The Number of k-Dimensional Corner-Free Subsets of Grids

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Submitted: Mar 10, 2020; Accepted: May 25, 2022; Published: Jun 17, 2022

Abstract

A subset A of the k-dimensional grid $\{1, 2, \dots, N\}^k$ is said to be k-*dimensional corner-free* if it does not contain a set of points of the form $\{\mathbf{a}\} \cup \{\mathbf{a} + de_i : 1 \leqslant i \leqslant k\}$ for some $\mathbf{a} \in \{1, 2, \dots, N\}^k$ and $d > 0$, where $e_1, e_2, \dots, e_k$ is the standard basis of $\mathbb{R}^k$. We define the maximum size of a k-dimensional corner-free subset of $\{1, 2, \dots, N\}^k$ as $c_k(N)$. In this paper, we show that the number of k-dimensional corner-free subsets of the k-dimensional grid $\{1, 2, \dots, N\}^k$ is at most $2^{O(c_k(N))}$ for infinitely many values of N. Our main tools for proof are the hypergraph container method and the supersaturation result for k-dimensional corners in sets of size $\Theta(c_k(N))$.

Mathematics Subject Classifications: 05D05

1 Introduction

In 1975, Szemerédi [25] proved that for every real number $\delta > 0$ and every positive integer k, there exists a positive integer N such that every subset A of the set $\{1, 2, \dots, N\}$ with $|A| \geqslant \delta N$ contains an arithmetic progression of length k. There has been a plethora of research related to Szemerédi's theorem mixing methods in many areas of mathematics. Szemerédi's original proof is a tour de force of involved combinatorial arguments. There have been now alternative proofs of Szemerédi's theorem by Furstenberg [9] using methods from ergodic theory, and by Gowers [12] using high order Fourier analysis. The case $k = 3$ was proven earlier by Roth [20].

*Supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2017R1A6A3A04005963).

 https://doi.org/10.37236/9424

A subset A of the set $\{1, 2, \ldots, N\}$ is said to be *k-AP-free* if it does not contain an arithmetic progression of length k. We define the maximum size of a k-AP-free subset of $\{1, 2, \ldots, N\}$ as $r_k(N)$. In 1990, Cameron and Erdős [6] were interested in counting the number of subsets of the set $\{1, 2, \ldots, N\}$ which do not contain an arithmetic progression of length k and asked the following question.

Question 1 (Cameron and Erdős [6])**.** For every positive integer k and N, is it true that the number of k-AP free subsets of $\{1, 2, \ldots, N\}$ is $2^{(1+o(1))r_k(N)}$?

Until recently, research on how to improve the bounds $r_k(N)$ has been studied by many authors [4, 5, 8, 18, 11, 12]. Despite much effort, the difference between the currently known lower and upper bounds of $r_3(N)$ is still quite large. The upper bound has improved gradually over the years, and the current best upper bound is due to Bloom and Sisask [5]:

$$r_3(N) \leqslant \frac{N}{(\log N)^{1+c}},$$

where $c > 0$ is an absolute constant.
For a lower bound of $r_3(N)$, the configuration of Behrend [4] shows:

$$r_3(N) = \Omega\Big(\frac{N}{2^{2\sqrt{2}\sqrt{\log_2 N}} \cdot \log^{\frac{1}{4}} N}\Big).$$

This has been improved by Elkin's modification [8] by a factor of $\sqrt{\log n}$.
The currently known lower and upper bounds for $r_k(N)$ are as follows:
Let $m = \lceil \log_2 k \rceil$. For $k \geqslant 4$, there exist $c_k, c_k' > 0$ such that

$$c_k \cdot N \cdot (\log N)^{1/2m} \cdot 2^{-m2^{(m-1)/2(\log n)^{1/m}}} \leqslant r_k(N) \leqslant \frac{N}{(\log\log N)^{c_k'}},$$

where the lower bound is due to O'Bryant [18] and the upper bound is due to Gowers [11, 12].

In 2017, Balogh, Liu, and Sharifzadeh [2] provided a weaker version of Cameron and Erdős's conjecture [6] that the number of subsets of the set $\{1, 2, \ldots, N\}$ without an arithmetic progression of length k is at most $2^{O(r_k(N))}$ for infinitely many values of N, which is optimal up to a constant factor in the exponent.

A triple of points in the 2-dimensional grid $\{1, 2, \ldots, N\}^2$ is called a *corner* if it is of the form $(a_1, a_2), (a_1 + d, a_2), (a_1, a_2 + d)$ for some $a_1, a_2 \in \{1, 2, \ldots, N\}$ and $d > 0$. In 1974, Ajtai and Szemerédi [1] discovered that for every number $\delta > 0$, there exists a positive integer N such that every subset A of the 2-dimensional grid $\{1, 2, \ldots, N\}^2$ with $|A| \geqslant \delta N^2$ contains a corner. In 1991, Fürstenberg and Katznelson [10] found that their more general theorem implied the result of Ajtai and Szemerédi [1], but did not specify an explicit bound for N as it uses ergodic theory. An easy consequence of their result is the case $k = 3$ of Szemerédi's theorem, which was first proved by Roth [20] using Fourier analysis. Afterward, in 2003, Solymosi [24] provided a simple proof for Ajtai and Szemerédi [1] theorem using the Triangle Removal Lemma.

A subset A of the 2-dimensional grid $\{1, 2, \ldots, N\}^2$ is called *corner-free* if it does not contain a corner. We define the maximum size of corner-free sets in $\{1, 2, \ldots, N\}^2$ as $c_2(N)$. The problem of improving the bounds for $c_2(N)$ has been studied by many authors [14, 16, 22, 23]. The current best lower bound of $c_2(N)$ is due to Green [14], based on Linial and Shraibman's construction [16]:

$$\frac{N^2}{2^{(l_1+o(1))\sqrt{\log_2 N}}} \leqslant c_2(N),$$

where $l_1 \approx 1.822$.
The current best upper bound of $c_2(N)$ is due to Shkredov [22]:

$$c_2(N) \leqslant \frac{N^2}{(\log\log N)^{l_2}},$$

where $l_2 \approx 0.0137$.

The higher dimensional analog of a corner in the 2-dimensional grid $\{1, 2, \ldots, N\}^2$ is the following. A subset A of the k-dimensional grid $\{1, 2, \ldots, N\}^k$ is called *k-dimensional corner* if it is a set of points of the form $\{\mathbf{a}\} \cup \{\mathbf{a} + de_i : 1 \leqslant i \leqslant k\}$ for some $\mathbf{a} \in \{1, 2, \ldots, N\}^k$ and $d > 0$, where $e_1, e_2, \ldots, e_k$ is the standard basis of $\mathbb{R}^k$. The following multidimensional version of Ajtai and Szemerédi theorem [1] was proved by Fürstenberg, Katznelson [10], and Gowers [13].

