \ldots, n\} \setminus (I(t) \setminus L(t))$ we have $|E(t)| = n - |I(t)| + |L(t)| = (n/2) + |L(t)|$.

By the definition of $m(t, v_t)$ there is at least one pair of indices of empty t-orthogonals at distance $m(t, v_t)$. If there were two such pairs, the lowest and highest of the indices involved would come from a pair of empty t-orthogonals at distance greater than $m(t, v_t)$, a contradiction.

Thus there is exactly one such pair, implying that the maximal arithmetic progressions of step $m(t, v_t)$ in $E(t)$ are $|E(t)| - 2$ singletons and one of length two, so $o(t, m(t, v_t)) = |E(t)| - 2$.

If there were three pairs of indices of empty t-orthogonals at distance $m(t, \overline{v_t})$, two pairs would have lower index of the same parity, and the lowest and highest indices involved in these pairs would come from a pair of empty t-orthogonals at distance of parity $\overline{v_t}$ and greater than $m(t, \overline{v_t})$, a contradiction.

If either there is one such pair, or there are two coming from a progression of length three, then $o(t, m(t, v_t)) = |E(t)| - 2$. Otherwise there are two pairs coming from two (maximal length) progressions of length two, so $o(t, m(t, v_t)) = |E(t)| - 4$. □

Proposition 11. *Let $k > 2$ be an integer and let D be a bichromatic dominating set of size k for Q_{2k}. One of the following two conditions holds:*

(A) $\{|D_0|, |D_1|\} = \{\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil\}$*;*

(B) *k is even and $\{|D_0|, |D_1|\} = \{\frac{k}{2} - 1, \frac{k}{2} + 1\}$. Then $m(t, 0) > m(t, 1)$ for each $t \in \{row, column\}$ and $W_{\text{orth}}(D) = 0$.*

Thus $||D_0| - |D_1|| \leqslant 2$, with strict inequality if $m(\text{col}, 1) > m(\text{col}, 0)$ or $m(\text{row}, 1) > m(\text{row}, 0)$.

Proof. Let $t \in \{\text{row}, \text{column}\}$. From Lemma 10 there is $h \in \{1, 2\}$ with $o(\bar{t}, m(\bar{t}, 1)) = |E(\bar{t})| - 2h$. Using this in Corollary 9 with $d = m(\bar{t}, 1)$, for $q \in \mathbb{Z}_2$ we have

$$|D_q| \geqslant \left\lceil \frac{|E(\bar{t})| + o(\bar{t}, m(\bar{t}, 1))}{4} \right\rceil = \left\lceil \frac{|E(\bar{t})| - h}{2} \right\rceil .$$

From Lemma 10, $|E(\bar{t})| = k + |L(\bar{t})|$ so for $q \in \mathbb{Z}_2$,

$$|D_q| \geqslant \left\lceil \frac{k + |L(\bar{t})| - h}{2} \right\rceil . \tag{6}$$

Let $H(\bar{t}) = |L(\bar{t})| - h$. As $|L(\bar{t})| \geqslant 0$ and $h \leqslant 2$, $H(\bar{t}) \geqslant -2$.

If $H(\bar{t}) \geqslant 1$, or $H(\bar{t}) = 0$ and k is odd, summing (6) for $q = 0,1$ gives $|D| = |D_0| + |D_1| > k$, a contradiction.

If $H(\bar{t}) \in \{-1, 0\}$ and k is even, (6) implies $\{|D_0|, |D_1|\} = \{\frac{k}{2}\}$.

If $H(\bar{t}) \in \{-2, -1\}$ and k is odd, (6) implies $\{|D_0|, |D_1|\} = \{\frac{k-1}{2}, \frac{k+1}{2}\}$.

Otherwise $H(\bar{t}) = -2$ and k is even. Here (6) implies $\{|D_0|, |D_1|\}$ is either $\{\frac{k}{2}\}$ or $\{\frac{k}{2} - 1, \frac{k}{2} + 1\}$. The first possibility and the preceding ones are included in (A), so assume the second occurs. By the definition of H, $|L(\bar{t})| = 0$ and $h = 2$. The latter implies by Lemma 10 that $m(\bar{t}, 1) = m(\bar{t}, \overline{v_{\bar{t}}})$. That is, $v_{\bar{t}} = 0$, which means $m(\bar{t}, 0) > m(\bar{t}, 1)$.

We may then reverse the roles of t and $\bar{t}$ to conclude that $|L(t)| = 0$, so $W_{\mathrm{orth}}(D) = 0$, and $m(t, 0) > m(t, 1)$. □

3 Monochromatic dominating sets of size k for Q_{2k}

Definition 12. For each positive integer n, let $\gamma_m(Q_n)$ be the minimum size of a monochromatic dominating set of Q_n. For $n \notin \{4, 5, 6\}$, let $i_m(Q_n)$ be the minimum size of an independent monochromatic dominating set of Q_n (such sets do not exist for $n \in \{4, 5, 6\}$).

