An Improved Planar Graph Product Structure Theorem

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Abstract

Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [J. ACM 2020] proved that for every planar graph G there is a graph H with treewidth at most 8 and a path P such that $G \subseteq H \boxtimes P$. We improve this result by replacing "treewidth at most 8" by "simple treewidth at most 6".

Mathematics Subject Classifications: 05C10, 05C76

1 Introduction

This paper is motivated by the following question: what is the global structure of planar graphs? Recently, Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [13] gave an answer to this question that describes planar graphs in terms of products of simpler graphs, in particular, graphs of bounded treewidth. In this note, we improve this result in two respects. To describe the result from [13] and our improvement, we need the following definitions.

A *tree-decomposition* of a graph G is a collection $(B_x \subseteq V(G) : x \in V(T))$ of subsets of $V(G)$ (called *bags*) indexed by the nodes of a tree T, such that:

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(a) for every edge $uv \in E(G)$, some bag B_x contains both u and v, and

(b) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty (connected) subtree of T.

The *width* of a tree decomposition is the size of the largest bag minus 1. The *treewidth* of a graph G, denoted by $\text{tw}(G)$, is the minimum width of a tree decomposition of G. These definitions are due to Robertson and Seymour [23]. Treewidth is recognised as the most important measure of how similar a given graph is to a tree. Indeed, a connected graph with at least two vertices has treewidth 1 if and only if it is a tree. See [3, 16, 22] for surveys on treewidth.

A tree-decomposition $(B_x : x \in V(T))$ of a graph G is *k-simple*, for some $k \in \mathbb{N}$, if it has width at most k, and for every set S of k vertices in G, we have $|\{x \in V(T) : S \subseteq B_x\}| \leqslant 2$. The *simple treewidth* of a graph G, denoted by $\text{stw}(G)$, is the minimum $k \in \mathbb{N}$ such that G has a k-simple tree-decomposition. Simple treewidth appears in several places in the literature under various guises [17–19, 24]. The following facts are well-known: A graph has simple treewidth 1 if and only if it is a linear forest. A graph has simple treewidth at most 2 if and only if it is outerplanar. A graph has simple treewidth at most 3 if and only if it has treewidth at most 3 and is planar [18]. The edge-maximal graphs with simple treewidth 3 are ubiquitous objects, called *planar 3-trees* or *stacked triangulations* in structural graph theory and graph drawing [2, 18], called *stacked polytopes* in polytope theory [8], and called *Apollonian networks* in enumerative and random graph theory [15]. It is also known and easily proved that $\text{tw}(G) \leqslant \text{stw}(G) \leqslant \text{tw}(G) + 1$ for every graph G (see [17, 24]).

The *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y) \in V(A \boxtimes B)$ are adjacent if (1) $v = w$ and $xy \in E(B)$, or (2) $x = y$ and $vw \in E(A)$, or (3) $vw \in E(A)$ and $xy \in E(B)$.

Dujmović et al. [13] proved the following theorem describing the global structure of planar graphs.

Theorem 1 ([13])**.** *Every planar graph G is isomorphic to a subgraph of $H \boxtimes P$, for some planar graph H with treewidth at most 8 and some path P.*

Theorem 1 has been used to solve several open problems regarding queue layouts [13], non-repetitive colourings [11], centered colourings [9], clustered colourings [12], adjacency labellings [4, 10, 14], vertex rankings [6], and twin-width [5].

We modify the proof of Theorem 1 to establish the following.

Theorem 2. *Every planar graph G is isomorphic to a subgraph of $H \boxtimes P$, for some planar graph H with simple treewidth at most 6 and some path P.*

Theorem 2 improves upon Theorem 1 in two respects. First it is for simple treewidth (although it should be said that the proof of Theorem 1 gives the analogous result for simple treewidth 8). The main improvement is to replace 8 by 6, which does require new ideas. The proof of Theorem 2 builds heavily on the proof of Theorem 1, which in turn builds on a result of Pilipczuk and Siebertz [21], who showed that every planar graph has a partition into geodesic paths whose contraction gives a graph with treewidth at most 8.

Dujmović et al. [13] also proved a variant of Theorem 1 in which they lowered the bound on the treewidth of H to 3 at the expense of adding K_3 (the complete graph on three vertices) as a third factor to the product.