Theorem 2 ([10, 13])**.** *For every number $\delta > 0$ and every positive integer k, there exists a positive integer N such that every subset A of the k-dimensional grid $\{1, 2, \ldots, N\}^k$ with $|A| \geqslant \delta N^k$ contains a k-dimensional corner.*

In 1991, Fürstenberg and Katznelson [10] showed that their more general theorem implied Theorem 2, but did not specify an explicit bound as it uses ergodic theory. Later, in 2007, Gowers [13] provided the first proof with explicit bounds and the first proof of Theorem 2 not based on Fürstenberg's ergodic-theoretic approach. They also proved that Theorem 2 implied the multidimensional Szemerédi theorem.

Another fundamental result in additive combinatorics is the multidimensional Szemerédi theorem, which was demonstrated for the first time by Fürstenberg and Katznelson [9] using the ergodic method, but provided no explicit bounds. In 2007, Gowers [13] yielded a combinatorial proof of the multidimensional Szemerédi theorem by establishing the Regularity and Counting Lemmas for the r-uniform hypergraph. This is the first proof to provide an explicit bound. Similar results were obtained independently by Nagle, Rödl, and Schacht [17].

Theorem 3 (Multidimensional Szemerédi theorem [9, 13, 17])**.** *For every real number $\delta > 0$, every positive integer k, and every finite set $X \subset \mathbb{Z}^k$, there exists a positive integer N such that every subset A of the k-dimensional grid $\{1, 2, \ldots, N\}^k$ with $|A| \geqslant \delta N^k$ contains a subset of the form $\mathbf{a} + dX$ for some $\mathbf{a} \in \{1, 2, \ldots, N\}^k$ and $d > 0$.*

A subset A of the k-dimensional grid $\{1, 2, \ldots, N\}^k$ is called *k-dimensional corner-free* if it does not contain a k-dimensional corner. We define the maximum size of a

k-dimensional corner-free subset of $\{1,2,\ldots,N\}^k$ as $c_k(N)$. In this paper, we study a natural higher dimensional version of the question of Cameron and Erdős, i.e. counting k-dimensional corner-free sets in $\{1,2,\ldots,N\}^k$ as follows.

Question 4. For every positive integer k and N, is it true that the number of k-dimensional corner-free subsets of the k-dimensional grid $\{1,2,\ldots,N\}^k$is $2^{(1+o(1))c_k(N)}$?

In addressing this question, we show the following theorem. Similar to the results of Balogh, Liu, and Sharifzadeh [2], despite not knowing the value of the extremal function $c_k(N)$, we can derive a counting result that is optimal up to a constant factor in the exponent.

Theorem 5. *The number of k-dimensional corner-free subsets of the k-dimensional grid $\{1,2,\ldots,N\}^k$ is $2^{O(c_k(N))}$ for infinitely many values of N.*

Our paper is organized as follows. In Section 2, we provide the two main tools for proof: the hypergraph container theorem and supersaturation results for k-dimensional corners. In Section 3, we provide proof of the saturation result for k-dimensional corners in sets of size $\Theta(c_k(N))$, which is specified in Section 2. In Section 4, we provide proof of our main result, Theorem 5.

2 Preliminaries

2.1 Hypergraph Container Method

The hypergraph container method [3, 21] is a very powerful technique for bounding the number of discrete objects avoiding certain forbidden structures. A graph is H-free if it does not have subgraphs that are isomorphic to H. For example, we use the container method when we count the family of H-free graphs or the family of sets without k term arithmetic progression. The r-uniform hypergraph $\mathcal{H}$ is defined as the pair $(V(\mathcal{H}), E(\mathcal{H}))$ where $V(\mathcal{H})$ is the set of vertices and $E(\mathcal{H})$ is the set of hyperedges that are the r-subset of the vertices of $V(\mathcal{H})$. Let $\Gamma(\mathcal{H})$ be a collection of independent sets of hypergraph $\mathcal{H}$, where the independent set of hypergraph $\mathcal{H}$ is the set of vertices inducing no hyperedge in $E(\mathcal{H})$. For a given hypergraph $\mathcal{H}$, we define the maximum degree of a set of l vertices of $\mathcal{H}$ as

$$\Delta_l(\mathcal{H}) = \max\{\ d_{\mathcal{H}}(A) : A \subset V(\mathcal{H}),\ |A| = l\ \},$$

where $d_{\mathcal{H}}(A)$ is the number of hyperedges in $E(\mathcal{H})$ containing the set A.

Let $\mathcal{H}$ be an r-uniform hypergraph of order n and average degree d. For any $0<\tau<1$, the *co-degree* $\Delta(\mathcal{H},\tau)$ is defined as

$$\Delta(\mathcal{H},\tau) = 2^{\binom{r}{2}-1} \sum_{j=2}^{r} 2^{\binom{-j-1}{2}} \frac{\Delta_j(\mathcal{H})}{\tau^{j-1} d}.$$

In this paper, we use the following hypergraph container lemma, which contains accurate estimates for the r-uniform hypergraph in Corollary 3.6 in [21].

Theorem 6 (Hypergraph Container Lemma [21]). *For every positive integer $r \in \mathbb{N}$, let $\mathcal{H} \subseteq \binom{V}{r}$ be an r-uniform hypergraph. Suppose that there exist $0 < \epsilon, \tau < 1/2$ such that*

- $\tau < 1/(200 \cdot r \cdot r!^2)$
- $\Delta(\mathcal{H}, \tau) \leqslant \frac{\epsilon}{12r!}$.

Then there exist $c = c(r) \leqslant 1000 \cdot r \cdot r!^3$ and a collection $\mathcal{C}$ of subsets of $V(\mathcal{H})$ such that the following holds:

- *for every independent set $I \in \Gamma(\mathcal{H})$, there exists $S \in \mathcal{C}$ such that $I \subset S$,*
- $\log |\mathcal{C}| \leqslant c \cdot |V| \cdot \ \tau \cdot \log(1/\epsilon) \cdot \log(1/\tau)$,
- *for every $S \in \mathcal{C}$, $e(\mathcal{H}[S]) \leqslant \epsilon \cdot e(\mathcal{H})$,*

where $\mathcal{H}[S]$ is a subhypergraph of $\mathcal{H}$ induced by S.

Let us consider a $(k+1)$-uniform hypergraph $\mathcal{G}$ encoding the set of all k-dimensional corners in the k-dimensional grid $[n]^k$. It means that $V(\mathcal{G}) = [n]^k$ and the edge set of $\mathcal{G}$ consists of all $(k+1)$-tuples forming k-dimensional corners. Note that the independent set in $\mathcal{G}$ is the k-dimensional corner-free set in $[n]^k$. Applying the Hypergraph Container Lemma to the hypergraph $\mathcal{G}$ gives the following theorem, which is an important result to prove our main result, Theorem 5.