Proposition 13. *Let n be a positive integer and D a dominating set of Q_n of size $\lfloor n/2 \rfloor$. Then D is monochromatic if and only if it is possible to place the board so that the squares of D include exactly one from every even-indexed orthogonal.*

Proof. The converse direction is clear, so assume that D is a monochromatic dominating set of Q_n of size $\lfloor n/2 \rfloor$. Choose a square in D and place the board so the center of that square is the origin. Then every square of D is even, so no odd square of the board is diagonally covered. If there is an empty orthogonal with even index, we may assume without loss of generality that it is a column. Then every odd square of that column is covered along its row, which has odd index. So each of the $\lfloor n/2 \rfloor$ or more odd-indexed rows is occupied, as well as row 0, but this cannot be done by $\lfloor n/2 \rfloor$ squares. Thus each of the even-indexed rows and columns is occupied by D, and since $|D| = \lfloor n/2 \rfloor$ there must be $\lfloor n/2 \rfloor$ of each, each occupied once. □

Definition 14. Let n be an even positive integer such that Q_n has a monochromatic dominating set D of size $k = n/2$. By Proposition 13, we may place the board in the Cartesian plane with upper right corner square (k, k) and with the orthogonals occupied by D exactly those with the same parity as k. We assume this *standard* placement of board and dominating set in what follows.

Each odd square lies in exactly one orthogonal whose index has the same parity p as $n/2$, thus is covered exactly once by D. Each square (x, y) with $x \equiv y \equiv p \pmod 2$ is covered twice orthogonally by D. It remains to see how D covers those squares (x, y) with $x \equiv y \equiv \overline{p} \pmod 2$. The set of these squares induces a copy U of $Q_{n/2}$ that is symmetric across row 0 and across column 0; this symmetry is useful later.

Proposition 15. *Let n be an even positive integer with $\gamma_m(Q_n) = n/2$. Then $\gamma(Q_n) = n/2$. If $n \notin \{4, 12\}$ then $\gamma(Q_{n-1}) = n/2$. If $n \notin \{2, 10\}$ then $\gamma(Q_{n+1}) = (n/2) + 1$.*

Proof. Suppose that $\gamma_m(Q_n) = n/2$. The first assertion is clear. If $n \notin \{4, 12\}$ then Theorem 2 implies $\gamma(Q_{n-1}) \geqslant n/2$. By Proposition 13 there exist an edge row and an edge column of Q_n that do not contain any squares of D. Thus D dominates a copy of Q_{n-1}, so $\gamma(Q_{n-1}) = n/2$. If $n \notin \{2, 10\}$ then $\gamma(Q_{n+1}) \geqslant (n/2) + 1$ by Theorem 2. Adding an edge row and edge column to Q_n and adding the new corner square to D then shows $\gamma(Q_{n+1}) = (n/2) + 1$. □

Definitions. Let n be an even positive integer and let D be a monochromatic dominating set of Q_n with size $n/2$ with standard orientation.

Among the integers congruent to 0 modulo 4, let d_0 be the least that is the absolute value of the index of an unoccupied difference diagonal and let s_0 be the least that is the absolute value of the index of an unoccupied sum diagonal.

Among the integers congruent to 2 modulo 4, let d_2 be the least that is the absolute value of the index of an unoccupied difference diagonal and let s_2 be the least that is the absolute value of the index of an unoccupied sum diagonal.

By the definitions of d_0 and d_2, D occupies at least the difference diagonals with indices $4-d_0, 8-d_0, \ldots, d_0-4$, and at least those with indices $4-d_2, 8-d_2, \ldots, d_2-4$. Say that the indices just listed are the *required* difference diagonal indices. Any further indices of difference diagonals occupied by D are *excess* difference diagonal indices; these may include repetitions of required difference diagonal indices. We similarly define required and excess indices of sum diagonals. Let e_d (respectively e_s) denote the number, with multiplicity, of indices of excess difference diagonals (respectively excess sum diagonals).

Lemma 16.

(A) *For all even $n > 2$, any minimum dominating set of Q_n with size $n/2$ that is monochromatic and has standard orientation occupies difference diagonal 0 and sum diagonal 0. Thus $d_0 \geqslant 4$ and $s_0 \geqslant 4$.*

(B) *For $h \in \{0, 2\}$, the number of required difference diagonal indices congruent to h modulo 4 is $\frac{d_h}{2} - 1$, and the number of required sum diagonal indices congruent to h modulo 4 is $\frac{s_h}{2} - 1$.*

(C) *We have $e_d = \frac{n}{2} - (\frac{d_0}{2} - 1) - (\frac{d_2}{2} - 1)$ and $e_s = \frac{n}{2} - (\frac{s_0}{2} - 1) - (\frac{s_2}{2} - 1)$.*

Proof. (A) Let $n > 2$ be an even integer and let D be a monochromatic dominating set of Q_n with size $k = n/2$ that has standard orientation.

As row $1-k$ and column $1-k$ are not occupied by D, the square $(1-k, 1-k)$ can only be covered by a square in difference diagonal 0, which thus contains a square of D.

Assume for purposes of contradiction that sum diagonal 0 does not contain a square of D. For $j = 1, \ldots, \lfloor (k+1)/2 \rfloor$, we examine how D covers the squares $\pm(2j-1-k, k+1-2j)$ of sum diagonal 0. These squares are in empty orthogonals, so must be covered along their difference diagonals.