Theorem 3 ([13])**.** *Every planar graph G is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_3$, for some planar graph H with simple treewidth at most 3 and some path P.*

Theorems 1 and 2 can be thought of as having an extra K_1-factor, since $H \boxtimes K_1 \cong H$. We show another variant of this trade-off between the (simple) treewidth of H and the size of a third clique factor.

Theorem 4. *Every planar graph G is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_2$, for some planar graph H with simple treewidth at most 4 and some path P.*

Dujmović et al. [13] generalised Theorem 1 for graphs on surfaces. The *Euler genus* of the surface with h handles and c cross-caps is $2h + c$. The *Euler genus* of a graph G is the minimum Euler genus of a surface in which G embeds; see [20] for more about graph embeddings in surfaces. Dujmović et al. [13] showed that every graph with Euler genus g is isomorphic to a subgraph of $H \boxtimes P$, for some graph H with treewidth at most $2g + 8$ and some path P. Their proof in conjunction with Theorem 2 instead of Theorem 1 shows the following result. Here $A + B$ is the *complete join* of graphs A and B (obtained from the disjoint union of A and B by adding all edges between A and B).

Theorem 5. *Every graph with Euler genus g is isomorphic to a subgraph of $(H + K_{2g}) \boxtimes P$, for some planar graph H with simple treewidth at most 6 and some path P.*

2 Proof of Theorem 2

Our goal is to find a given planar graph G as a subgraph of $H \boxtimes P$ for some graph H of small treewidth and path P. Dujmović et al. [13] showed this can be done by partitioning the vertices of G into so-called vertical paths in a BFS spanning tree so that contracting each path into a single vertex gives the graph H (see Lemma 6 and Figure 1 below).

To formalise this idea, we need the following terminology and notation. A *partition* $\mathcal{P}$ of a graph G is a set of connected subgraphs of G, such that each vertex of G is in exactly one subgraph in $\mathcal{P}$. The *quotient* of $\mathcal{P}$, denoted $G/\mathcal{P}$, is the graph with vertex set $\mathcal{P}$, where distinct elements $A, B \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if there is an edge of G with endpoints in A and B. Note that $G/\mathcal{P}$ is a minor of G, so if G is planar then $G/\mathcal{P}$ is planar.

If T is a tree rooted at a vertex r, then a non-empty path $(x_0, \dots, x_p)$ in T is *vertical* if $\mathrm{dist}_T(x_i, r) = \mathrm{dist}_T(x_0, r) + i$ for all $i \in [0, p]$.

Lemma 6 ([13])**.** *Let T be a BFS spanning tree in a connected graph G. Let $\mathcal{P}$ be a partition of G into vertical paths in T. Then G is isomorphic to a subgraph of $(G/\mathcal{P}) \boxtimes P$, for some path P.*

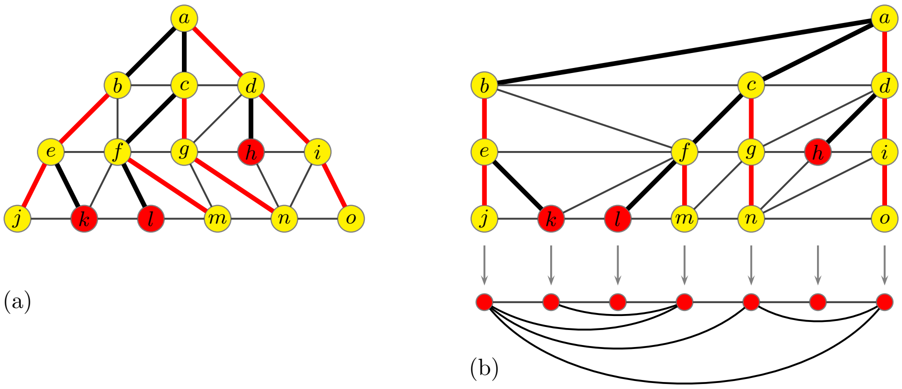

Figure 1: (a) A partition $\mathcal{P}$ of a planar graph G into red vertical paths in a BFS spanning tree. (b) Illustration of G as a subgraph of $(G/\mathcal{P}) \boxtimes P$.

The heart of this paper is Lemma 8 below, which is an improved version of the key lemma from [13]. The statement of Lemma 8 is identical to Lemma 13 from [13], except that we require F to be partitioned into at most 5 instead of 6 paths and that the tree-decomposition of H is 6-simple.