Theorem 7. *For every positive integer $k \in \mathbb{N}$, let $\mathcal{G}$ be a $(k+1)$-uniform hypergraph encoding the set of all k-dimensional corners in $[n]^k$. Suppose that there exists $0 < \epsilon, \tau < 1/2$ satisfying that*

- $\tau < 1/(200 \cdot (k+1) \cdot (k+1)!^2)$
- $\Delta(\mathcal{G}, \tau) \leqslant \frac{\epsilon}{12(k+1)!}$.

Then there exist $c = c(k+1) \leqslant 1000 \cdot (k+1) \cdot (k+1)!^3$ and a collection $\mathcal{C}$ of subsets of $V(\mathcal{G})$ such that the following holds.

(i) every k-dimensional corner-free subset of $[n]^k$ is contained in some $S \in \mathcal{C}$,

(ii) $\log |\mathcal{C}| \leqslant c \cdot |V(\mathcal{G})| \cdot \ \tau \cdot \log(1/\epsilon) \cdot \log(1/\tau)$,

(iii) for every $S \in \mathcal{C}$, the number of k-dimensional corners in S is at most $\epsilon \cdot e(\mathcal{G})$.

2.2 Supersaturation Results

In this section, we present the supersaturation result for k-dimensional corners, which is the second main ingredient for proof of our main result. A supersaturation result says that sufficiently dense subsets of a given set contain many copies of certain structures. For the arithmetic progression, the supersaturation result concerned only sets of size linear in n was first demonstrated by Varnavides [26] by showing that any subset of $[n]$ of size $\Omega(n)$ has $\Omega(n^2)$ k-APs. In 2008, Green and Tao [15] obtained the supersaturation result by proving that any subset of $\mathbf{P}_{\leqslant n}$ of size $\Omega\left(|\mathbf{P}_{\leqslant n}|\right)$ has $\Theta(n^2/\log^k n)$ k-APs, where $\mathbf{P}_{\leqslant n}$ is the set of prime numbers up to n. Later, Croot and Sisask [7] provided a quantitative version of Varnavides [26] by proving that for every $1 \leqslant M \leqslant n$, the number of 3-AP in A is at least

$$\left(\frac{|A|}{n} - \frac{r_3(M)+1}{M}\right) \cdot \frac{n^2}{M^4}.$$

To prove Theorem 5, we need the supersaturation result of the minimum value of the number of k-dimensional corners for any set A in the k-dimensional grid $[n]^k$ of size $\Theta(c_k(N))$. To explain the supersaturation results, we introduce the following definitions. Recall that we define the maximum size of a k-dimensional corner-free subset of the k-dimensional grid $[n]^k$ as $c_k(n)$. Let $\Gamma_k(A)$ denote the number of k-dimensional corners in the set $A \subseteq [n]^k$. The following theorem shows that the number of k-dimensional corners in any set $A \subseteq [n]^k$ of size constant factor times larger than $c_k(n)$ is superlinear in n. In Section 3, we provide proof of Theorem 8.

Theorem 8. *For the given $k \geqslant 3$, there exist $C' := C'(k)$ and an infinite sequence $\{n_i\}_{i=1}^{\infty}$ such that the following holds. For all $n \in \{n_i\}_{i=1}^{\infty}$ and any set A in the k-dimensional grid $[n]^k$ of size $C' \cdot c_k(n)$, we have*

$$\Gamma_k(A) \geqslant \log^{(3k+1)} n \cdot \left(\frac{n^k}{c_k(n)}\right)^k \cdot n^{k-1} = \Upsilon(n) \cdot n^k,$$

where $\Upsilon(n) = \frac{\log^{3k+1} n}{n} \cdot \left(\frac{n^k}{c_k(n)}\right)^k$.

2.2.1 Supersaturation Lemmas

In this section, we present more supersaturation results for the minimum value of the number of k-dimensional corners to obtain a superlinear bound in Theorem 8. First, we provide the following simple supersaturation result using the greedy algorithm.

Lemma 9. *For the positive integer $k \geqslant 2$, let A be any set in the k-dimensional grid $[n]^k$ of size $K \cdot c_k(n)$, where $K \geqslant 2$ is a constant. Then we get*

$$\Gamma_k(A) \geqslant (K-1) \cdot c_k(n).$$

Proof. We use the greedy algorithm to determine the minimum value of the number of k-dimensional corners in a set A of size $K \cdot c_k(n)$, where $K \geqslant 2$. We consider the following process iteratively. As $|A| > c_k(n)$, there exists a k-dimensional corner C in the set A. It then updates the set A by removing an arbitrary element from C. By repeating this process $(K-1) \cdot c_k(n)$ times, we have

$$\Gamma_k(A) \geqslant (K-1) \cdot c_k(n). \qquad \square$$

Next, we use Lemma 9 to give the following improved supersaturation result.

Lemma 10. *For the positive integer $k \geqslant 2$, let A be any set in the k-dimensional grid $[n]^k$ of size at least $K \cdot c_k(n)$, where $K \geqslant 2$ is a constant. Then we obtain*

$$\Gamma_k(A) \geqslant \left(\frac{K}{2}\right)^{k+1} \cdot c_k(n).$$

Proof. Let A be any set of $[n]^k$ and have a size greater than equal to $K \cdot c_k(n)$. We consider the set S, which is one of all subsets of A of size $2 \cdot c_k(n)$. With Lemma 9, we have $\Gamma_k(S) \geqslant c_k(n)$ for every S. Therefore we get

$$\binom{|A|}{2 \cdot c_k(n)} \cdot c_k(n) \leqslant \sum_{S \subseteq A, |S| = 2 \cdot c_k(n)} \Gamma_k(S) \leqslant \Gamma_k(A) \cdot \binom{|A|-k-1}{2 \cdot c_k(n)-k-1}.$$

Then we conclude that

$$\begin{aligned} \Gamma_k(A) &\geqslant \frac{\binom{|A|}{2 \cdot c_k(n)}}{\binom{|A|-k-1}{2 \cdot c_k(n)-k-1}} \cdot c_k(n) \\ &\geqslant \left(\frac{|A|}{2 \cdot c_k(n)}\right)^{k+1} \cdot c_k(n) \\ &\geqslant \left(\frac{K}{2}\right)^{k+1} \cdot c_k(n). \qquad \square \end{aligned}$$

Note that the bounds of Lemma 9 and Lemma 10 are linear in the set A of $[n]^k$. In the following lemma, we provide a superlinear bound for the minimum value of the number of k-dimensional corners by applying Lemma 10 to the set of carefully chosen k-dimensional corners with prime common differences. The following lemma is an important result for proving the supersaturation result for k-dimensional corners in sets of size $\Theta(c_k(N))$ with superlinear bounds, which is specified in Theorem 8.