With $j = 1$ we get the squares $\pm(1-k, k-1)$ which are in difference diagonals $\pm(2k-2)$. Each of these diagonals has only one square in orthogonals with the parity of k, so those squares, which are $(k, 2-k)$ and $(2-k, k)$, are in D. If $k > 2$ we go to $j = 2$, and see that D must occupy difference diagonals $\pm(2k-6)$. However, the only squares of those diagonals in orthogonals of the parity of k and not already occupied by D are $(k-2, 4-k)$ and $(4-k, k-2)$, which thus are in D.

Continuing, we find that for successive values of j, the difference diagonals $\pm(2k+2-4j)$ that must be occupied each have only one square that lies in orthogonals of the parity of k that are not already occupied by D, namely $(2j-k, k+2-2j)$ and $(k+2-2j, 2j-k)$. So these squares are in D.

This finally implies that the members of D are exactly the squares in sum diagonal 2 whose orthogonals have the parity of k. Then if k is even, all difference diagonals of squares of D have indices congruent to 2 modulo 4, so D does not contain a square of difference diagonal 0, contradicting a previous conclusion. If k is odd, all difference diagonals of squares of D have indices congruent to 0 modulo 4. The needy square $(-2, 0)$ (on the board since $k \geqslant 3$) is thus not covered by D, a contradiction. Therefore D contains a square of sum diagonal 0.

(B) This is easily verified.

(C) As there are $n/2$ squares in D, this follows from (B). $\square$

By Lemma 16 (C), $\frac{n}{2} - (\frac{d_0}{2} - 1) - (\frac{d_2}{2} - 1) \geqslant 0$, and similarly for sum diagonals; clearing denominators, we have

$$d_0 + d_2 \leqslant n + 4 \text{ and } s_0 + s_2 \leqslant n + 4. \tag{7}$$

For any even n, the definitions of d_0 and d_2 imply that the left sides of the inequalities (7) are congruent to 2 modulo 4. But for $n \equiv 0 \pmod 4$, the right sides of those inequalities are congruent to 0 modulo 4. Thus we have

$$d_0 + d_2 \leqslant n + 2 \text{ and } s_0 + s_2 \leqslant n + 2 \text{ if } n \equiv 0 \pmod 4. \tag{8}$$

We first consider the case $n \equiv 0 \pmod 4$. Then $k = \frac{n}{2} \equiv 0 \pmod 2$, so we need to see how the odd-odd squares are covered. By the definitions of d_0 and s_2, at least one of the difference diagonals with indices $\pm d_0$ and at least one of the sum diagonals with indices $\pm s_2$ are unoccupied by D. If difference diagonal d_0 and sum diagonal s_2 are unoccupied, then the odd-odd square $(\frac{s_2-d_0}{2}, \frac{s_2+d_0}{2})$ at which these diagonals meet must be off the board, implying $\frac{s_2+d_0}{2} \geqslant k$. However the two sides of this inequality have different parity, implying that $\frac{s_2+d_0}{2} \geqslant k+1$, so $d_0 + s_2 \geqslant n+2$. By the symmetry of U, the other possibilities of unoccupied diagonals imply the same result. An argument with d_2 and s_0 works similarly, so we have

$$d_0 + s_2 \geqslant n + 2 \text{ and } d_2 + s_0 \geqslant n + 2 \text{ for } n \equiv 0 \pmod 4. \tag{9}$$

Theorem 17. *Among the positive integers n, $n \equiv 0 \pmod 4$, only for $n = 4, 12$ does Q_n have a monochromatic dominating set of size $n/2$.*

Proof. From the first inequalities in (8), (9) we have $d_2 \leqslant s_2$. The other pairs of inequalities from (8), (9) then imply $d_0 = s_0$ and $d_2 = s_2$, and $d_0 + d_2 = n+2$ and $s_0 + s_2 = n+2$. From Lemma 16(C), $e_d = \frac{n}{2} - (\frac{d_0}{2} - 1) - (\frac{d_2}{2} - 1)$, which here reduces to 1. Similarly, this is the value of e_s.

By the linear constraint (1), the sum over D of all difference diagonal indices is zero. Also the sum of the required difference diagonal indices is zero, so the single excess difference diagonal index is zero, repeating a required difference diagonal index. The sum of all sum diagonal indices is $2k$ and the sum of the required sum diagonal indices is zero, so the single excess sum diagonal index is $2k$. The only board square on sum diagonal $2k$ is (k,k), so this square is in D. It also is in difference diagonal 0.

However, as D also contains another square in difference diagonal 0, the set $D' = D \setminus \{(k,k)\}$ dominates the copy of Q_{2k-1} obtained by removing the rightmost column and top row of Q_{2k}. By Theorem 2, this implies that $2k-1$ is either 3 or 11, so $n = 2k$ is either 4 or 12.