For a cycle C, we write $C = [P_1, \dots, P_k]$ if $P_1, \dots, P_k$ are pairwise disjoint non-empty paths in C, and the endpoints of each path P_i can be labelled x_i and y_i so that $y_i x_{i+1} \in E(C)$ for $i \in [k]$, where x_{k+1} means x_1. This implies that $V(C) = \bigcup_{i=1}^{k} V(P_i)$.

The proof of Lemma 8 employs the following well-known variation of Sperner's Lemma (see [1]). A *near-triangulation* is a 2-connected plane graph in which every internal face is a triangle.

Lemma 7 (Sperner's Lemma)**.** *Let G be a near-triangulation whose vertices are coloured $1, 2, 3$, with the outerface $F = [P_1, P_2, P_3]$ where each vertex in P_i is coloured i. Then G contains an internal face whose vertices are coloured $1, 2, 3$.*

Lemma 8. *Let G^+ be a plane triangulation, let T be a spanning tree of G^+ rooted at some vertex r on the outerface of G^+, and let $P_1, \dots, P_k$ for some $k \in [5]$, be pairwise disjoint vertical paths in T such that $F = [P_1, \dots, P_k]$ is a cycle in G^+. Let G be the near-triangulation consisting of all the edges and vertices of G^+ contained in F and the interior of F. Then G has a partition $\mathcal{P}$ into paths in G that are vertical in T, such that $P_1, \dots, P_k \in \mathcal{P}$ and the quotient $H := G/\mathcal{P}$ has a 6-simple tree-decomposition such that some bag contains all the vertices of H corresponding to $P_1, \dots, P_k$.*

Proof. The proof is by induction on $n = |V(G)|$. If $n = 3$, then G is a 3-cycle and $k \leqslant 3$. The partition into vertical paths is $\mathcal{P} = \{P_1, \dots, P_k\}$. The tree-decomposition of H consists of a single bag that contains the $k \leqslant 3$ vertices corresponding to $P_1, \dots, P_k$. Now assume that $n > 3$.

We now set up an application of Sperner's Lemma to the near-triangulation G. We begin by colouring the vertices in $k \leqslant 5$ colours. For $i \in \{1, \dots, k\}$, colour each vertex in P_i by i. Now, for each remaining vertex v in G, consider the path P_v from v to the root

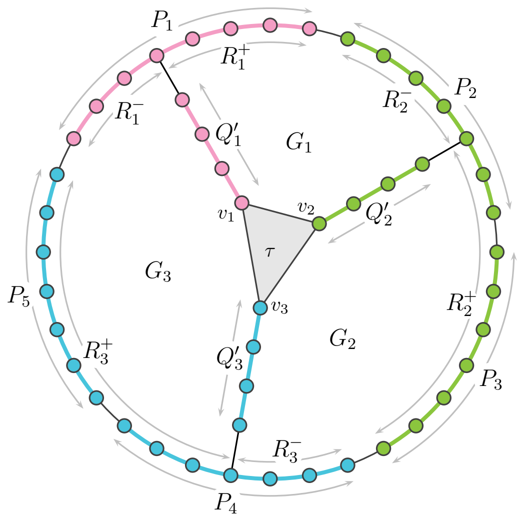

Figure 2: Example of the proof of Lemma 8 with $k = 5$.

of T. Since r is on the outerface of G^+, P_v contains at least one vertex of F. If the first vertex of P_v that belongs to F is in P_i, then assign the colour i to v. The set V_i of all vertices of colour i induces a connected subgraph of G for each $i \in \{1, \dots, k\}$. Consider the graph $M = G/\{V_1, \dots, V_k\}$ obtained by contracting each colour class V_i into a single vertex c_i. Since G is planar, M is planar. (In fact, M is outerplanar, although we will not use this property.) Moreover, if $k \geqslant 3$ then $[c_1, \dots, c_k]$ is a (not necessarily induced) cycle in M. Since $M \not\cong K_5$, we may assume without loss of generality that either $k \leqslant 4$ or $k = 5$ and c_2c_5 is not an edge in M; that is, no vertex coloured 2 is adjacent to a vertex coloured 5.