Lemma 11. *For the positive integer $k \geqslant 2$, let A be any set in the k-dimensional grid $[n]^k$ such that there exists a positive constant M satisfying $\frac{|A|}{2^{k+1} M n^{k-1}}$ is sufficiently large and $\frac{|A|}{n^k} \geqslant \frac{8K \cdot c_k(M)}{M^k}$, where $K \geqslant 2$ is a constant. Then we obtain*

$$\Gamma_k(A) \geqslant \frac{|A|^2}{2^{2k+4}} \cdot \frac{(K)^{k+1} \cdot c_k(M)}{M^{k+1} n^{k-1} \log^2 n}.$$

Proof. Given the set A of $[n]^k$, we let $x = \frac{|A|}{2^{k+1}Mn^{k-1}}$ which is sufficiently large. Let $\mathcal{G}_d$ be the set of $M \times \cdots \times M$ grids in $[n]^k$, whose consecutive layers are of distance d apart, for a prime $d \leqslant x$. Let us consider $\mathcal{G} = \bigcup_{d \leqslant x} \mathcal{G}_d$. For any k-dimensional corner $C = \{\mathbf{a}\} \cup \{\mathbf{a} + d'e_i : 1 \leqslant i \leqslant k\}$ for some $\mathbf{a} \in [n]^k$ and $d' > 0$, where $e_1, e_2, \ldots, e_k$ are the standard bases of $\mathbb{R}^k$, we consider a grid $G \in \mathcal{G}_d$ containing C. This means that d must be a prime divisor of d'. The number of prime divisors of d' is at most $\log d' \leqslant \log n$, so the number of these choices is at most $\log n$. Since every corner can occur in at most $(M-1)^k$ grids from each fixed $\mathcal{G}_d$ and the length of the corner has at most $\log n$ distinct prime factors, we get

$$\Gamma_k(A) \geqslant \frac{1}{M^k \cdot \log n} \sum_{G \in \mathcal{G}} \Gamma_k(A \cap G). \tag{1}$$

Let us consider $\mathcal{R} \subseteq \mathcal{G}$ consisting of all $G \in \mathcal{G}$ such that $|A \cap G| \geqslant K \cdot c_k(M)$, where $K \geqslant 2$ is a constant. Applying Lemma 10 to $A \cap G$ gives:

$$\Gamma_k(A \cap G) \geqslant \left(\frac{K}{2}\right)^{k+1} \cdot c_k(M). \tag{2}$$

for all $G \in \mathcal{R}$. Combining the inequalities (1) and (2), we obtain

$$\begin{aligned} \Gamma_k(A) &\geqslant \frac{1}{M^k \cdot \log n} \sum_{G \in \mathcal{G}} \Gamma_k(A \cap G) \\ &= \frac{1}{M^k \cdot \log n} \left(\sum_{G \in \mathcal{R}} \Gamma_k(A \cap G) + \sum_{G \in \mathcal{G}/\mathcal{R}} \Gamma_k(A \cap G) \right) \\ &\geqslant |\mathcal{R}| \cdot \left(\frac{K}{2}\right)^{k+1} \cdot \frac{c_k(M)}{M^k \cdot \log n}. \end{aligned} \tag{3}$$

Next, let us prove the lower bound for $|\mathcal{R}|$. For a prime number $d \leqslant x = \frac{|A|}{2^{k+1}Mn^{k-1}}$, we define $\zeta_d := [(M-1)d + 1, n - (M-1)d]^k$. Then we get the following inequality:

$$\begin{aligned} |A \cap \zeta_d| &\geqslant |A| - 2^k M d n^{k-1} \\ &\geqslant |A| - 2^k M n^{k-1} \frac{|A|}{2^{k+1} M n^{k-1}} = \frac{|A|}{2}. \end{aligned}$$

Note that the number of primes less than or equal to x is at least $\frac{x}{\log x}$ and at most $\frac{2x}{\log x}$ by the Prime Number Theorem. Since every $z \in \zeta_d$ appears exactly in the M^k members of $\mathcal{G}_d$, we derive that

$$\begin{aligned} \sum_{G \in \mathcal{G}} |A \cap G| &= \sum_{d \leqslant x} \sum_{G \in \mathcal{G}_d} |A \cap G| \\ &\geqslant M^k \sum_{d \leqslant x} |A \cap \zeta_d| \geqslant M^k \cdot \frac{x}{\log x} \cdot \frac{|A|}{2}. \end{aligned} \tag{4}$$

Obviously the inequality $|\mathcal{G}_d| \leqslant n^k$ is held for each prime number $d \leqslant x$. Then we get the following equation:

$$|\mathcal{G}| = |\bigcup_{d\leqslant x} \mathcal{G}_d| \leqslant \frac{2x}{\log x} \cdot n^k. \tag{5}$$

Since $\mathcal{R} \subseteq \mathcal{G}$ consists of all $G \in \mathcal{G}$ such that $|A \cap G| \geqslant K \cdot c_k(M)$, using the equation (5) we get

$$\begin{aligned}
\sum_{G\in\mathcal{G}} |A\cap G| &= \sum_{G\in\mathcal{R}} |A\cap G| + \sum_{G\in\mathcal{G}\backslash\mathcal{R}} |A\cap G| \\
&\leqslant M^k|\mathcal{R}| + K\cdot c_k(M)\cdot|\mathcal{G}\backslash\mathcal{R}| \\
&\leqslant M^k|\mathcal{R}| + K\cdot c_k(M)\cdot|\mathcal{G}| \\
&\overset{(5)}{\leqslant} M^k|\mathcal{R}| + K\cdot c_k(M)\cdot\frac{2x}{\log x}\cdot n^k.
\end{aligned} \tag{6}$$

Using the equations (4) and (6), we obtain

$$\begin{aligned}
|\mathcal{R}| &\overset{(6)}{\geqslant} \frac{1}{M^k}\cdot\left(\sum_{G\in\mathcal{G}}|A\cap G| - K\cdot c_k(M)\cdot\frac{2x}{\log x}\cdot n^k\right) \\
&\overset{(4)}{\geqslant} \frac{1}{M^k}\cdot\left(M^k\cdot\frac{x}{\log x}\cdot\frac{|A|}{2} - K\cdot c_k(M)\cdot\frac{2x}{\log x}\cdot n^k\right) \\
&= \frac{x}{\log x}\cdot\frac{|A|}{2} - \frac{K\cdot c_k(M)}{M^k}\cdot\frac{2x}{\log x}\cdot n^k \\
&= \frac{x}{\log x}\cdot\left(\frac{|A|}{2} - \frac{2K\cdot c_k(M)}{M^k}\cdot n^k\right).
\end{aligned}$$