Conversely, $D_1 = \{(0,0)(2,2)\}$ is a monochromatic dominating set of size 2 for Q_4, and $D_2 = \{(-4,2), (-2,-4), (0,0), (2,4), (4,-2), (6,6)\}$ is a monochromatic dominating set of size 6 for Q_{12}. □

From this point we will assume $n \equiv 2 \pmod 4$, with $k = \frac{n}{2} \equiv 1 \pmod 2$. We ask how even-even squares are covered here. By the definitions of d_0 and s_0, at least one of the difference diagonals with indices $\pm d_0$ and at least one of the sum diagonals with indices $\pm s_0$ are unoccupied by D. If difference diagonal d_0 and sum diagonal s_0 are unoccupied, then the even-even square $(\frac{s_0 - d_0}{2}, \frac{d_0 + s_0}{2})$ at which these diagonals meet must be off the board, implying $\frac{d_0 + s_0}{2} \geqslant k$. The two sides of this inequality have different parity, so $\frac{d_0 + s_0}{2} \geqslant k+1$, implying $d_0 + s_0 \geqslant n+2$. By symmetry, the other possibilities of unoccupied diagonals imply the same result. An argument with d_2 and s_2 works similarly, so we have

$$d_0 + s_0 \geqslant n+2 \text{ and } d_2 + s_2 \geqslant n+2 \text{ for } n \equiv 2 \pmod 4. \tag{10}$$

Theorem 18. *Let $n > 2$ be an integer, $n \equiv 2 \pmod 4$, and let D be a monochromatic dominating set of Q_n of size $n/2$ with standard orientation. Then D has no excess difference diagonals; D has two excess sum diagonals, say with indices a_0 and a_2, where $a_i \equiv i \pmod 4$ and $a_0 + a_2 = n$. Here d_0, d_2, s_0, s_2 are related by the equations $s_0 + s_2 = n, d_0 = s_2 + 2, d_2 = s_0 + 2$. Let $k = n/2$, $d = |a_0 - a_2|/2$, and $e = |s_0 - s_2|/2$. Then*

$$d^2 + (k-1)e^2 = \frac{k(k^2+2)}{3}. \tag{11}$$

Proof. Since $n \equiv 2 \pmod 4$, Lemma 16(C) implies e_d and e_s are even. Using (10) with Lemma 16(C), we see that $e_d + e_s = \frac{n+4-d_0-d_2}{2} + \frac{n+4-s_0-s_2}{2} = n + 4 - \frac{d_0+s_0+d_2+s_2}{2} \leqslant 2$. As the sum of the required sum diagonal indices is zero, and the sum of all sum diagonal indices equals n by (1), there exist excess sum diagonal indices and their sum is n. So we have two nonnegative even integers e_d and e_s whose sum does not exceed 2, and e_s is not zero: we may conclude that $e_d = 0$ and $e_s = 2$, and the two excess sum diagonal indices a_0, a_2 have sum n. As all indices of occupied orthogonals are odd, all indices of occupied diagonals are even, so the indices a_0, a_2 of the excess sum diagonals are even. From $a_0 + a_2 = n \equiv 2 \pmod 4$, we see that just one of them is congruent to 0 modulo 4. We will assume henceforth that $a_0 \equiv 0 \pmod 4$ and then $a_2 \equiv 2 \pmod 4$. As n is the greatest possible index for a sum diagonal, necessarily $a_0, a_2 \leqslant n$, which implies $a_0, a_2 \geqslant 0$.

Then (10) and Lemma 16(C) imply the equations claimed involving d_0, d_2, s_0, s_2.

The sum of the squares of the indices of the occupied orthogonals is $n(n^2 + 8)/12$. Using the identity $8\binom{m/2}{3} = (4-m)^2 + (8-m)^2 + \cdots + (m-4)^2$, (2) becomes

$$\frac{n(n^2+8)}{6} = 8\left[\binom{s_2/2}{3} + \binom{s_0/2}{3} + \binom{d_2/2}{3} + \binom{d_0/2}{3}\right] + a_0^2 + a_2^2.$$

Then using the facts $\{a_0, a_2\} = \{k \pm d\}$ and $\{s_0, s_2\} = \{k \pm e\}$ leads to (11). □

Proposition 19. *The only positive even integers n for which $i_m(Q_n) = n/2$ are $n = 2, 10$.*

Proof. For $n \equiv 0 \pmod 4$, Theorem 17 says that only for $n = 4, 12$ does Q_n have a monochromatic dominating set of size $n/2$, and the proof of Theorem 17 shows that each such set has two squares on difference diagonal 0, so is dependent.

Suppose then that $n \equiv 2 \pmod 4$ and D is a monochromatic independent dominating set of Q_n with standard orientation. If $n = 2$ then $D = \{(1, 1)\}$, so we will assume $n \geqslant 6$. By independence the excess sum diagonal indices a_0, a_2 are distinct from the required sum diagonal indices, which implies $a_0 \geqslant s_0$ and $a_2 \geqslant s_2$. Since $a_0 + a_2 = n = s_0 + s_2$, we have $a_0 = s_0$ and $a_2 = s_2$.

Let $l = (n/2) + 1$. Make a copy of Q_{n+1} by adjoining a row and a column, each indexed l, to Q_n. We show that D dominates this Q_{n+1}.

Change the coordinates by subtracting one from each row and column index, thus moving the origin of the coordinate system to the square formerly labeled $(1, 1)$. It is then not difficult to verify that for each of the four types of lines, the sets of indices of occupied lines are symmetric across zero. This implies that the set of undominated squares is symmetric under rotation by a half-turn about the origin. But we know that the left and bottom edges are covered, so also the top and right-hand edges are.