Group consecutive paths from $P_1, \dots, P_k$ as follows:

- If $k = 1$ then, since F is a cycle, P_1 has at least three vertices, so $P_1 = [v, P_1', w]$ for two distinct vertices v and w. Let $R_1 := v$, $R_2 := P_1'$ and $R_3 := w$.
- If $k = 2$ then, without loss of generality, P_1 has at least two vertices, say $P_1 = [v, P_1']$. Let $R_1 := v$, $R_2 := P_1'$ and $R_3 := P_2$.
- If $k = 3$ then let $R_1 := P_1$, $R_2 := P_2$ and $R_3 := P_3$.
- If $k = 4$ then let $R_1 := P_1$, $R_2 := P_2$ and $R_3 := [P_3, P_4]$.
- If $k = 5$ then let $R_1 := P_1$, $R_2 := [P_2, P_3]$ and $R_3 := [P_4, P_5]$.

We now derive a 3-colouring from the k-colouring above. For $i \in \{1, 2, 3\}$, colour each vertex in R_i by i. Now, for each remaining vertex v in G, consider again the path P_v from v to the root of T and if the first vertex of P_v that belongs to F is in R_i, then assign the colour i to v. Hence, for $k = 3$ we obtain exactly the same 3-colouring as above, while for $k \in \{4, 5\}$ some pairs of colour classes from the k-colouring are merged into one colour class in the 3-colouring. In each case, we obtain a 3-colouring of $V(G)$ that satisfies the conditions of Lemma 7. Therefore there exists a triangular face $\tau = v_1v_2v_3$ of G whose vertices are coloured $1, 2, 3$ respectively; see Figure 2.

For each $i \in \{1,2,3\}$, let Q_i be the path in T from v_i to the first ancestor v_i' of v_i in T that is in F. Observe that Q_1, Q_2, and Q_3 are disjoint since Q_i consists only of vertices coloured i. Note that Q_i may consist of the single vertex $v_i = v_i'$. Let Q_i' be Q_i minus its final vertex v_i'. Imagine for a moment that the cycle F is oriented clockwise, which defines an orientation of R_1, R_2 and R_3. Let R_i^- be the subpath of R_i that contains v_i' and all vertices that precede it, and let R_i^+ be the subpath of R_i that contains v_i' and all vertices that succeed it.

Consider the subgraph of G that consists of the edges and vertices of F, the edges and vertices of τ, and the edges and vertices of $Q_1 \cup Q_2 \cup Q_3$. This graph has an outerface, an inner face τ, and up to three more inner faces F_1, F_2, F_3 where $F_i = [Q_i', R_i^+, R_{i+1}^-, Q_{i+1}']$, where we use the convention that $Q_4 = Q_1$ and $R_4 = R_1$. Note that F_i may be *degenerate* in the sense that $[Q_i', R_i^+, R_{i+1}^-, Q_{i+1}']$ may consist only of a single edge $v_i v_{i+1}$.

Consider any non-degenerate $F_i = [Q_i', R_i^+, R_{i+1}^-, Q_{i+1}']$. Note that these four paths are pairwise disjoint, and thus F_i is a cycle. If Q_i' and Q_{i+1}' are non-empty, then each is a vertical path in T. Furthermore, each of R_i^+ and R_{i+1}^- consists of at most two vertical paths in T. Thus, F_i is the concatenation of at most six vertical paths in T. Let k_i be the actual number of (non-empty) vertical paths whose concatenation gives F_i. Then $k_1 \leqslant 5$ and $k_3 \leqslant 5$ since R_1^- and R_1^+ consist of only one vertical path in T. Also, if $k \leqslant 4$ then R_2^+ consists of only one vertical path in T, implying $k_2 \leqslant 5$. If $k = 5$, then in our preliminary k-colouring no vertex coloured 2 is adjacent to a vertex coloured 5. Since $v_2 v_3$ is an edge, this means that either v_2' lies on P_3 or v_3' lies on P_4 or both. In any case, at least one of R_2^+ and R_3^- consists of only one vertical path in T, which again gives $k_2 \leqslant 5$.

So F_i is the concatenation of $k_i \leqslant 5$ vertical paths in T for each $i \in \{1,2,3\}$. Let G_i be the near-triangulation consisting of all the edges and vertices of G^+ contained in F_i and the interior of F_i. Observe that G_i contains v_i and v_{i+1} but not the third vertex of τ. Therefore G_i satisfies the conditions of the lemma and has fewer than n vertices. By induction, G_i has a partition $\mathcal{P}_i$ into vertical paths in T, such that $H_i := G_i/\mathcal{P}_i$ has a 6-simple tree-decomposition $(B_x^i : x \in V(J_i))$ in which some bag $B_{u_i}^i$ contains the vertices of H_i corresponding to the at most five vertical paths that form F_i. Do this for each non-degenerate F_i.