From the condition $\frac{|A|}{n^k} \geqslant \frac{8K\cdot c_k(M)}{M^k}$, we have

$$\begin{aligned}
|\mathcal{R}| &\geqslant \frac{x}{\log x}\cdot\left(\frac{|A|}{2} - \frac{2K\cdot c_k(M)}{M^k}\cdot n^k\right) \\
&\geqslant \frac{x}{\log x}\cdot\left(\frac{|A|}{2} - \frac{1}{4}\frac{|A|}{n^k}\cdot n^k\right) \\
&\geqslant \frac{x}{\log x}\cdot\frac{|A|}{4} \geqslant \frac{|A|}{4}\cdot\frac{|A|}{2^{k+1}Mn^{k-1}}\cdot\frac{1}{\log n}.
\end{aligned} \tag{7}$$

Using the equations (3) and (7), we conclude that

$$\begin{aligned}
\Gamma_k(A) &\overset{(3)}{\geqslant} |\mathcal{R}|\cdot(\frac{K}{2})^{k+1}\cdot\frac{c_k(M)}{M^k\log n} \\
&\overset{(7)}{\geqslant} \frac{|A|^2}{4}\cdot\frac{1}{2^{k+1}Mn^{k-1}}\cdot\frac{1}{\log n}\cdot(\frac{K}{2})^{k+1}\cdot\frac{c_k(M)}{M^k\log n} \\
&= \frac{|A|^2}{2^{2k+4}}\cdot\frac{(K)^{k+1}\cdot c_k(M)}{M^{k+1}n^{k-1}\log^2 n}.
\end{aligned}$$

□

3 Proof of Theorem 8

The supersaturation result of k-dimensional corners in sets of size $\Theta(c_k(N))$, which is specified in Theorem 8, is the main tool for proof of Theorem 5. In this section, we prove Theorem 8 using Lemma 11 and the following relationship between $f(n_i)$ and $f(\Lambda(n_i))$ for some infinite sequence $\{n_i\}_{i=1}^{\infty}$.
For every $n \in \{n_i\}_{i=1}^{\infty}$, we define the following functions:

$$\Lambda(n) = \frac{n}{\log^{3k+3} n} \cdot \left(\frac{c_k(n)}{n^k}\right)^{k+3}, \qquad f(n) = \frac{c_k(n)}{n^k},$$

where $c_k(n)$ is the maximum size of a k-dimensional corner-free subset of $[n]^k$.

Lemma 12. *For the given $k \geqslant 3$, there exist $b := b(k) > 2^{2k}$ and an infinite sequence $\{n_i\}_{i=1}^{\infty}$ such that*

$$bf(n_i) \geqslant f(\Lambda(n_i))$$

for all $i \geqslant 1$.

First, we give the following relationship between $f(n)$ and $f(m)$ for any $m < n$, which is what we need to get Lemma 12.

Lemma 13. *For every $m < n$, we obtain $f(n) < 2^k \cdot f(m)$.*

Proof. For every $m < n$, we divide the k-dimensional grid $[n]^k$ into consecutive grids of size m^k because the corner-free property is invariant under translation. Since any given k-dimensional corner free subset of $[n]^k$ contains at most $c_k(m)$ elements in each grid of size m^k, for any $m < n$ we have

$$c_k(n) \leqslant \lceil \frac{n}{m} \rceil^k \cdot c_k(m).$$

Since $\frac{1}{n^k} \cdot \lceil \frac{n}{m} \rceil^k < \frac{2^k}{m^k}$ for every $m < n$, we conclude that

$$f(n) = \frac{c_k(n)}{n^k} \leqslant \lceil \frac{n}{m} \rceil^k \cdot \frac{c_k(m)}{n^k} < \frac{2^k}{m^k} \cdot c_k(m) = 2^k \cdot f(m).$$

This completes the proof of Lemma 13. □

To get Lemma 12, we also need a lower bound on $c_k(n)$, which follows from Rankin [19]'s result that is a generalization of Behrend [4]'s construction of dense 3-AP-free subset of integers to the case of arbitrary $k \geqslant 3$.

Lemma 14. *For the given $k \geqslant 2$, there exists α_k such that*

$$\frac{c_k(n)}{n^k} > 2^{-\alpha_k (\log n)^{\beta_k}}$$

for all sufficiently large n, where α_k is a positive absolute constant that depends only on k and $\beta_k = \dfrac{1}{\lceil \log k \rceil}$.

Proof. Let us first consider the case when $k = 2$. Let A be the 3-AP-free subset of $[n]$ with size $n \cdot 2^{-\alpha\sqrt{\log n}}$ from Behrend [4]'s construction. We construct a dense 2-dimensional corner-free subset B of $[n]^2$ of size $\Omega(|A|n)$ as follows: Let L be the collection of all lines of the form $y = x + a$ for every $a \in A$, and B be the intersection of L and $[n]^2$. It is easy to see that $|B| = \Omega(|A|n)$. It remains to prove that B is 2-dimensional corner-free. Let us assume otherwise, i.e. there exists a 2-dimensional corner in the set B, say $(x, y), (x + d, y), (x, y + d)$. Then, depending on the configuration, the three elements $y - x = a_1, y - (x + d) = a_2$, and $(y + d) - x = a_3$ are all in the set A forming 3-AP with $a_2 + a_3 = 2a_1$. This is a contradiction. Since the case of $k \geqslant 3$ is similar, the result of Rankin [19] is used instead, so details are omitted. □

Now we use Lemma 13 and Lemma 14 to prove Lemma 12.

Proof of Lemma 12. Fix $b := b(k) > 2^{2k}$ a large enough constant. Let us assume otherwise, i.e. there exists n_0 for all $n \geqslant n_0$ satisfying

$$f(n) < b^{-1} f(\Lambda(n)). \tag{8}$$

Using Lemma 14, there exists α_k such that $f(n) > 2^{-\alpha_k (\log n)^{\beta_k}}$ for every sufficiently large n, where $\beta_k = \dfrac{1}{\lceil \log k \rceil}$ and α_k is a positive absolute constant depending only on k. Using these α_k and β_k, for all $x \geqslant 1$, we define the decreasing function $g(x)$ as

$$g(x) = 2^{-(k\alpha_k + 3\alpha_k + 1)(\log x)^{\beta_k}}.$$

Then we get the following inequality for every $n \geqslant n_0$:

$$\begin{aligned} \Lambda(n) &= \frac{n}{\log^{3k+3} n} \cdot \left(\frac{c_k(n)}{n^k} \right)^{k+3} \\ &= \frac{n}{\log^{3k+3} n} \cdot (f(n))^{k+3} \\ &\overset{\textit{Lemma 14}}{>} \frac{n}{\log^{3k+3} n} \cdot \left(2^{-\alpha_k (\log n)^{\beta_k}} \right)^{k+3} \\ &> n \cdot 2^{-(k\alpha_k + 3\alpha_k + 1)(\log n)^{\beta_k}} = n \cdot g(n). \end{aligned} \tag{9}$$

From the equation (9), if we apply Lemma 13 to $\Lambda(n)$ and $n \cdot g(n)$ then we derive

$$f(n) \overset{(8)}{<} b^{-1} f(\Lambda(n)) \overset{\textit{Lemma 13}}{<} b^{-1} 2^k \cdot f(n \cdot g(n)) = \left(\frac{b}{2^k} \right)^{-1} \cdot f(n \cdot g(n)), \tag{10}$$

for all $n \geqslant n_0$.