Thus D is a dominating set of Q_{n+1} of size $\frac{(n+1)-1}{2}$, which by Theorem 2 implies $n+1 \in \{3, 11\}$, so $n \in \{2, 10\}$. As $\{(-3,-1), (-1, 5) (1, 1), (3, -3), (5, 3)\}$ is an independent monochromatic dominating set of Q_{10}, we are done. □

Since a_0, a_2 are even and not congruent modulo 4, $d = |a_0 - a_2|/2$ is odd. Similarly, $e = |s_0 - s_2|/2$ is odd. We may regard each triple (k, e, d) satisfying (11), with k, e, d each an odd integer and $1 \leqslant d, e \leqslant k$, as a *candidate* for solutions of $\gamma_m(Q_{2k}) = k$. From (11) and $1 \leqslant d \leqslant k$ it follows that

$$\left(\frac{k-1}{\sqrt{3}}\right)^2 - \frac{1}{3} \leqslant e^2 \leqslant \left(\frac{k+(1/2)}{\sqrt{3}}\right)^2 + \frac{11}{12}.$$

This implies that for any odd integer $k \geqslant 3$ there is at most one value of e giving a candidate (k, e, d), and if there is one, it is $\lceil \sqrt{(k^2-2k)/3}\, \rceil$.

k	1	3	5	9	15	19	27	65	71	117	119	215	363	435	469
e	1	1	3	5	9	11	15	37	41	67	69	153	209	251	271
d	1	3	3	7	1	11	27	63	41	115	1	153	363	309	131

Table 1: This shows the 15 candidates with least values of k; the first 13 come from Fermat-Pell equations.

The argument leading to (11) is not valid for $k = 1$, but we can take the solution $(k, e, d) = (1, 1, 1)$ of (11) as corresponding to the dominating set $\{(1, 1)\}$ of Q_2.

The fifteen candidates with least values of k are shown in Table 1.

Below we identify six infinite sequences of candidates. Each sequence is derived from solutions of an equation of Fermat-Pell type. The following lemma collects the facts we need about such equations; proofs may be found in [1].

Lemma 20. *Let C be a fixed integer and consider the Fermat-Pell equation*

$$X^2 - 3Y^2 = C. \tag{12}$$

(A) *Pair (x, y) is a solution of (12) if and only if $H(x, y) = (2x + 3y, x + 2y)$ is;*

(B) *There are only finitely many positive integral solutions of (12) (the* minimal *solutions) that cannot be obtained by (A) from smaller positive integral solutions.*

Thus the positive integral solutions of (12) are the members of a finite number of infinite sequences derived from the minimal solutions by (A).

We will only need to examine $C \in \{1, -2, -11\}$.

Case $d = e$. Putting $d = e$ in (11) gives $k^2 - 3e^2 = -2$. The unique minimal solution of $X^2 - 3Y^2 = -2$ is $(x, y) = (1, 1)$. Then the desired candidates are those obtained from $(1, 1)$ by iterating H from Lemma 20(A). This gives a sequence $(k, e, d) = (1, 1, 1), (5, 3, 3), (19, 11, 11), (71, 41, 41), \dots$. As $d = e$ implies the excess sum diagonals are distinct from the required sum diagonals, any dominating set of this kind is independent. Thus Proposition 19 implies that in this sequence only the first two terms give dominating sets.

For the next two cases, we need another version of (11). Since k and d are odd and $1 \leqslant d \leqslant k$, there is an integer j, $0 \leqslant j \leqslant (k-1)/2$, such that $d = k - 2j$. Replacing d with $k - 2j$ in (11) leads to the equation

$$(k-1)^2 - 3e^2 = 1 - 12j + 12j(j-1)/(k-1). \tag{13}$$

Case $d = k$. Putting $j = 0$ in (13) yields $(k-1)^2 - 3e^2 = 1$. The unique minimal solution of $X^2 - 3Y^2 = 1$ is $(2, 1)$. The positive integral solutions (x, y) of this equation can be obtained as in the case $d = e$. However, the values of x alternate in parity, and we are interested in odd k, so we only want every other solution: those with even x. Thus we obtain the desired solutions from $(2,1)$ by doubling the iteration of Lemma 20(A). That is, we begin with $(x,y)=(2,1)$ and iterate $H^2(x,y)=(7x+12y,4x+7y)$. This gives a sequence $(k,e,d)=(3,1,3),(27,15,27),(363,209,363),\ldots$

Case $\boldsymbol{d=k-2}$. Putting $j=1$ in (13) yields $(k-1)^2-3e^2=-11$. The equation $X^2-3Y^2=-11$ has two minimal positive solutions $(x,y)=(1,2)$ and $(4,3)$, and again iteration with H gives solutions whose x-values alternate in parity, so we obtain the desired solutions with even x by applying the iteration of the case $d=k$. Here we want $H^{2i+1}(1,2)$ and $H^{2i}(4,3)$ for $i\geqslant 0$.

The first gives a sequence $(k,e,d)=(9,5,7),(117,67,115),(1617,933,1615),\ldots$

The second gives a sequence $(k,e,d)=(5,3,3),(65,37,63),(893,515,891),\ldots$

Case $\boldsymbol{d=1}$. Putting $d=1$ in (11) and multiplying by 12 gives $(2k+1)^2-3(2e)^2=-11$. The equation $X^2-3Y^2=-11$ was just considered; here we need solutions (x,y) with $x=2k+1\equiv 3 \pmod 4$ and $y=2e\equiv 2 \pmod 4$. These are given by $H^{4i+2}(1,2)$ and $H^{4i+3}(4,3)$ for $i\geqslant 0$.