We now construct the desired partition $\mathcal{P}$ of G. Initialise $\mathcal{P} := \{P_1, \dots, P_k\}$. Then add each non-empty Q_i' to $\mathcal{P}$. Now for each non-degenerate F_i, classify each path in $\mathcal{P}_i$ as either *external* (that is, fully contained in F_i) or *internal* (with no vertex in F_i). Add all the internal paths of $\mathcal{P}_i$ to $\mathcal{P}$. By construction, $\mathcal{P}$ partitions $V(G)$ into vertical paths in T and $\mathcal{P}$ contains $P_1, \dots, P_k$.

Let $H := G/\mathcal{P}$. Next we construct a tree-decomposition of H. Let J be the tree obtained from the disjoint union of J_i, taken over the $i \in \{1,2,3\}$ such that F_i is non-degenerate, by adding one new node u adjacent to each u_i. (Recall that u_i is the node of J_i for which the bag $B_{u_i}^i$ contains the vertices of H_i corresponding to the paths that form F_i.) Let the bag B_u contain all the vertices of H corresponding to $P_1, \dots, P_k, Q_1', Q_2', Q_3'$. For each non-degenerate F_i, and for each node $x \in V(J_i)$, initialise $B_x := B_x^i$. Recall that vertices of H_i correspond to contracted paths in $\mathcal{P}_i$. Each internal path in $\mathcal{P}_i$ is in $\mathcal{P}$. Each external path P in $\mathcal{P}_i$ is a subpath of P_j for some $j \in [k]$ or is one of Q_1', Q_2', Q_3'.

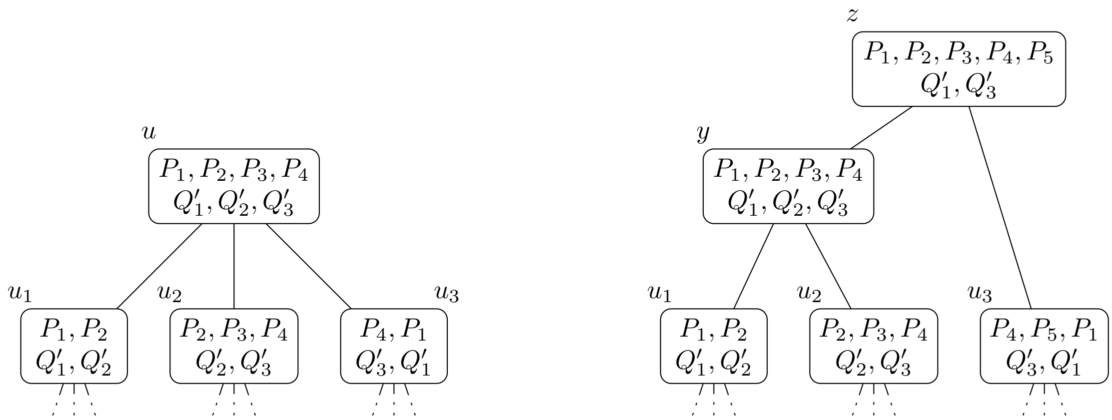

Figure 3: Illustration of 6-simple tree-decomposition for a possible scenario with $k = 4$ (left) and $k = 5$ (right).

For each such path P, for every $x \in V(J)$, in bag B_x, replace each instance of the vertex of H_i corresponding to P by the vertex of H corresponding to the path among $P_1, \ldots, P_k, Q'_1, Q'_2, Q'_3$ that contains P. This completes the description of $(B_x : x \in V(J))$. By construction, $|B_x| \leqslant k + 3 \leqslant 8$ for every $x \in V(J)$.

First we show that for each vertex a in H, the set $X := \{x \in V(J) : a \in B_x\}$ forms a subtree of J. If a corresponds to a path distinct from $P_1, \ldots, P_k, Q'_1, Q'_2, Q'_3$ then X is fully contained in J_i for some $i \in \{1, 2, 3\}$. Thus, by induction X is non-empty and connected in J_i, so it is in J. If a corresponds to P which is one of the paths among $P_1, \ldots, P_k, Q'_1, Q'_2, Q'_3$ then $u \in X$ and whenever X contains a vertex of J_i it is because some external path of $\mathcal{P}_i$ was replaced by P. In particular, we would have $u_i \in X$ in that case. Again by induction each $X \cap J_i$ is connected and since $uu_i \in E(T)$, we conclude that X induces a (connected) subtree of J.