To prove Lemma 12, we need the following claim.

Claim 15. *Let us write* $t = \lfloor \frac{1}{2} \frac{(\log n)^{\beta_k}}{k\alpha_k + 3\alpha_k + 1} \rfloor$ *with* α_k *satisfying* $f(n) > 2^{-\alpha_k (\log n)^{\beta_k}}$. *Then for all* $n > n_0^{1/(1-\beta_k)}$ *we obtain that*

$$f(n) < \left(\frac{b}{2^{2k}} \right)^{-j} f\left(n \cdot (g(n))^j \right)$$

for all $1 \leqslant j \leqslant t$.

Proof of Claim 15. We proceed by induction on j. The base case $j = 1$ is done by the equation (10). Assume that the statement of Claim 15 holds for every $1 \leqslant j < t$. Now we consider $n' = n \cdot (g(n))^j$ for all $1 \leqslant j < t$. Since $g(n)$ is a decreasing function, for each $j < t$ we have

$$\begin{aligned} n' = n \cdot (g(n))^j > n \cdot (g(n))^t &= n \cdot 2^{-(k\alpha_k+3\alpha_k+1)(\log n)^{\beta_k} \cdot \lfloor \frac{1}{2}\frac{(\log n)^{\beta_k}}{k\alpha_k+3\alpha_k+1}\rfloor} \\ &= n \cdot \left(\frac{1}{2}\right)^{(k\alpha_k+3\alpha_k+1)(\log n)^{\beta_k} \cdot \lfloor \frac{1}{2}\frac{(\log n)^{\beta_k}}{k\alpha_k+3\alpha_k+1}\rfloor} \\ &\geqslant n \cdot \left(\frac{1}{2}\right)^{(k\alpha_k+3\alpha_k+1)(\log n)^{\beta_k} \cdot \left(\frac{1}{2}\frac{(\log n)^{\beta_k}}{k\alpha_k+3\alpha_k+1}\right)} \\ &\geqslant n \cdot \left(\frac{1}{2}\right)^{\frac{1}{2}(\log n)^{2\beta_k}} = n \cdot 2^{-\frac{1}{2}(\log n)^{2\beta_k}} = n^{1-\beta_k} > n_0, \end{aligned} \tag{11}$$

for all $n > n_0^{1/(1-\beta_k)} \geqslant n_0$.
Note that $n' > n_0$ in the equation (11). Then we use the equation (10) to get

$$f(n') \overset{(10)}{<} \left(\frac{b}{2^k}\right)^{-1} \cdot f(n' \cdot g(n')). \tag{12}$$

Since $n' < n$ and $g(n)$ is a decreasing function, we have $n' \cdot g(n') > n' \cdot g(n)$. Applying Lemma 13 to $n' \cdot g(n')$ and $n' \cdot g(n)$ gives:

$$f(n' \cdot g(n')) < 2^k f(n' \cdot g(n)). \tag{13}$$

Using the equations (12) and (13), we obtain that

$$f(n') \overset{(12)}{<} \left(\frac{b}{2^k}\right)^{-1} \cdot f(n' \cdot g(n')) \overset{(13)}{<} \left(\frac{b}{2^k}\right)^{-1} \cdot 2^k f(n' \cdot g(n)) = \left(\frac{b}{2^{2k}}\right)^{-1} f(n' \cdot g(n)), \tag{14}$$

for all $n > n_0^{1/(1-\beta_k)}$. According to the inductive hypothesis, for every $1 \leqslant j < t$, we get

$$f(n) < \left(\frac{b}{2^{2k}}\right)^{-j} \cdot f(n \cdot (g(n))^j). \tag{15}$$

From the equations (14) and (15), for every $1 \leqslant j < t$, we observe that

$$\begin{aligned} f(n) &\overset{(15)}{<} \left(\frac{b}{2^{2k}}\right)^{-j} \cdot f(n \cdot (g(n))^j) \\ &= \left(\frac{b}{2^{2k}}\right)^{-j} \cdot f(n') \\ &\overset{(14)}{<} \left(\frac{b}{2^{2k}}\right)^{-j} \cdot \left(\frac{b}{2^{2k}}\right)^{-1} f(n' \cdot g(n)) \\ &= \left(\frac{b}{2^{2k}}\right)^{-j-1} \cdot f(n \cdot (g(n))^{j+1}), \end{aligned} \tag{16}$$

when $n > n_0^{1/(1-\beta_k)}$.
From the equation (16), we see that the statement of Claim 15 also holds for $j+1$. By the Induction axiom, the statement of Claim 15 holds for every $1 \leqslant j \leqslant t$. This completes the proof of Claim 15. □

Let $t = \lfloor \frac{1}{2} \frac{(\log n)^{\beta_k}}{k\alpha_k + 3\alpha_k + 1} \rfloor$ be an integer when α_k satisfies the inequality $f(n) > 2^{-\alpha_k (\log n)^{\beta_k}}$. Assume that $n > n_0^{1/(1-\beta_k)} \geqslant n_0$. Applying Claim 15, we get

$$f(n) < \left(\frac{b}{2^{2k}}\right)^{-t} f\left(n \cdot (g(n))^t\right). \tag{17}$$

Note that $n \cdot (g(n))^t \geqslant n^{1-\beta_k}$ from the equation (11). Applying Lemma 13 to $n \cdot (g(n))^t$ and $n^{1-\beta_k}$ gives:

$$f\left(n \cdot (g(n))^t\right) < 2^k \cdot f\left(n^{1-\beta_k}\right). \tag{18}$$

Using the equations (17) and (18), we draw the following conclusion.

$$\begin{aligned} f(n) &\overset{(17)}{<} \left(\frac{b}{2^{2k}}\right)^{-t} \cdot f\left(n \cdot (g(n))^t\right) \\ &\overset{(18)}{<} \left(\frac{b}{2^{2k}}\right)^{-t} \cdot 2^k \cdot f\left(n^{1-\beta_k}\right) \\ &\leqslant \left(\frac{b}{2^{2k}}\right)^{-t} \cdot 2^k \\ &= 2^k \cdot \left(\frac{b}{2^{2k}}\right)^{-\lfloor \frac{1}{2} \frac{(\log n)^{\beta_k}}{k\alpha_k + 3\alpha_k + 1} \rfloor} < 2^{-\alpha_k (\log n)^{\beta_k}}, \end{aligned} \tag{19}$$

where $b := b(k) > 2^{2k}$ is a sufficiently large constant. The equation (19) contradicts the definition of α_k. This completes the proof of Lemma 12. □

Now we use Lemma 11 and Lemma 13 to provide a proof of Theorem 8.