The first gives a sequence $(k,e,d)=(15,9,1),(3015,1741,1),\ldots$ of candidates, and the second gives a sequence $(k,e,d)=(119,69,1),(23183,13385,1),\ldots$

Following the approach of [12], we can use the identity

$$|x|+|y|=\max\{|y-x|,|y+x|\} \tag{14}$$

to restrict the search for those $n\equiv 2 \pmod 4$ for which $\gamma_m(Q_n)=n/2$. Let D be a monochromatic dominating set of size $n/2$ for Q_n and let S be the sum of $|x|+|y|$ over (x,y) in D. By our assumption of standard orientation,

$$\sum_{(x,y)\in D}(|x|+|y|)=2\left(|2-k|+|4-k|+\cdots+|k-2|+|k|\right)=k^2+1. \tag{15}$$

By (14) and (15),

$$\left[\sum_{(x,y)\in D}\max\{|y-x|,|y+x|\}\right]-(k^2+1)=0. \tag{16}$$

For each candidate (k,e,d) with $k\geqslant 3$, we will construct an upper bound $F(k)$ for the left side of (16). For some k we get $F(k)<0$, implying $\gamma_m(Q_{2k})>k$.

We begin by constructing a sum $F_1(k)$ involving the absolute values of the indices of the required diagonals. Note that since all occupied orthogonals have odd indices, for each occupied square one of its diagonals has index congruent to 0 modulo 4 and the other has index congruent to 2 modulo 4. So in what follows we seek to pair off absolute sum diagonal indices congruent to 0 mod 4 (respectively 2 mod 4) with absolute difference diagonal indices congruent to 2 mod 4 (respectively 0 mod 4) to achieve an upper bound for $\sum_{(x,y)\in D}\max\{|y-x|,|y+x|\}$.

We examine four cases. As they are similar, we only describe the first in detail.

Case $s_2 \equiv -2 \pmod 8$. Then there is a positive integer h with $s_2 = 8h-2$ and $d_0 = 8h$. The number of indices of occupied difference diagonals that are congruent to 0 modulo 4 is $4h-1$, and necessarily this is also the number of indices of occupied sum diagonals that are congruent to 2 modulo 4; this includes the extra sum diagonal index a_2. In this and the next case, we treat a_2 as the least absolute sum diagonal index for the moment.

The absolute values of the difference diagonal indices in descending order are: $2 \times (8h-4), 2 \times (8h-8), \ldots, 2 \times 4, 0$.

The absolute values of the sum diagonal indices in descending order are: $2 \times (8h-6), 2 \times (8h-10), \ldots, 4h-2, 4h-2, 2 \times (4h-6), \ldots, 2 \times 2$, and a_2.

To maximize the terms contributed to the bound $S_{\max}$ here, we pair off the top $2h$ absolute difference diagonal indices with the bottom $2h$ absolute sum diagonal indices (including a_2), and the top $2h-1$ absolute sum diagonal indices with the bottom $2h-1$ absolute difference diagonal indices. This gives

$$2\sum_{i=1}^{h}(8h-4i) + \sum_{i=1}^{h}(8h-2-4i) + \sum_{i=1}^{h-1}(8h-2-4i) = \frac{(s_2-2)(3s_2+2)}{8}.$$

The least value used comes from a sum diagonal with absolute index $(s_2/2)-1$.

Case $s_2 \equiv 2 \pmod 8$. There is a nonnegative integer h with $s_2 = 8h+2$ and $d_0 = 8h+4$. To maximize the terms contributed to the bound $S_{\max}$ here, we pair off the top $2h+1$ absolute difference diagonal indices with the bottom $2h+1$ absolute sum diagonal indices (including a_2), and the top $2h$ absolute sum diagonal indices with the bottom $2h$ absolute difference diagonal indices. This gives the sum $(s_2-2)(3s_2+2)/8$ as in the previous case. The least value used comes from a difference diagonal with absolute index $(s_2/2)-1$.

Case $s_0 \equiv 0 \pmod 8$. There is a positive integer h with $s_0 = 8h$ and $d_2 = 8h+2$. In this and the next case, we treat a_0 as the least absolute sum diagonal index temporarily. To maximize the terms contributed to the bound $S_{\max}$ here, we pair off the top $2h$ absolute difference diagonal indices with the bottom $2h$ absolute sum diagonal indices (including a_0), and the top $2h$ absolute sum diagonal indices with the bottom $2h$ absolute difference diagonal indices. This gives

$$2\sum_{i=1}^{h+1}(8h+2-4i) + 2\sum_{i=1}^{h}(8h-4i) = \frac{s_0(3s_0-4)}{8}.$$

The least value used comes from a sum diagonal with absolute index $s_0/2$.