Now we show that, for every edge ab of H, there is a bag B_x that contains a and b. If a and b are both obtained by contracting any of $P_1, \ldots, P_k, Q'_1, Q'_2, Q'_3$, then a and b both appear in B_u. If a and b are both in H_i for some $i \in \{1, 2, 3\}$, then some bag B^i_x contains both a and b. Finally, when a is obtained by contracting a path P_a in $G_i - V(F_i)$ and b is obtained by contracting a path P_b not in G_i, then the cycle F_i separates P_a from P_b so the edge ab is not present in H. This concludes the proof that $(B_x : x \in V(J))$ is a tree-decomposition of H. Note that B_u contains the vertices of H corresponding to $P_1, \ldots, P_k$.

By assumption the tree-decomposition $(B^i_x : x \in V(J_i))$ of H_i is 6-simple for $i \in \{1, 2, 3\}$. Since $|B_u \cap B_{u_i}| \leqslant 5$ for each $i \in \{1, 2, 3\}$, the tree-decomposition $(B_x : x \in V(J))$ of H is 6-simple, unless $|B_u| = 8$, which only occurs if $k = 5$ (since $|B_u| \leqslant k + 3$). Now assume that $k = 5$. Recall again that either v'_2 lies on P_3 or v'_3 lies on P_4 or both. Without loss of generality, v'_3 lies on P_4, and thus there is no edge between Q'_2 and P_5.

We now modify the above tree-decomposition of H in the $k = 5$ case. See Figure 3 for an illustration. First delete node u from J and the corresponding bag B_u. Add a new node y to J adjacent to u_1 and u_2, where B_y consists of the vertices of H corresponding to $P_1, \dots, P_4, Q_1', Q_2', Q_3'$. Thus $|B_y| = 7$. Add a node z to J adjacent to y and u_3, where B_z consists of the vertices of H corresponding to $P_1, \dots, P_5, Q_1', Q_3'$. Thus $|B_z| = 7$ and $(B_x : x \in V(J))$ is a tree-decomposition of H with width 6. Since P_5 has no vertex in $G_1 \cup G_2$, the vertex of H corresponding to P_5 is not in $B_{u_1} \cup B_{u_2}$, and thus the nodes of J whose bags contain this vertex form a connected subtree of J. Similarly, the vertex of H corresponding to Q_2' is not in B_{u_3} and thus the nodes of J whose bags contain this vertex form a connected subtree of J. The argument for the other vertices of H is identical to that above. This completes the proof that $(B_x : x \in V(J))$ is a tree-decomposition of H with width at most 6. It is 6-simple since the tree-decompositions of G_1, G_2 and G_3 are 6-simple, and $|B_y \cap B_{u_1}| \leqslant 5$ and $|B_y \cap B_{u_2}| \leqslant 5$ and $|B_z \cap B_{u_3}| \leqslant 5$. Moreover, B_z contains the vertices of H corresponding to $P_1, \dots, P_5$ as desired. □

The following corollary of Lemma 8 is a direct analogue of the corresponding result in [13, Theorem 12].

Corollary 9. *Let T be a rooted spanning tree in a connected planar graph G. Then G has a partition $\mathcal{P}$ into vertical paths in T such that* $\operatorname{stw}(G/\mathcal{P}) \leqslant 6$.

Proof. The result is trivial if $|V(G)| < 3$. Now assume $|V(G)| \geqslant 3$. Let r be the root of T. Let G^+ be a plane triangulation containing G as a spanning subgraph with r on the outerface of G^+. The three vertices on the outerface of G^+ are vertical (singleton) paths in T. Thus, G^+ satisfies the assumptions of Lemma 8 with $k = 3$ and F being the outerface, which implies that G^+ has a partition $\mathcal{P}$ into vertical paths in T such that $\operatorname{stw}(G^+/\mathcal{P}) \leqslant 6$. Note that $G/\mathcal{P}$ is a subgraph of $G^+/\mathcal{P}$. Hence $\operatorname{stw}(G/\mathcal{P}) \leqslant 6$. □

Corollary 9 and Lemma 6 imply Theorem 2 (since we may assume that G is connected).