Proof of Theorem 8. Let $b(k)$ and an infinite sequence $\{n_i\}_{i=1}^{\infty}$ obtained from Lemma 12. For all $n \in \{n_i\}_{i=1}^{\infty}$, we let A be any set in the k-dimensional grid $[n]^k$ of size $8K \cdot b(k) \cdot c_k(n)$. Using Lemma 12, we get

$$\frac{|A|}{n^k} = \frac{8K \cdot b(k) \cdot c_k(n)}{n^k} \geqslant \frac{8K \cdot c_k(\Lambda(n))}{(\Lambda(n))^k}, \tag{20}$$

and

$$\frac{|A|}{2^{k+1} \cdot \Lambda(n) \cdot n^{k-1}} = \frac{8K \cdot b(k) \cdot c_k(n)}{2^{k+1} \cdot \Lambda(n) \cdot n^{k-1}} \geqslant 8K \cdot b(k) \cdot \left(\frac{\log^3 n}{2}\right)^{k+1}, \tag{21}$$

where $\Lambda(n) = \frac{n}{\log^{3k+3} n} \cdot \left(\frac{c_k(n)}{n^k}\right)^{k+3}$. Applying Lemma 13 to the the inequality $\Lambda(n) \leqslant n$ gives:

$$\frac{c_k(n)}{n^k} < 2^k \cdot \frac{c_k(\Lambda(n))}{(\Lambda(n))^k}. \tag{22}$$

From the inequality $\sqrt{n} \leqslant \Lambda(n)$, we get

$$\frac{n}{\Lambda(n)^2} \leqslant 1. \tag{23}$$

From the equations (20) and (21), we can apply Lemma 11 with $M = \Lambda(n)$ and derive that

$$\begin{aligned} \Gamma_k(A) &\geqslant \frac{|A|^2}{2^{2k+4}} \cdot \frac{(K)^{k+1} \cdot c_k(\Lambda(n))}{(\Lambda(n))^{k+1} \cdot n^{k-1} \cdot \log^2 n} \\ &= \frac{8^2 \cdot K^2 \cdot (b(k))^2 \cdot (c_k(n))^2}{(\Lambda(n)) \cdot \log^2 n} \cdot \frac{c_k(\Lambda(n))}{(\Lambda(n))^k} \cdot \frac{(K)^{k+1}}{n^{k-1} \cdot 2^{2k+4}}, \end{aligned} \tag{24}$$

where $|A| = 8K \cdot b(k) \cdot c_k(n)$. The following conclusion is drawn using the equations (21), (22), (23), and (24):

$$\begin{aligned} \Gamma_k(A) &\overset{(24)}{\geqslant} \frac{8^2 \cdot K^2 \cdot (c(k))^2 \cdot (c_k(n))^2}{(\Lambda(n)) \cdot \log^2 n} \cdot \frac{c_k(\Lambda(n))}{(\Lambda(n))^k} \cdot \frac{(K)^{k+1}}{n^{k-1} \cdot 2^{2k+4}} \\ &\overset{(22)}{\geqslant} \frac{8^2 \cdot K^2 \cdot (b(k))^2 \cdot (c_k(n))^2}{(\Lambda(n)) \cdot \log^2 n} \cdot \frac{c_k(n)}{2^k \cdot n^k} \cdot \frac{(K)^{k+1}}{n^{k-1} \cdot 2^{2k+4}} \\ &\overset{(23)}{\geqslant} \frac{\log^{3k+1} n \cdot (n^k)^{k+3} \cdot n \cdot 8^2 \cdot K^2 \cdot (b(k))^2 \cdot (c_k(n))^2}{n^2 \cdot (c_k(n))^{k+3} \cdot (\Lambda(n))^2 \cdot 2^{2k+2} \cdot n^{2k-2}} \cdot \frac{(K)^{k+1} \cdot c_k(n)}{2^{k+2}} \\ &\overset{(21)}{\geqslant} \log^{3k+1} n \cdot \left(\frac{n^k}{c_k(n)}\right)^{k+2} \cdot n^{k-1} \cdot 8^2 \cdot K^2 \cdot (b(k))^2 \cdot \left(\frac{\log^3 n}{2}\right)^{2k+2} \cdot \frac{(K)^{k+1}}{2^{k+2}} \\ &\geqslant \log^{3k+1} n \cdot \left(\frac{n^k}{c_k(n)}\right)^{k} \cdot n^{k-1} = \Upsilon(n) \cdot n^k. \end{aligned}$$

This completes the proof of Theorem 8. ∎

4 Proof of Theorem 5

In this section, we prove the main result Theorem 5 using the hypergraph container method(Theorem 7) and supersaturation result for k-dimensional corners in sets of size $\Theta(c_k(N))$(Theorem 8).

Proof of Theorem 5. Let $b(k)$ and the infinite sequence $\{n_i\}_{i=1}^{\infty}$ obtained from Lemma 12. For every $n \in \{n_i\}_{i=1}^{\infty}$, we define the following functions:

$$\Upsilon(n) = \frac{\log^{3k+1} n}{n} \cdot \left(\frac{n^k}{c_k(n)}\right)^k,$$
$$\Psi(n) = \frac{c_k(n)}{n^k} \cdot \frac{1}{\log^3 n},$$

where $c_k(n)$ is the maximum size of a k-dimensional corner-free subset of $[n]^k$. For sufficiently large n, we have

$$\Psi(n) < \frac{1}{200 \cdot (k+1)^{2(k+1)}} < \frac{1}{200 \cdot ((k+1)!)^2 \cdot (k+1)}, \tag{25}$$

and

$$\begin{aligned}\Upsilon(n) \cdot n \cdot \Psi(n)^k &= \frac{\log^{3k+1} n}{n} \cdot \left(\frac{n^k}{c_k(n)}\right)^k \cdot n \cdot \left(\frac{c_k(n)}{n^k} \cdot \frac{1}{\log^3 n}\right)^k \\ &= \log n \\ &> (k+1)^{3(k+1)}.\end{aligned} \tag{26}$$