Case $s_0 \equiv 4 \pmod 8$. There is a nonnegative integer h with $s_0 = 8h+4$ and $d_2 = 8h+6$. To maximize the terms contributed to the bound $S_{\max}$ here, we pair off the top $2h+2$ absolute difference diagonal indices with the bottom $2h+2$ absolute sum diagonal indices (including a_0), and the top $2h$ absolute sum diagonal indices with the bottom $2h$ absolute difference diagonal indices. This gives the sum $s_0(3s_0-4)/8$ as in the previous case. The least value used comes from a difference diagonal with absolute index $s_0/2$.

As the final values in the first two cases above are equal, as are those in the latter two, addition gives $F_1(k) = (12k^2 - 8k - 4 - 6s_0s_2)/8 = (6k^2 - 8k - 4 + 6e^2)/8$. Define $F_2(k) = F_1(k) - (k^2 + 1)$. Then using (13),

$$F_2(k) = \frac{-(k^2 - 3e^2 + 2)}{4} - (k + 1) = \frac{-3(k + 1)}{2} + 3\left[j - \frac{j(j - 1)}{k - 1}\right]. \tag{17}$$

Then with $d = k$ (so $j = 0$) and $d = 1$ (so $j = (k - 1)/2$), (17) gives $-3(k + 1)/2 \leqslant F_2(k) \leqslant -3(k + 1)/4$ for all k. Thus $F_2(k) < 0$ always.

In our construction of the sum $F_1(k)$ we ignored the extra sum diagonal indices a_0, a_2 by assuming they had smaller absolute values than all those used in the sum. Now we consider a_0, a_2.

If $a_0 > (s_2/2) - 1$ then we need to replace the lowest index used in the sum in the appropriate one of the first two cases above, which is $(s_2/2) - 1$, with a_0. (In the first case, we just replace the absolute sum diagonal index $(s_2/2) - 1$ with absolute sum diagonal index a_0. In the second case, we can replace the absolute sum diagonal index which was paired with, and less than, the absolute difference diagonal index $(s_2/2) - 1$, with absolute sum diagonal index a_0.) And if $a_2 > s_0/2$ then we need to replace the lowest index used in the sum in the appropriate one of the latter two cases above, which is $s_0/2$, with a_2. At least one of these changes is necessary, as the inequalities $a_0 \leqslant (s_2/2) - 1$ and $a_2 \leqslant s_0/2$ would imply $n = a_0 + a_2 \leqslant ((s_2/2) - 1) + (s_0/2) = (n/2) - 1$, which is not possible.

With the changes described, we can find the desired function F and know that those candidates (k, e, d) with $F(k) < 0$ have $\gamma_m(Q_{2k}) > k$.

We examine the candidates with extreme values of d: $d \in \{1, k - 2, k\}$.

Let $d = k$; the excess sum diagonals have indices $2k$ and 0. This implies that $F(k) = F_2(k) + 2k - ((s_2/2) - 1)$ here. Set $(k_0, e_0) = (3, 1)$. Using H^2, define a recursion by

$$(k_{i+1}, e_{i+1}) = (7k_i + 12e_i - 6, 4k_i + 7e_i - 4) \text{ for } k \geqslant 0. \tag{18}$$

This gives all candidates with $d = k$, and implies that modulo 4, each of the infinite sequences $(k_i + e_i)$ and $(k_i - e_i)$ satisfies the recursion $z_{i+1} = -z_i - 2$. Then starting with $(k_0, e_0) = (3, 1)$, we see $(s_2)_i = k_i + (-1)^{i+1}e_i$ for all i. Using (17) with $j = 0$, we have $F(k_i) = -3(k_i+1)/2+2k_i-((k_i+(-1)^{i+1}e_i)/2+1 = [(-1)^ie_i-1]/2$. Thus $\gamma_m(Q_{2k_i}) > k_i$ for odd i here, in particular for $k_1 = 27$.

Let $d = k-2$. We will need to consider two possibilities for the initial pair (k_0, e_0); from each, the recursion (18) allows us to find an infinite sequence of candidates, and together these include all candidates with $d = k - 2$. With $j = 1$ in (17), $F_2(k) = -3(k - 1)/2$. As $d + k = 2k - 2 \equiv 0 \pmod 4$, we have $a_0 = 2k - 2$. Then $F_2(k) = -3(k - 1)/2 + 2k - 2 - (s_0/2) = (k - 1 - s_0)/2$.

Starting with $(k_0, e_0) = (9, 5)$, we may use (18) to obtain $(s_0)_i = k_i + (-1)^{i+1}e_i$, and then $F(k_i) = [(-1)^ie_i - 1]/2$ for $i \geqslant 0$. So here $\gamma_m(Q_{2k_i}) > k_i$ for odd i.

Starting with $(k_0, e_0) = (5, 3)$, we similarly obtain $F(k_i) = [(-1)^{i+1}e_i - 1]/2$, but for $i \geqslant 1$. So here $\gamma_m(Q_{2k_i}) > k_i$ for even $i \geqslant 2$.

m	1	2	3	4	5	6	7	8	9
Candidates with $k < 10^m$	4	9	21	32	47	72	97	133	195
$F(k) < 0$, or $d = e$ but $k > 5$	0	3	8	11	15	24	29	39	53
Fermat-Pell candidates	3	8	14	18	24	30	35	39	46

Table 2: For each m, $1 \leqslant m \leqslant 9$, the number of candidates (k, e, d) with $k < 10^m$ is shown; next, the number of candidates already known to give $\gamma(Q_{2k}) > k$; finally, the number of candidates with $d \in \{1, e, k-2, k\}$.