3 Proof of Theorem 4

Here we show that every planar graph is a subgraph of $H \boxtimes P \boxtimes K_2$ for some planar graph H with simple treewidth at most 4 and some path P. The proof follows the same approach as before. We consider a connected planar graph G and a rooted spanning tree T of G. We then show that G has a partition $\mathcal{P}$, each part being the union of up to two vertical paths whose lower endpoints are adjacent in G. Call such a subgraph of G a *bipod*. The same idea except with up to three vertical paths with pairwise adjacent lower endpoints (so called *tripods*) is used by Dujmović et al. [13] to show Theorem 3. The following result is analogous to the key lemma from [13].

Lemma 10. *Let G^+ be a plane triangulation, let T be a spanning tree of G^+ rooted at some vertex r on the boundary of the outerface of G^+, and for some $k \in [4]$, let $P_1, \dots, P_k$ be pairwise disjoint bipods such that $F = [P_1, \dots, P_k]$ is a cycle in G^+. Let G be the near-triangulation consisting of all the edges and vertices of G^+ contained in F and the interior of F. Then G has a partition $\mathcal{P}$ into bipods such that $P_1, \dots, P_k \in \mathcal{P}$, and the quotient $H := G/\mathcal{P}$ has a* 4*-simple tree-decomposition such that exactly one bag contains all the vertices of H corresponding to $P_1, \dots, P_k$.*

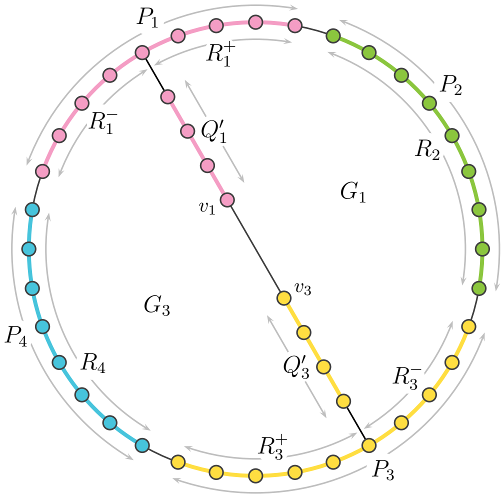

Figure 4: Example of the proof of Lemma 10 with $k = 4$.

Proof. The argument is in large part analogous to the proof of Lemma 8. We proceed by induction on the number n of vertices in G; the base case $n \leqslant 4$ being trivial.

As before, assign each vertex $v \in V(G)$ colour $i \in [k]$ if the first vertex in path P_v from v to the root of T that belongs to F lies in P_i (this first vertex might be v itself). The set V_i of all vertices coloured i induces a connected subgraph in G, and the graph $M = G/\{V_1, \dots, V_k\}$ obtained from G by contracting each V_i into a single vertex c_i is outerplanar and $|V(M)| = k$. Since K_4 is not outerplanar, $M \ncong K_4$. Thus $k \leqslant 3$, or $k = 4$ and we may assume that c_2c_4 is not an edge in M. In the latter case observe that M is inner triangulated and hence c_1c_3 is an edge in M.

If $k = 1$ then $|P_1| \geqslant 3$ and we recolour one endpoint of P_1 in colour 2 and the other endpoint in colour 3. If $k = 2$ then, without loss of generality, $|P_1| \geqslant 2$ and we recolour one endpoint of P_1 in colour 3. In every case, we now have a vertex colouring of G with $\max(3, k)$ colours such that at least one internal edge e of G has its two endpoints of different colours. By potentially renaming colours in the $k \leqslant 3$ case, and the assumption that c_1c_3 is an edge of M in the $k = 4$ case, we may assume that $e = v_1v_3$ with v_1 of colour 1 and v_3 of colour 3.