Let us consider $(k+1)$-uniform hypergraph $\mathcal{G}$ encoding the set of all k-dimensional corners in $[n]^k$. For a given hypergraph $\mathcal{G}$, the maximum degree of a set of j vertices of $\mathcal{G}$ is $\Delta_j(\mathcal{G}) = \max\{\ d_{\mathcal{G}}(A) : A \subset V(\mathcal{G}),\ |A| = j\ \}$, where $d_{\mathcal{G}}(A)$ is the number of hyperedges in $E(\mathcal{G})$ containing the set A. Then the co-degree of a $(k+1)$-uniform hypergraph $\mathcal{G}$ of order n and average degree d is written as

$$\begin{aligned}\Delta(\mathcal{G}, \Psi) &= 2^{\binom{k+1}{2}-1} \sum_{j=2}^{k+1} 2^{-\binom{j-1}{2}} \Psi(n)^{-(j-1)} \cdot \frac{\Delta_j(\mathcal{G})}{d} \\ &= 2^{\binom{k+1}{2}-1} \sum_{j=2}^{k+1} \beta_j \cdot \frac{\Delta_j(\mathcal{G})}{d},\end{aligned} \tag{27}$$

where $\beta_j = 2^{-\binom{j-1}{2}} \Psi(n)^{-(j-1)}$ for all $2 \leqslant j \leqslant k+1$. Since $\Psi(n) < \frac{1}{200 \cdot (k+1)^{2(k+1)}} < 2^{-3(k+1)}$, we have

$$\frac{\beta_j}{\beta_{j+1}} = \frac{2^{\binom{j}{2}} \Psi(n)^j}{2^{\binom{j-1}{2}} \Psi(n)^{j-1}} = 2^{j-1}\Psi(n) < 2^{(k+1)} \cdot \Psi(n) < 1, \tag{28}$$

for all $2 \leqslant j \leqslant k-1$. For the case $j = k$, we obtain the following inequality:

$$(k-1)(k+1)^2 \cdot \frac{\beta_k}{\beta_{k+1}} = (k-1)(k+1)^2 \cdot 2^{k-1}\Psi(n) < 1. \tag{29}$$

Using the equations (26), (28) and (29), we derive that

$$\begin{aligned}
\Delta(\mathcal{G}, \Psi) &= 2^{\binom{k+1}{2}-1} \sum_{j=2}^{k+1} \beta_j \frac{\Delta_j(\mathcal{G})}{d} \\
&\leqslant 2^{\binom{k+1}{2}-1} \left(\sum_{j=2}^{k} \beta_j \frac{(k+1)^2}{d} + \frac{\beta_{k+1}}{d} \right) \\
&\overset{(28)}{\leqslant} 2^{\binom{k+1}{2}-1} \left((k-1) \cdot \beta_k \cdot \frac{(k+1)^2}{d} + \frac{\beta_{k+1}}{d} \right) \\
&\overset{(29)}{\leqslant} 2^{\binom{k+1}{2}-1} \left(\frac{2\beta_{k+1}}{d} \right) = \frac{2^k}{d \cdot (\Psi(n))^k} \\
&\leqslant \frac{(k+1)^{k+1}}{n \cdot (\Psi(n))^k} \overset{(26)}{<} \frac{\Upsilon(n)}{12 \cdot (k+1)!}.
\end{aligned} \tag{30}$$

From the equations (25) and (30), we can apply the Hypergraph Container Lemma (Theorem 7) on the hypergraph $\mathcal{G}$ with $\epsilon = \Upsilon(n), \tau = \Psi(n)$ as a function of n to get the collection $\mathcal{C}$ of containers such that all k-dimensional corner-free subsets of the k-dimensional grid $[n]^k$ are contained in some container in $\mathcal{C}$. Using Theorem 7, there exist $c = c(k+1) \leqslant 1000 \cdot (k+1) \cdot ((k+1)!)^3$ and a collection $\mathcal{C}$ of containers such that the followings hold:

- for every k-dimensional corner free subset of the k-dimensional grid $[n]^k$ is contained in some container in $\mathcal{C}$,
- $\log |\mathcal{C}| \leqslant c \cdot n \cdot \Psi(n) \cdot \log \frac{1}{\Upsilon(n)} \cdot \log \frac{1}{\Psi(n)}$,
- for every container $A \in \mathcal{C}$ the number of k-dimensional corners in A is at most $\Upsilon(n) \cdot n^k$.

The definitions of $\Upsilon(n)$ and $\Psi(n)$ give the following inequality:

$$\begin{aligned}
\log \frac{1}{\Upsilon(n)} \cdot \log \frac{1}{\Psi(n)} &= \log \left(\frac{n}{\log^{3k+1} n} \cdot \left(\frac{c_k(n)}{n^k} \right)^k \right) \cdot \log \left(\frac{n^k}{c_k(n)} \cdot \log^3 n \right) \\
&\leqslant \log n \cdot ((k+3) \log n) = (k+3) (\log n)^2 .
\end{aligned} \tag{31}$$

Using the equation (31) for the collection $\mathcal{C}$ of containers gives:

$$\begin{aligned}\log|\mathcal{C}| &\leqslant c\cdot n\cdot\Psi(n)\cdot\log\frac{1}{\Upsilon(n)}\cdot\log\frac{1}{\Psi(n)}\\ &\leqslant 1000\cdot(k+1)\cdot((k+1)!)^3\cdot n\cdot\Psi(n)\cdot\log\frac{1}{\Upsilon(n)}\cdot\log\frac{1}{\Psi(n)}\\ &\overset{(31)}{\leqslant} 1000\cdot(k+1)\cdot((k+1)!)^3\cdot n\cdot\frac{c_k(n)}{n^k}\cdot\frac{1}{\log^3 n}\cdot(k+3)\,(\log n)^2=o(c_k(n)).\end{aligned}\tag{32}$$

Note that for every container $A\in\mathcal{C}$, the number of k-dimensional corners in A is at most $\Upsilon(n)\cdot n^k$. Now applying Theorem 8 gives:

$$|A|<C'\cdot c_k(n),\tag{33}$$

for every container $A\in\mathcal{C}$. Since every k-dimensional corner free subset of the k-dimensional grid $[n]^k$ is contained in some container in $\mathcal{C}$, we conclude that the number of k-dimensional corner free subsets of $[n]^k$ is at most

$$\begin{aligned}\sum_{A\in\mathcal{C}}2^{|A|}&\leqslant|\mathcal{C}|\cdot\max_{A\in\mathcal{C}}2^{|A|}\\ &\overset{(32)\&(33)}{<}2^{o(c_k(n))}\cdot 2^{C'\cdot c_k(n)}=2^{O(c_k(n))},\end{aligned}$$

using the equations (32) and (33). This completes the proof of Theorem 5. □

Acknowledgements

I would like to thank Dong Yeap Kang and Hong Liu for their helpful discussions. I would particularly like to thank Hong Liu for providing many helpful comments.

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