Let $d = 1$. Again there are two possibilities for the initial pair (k_0, e_0); from each, the recursion $(k_{i+1}, e_{i+1}) = (97k_i + 168e_i + 48, 56k_i + 97e_i + 28)$, derived from H^4, allows us to find an infinite sequence of candidates, and together these include all candidates with $d = 1$. Here putting $j = (k_i - 1)/2$ in (17) gives $F_2(k_i) = -3(k_i + 1)/4$. As $\{a_0, a_2\} = \{k_i \pm 1\}$ and $\max\{s_0/2, (s_2/2) - 1\} \approx (3 + \sqrt{3})k/6 > k$, both of the changes we consider to obtain F from F_2 need to be made. Thus $F(k_i) = -3(k_i + 1)/4 + a_0 + a_2 - [(s_0/2) + (s_2/2) - 1] = -3(k_i + 1)/4 + (k_i + 1) = (k_i + 1)/4 > 0$, so this test gives no information here.

In Table 2 we show the number of candidates in certain ranges, along with the number for which we can already conclude $\gamma_m(Q_{2k}) = k$ is not possible (because either $F(k) < 0$, or $d = e$ but $k > 5$), and the number coming from Fermat-Pell equations.

4 Domination numbers and sets

For each of the sets mentioned below to show that a parameter of Q_n has value $(n/2) + 1$ or is in the set $\{n/2, (n/2) + 1\}$, there is $p \in \mathbb{Z}_2$ such that $n/2$ of the squares (x, y) in the set have $x \equiv y \equiv p \pmod 2$ and jointly occupy all orthogonals with indices of parity p, and one square has $x \equiv y \equiv \overline{p} \pmod 2$.

Proposition 21. *For $n = 2, 10$, $i_m(Q_n) = n/2$. For even n, $n \leqslant 120$, except $n = 2, 4, 6, 10$, $i_m(Q_n) = (n/2) + 1$. (As mentioned previously, $i_m(Q_n)$ is not defined for $n = 4, 5, 6$.)*

For $n = 4, 6, 8, 12, 14, 16, 18, 20, 22, 24$, $i(Q_n) = (n/2) + 1$. For even n, $26 \leqslant n \leqslant 120$, $i(Q_n) \in \{n/2, (n/2) + 1\}$.

For $n = 2, 4, 6, 10, 12, 18, 30, 130$, $\gamma(Q_n) = \gamma_m(Q_n) = n/2$.

For $n = 8, 14, 16, 20, 22, 24$, $\gamma(Q_n) = \gamma_m(Q_n) = (n/2) + 1$.

For even n, $26 \leqslant n \leqslant 122$ and $n = 126, 132$, $\gamma(Q_n), \gamma_m(Q_n) \in \{n/2, (n/2) + 1\}$.

Proof. About $i_m(Q_n)$: For $n = 2, 10$, independent monochromatic dominating sets of size $n/2$ are given in the proof of Proposition 19. By Theorem 2 these are minimum.

For $n = 8, 12, 14, 16, 18, 20, 24, 32$, independent monochromatic dominating sets of size $\frac{n}{2} + 1$ are given in [22]. For $n = 26, 28, 30$ and even n from 34 to 120, independent monochromatic dominating sets of size $(n/2) + 1$ are in [17]. These sets are minimum by Proposition 19.

About $i(Q_n)$: For $n = 4, 6$, independent dominating sets of size $(n/2) + 1$ are easily found by trial. For $n = 8, 12, 14, 16, 18, 20, 24$, see above.

For $n = 22$: for odd x from -9 to 11, y-values are $9, 1, -5, 11, -3, 3, 7, -7, -1, 5, -9$, with additional square $(6, 12)$. Due to W. Bird [4]. Exhaustive search (early work for $n = 8, 12$; [15] for $n = 14, 16, 18$; [3] for $n = 20, 22, 24$) has shown these sets are minimum. For n from 26 to 120 see above.

About $\gamma_m(Q_n)$ and $\gamma(Q_n)$: For $n = 2, 4, 6, 10, 12$ see above.

For $\gamma(Q_{18})$: for odd x from -7 to 9, y-values are $1, 7, -7, 3, -1, -5, 5, 9, -3$. Found by A. McRae [16].

For $\gamma(Q_{30})$: for odd x from -13 to 15, y-values are $1, -9, 9, 15, -11, -5, 3, -3, 13, 11, 7, -13, -7, -1, 5$.

For $\gamma(Q_{130})$: for odd x from -63 to 65, y-values are $29, 23, -15, 15, -23, -1, -11, -21, 1, 55, 25, 47, -55, 59, -63, -57, 33, 27, -59, -33, 37, -61, 57, 63, -51, -7, 11, 5, 19, 13, -9, -49, -25, -19, -13, 17, -5, 9, 3, 39, -47, -41, -31, 43, 61, -53, -43, -37, 49, 35, 41, 51, 21, 31, 53, -45, 45, 7, -3, -29, -39, -17, -27, 65, -35$. This was given in a different form in [17, page 17].

The final claim is established by dominating sets in [17]. □

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