For $i \in \{1, 3\}$, let Q_i be the path in T from v_i to the first ancestor of v_i in T that belongs to F, and Q_i' be its (possibly empty) subpath minus the final vertex. The graph consisting of $e = v_1v_3$ and all vertices and edges of F, Q_1 and Q_3 has two inner faces F_1, F_3. Let R_i be the part of F in colour i. Let R_i^+, R_i^- be the sub-paths of R_i defined in Lemma 8. Then $F_1 = [Q_1', R_1^+, R_2, R_3^-, Q_3']$ and $F_3 = [Q_3', R_3^+, R_4, R_1^-, Q_1']$; see Figure 4 for an illustration. Note that $R_4 = \varnothing$ if $k \leqslant 3$. Since $Q_1' \cup Q_3'$ is a bipod, each of F_1 and F_3 is composed of at most four bipods. As before, for $i \in \{1, 3\}$, let G_i be the subgraph of G on all vertices and edges of F_i and the interior of F_i. By planarity, no vertex in the interior of F_1 is adjacent to any vertex in the interior of F_3. For $k = 4$ this relates to the fact that c_2c_4 is not an edge in M.

By induction, there is a partition $\mathcal{P}_i$ of G_i such that $H_i := G_i/\mathcal{P}_i$ has a 4-simple tree-decomposition $(B^i_x \colon x \in V(J_i))$ in which exactly one bag $B^i_{u_i}$ contains the vertices of H_i corresponding to the bipods that form F_i. We construct the partition $\mathcal{P}$ for G as before by initializing $\mathcal{P} := \{P_1, \dots, P_k\}$ and adding the bipod $Q'_1 \cup Q'_3$ to $\mathcal{P}$. For each F_i, add all internal bipods (those with no vertex in F_i) of $\mathcal{P}_i$ to $\mathcal{P}$. This concludes the definition of $\mathcal{P}$.

For a 4-simple tree-decomposition of $H := G/\mathcal{P}$, let J be the tree obtained from J_1 and J_3 by adding one new node u adjacent to u_1 and u_3. Let the bag B_u contain all vertices of H corresponding to the bipods that form F_1 and F_3. These are $P_1, \dots, P_k$ and $Q'_1 \cup Q'_3$; that is, exactly $k + 1 \leqslant 5$ bipods. For each external bipod P (those with some vertex in F_i) of $\mathcal{P}_i$, replace each instance of the vertex of H_i corresponding to P by the vertex of H corresponding to the bipod among $P_1, \dots, P_k, (Q'_1 \cup Q'_3)$ that contains P. By construction, $|B_x| \leqslant 5$ for each node $x \in V(J)$.

The proof that $(B_x \colon x \in V(J))$ is a tree-decomposition of H is analogous to the proof in Lemma 8. To see that it is 4-simple, i.e., any set of 4 nodes appears in at most two bags, first note that the tree-decompositions of G_1 and G_3 are 4-simple, and $B_u \cap B_{u_1}$ is a subset of the vertices of H corresponding to $Q'_1 \cup Q'_3, P_1, P_2, P_3$, while $B_u \cap B_{u_3}$ corresponds to a subset of the four bipods $Q'_1 \cup Q'_3, P_2, P_3, P_4$. As (by induction hypothesis) no other bag of G_1 contains the vertices corresponding to $Q'_1 \cup Q'_3, P_1, P_2, P_3$, and P_1 is disjoint from G_3, this 4-tuple of nodes appears only in the two bags B_u and B_{u_1}. Similarly, $Q'_1 \cup Q'_3, P_2, P_3, P_4$ appears only in B_u and B_{u_3}, as this 4-tuple appears only once in the tree-decomposition of G_3 and not at all in the tree-decomposition of G_1, since P_2 is disjoint from G_3. Hence $(B_x \colon x \in V(J))$ is a 4-simple tree-decomposition of H. Moreover, B_u contains the vertices of H corresponding to $P_1, \dots, P_k$ and is the unique bag with that property, as desired. $\square$

Finally, Theorem 4 follows as a corollary from Lemma 10 in the same way as Lemma 8 gives Theorem 2.

4 Discussion

We conclude with an open problem. Bose, Dujmović, Javarsineh, Morin, and Wood [7] defined the *row treewidth* of a graph G to be the minimum integer k such that G is isomorphic to a subgraph of $H \boxtimes P$ for some graph H with treewidth k and for some path P. Theorem 1 by Dujmović et al. [13] says that planar graphs have row treewidth at most 8. Our Theorem 2 improves this upper bound to 6. Dujmović et al. [13] proved a lower bound of 3. In fact, they showed that for every integer ℓ there is a planar graph G such that for every graph H and path P, if G is isomorphic to a subgraph of $H \boxtimes P \boxtimes K_\ell$, then H contains K_4 and thus has treewidth at least 3. Determining the maximum row treewidth of a planar graph is a tantalising open problem.

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